Isospin constraints on angular momentum truncated wave functions

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In this report we examine two topics relating to previous work. We feel that there are points to be made which we have not made before. A common thread in the two problems is that they both involve the isospin variable in an important way.

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In a publication by Devi *et al.* [1] we calculated the excitation energies of $T=T_{\min}+1$ states in odd A nuclei and of $T=T_{\min}+2$ states of even-even nuclei in the f-p shell, where $T_{\min}=|N-Z|/2$. We performed a linear fit to these excitation energies

$$E(SA)=b(T+X),$$

$$E(DA)=2b\left(T+X+\frac{1}{2}\right). \tag{1}$$

For a simple interaction of the form $a+b\ t(1)t(2)$, the value of X is unity.

We point out that with a simple adjustment we can convert this to a formula for an isospin dependent term in the binding energy. This is due to the fact that the excitation energy expressions in Eq. (1) arise from differences of binding energy isospin dependence. We assume that the isospin-dependent term is of the form

$$E(T)=\frac{b}{2}T(T+Y). \tag{2}$$

Hence we obtain, for example, in the double analog case

$$E(DA)=\frac{b}{2}(T+2)(T+2+Y)-\frac{b}{2}T(T+Y). \tag{3}$$

We thus obtain the result $Y=2X-1$. We obtain the same result if we use the formula for the $E(SA)$. For the $t(1)t(2)$ interaction we have $X=1$, $Y=1$. For the Wigner [2,3] SU(4) limit, we have $X=2.5$, $Y=4$. It is worthwhile to note that in mean-field theories we cannot obtain a linear term in T, but in shell model calculations it is impossible to avoid such a term.

In Ref. [1] we performed a fit to the single j shell calculation. We found that a good fit was obtained with $b=2.32$ and $X=1.3$. This leads to an isospin dependent binding term in the single j shell

$$E=\frac{b}{2}T(T+1.6). \tag{4}$$

In Talmi's book [4], expressions for the binding energy in both the SU(4) limit and the seniority conserving limit are shown. In the former case the binding energy goes as $T(T+4)$ and in the latter as $T(T+1)$ [5]. It has been pointed out by McCullen *et al.* [6] that although seniority may be a pretty good quantum number for a system of identical nucleons, e.g., the calcium isotopes, seniority is badly broken when we have both protons and neutrons in open shells. This point has also been discussed in Lawson's book [7]. This formula $T(T+1.6)$ lies in between the two extremes—one of seniority conservation for mixed protons and neutrons and the other of the SU(4) limit in spin and isospin variables.

The next problem we consider takes note of the fact that the angular momentum $I=0$ ground-state wave functions of even-even Ti isotopes are dominated by terms in which the protons couple to angular momentum zero and two, likewise the neutrons. In Table I we show the MBZ wave functions for the $J=0^+T_{\min}$ ground states of ^{44,46,48}Ti as well as the unique (in the single j shell model) $T_{\min}+2$ states. The wave function for a Ti isotope is written as

TABLE I. Wave functions of $I=0_1$, $T_{\min}$ and $I=0$, $T_{\min}+2$ states of ^{44}Ti, ^{46}Ti, and ^{48}Ti. The asterisk means $v=4$ (The phases have been adjusted to fit with the CFP conventions of Ref. [4] and differ in some way with those in Ref. [6]).

^{44}Ti	J_P	J_N	$I=0$ $T=0$	$I=0$ $T=2$
	0	0	0.7608	0.5000
	2	2	0.6090	−0.3727
	4	4	0.2093	−0.5000
	6	6	0.0812	−0.6009
^{46}Ti	J_P	J_N	$I=0$ $T=1$	$I=0$ $T=3$
	0	0	0.8224	0.3162
	2	2	0.5420	−0.4082
	2	2*	0.0563	0.0
	4	4	0.0861	−0.5477
	4	4*	−0.1383	0.0
	6	6	−0.0127	−0.6583
^{48}Ti	J_P	J_N	$I=0$ $T=2$	$I=0$ $T=4$
	0	0	0.9136	0.1890
	2	2	0.4058	−0.4226
	4	4	0.0196	−0.5669
	6	6	−0.0146	−0.6814

$$\psi^{I\alpha} = \Sigma D^{I\alpha}(J_pJ_n)[(j^2)^{J_p}(j^n)^{J_n}]^I, \tag{5}$$

where I is the total angular momentum and $D^I(J_pJ_n)$ is the probability amplitude that the protons couple to J_p and the neutrons couple to J_n. For $I=0$, $J_p=J_n=J$. The label α is inserted to remind us that there are several states of the same angular momentum I. For example in ^{44}Ti there are three $I=0$ states with isospin $T=0$ and one with isospin $T=2$; in ^{46}Ti there are five $I=0$ states with isospin $T=1$ and one with isospin $T=3$; in ^{48}Ti there are three with isospin $T=2$ and one with isospin $T=4$. Note that in the single j shell the allowed isospins for the even-even Ti isotopes are $T_{\rm min}=|N-Z|/2$ and $T_{\rm max}=T_{\rm min}+2$. There are no $I=0$ states with isospin $T=T_{\rm min}+1$.

As seen in Table I the probability amplitude for $J_p=0$, $J_n=0$ in ^{44}Ti is 0.7608 and for $J_p=2$, $J_n=2$ it is 0.6090. Hence the probability of s and d couplings only is 95%. Similar results are obtained for ^{46}Ti and ^{48}Ti—indeed the percentages are even higher in these nuclei. Also in ^{46}Ti the percentage of $J_n=2$, seniority $v=2$ is much larger than that of $J_n=2$, $v=4$. This serves as a motivation for truncating the ground state wave functions to $J_p=0$ and 2 and $J_n=0$ and 2, $v=2$. We can say then that we have a model in which only s and d couplings of fermions are considered. In the single j shell such model can serve as a starting point for the justification of those IBA models which involve only s and d bosons. There are various versions of the interacting boson approximation IBA1 [8], IBA2 [9], and IBA3 [10]. The format of the Ti wave functions in MBZ [6] most closely resembles that of IBA2.

In the single j model space, the states with the higher isospin $T_{\rm max}=T_{\rm min}+2$ are not affected by any isospin conserving two nucleon interaction. In fact for these states the coefficients $D^I(J_pJ_n)$ are two particle coefficients of fractional parentage (CFP). The reason for this is that these states in Ti are double analogs of corresponding states in Ca, and for Ca we are dealing with a system of identical particles, i.e., only $f_{7/2}$ neutrons. A two particle cfp will be an expansion in which $(n+2)$ neutrons are separated into n and 2. We can then easily see the following for $I=0$:

$$D^{IT_{\rm max}}(JJ) = (j^nJ;j^2J|\}j^{n+2}0)\ . \tag{6}$$

From the fact that wave functions satisfy the orthonormality conditions we obtain

$$\Sigma_{J_nJ_p}D^{I\alpha}(J_pJ_n)D^{I\alpha'}(J_pJ_n) = \delta_{\alpha\alpha'} \tag{7}$$

so that in particular any $T_{\rm min}$ state is orthogonal to a state with $T=T_{\rm min}+2$.

For brevity we will drop the superscript α on the $D(JJ)$'s. We now truncate to only $J=0$ and $J=2$ couplings for the neutrons and protons for the ground-state wave function i.e., go to the s-d pair model. We now have

$$\psi \approx D^0(00)[(j^2)^0(j^n)^0]^0 + D^0(22)[(j^2)^2(j^n)^2]^0. \tag{8}$$

With the conditions that

$$D^0(00)^2 + D^0(22)^2 = 1. \tag{9}$$

We also impose the condition that the above wave function is orthonormal to the $T_{\rm max}$ state

$$D^0(00)(j^n0;j^20|\}j^{n+2}0)\ + D^0(22)(j^n2;j^22|\}j^{n+2}0)\ = 0. \tag{10}$$

But these two conditions mean that $D^0(00)$ and $D^0(22)$ are completely determined—there is no freedom. We can show that the wavefunctions, written as two component vectors for the various Ti isotopes are

$$\psi_{^{44}\rm Ti} = \frac{1}{\sqrt{14}}(\sqrt{5},3) = (0.5976, 0.8018),$$

$$\psi_{^{46}\rm Ti} = \frac{1}{\sqrt{8}}(\sqrt{5},\sqrt{3}) = (0.7906, 0.6124),$$

$$\psi_{^{48}\rm Ti} = \frac{1}{\sqrt{6}}(\sqrt{5},1) = (0.9129, 0.4082).$$

This comes from a more general expression [11,12] of Zamick, Mekjian, and Lee:

$$\sqrt{\frac{(2j+1-n)}{(n+1)(2j+1)}}D^0(00) - M\sqrt{\frac{2n}{(n+1)(2j+1)(2j-1)}} = \begin{cases} 0, & T=T_{\rm min}, \\ 1, & T=T_{\rm min}+2, \end{cases} \tag{11}$$

where $M=\Sigma_{J\geqslant 2}D^0(JJ)\sqrt{(2J+1)}$.

Comparing with the results of Table I we see that for ^{44}Ti there is too much $J=2$ coupling—more than $J=0$. However, the trend as one goes through the Ti isotopes is quite reasonable and the wave functions for ^{48}Ti are remarkably similar.

The coefficients $D(22)$ play an important role in the calculations of $M1$ transitions in the single j shell. The expression for $B(M1)\uparrow$ from a $J=0^+$ to $J=1^+$ in units of μ_N^2 is given by [13]

$$B(M1{:}0^+ \to 1^{+\alpha}) = \frac{3}{4\pi}(g_p - g_n)^2|\Sigma_{JV}D^0(J,J_V)D^{1\alpha}(J,J_V) \times \sqrt{J(J+1)}|^2. \tag{12}$$

Here g_p and g_n are the Schmidt values. If we sum over all $J=1^+$ final states we obtain [14]

$$\Sigma_\alpha B(M1) = \frac{3}{4\pi}(g_p - g_n)^2[\Sigma_J D^0(J,J)^2J(J+1)]. \tag{13}$$

Here we make a comparison of the MBZ and the *s*-*d* truncation result for the summed strengths, using the effective value for $(g_p - g_n) = 1.89$ as in Ref. [12].

ΣB(M1)	MBZ	*s*-*d* only
^{44}Ti	2.881	3.289
^{46}Ti	1.977	1.919
^{48}Ti	0.857	0.853

Note that in the *s*-*d* model only one term, corresponding to $J=2$ contributes.

The values for ^{46}Ti and ^{48}Ti are remarkably similar for MBZ and *s*-*d* truncation even though the values of $D(22)$ are quite different. It appears that the higher J contributions conspire to make the summed $B(M1)$'s for MBZ about the same as for *s*-*d* truncation.

The important point we wish to make here is that for the even-even Ti isotopes in the single j shell model, once we make the assumption that the $T=T_{\text{min}}$ state consists of only $J_p=J_n=0$ and $J_p=J_n=2$ couplings, the relative amounts of the couplings is fixed. There is no freedom. The reason for this is that the states with T_{min} must be orthogonal to the states with $T_{\text{min}}+2$. A small amount of the higher J couplings restores the freedom to adjust the relative amounts of $J=0$ and $J=2$.

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