General structure of a two-body operator for spin-$\frac{1}{2}$ particles

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A direct derivation of the operator structure for two spin-$\frac{1}{2}$ particles is presented subject to invariance under basic symmetries and Galilean frame transformation. The partial wave decomposition for coefficient functions, valid on- and off-shell, is explicitly deduced. The momentum transfer representation and angular momentum decomposition for general spin-dependent potentials are obtained.

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I. INTRODUCTION

Two-body operators are the basic elements defining the properties of interacting systems. The on- and off-shell scattering operators ($S,-T$ matrices), the two-body interaction, the optical potential between particles, etc., supply examples of such operators. Figure 1 shows a schematic representation of a two-body operator $U(\boldsymbol{k}_1',\boldsymbol{k}_2';\boldsymbol{k}_1,\boldsymbol{k}_2)$ in momentum space. The initial and final momenta (spin projections) of particle i are denoted by $\boldsymbol{k}_i$ (ν_i) and $\boldsymbol{k}_i'$ (ν_i'), respectively. The operator dependence on any parameters (for example, energy E) is not shown explicitly. The particle momenta or their combinations are the operator arguments and the operator $U(\boldsymbol{k}_1',\boldsymbol{k}_2';\boldsymbol{k}_1,\boldsymbol{k}_2)$ is a matrix in the spin space of particles. It is convenient to choose relative and total momenta of two particles in the initial ($\boldsymbol{k},\boldsymbol{P}$) and final ($\boldsymbol{k}',\boldsymbol{P}'$) states as independent operator variables

$$\boldsymbol{k}=\frac{m_1m_2}{m_1+m_2}\left(\frac{\boldsymbol{k}_1}{m_1}-\frac{\boldsymbol{k}_2}{m_2}\right),\quad \boldsymbol{P}=\boldsymbol{k}_1+\boldsymbol{k}_2$$

$$\boldsymbol{k}'=\frac{m_1m_2}{m_1+m_2}\left(\frac{\boldsymbol{k}_1'}{m_1}-\frac{\boldsymbol{k}_2'}{m_2}\right),\quad \boldsymbol{P}'=\boldsymbol{k}_1'+\boldsymbol{k}_2',$$

where m_i are the particle masses. Thus, the operator $U(\boldsymbol{k}_1',\boldsymbol{k}_2';\boldsymbol{k}_1,\boldsymbol{k}_2)\equiv U(\boldsymbol{P}',\boldsymbol{k}';\boldsymbol{P},\boldsymbol{k})$ must be constructed from its arguments and a complete set of spin matrices. In order to be physically relevant, the operator structure must be compatible with the restrictions imposed upon it by general symmetry principles and frame transformations. Implementation of symmetries reduces a number of independent arguments and constrains the allowed operator forms.

The general structure of a two-body operator in a nonrelativistic case is usually constructed in the following way [1,2]. The condition of invariance under rotations requires that the operator U should be a scalar. For spin $\frac{1}{2}$ particles, U is a matrix in the four-dimensional spin space. This matrix can be represented by a combination of any sixteen linearly independent matrices. A convenient set of such matrices can be formed from the unit and Pauli spin matrices $\boldsymbol{\sigma}_i$ arranged into scalars, axial vectors, and symmetric tensors of the second rank

$$1,\ (\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2),(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2),(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2),[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_{1m},[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_{2m}$$

where $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$ and $[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_{lm}$ denote scalar and direct products of the $\boldsymbol{\sigma}$-operators

$$(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)=\sum_{\nu}(-1)^{\nu}(\boldsymbol{\sigma}_1)_{\nu}(\boldsymbol{\sigma}_2)_{-\nu}$$

$$[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_{lm}=\sum_{\nu_1,\nu_2}(1\nu_1 1\nu_2|lm)(\boldsymbol{\sigma}_1)_{\nu_1}(\boldsymbol{\sigma}_2)_{\nu_2}.$$

Overall scalars can be formed by contraction of the spin vectors with appropriate momentum vectors and also by contraction of the spin tensors with second-rank tensors constructed from the available momentum vectors. Any of these terms may be multiplied by arbitrary scalar functions that depend on scalars formed from momenta. Additional restrictions on the operator structure follow from the implications of symmetries like the translational, parity, and reciprocity (time reversal) invariance, and the Pauli principle. The operator structure compatible with general symmetries was deduced [1,2] for on-shell operators and a broad variety of applications in nuclear physics was found. The generalization to the case of relativistic scattering operators for Dirac particles can be found in Ref. [3]. So five (six) terms are necessary to define the on-shell operator completely if particles are identical (not identical). The number of independent terms is increased off shell to six and eight for identical and nonidentical particles, respectively [4,5]. Nevertheless, the two drawbacks are present at approach like this. One is that the structure of arbitrary functions remains undefined. Taking the traces over the Pauli spin matrices $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$

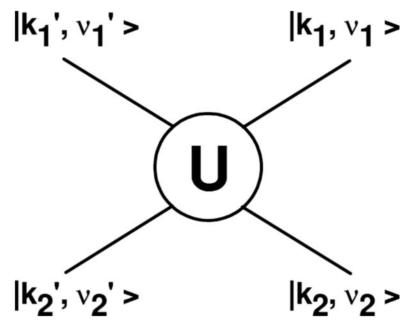

FIG. 1. Diagrammatic representation of a two-body operator.

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from products of the U-operator with spin tensors, these functions can be expressed for the *on-shell* case by different combinations of the operator matrix elements with the known partial wave decomposition. Thus, arbitrary functions can be related with the partial wave structure of the operator U. The NN on-shell amplitude is the well-known example [6]. For off-shell momenta the set of unit momentum vectors usually used in spin-tensors becomes nonorthogonal. Also the time reversal invariance gives only some relations between the amplitudes, but it does not restrict the number of amplitudes as for instance takes place on the energy shell. Thus the off-shell relations become rather complicated and the numerical matrix inversion is used [7]. The second disadvantage is that for off-shell situation the above derivation does not exclude appearance of arbitrary scalar functions becoming zero at on-shell momenta.

Below the general structure of the two-body operator U, valid for the on- and off-shell momenta, is directly derived from the operator partial-wave decomposition within the framework of nonrelativistic dynamics. This derivation is rather transparent, follows straightforwardly from algebraic manipulations and gives a complete structure. The method does not require the special selection of a coordinate system and is equally applicable to on- and off-shell situations. As result, the operator splits into two parts, one conserves parity and the other does not. The scalar functions of the operator structure are explicitly defined by the partial wave components of the U operator. The momentum transfer representation of the operator structure is also given and the angular momentum decomposition for spin-dependent potentials is developed in the analytical form. In the Appendix, the method is applied to derive the operator structure in the case of particles with spin-0 and spin-$\frac{1}{2}$.

II. GENERAL STRUCTURE OF A TWO-BODY OPERATOR

A. Translation and Galilean invariance

The translation invariance implies that a coordinate space representation of a two-body operator $U(\boldsymbol{r}_1',\boldsymbol{r}_2';\boldsymbol{r}_1,\boldsymbol{r}_2)$ is independent of a shift of all space coordinates on an arbitrary vector $\boldsymbol{a}$

$$U(\boldsymbol{r}_1',\boldsymbol{r}_2';\boldsymbol{r}_1,\boldsymbol{r}_2) = U(\boldsymbol{r}_1'+\boldsymbol{a},\boldsymbol{r}_2'+\boldsymbol{a};\boldsymbol{r}_1+\boldsymbol{a},\boldsymbol{r}_2+\boldsymbol{a}).$$

In momentum space this relation dictates conservation of the total momentum of two particles $\boldsymbol{P}'=\boldsymbol{P}$. Therefore, only three momenta of the four are independent in the two-body operator $U(\boldsymbol{P}',\boldsymbol{k}';\boldsymbol{P},\boldsymbol{k})\to U(\boldsymbol{P};\boldsymbol{k}',\boldsymbol{k})$. The operator dependence on total momentum $\boldsymbol{P}$ is excluded by the invariance with respect to Galilean frame transformations [4]. Hence, the relative momenta of two particles in the initial $\boldsymbol{k}$ and final $\boldsymbol{k}'$ states are the independent operator variables $U(\boldsymbol{P};\boldsymbol{k}',\boldsymbol{k})\to U(\boldsymbol{k}',\boldsymbol{k})$.

B. Rotation invariance

The condition of invariance under rotations requires that an operator $U(\boldsymbol{k}',\boldsymbol{k})$ should be a scalar. Assuming the translation, rotation, and Galilean invariance the general partial wave decomposition of a two-body operator $U(\boldsymbol{k}',\boldsymbol{k})$ can be written as

$$U(\boldsymbol{k}',\boldsymbol{k}) = \sum_{LSL'S'JM} \imath^{L-L'} \Phi^{JM}_{L'S'}(\hat{\boldsymbol{k}}') U^{J,S'S}_{L'L}(k',k) \Phi^{JM}_{LS}(\hat{\boldsymbol{k}})^{+} \tag{1}$$

where $k=|\boldsymbol{k}|$ is the absolute value of the momentum vector, $\hat{\boldsymbol{k}}=\boldsymbol{k}/k$ denotes the unit vector. The quantum numbers J and M are the total angular momentum and its projection on the quantization axis z, $L(L')$ and $S(S')$ are the orbital angular momentum of relative motion and the total spin of two particles before (after) interaction, respectively. The tensor spherical harmonics $\Phi^{JM}_{LS}(\hat{\boldsymbol{k}})$ are given as

$$\Phi^{JM}_{LS}(\hat{\boldsymbol{k}}) = \sum_{M_L,\,M_S} (LM_LSM_S|JM) Y_{LM_L}(\hat{\boldsymbol{k}})|SM_S\rangle \tag{2}$$

$$|SM_S\rangle = \sum_{m_1,\,m_2} \left(\frac{1}{2}m_1\frac{1}{2}m_2|SM_S\right)\left|\frac{1}{2}m_1\right\rangle_1\left|\frac{1}{2}m_2\right\rangle_2, \tag{3}$$

where the wave function $|SM_S\rangle$ describes the spin state of two particles with the total spin S and its projection M_S, $|\frac{1}{2}m_i\rangle_i$ is the spin state of particle i. The spherical harmonics $Y_{LM_L}(\hat{\boldsymbol{k}})$, the Clebsch-Gordon and all other recoupling coefficients below are defined according to Ref. [8]. $U^{J,S'S}_{L'L}(k',k)$ depends only on the absolute values of relative momenta $|\boldsymbol{k}|$ and $|\boldsymbol{k}'|$. Below we will suppress this dependence and for $S'=S$ will show only one index $U^{J,SS}_{L'L}\equiv U^{J,S}_{L'L}$. For any value of the total momentum J there are sixteen different combinations of allowed values of quantum numbers L,L',S and S': one with $S=S'=0$, nine with $S=S'=1$, and six with $S\neq S'$. Since the parity of each term is equal to $(-1)^{L+L'}$, there are eight terms that are even under the parity transformation and eight that are odd. The spherical harmonics $Y_{LM_L}(\hat{\boldsymbol{k}})$ can be coupled in the bipolar harmonics $\Upsilon^{L,L'}_{lm_l}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')$. Thus, the decomposition (1) takes the form

$$U(\boldsymbol{k}',\boldsymbol{k}) = \sum_{LSL'S'Jlm_lM_SM_{S'}} \imath^{L-L'} \frac{\hat{l}\hat{J}^2}{\hat{S}} \begin{Bmatrix} L & S & J \\ S' & L' & l \end{Bmatrix} \times U^{J,S'S}_{L'L} \Upsilon^{L,L'}_{lm_l}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}') * (-1)^{J+L+L'+S'} \times (lm_lS'M_{S'}|SM_S)|S'M_{S'}\rangle\langle SM_S| \tag{4}$$

$$\Upsilon^{L,L'}_{lm_l}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}') = \sum_{M_L,M_{L'}} (LM_LL'M_{L'}|lm_l) Y_{LM_L}(\hat{\boldsymbol{k}}) Y_{L'M_{L'}}(\hat{\boldsymbol{k}}'), \tag{5}$$

where the caret over a quantum number $\hat{J}$ is a standard abbreviation $\hat{J}=\sqrt{2J+1}$. The product operator $|S'M_{S'}\rangle\langle SM_S|$ is a matrix in the two-body spin space. It is convenient to express this matrix as a combination of the Pauli spin matrices $\boldsymbol{\sigma}_i$ which act on the spin coordinate of particle i. In the one-body spin space the expansion of the product operator in terms of spin matrices is well known [8]

$$\left|\frac{1}{2}\nu'\right\rangle\left\langle\frac{1}{2}\nu\right| = \frac{1}{2}\left\{\delta_{\nu,\nu'} - \sqrt{3}\left(1\nu' - \nu\frac{1}{2}\nu\middle|\frac{1}{2}\nu'\right)(\boldsymbol{\sigma})_{\nu'-\nu}\right\}. \tag{6}$$

Using this relation, after some straightforward algebra one finds

$$\sum_{M_S M_{S'}} (lmS'M_{S'}|SM_S)|S'M_{S'}\rangle\langle SM_S|$$
$$= \delta_{S,S'}\delta_{l,0}(P_{S=0} + P_{S=1}) + \delta_{S,S'}\delta_{S,1}\frac{(-1)^m}{2} \times\left\{-\delta_{l,1}\frac{1}{\sqrt{2}}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)_{-m} + \delta_{l,2}\sqrt{\frac{3}{5}}[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_{2-m}\right\}$$
$$+ \delta_{l,1}\frac{(-1)^m}{4}\left\{(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)_{-m}\left(\delta_{S,1}\delta_{S',0} - \frac{\delta_{S,0}\delta_{S',1}}{\sqrt{3}}\right) - \sqrt{2}[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_{1-m}\left(\delta_{S,1}\delta_{S',0} + \frac{\delta_{S,0}\delta_{S',1}}{\sqrt{3}}\right)\right\}, \tag{7}$$

where $P_{S=0}$ and $P_{S=1}$ are the spin singlet and triplet projection operators

$$P_{S=0} = \tfrac{1}{4}(1 - \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2) \tag{8}$$

$$P_{S=1} = \tfrac{1}{4}(3 + \boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2). \tag{9}$$

Introducing the notation $\mu=(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}')$ and substituting Eq. (7) into Eq. (4) gives

$$U(\boldsymbol{k}',\boldsymbol{k}) = \alpha(k,k',\mu)P_{S=0} + \beta(k,k',\mu)P_{S=1} + \frac{1}{2\sqrt{2}}\sum_{J,L,L'}\imath^{L-L'}(-1)^{J+L+L'}\hat{J}^2\begin{Bmatrix}L & 1 & J\\ 1 & L' & 1\end{Bmatrix}U^{J,1}_{L'L}((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\Upsilon^{L,L'}_1(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'))$$
$$-\frac{1}{2}\sum_{J,L,L'}\imath^{L-L'}(-1)^{J+L+L'}\hat{J}^2\begin{Bmatrix}L & 1 & J\\ 1 & L' & 2\end{Bmatrix}U^{J,1}_{L'L}([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_2\cdot\Upsilon^{L,L'}_2(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')) - \frac{1}{4\sqrt{3}}\sum_{J,L'}\imath^{J-L'}(-1)^J\hat{J}U^{J,10}_{L'J}((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2$$
$$+\sqrt{2}[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1)\cdot\Upsilon^{J,L'}_1(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')) - \frac{1}{4\sqrt{3}}\sum_{J,L}\imath^{L-J}(-1)^J\hat{J}U^{J,01}_{JL}((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2-\sqrt{2}[\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1)\cdot\Upsilon^{L,J}_1(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')) \tag{10}$$

where α and β are scalar functions

$$\alpha(k,k',\mu) = \frac{1}{4\pi}\sum_J \hat{J}^2 U^{J,0}_{JJ}P_J(\mu) \tag{11}$$

$$\beta(k,k',\mu) = \frac{1}{4\pi}\sum_{L,J}\frac{\hat{J}^2}{3}U^{J,1}_{LL}P_L(\mu) \tag{12}$$

that are invariant under rotations, and $P_L(\mu)$ is the Legendre polynomials. In the last four lines of Eq. (10), in contrast to the first one, the spin and angular degrees of freedom are intertwined in a complex way. We will simplify these structures and single out the spin dependence in the form of different tensor operators multiplied by scalar functions. To achieve this goal, the bipolar harmonics $\Upsilon^{L,L'}_{lm}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')$ are expressed in terms of a basis set of bipolar harmonics with a minimal angular index, times the derivatives of the Legendre polynomials of argument μ [9,10]. Using the relation $\hat{\boldsymbol{k}}_m = \sqrt{4\pi/3}Y_{1m}(\hat{\boldsymbol{k}})$ and defining the vector $\boldsymbol{n}$ by the vector product of $\hat{\boldsymbol{k}}$ and $\hat{\boldsymbol{k}}'$ ($\boldsymbol{n}=[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}']$, $(\boldsymbol{n}\cdot\boldsymbol{n})=1-\mu^2$), the reduction formulas of Ref. [9] for the total orbital momentum l equal to 1 and 2 can be rewritten as

$$\Upsilon^{L,L'}_{1m}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}') = \delta_{L,L'}\frac{(-1)^{L+1}}{4\pi}\sqrt{\frac{6(2L+1)}{L(L+1)}}P'_L(\mu)[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}']_{1m} + \delta_{L\pm1,\,L'}\frac{(-1)^{L+1}}{4\pi}\sqrt{\frac{3}{\tilde{M}}}(\hat{\boldsymbol{k}}_{1m}P'_L(\mu) - \hat{\boldsymbol{k}}'_{1m}P'_{L'}(\mu)) \tag{13}$$

$$\Upsilon^{L,L'}_{2m}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}') = \delta_{L,L'}\frac{(-1)^L}{4\pi}\sqrt{\frac{30(2L+1)}{L(L+1)(2L-1)(2L+3)}} * \{P''_L(\mu)([\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}]_{2m} + [\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{k}}']_{2m}) - (2\mu P''_L(\mu) + P'_L(\mu))[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}']_{2m}\}$$
$$+ \imath\delta_{L\pm1,\,L'}\frac{(-1)^L}{4\pi}\sqrt{\frac{10}{\tilde{M}(\tilde{M}-1)(\tilde{M}+1)}}([\hat{\boldsymbol{k}}\times\boldsymbol{n}]_{2m}P''_L(\mu) - [\hat{\boldsymbol{k}}'\times\boldsymbol{n}]_{2m}P''_{L'}(\mu))$$
$$+ \delta_{L\pm2,\,L'}\frac{(-1)^L}{4\pi}\sqrt{\frac{5}{L(L+1)(2L+1)}} * (P''_L(\mu)[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}]_{2m} + P''_{L'}(\mu)[\hat{\boldsymbol{k}}'\times\hat{\boldsymbol{k}}']_{2m} - 2P''_{\tilde{L}}(\mu)[\hat{\boldsymbol{k}}\times\hat{\boldsymbol{k}}']_{2m}), \tag{14}$$

where $\widetilde{M}=\max\{L,L'\}$, $\widetilde{L}=\frac{1}{2}(L+L')$, $P'_L(\mu)=(d/d\mu)P_L(\mu)$, etc. It is useful to define the tensor operator $S(\boldsymbol{a},\boldsymbol{b})$ by the relation

$$S(\boldsymbol{a},\boldsymbol{b})=([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_2\cdot[\boldsymbol{a}\times\boldsymbol{b}]_2)=\frac{1}{2}((\boldsymbol{\sigma}_1\cdot\boldsymbol{a})(\boldsymbol{\sigma}_2\cdot\boldsymbol{b})+(\boldsymbol{\sigma}_1\cdot\boldsymbol{b})(\boldsymbol{\sigma}_2\cdot\boldsymbol{a}))-\tfrac{1}{3}(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$$

that for identical unit vectors reduces to the usual tensor operator $S(\hat{\boldsymbol{a}})$

$$S(\hat{\boldsymbol{a}})\equiv S(\hat{\boldsymbol{a}},\hat{\boldsymbol{a}})=(\boldsymbol{\sigma}_1\cdot\hat{\boldsymbol{a}})(\boldsymbol{\sigma}_2\cdot\hat{\boldsymbol{a}})-\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2).$$

Substituting the reduction formulas into Eq. (10), after some algebra we find the general decomposition of the two-body operator U. For convenience, we present this decomposition as a sum of two parts which are even U_{even} or odd U_{odd} under the parity transformation

$$U(\boldsymbol{k}',\boldsymbol{k})=U_{\text{even}}(\boldsymbol{k}',\boldsymbol{k})+U_{\text{odd}}(\boldsymbol{k}',\boldsymbol{k}).\tag{15}$$

The even part $U_{\text{even}}(\boldsymbol{k}',\boldsymbol{k})$ has eight terms

$$U_{\text{even}}(\boldsymbol{k}',\boldsymbol{k})=\alpha P_{S=0}+\beta P_{S=1}+i\gamma((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{n})+\delta S(\hat{\boldsymbol{k}})+\epsilon S(\hat{\boldsymbol{k}}')+\eta S(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')+i\kappa((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n})+i\lambda([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n})\tag{16}$$

where the coefficients α, β, γ, etc., are functions of the scalars k, k', μ (mention of this dependence will be often omitted). The coefficients have the explicit representation

$$\gamma=\frac{1}{16\pi}\sum_J\{D^J+2(U^{J+1,1}_{J\ J}-U^{J-1,1}_{J\ J})\}P'_J(\mu)\tag{17}$$

$$\delta=\frac{1}{8\pi}\sum_J\left\{D^J-\frac{1}{\sqrt{(J-1)J}}U^{J-1,1}_{J-2\ J}-\frac{1}{\sqrt{(J+1)(J+2)}}U^{J+1,1}_{J+2\ J}\right\}P''_J(\mu)\tag{18}$$

$$\epsilon=\frac{1}{8\pi}\sum_J\left\{D^J-\frac{1}{\sqrt{(J-1)J}}U^{J-1,1}_{J\ J-2}-\frac{1}{\sqrt{(J+1)(J+2)}}U^{J+1,1}_{J\ J+2}\right\}P''_J(\mu)\tag{19}$$

$$\eta=\frac{1}{8\pi}\sum_J\{2(S^J-\mu D^J)P''_J(\mu)-D^JP'_J(\mu)\}\tag{20}$$

$$\kappa=\frac{1}{16\pi}\sum_J\frac{2J+1}{\sqrt{J(J+1)}}\{U^{J,10}_{J\ J}+U^{J,01}_{J\ J}\}P'_J(\mu)\tag{21}$$

$$\lambda=\frac{\sqrt{2}}{16\pi}\sum_J\frac{2J+1}{\sqrt{J(J+1)}}\{U^{J,10}_{J\ J}-U^{J,01}_{J\ J}\}P'_J(\mu),\tag{22}$$

where D^J and S^J denote the next combinations of the matrix elements

$$D^J=\frac{1}{J}U^{J-1,1}_{J\ J}-\frac{2J+1}{J(J+1)}U^{J,1}_{J\ J}+\frac{1}{J+1}U^{J+1,1}_{J\ J}$$

$$S^J=\frac{1}{\sqrt{J(J+1)}}(U^{J,1}_{J+1\ J-1}+U^{J,1}_{J-1\ J+1}).$$

Correspondingly, the odd part U_{odd} of the two-body operator has the following structure:

$$U_{\text{odd}}(\boldsymbol{k}',\boldsymbol{k})=i((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\nu\hat{\boldsymbol{k}}+\xi\hat{\boldsymbol{k}}'))+i((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot(\varrho\hat{\boldsymbol{k}}+\phi\hat{\boldsymbol{k}}'))+i([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot(\varphi\hat{\boldsymbol{k}}+\chi\hat{\boldsymbol{k}}'))+\psi S(\hat{\boldsymbol{k}},\boldsymbol{n})+\omega S(\hat{\boldsymbol{k}}',\boldsymbol{n})\tag{23}$$

where for the scalar functions ν, ξ, etc., we find

$$\nu=\frac{-1}{16\pi}\sum_J\left\{\hat{J}\left(\frac{\sqrt{J+1}}{J}U^{J,1}_{J-1\ J}+\frac{\sqrt{J}}{J+1}U^{J,1}_{J+1\ J}\right)+\frac{\sqrt{(2J+3)(J+2)}}{J+1}U^{J+1,1}_{J+1\ J}+\frac{\sqrt{(2J-1)(J-1)}}{J}U^{J-1,1}_{J-1\ J}\right\}P'_J(\mu)$$

$$\xi=\frac{1}{16\pi}\sum_J\left\{\hat{J}\left(\frac{\sqrt{J+1}}{J}U^{J,1}_{J\ J-1}+\frac{\sqrt{J}}{J+1}U^{J,1}_{J\ J+1}\right)+\frac{\sqrt{(2J+3)(J+2)}}{J+1}U^{J+1,1}_{J\ J+1}+\frac{\sqrt{(2J-1)(J-1)}}{J}U^{J-1,1}_{J\ J-1}\right\}P'_J(\mu)$$

$$\rho=\frac{1}{16\pi}\sum_J\left\{\hat{J}\left(\frac{U^{J,10}_{J-1\ J}}{\sqrt{J}}-\frac{U^{J,10}_{J+1\ J}}{\sqrt{J+1}}\right)+\sqrt{\frac{2J+3}{J+1}}U^{J+1,01}_{J+1\ J}-\sqrt{\frac{2J-1}{J}}U^{J-1,01}_{J-1\ J}\right\}P'_J(\mu)$$

$$\phi=\frac{-1}{16\pi}\sum_J\left\{\hat{J}\left(\frac{U^{J,01}_{J\ J-1}}{\sqrt{J}}-\frac{U^{J,01}_{J\ J+1}}{\sqrt{J+1}}\right)+\sqrt{\frac{2J+3}{J+1}}U^{J+1,10}_{J\ J+1}-\sqrt{\frac{2J-1}{J}}U^{J-1,10}_{J\ J-1}\right\}P'_J(\mu)$$

$$\varphi=\frac{\sqrt{2}}{16\pi}\sum_J\left\{\hat{J}\left(\frac{U^{J,10}_{J-1\ J}}{\sqrt{J}}-\frac{U^{J,10}_{J+1\ J}}{\sqrt{J+1}}\right)-\sqrt{\frac{2J+3}{J+1}}U^{J+1,01}_{J+1\ J}+\sqrt{\frac{2J-1}{J}}U^{J-1,01}_{J-1\ J}\right\}P'_J(\mu)$$

$$\chi = \frac{\sqrt{2}}{16\pi}\sum_J \left\{ \hat{J}\left(\frac{U^{J,01}_{J\,J-1}}{\sqrt{J}} - \frac{U^{J,01}_{J\,J+1}}{\sqrt{J+1}}\right) - \sqrt{\frac{2J+3}{J+1}}U^{J+1,10}_{J\,J+1} + \sqrt{\frac{2J-1}{J}}U^{J-1,10}_{J\,J-1}\right\} P'_J(\mu)$$

$$\psi = \frac{1}{8\pi}\sum_J \left\{ \frac{\hat{J}}{J(J+1)}(\sqrt{J+1}U^{J,1}_{J-1\;J} - \sqrt{J}U^{J,1}_{J+1\;J}) + \frac{1}{J+1}\sqrt{\frac{2J+3}{J+2}}U^{J+1,1}_{J+1\;J} - \frac{1}{J}\sqrt{\frac{2J-1}{J-1}}U^{J-1,1}_{J-1\;J}\right\} P''_J(\mu)$$

$$\omega = \frac{-1}{8\pi}\sum_J \left\{ \frac{\hat{J}}{J(J+1)}(\sqrt{J+1}U^{J,1}_{J\;J-1} - \sqrt{J}U^{J,1}_{J\;J+1}) + \frac{1}{J+1}\sqrt{\frac{2J+3}{J+2}}U^{J+1,1}_{J\;J+1} - \frac{1}{J}\sqrt{\frac{2J-1}{J-1}}U^{J-1,1}_{J\;J-1}\right\} P''_J(\mu). \tag{24}$$

Notice that the operator $([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n})$ in Eq. (16) has a different representation $(-\imath\sqrt{2})([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n}) =(\boldsymbol{\sigma}_1\cdot\boldsymbol{k})(\boldsymbol{\sigma}_2\cdot\boldsymbol{k}')-(\boldsymbol{\sigma}_1\cdot\boldsymbol{k}')(\boldsymbol{\sigma}_2\cdot\boldsymbol{k})$. The derived formulas present the general structure of a two-body operator as a sum of products of simple spin operators and scalar functions with the known structure. In the case of nuclear forces, terms with the spin singlet $P_{S=0}$ and spin triplet $P_{S=1}$ projectors in the even part of the operator (16) correspond to central forces acting between the two-particle singlet and triplet states, respectively. The $(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{n}$ term is the spin-orbital part; $S(\hat{\boldsymbol{k}})$, $S(\hat{\boldsymbol{k}}')$ and $S(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')$ are tensor forces, while the last two terms $(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n}$ and $([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n})$ mix the two-particle singlet and triplet states. In the odd part of operator (23) four terms $((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{k}}),((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{k}}')$, $S(\hat{\boldsymbol{k}},\boldsymbol{n})$, and $S(\hat{\boldsymbol{k}}',\boldsymbol{n})$ have nonzero matrix elements only between the triplet states while the others mix the states with different total spins.

The direct method used above for derivation of the operator structure for spin-$\frac{1}{2}$ particles can be useful in many other cases. In the Appendix, for example, it is employed to get the operator structure for particles with spin-0 and spin-$\frac{1}{2}$.

C. Reciprocity invariance

The reciprocity (for non-Hermitian operators) or time-reversal (for Hermitian operators) invariance suggests [11,12] that an operator should satisfy the following relation:

$$U(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{k},\boldsymbol{k}') = U(-\boldsymbol{\sigma}_1,-\boldsymbol{\sigma}_2,-\boldsymbol{k}',-\boldsymbol{k}), \tag{25}$$

where operator dependence on particle spins is shown explicitly. This relation gives the constraints on scalar functions. The functions α, β, γ, η, and κ of the even operator (16) are symmetric with respect to exchange of k and k'; λ must change the sign, while δ is equal to ϵ

$$\alpha(k,k',\mu) = \alpha(k',k,\mu),\ldots$$

$$\lambda(k,k',\mu) = -\lambda(k',k,\mu) \tag{26}$$

$$\delta(k,k',\mu) = \epsilon(k',k,\mu).$$

From expressions (11), (12), and (17)–(22) of these functions there follows that relations (26) are satisfied if the even $(L'=L,L\pm2)$ partial wave components have the following symmetry:

$$U^{J,S'S}_{L'\;L}(k,k') = U^{J,SS'}_{L\;L'}(k',k). \tag{27}$$

The scalar functions of the odd operator (23) have the following constraints under the reciprocity invariance

$$\nu(k,k',\mu) = \xi(k',k,\mu)$$

$$\rho(k,k',\mu) = \phi(k',k,\mu)$$

$$\varphi(k,k',\mu) = -\chi(k',k,\mu) \tag{28}$$

$$\psi(k,k',\mu) = \omega(k',k,\mu).$$

To comply with these relations, the odd $(L'=L\pm1)$ partial wave components, as follows from decompositions (24), must transform in the following way

$$U^{J,S'S}_{L'\;L}(k,k') = -U^{J,SS'}_{L\;L'}(k',k). \tag{29}$$

The difference in sign for the even (27) and odd (29) components with respect to the reciprocity invariance is due to the factor $\imath^{L-L'}$ in our definition of the operator partial wave decomposition (1).

D. Identical particles

For two identical particles an operator is symmetric with respect to the particle exchange

$$U(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{k},\boldsymbol{k}') = U(\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_1,-\boldsymbol{k},-\boldsymbol{k}'). \tag{30}$$

This symmetry constrains the even and odd operators in a different way. In the even part (16), two structures $((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n})$ and $([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n})$ change sign under this operation. Therefore, the functions κ and λ must be identically zero. From the partial wave decompositions (21) and (22) of these functions we get that $U^{J,S'S}_{L'\;L}=0$ for $S\neq S'$. Hence, the total spin S is conserved by the even operator. For the odd operator (23) four structures $((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{k}}),((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{\boldsymbol{k}}'),S(\hat{\boldsymbol{k}},\boldsymbol{n})$, and $S(\hat{\boldsymbol{k}}',\boldsymbol{n})$ change sign, and the functions ν, ξ, ψ, and ω must be zero, respectively. It follows from Eq. (24) that the odd components $U^{J,S'S}_{L'\;L}=0$ for $S=S'$. Hence, only transitions between states with different spins are allowed in odd operators between identical particles.

In total, if a two-body operator is Galilean, translation, parity, and reciprocity (time reversal) invariant, then the operator has eight independent terms (16) for off-shell momenta. For the on-shell case the function λ is identically zero, and two terms (δ and ϵ) are equal. So only six terms remain to be independent for on-shell momenta. If particles are identical, then the κ and λ coefficients are zero (total spin S is conserved) and six terms are independent for off-shell momenta. This number reduces to five for the on-shell case ($\delta=\epsilon$).

E. Momentum transfer representation

Another set of momenta is often used in applications

$$\boldsymbol{q}=\boldsymbol{k}-\boldsymbol{k}',\ \boldsymbol{P}=\boldsymbol{k}+\boldsymbol{k}',\ \boldsymbol{n}=[\boldsymbol{k}\times\boldsymbol{k}']=\tfrac{1}{2}[\boldsymbol{q}\times\boldsymbol{P}], \qquad (31)$$

where $\boldsymbol{q}$ is the momentum transfer. For on-shell scattering ($k=k'$) the vectors $\boldsymbol{q}$ and $\boldsymbol{P}$ are orthogonal, $(\boldsymbol{q}\cdot\boldsymbol{P})=0$. Below we restrict our discussion to operators invariant under space inversion. Extension to the odd parity operators is straightforward. Under the transformation of momenta the scalars k, k', and μ are the functions of the scalar variables q, P and $(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{P}})$. Hence, only the structure of the tensor parts are modified. In the new coordinates the tensor operators take the following form:

$$S(\hat{\boldsymbol{k}})=\frac{1}{4k^2}(P^2S(\hat{\boldsymbol{P}})+q^2S(\hat{\boldsymbol{q}})+2qPS(\hat{\boldsymbol{q}},\hat{\boldsymbol{P}}))$$

$$S(\hat{\boldsymbol{k}}')=\frac{1}{4k'^2}(P^2S(\hat{\boldsymbol{P}})+q^2S(\hat{\boldsymbol{q}})-2qPS(\hat{\boldsymbol{q}},\hat{\boldsymbol{P}})) \qquad (32)$$

$$S(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')=\frac{1}{4kk'}(P^2S(\hat{\boldsymbol{P}})-q^2S(\hat{\boldsymbol{q}})).$$

Using these relations Eq. (16) can be rewritten as

$$U_{\text{even}}(\boldsymbol{q},\boldsymbol{P})=\alpha P_{S=0}+\beta P_{S=1}+i\gamma((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{n})+\tilde{\delta}S(\hat{\boldsymbol{q}}) + \tilde{\epsilon}S(\hat{\boldsymbol{P}})+\tilde{\eta}S(\hat{\boldsymbol{q}},\hat{\boldsymbol{P}})+i\kappa((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n}) + i\lambda([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n}), \qquad (33)$$

where the new coefficients $\tilde{\delta}$, $\tilde{\epsilon}$ and $\tilde{\eta}$ are given by

$$\tilde{\delta}=\left(\frac{q}{kk'}\right)^2\frac{1}{32\pi}\sum_J\left\{D^J(P^2P_J''(\mu)+kk'P_J'(\mu))-\left[\frac{1}{4}(q^2+P^2)\times(S^{J-1}+S^{J+1})+\frac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{P})(A^{J-1}-A^{J+1})+2kk'S^J\right]P_J''(\mu)\right\} \qquad (34)$$

$$\tilde{\epsilon}=\left(\frac{P}{kk'}\right)^2\frac{1}{32\pi}\sum_J\left\{D^J(q^2P_J''(\mu)+kk'P_J'(\mu))-\left[\frac{1}{4}(q^2+P^2)\times(S^{J-1}+S^{J+1})+\frac{1}{2}(\boldsymbol{q}\cdot\boldsymbol{P})(A^{J-1}-A^{J+1})-2kk'S^J\right]P_J''(\mu)\right\} \qquad (35)$$

$$\tilde{\eta}=\frac{qP}{32\pi(kk')^2}\sum_J\left\{(\boldsymbol{q}\cdot\boldsymbol{P})(S^{J-1}+S^{J+1}-2D^J)+\frac{1}{2}(q^2+P^2)\times(A^{J-1}-A^{J+1})\right\}P_J''. \qquad (36)$$

The coefficient A^J denotes the following combination of partial wave elements $U^{J,S}_{L'L}$

$$A^J=\frac{1}{\sqrt{J(J+1)}}(U^{J,1}_{J+1\ J-1}-U^{J,1}_{J-1\ J+1}).$$

If we assume the reciprocity (time reversal) invariance, then the coefficients S^J and A^J are symmetric and antisymmetric under an interchange of momenta k and k', respectively. Hence, on-shell the $\tilde{\eta}$ and λ terms are identically equal zero. It is also interesting to examine in detail the mechanism how the function $\tilde{\eta}$ becomes zero at on-shell momenta. We see from Eq. (36) that $\tilde{\eta}$ has two parts that are proportional to the $(\boldsymbol{q}\cdot\boldsymbol{P})$ and (q^2+P^2) coefficients, respectively. When $k=k'$, the first part is equal to zero due to the coefficient $(\boldsymbol{q}\cdot\boldsymbol{P})$ while the second is zero since the $A^J(k,k)=0$ by construction. Sometimes only the tensor part with the $(\boldsymbol{q}\cdot\boldsymbol{P})$ coefficient is introduced [15] as off-shell extension of a two-body interaction and the off-shell behavior similar to one in the second part of Eq. (36) is not taken into account.

Equation (33) with eight terms describes a general structure of the operator for two spin-$\frac{1}{2}$ particles. In applications, the most frequent case is the nucleon-nucleon scattering. For identical nucleons (*nn* or *pp*), six terms in the first line of Eq. (33) (or a set of equivalent operators) characterize completely the off-shell scattering [4]. Two terms, $((\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\boldsymbol{n})$ and $([\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2]_1\cdot\boldsymbol{n})$ that mix the spin-singlet and spin-triplet states, must be added for the neutron-proton interaction. These small terms are usually neglected in practice due to the additional assumption related to the isotopic invariance of nuclear forces. Only if a high accuracy is required and a breaking of the charge independence or charge symmetry becomes important then the spin mixing terms are included. An example, where the full structure of Eq. (33) must be considered and plays an important role, is an elastic scattering of polarized proton from a polarized ^{3}He at intermediate energies. The on-shell p-^{3}He T-matrix depends on six amplitudes (α, β, γ, $\tilde{\delta}$, $\tilde{\epsilon}$, and κ) and their knowledge defines elastic differential cross section and all spin observables. Calculations of the on-shell T-matrix require an off-shell optical potential, hence the two extra terms ($\tilde{\eta}$ and λ) must be added to completely define the optical potential structure. The experiments [13] show the large differences in proton and target-related asymmetries which is unambiguous evidence of a large κ amplitude in the p-^{3}He T-matrix even though this amplitude practically vanishes in nucleon-nucleon scattering. This experimentally observed significant mixing of the spin-singlet and spin-triplet states suggests that the off-shell λ-term in the optical potential may also be important for theoretical descriptions of elastic proton scattering. The microscopical optical potential, where the full structure is present, can be calculated within a full-folding approach [14]. This model is especially interesting for the

^{3}He nucleus since nuclear structure uncertainties are minimized and reliable Faddeev wave functions are available. Still the full-folding model for scattering of two spin-$\frac{1}{2}$ particles is not developed up to now and the approach shown here can be useful for a development of such models.

F. Angular momentum decomposition for spin-dependent potentials

Above, the general decomposition of the two-body operator U for particles with spin $s=1/2$ has been derived. The inverse problem often presents interest in practical applications when, for example, the right-hand side of Eq. (16) is known and the partial wave elements $U^{J,S'S}_{L'L}$ must be constructed. The calculation of elastic scattering of protons from ^{3}He with the microscopic, momentum space optical potential U is a typical example [7,16]. An ordinary procedure for obtaining the $U^{J,S'S}_{L'L}$ matrix elements reduces to a numerical matrix inversion [7]. Below we construct analytical expressions for $U^{J,S'S}_{L'L}$. We assume that the potential U has the form of Eq. (16) and the functions α, β, etc., have been derived within some nuclear models and can be calculated. Since these scalar functions depend only on k, k' and μ, they have the following partial wave decomposition:

$$\alpha(k',k,\mu) = \sum_{l,\,m} Y_{lm}(\hat{\boldsymbol{k}}')\alpha_l(k',k)Y^*_{lm}(\hat{\boldsymbol{k}}) = \frac{1}{4\pi}\sum_l (2l+1)\alpha_l(k',k)P_l(\mu). \tag{37}$$

The partial wave elements $\alpha_l(k',k)$ for any values of (k',k) can be calculated by the one-dimensional integration over μ. The other functions, β, etc., have analogous decompositions. Our aim is to find the expression for $U^{J,S'S}_{L'L}$ in terms of partial wave elements α_l, β_l, etc. With the help of the orthonormality condition for tensor spherical harmonics $\Phi^{JM}_{LS}(\hat{\boldsymbol{k}})$ [8] this expression can be directly obtained from Eq. (1) by the four-dimensional integration over the $(\hat{\boldsymbol{k}}',\hat{\boldsymbol{k}})$ directions

$$\imath^{L-L'}U^{J,S'S}_{L'L}(k',k) = \frac{1}{\hat{J}^2}\sum_M \langle\Phi^{JM}_{L'S'}(\hat{\boldsymbol{k}}')|U(\boldsymbol{k}',\boldsymbol{k})|\Phi^{JM}_{LS}(\hat{\boldsymbol{k}})\rangle. \tag{38}$$

By substituting the terms of U with the explicit spin dependence from the right-hand side of Eq. (16) into Eq. (38), the contributions from different parts can be calculated. As an example we calculate the contribution from the spin-orbit potential

$$[\imath^{L-L'}U^{J,S'S}_{L'L}]_\gamma = \frac{1}{\hat{J}^2}\sum_M \langle\Phi^{JM}_{L'S'}(\hat{\boldsymbol{k}}')|i\gamma((\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\boldsymbol{n})|\Phi^{JM}_{LS}(\hat{\boldsymbol{k}})\rangle.$$

The spin and space matrix elements can be easily calculated

$$\langle S'M'_S|(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)_{-\nu}|SM_S\rangle = \delta_{S,S'}\delta_{S,1}2\sqrt{2}(1M_S1-\nu|1M'_S)$$

$$\langle Y_{L'M'_L}(\hat{\boldsymbol{k}}')|\gamma\boldsymbol{n}_\nu|Y_{LM_L}(\hat{\boldsymbol{k}})\rangle = -i\sqrt{6}\hat{L}(1\nu LM_L|L'M'_L)\times\sum_l\gamma_l(L'0\,10|l0)(L0\,10|l0)\times\begin{Bmatrix}1 & 1 & 1\\ L' & L & l\end{Bmatrix}.$$

Combining together these results we get

$$[\imath^{L-L'}U^{J,S'S}_{L'L}]_\gamma = \delta_{S,S'}\delta_{S,1}12\hat{L}\hat{L}'(-1)^{J-L'}\begin{Bmatrix}1 & 1 & 1\\ L' & L & J\end{Bmatrix}\times\sum_l\gamma_l(L'0\,10|l0)(L0\,10|l0)\begin{Bmatrix}1 & 1 & 1\\ L' & L & l\end{Bmatrix}.$$

For any combination of quantum numbers it is straightforward to calculate this relation since the analytical expressions of Clebsch-Gordon coefficients and 6j-symbols are known [8]. Similarly, the contributions from other terms in Eq. (16) can be obtained. Finally, we have

$$U^{J,0}_{JJ} = \alpha_J \tag{39}$$

$$U^{J,1}_{JJ} = \beta_J + \frac{2}{\hat{J}^2}(\gamma_{J+1}-\gamma_{J-1}) + \frac{2}{3}(\delta_J+\varepsilon_J) + \frac{1}{3}\left(\frac{2J-1}{2J+1}\eta_{J+1} + \frac{2J+3}{2J+1}\eta_{J-1}\right) \tag{40}$$

$$U^{J,1}_{J-1\;J-1} = \beta_{J-1} - 2\frac{J-1}{2J-1}(\gamma_J-\gamma_{J-2}) - \frac{2}{3}\frac{J-1}{2J+1}(\delta_{J-1}+\varepsilon_{J-1}) - \frac{1}{3}\frac{J-1}{2J-1}\left(\eta_{J-2}+\frac{2J-3}{2J+1}\eta_J\right) \tag{41}$$

$$U^{J,1}_{J+1\;J+1} = \beta_{J+1} + 2\frac{J+2}{2J+3}(\gamma_{J+2}-\gamma_J) - \frac{2}{3}\frac{J+2}{2J+1}(\delta_{J+1}+\varepsilon_{J+1}) - \frac{1}{3}\frac{J+2}{2J+3}\left(\eta_{J+2}+\frac{2J+5}{2J+1}\eta_J\right) \tag{42}$$

$$U^{J,1}_{J-1\;J+1} = -2\frac{\sqrt{J(J+1)}}{2J+1}(\delta_{J-1}+\varepsilon_{J+1}+\eta_J) \tag{43}$$

$$U^{J,1}_{J+1\;J-1} = -2\frac{\sqrt{J(J+1)}}{2J+1}(\delta_{J+1}+\varepsilon_{J-1}+\eta_J) \tag{44}$$

$$U^{J,10}_{JJ} = -\frac{\sqrt{2J(J+1)}}{2J+1}(\sqrt{2}(\kappa_{J+1}-\kappa_{J-1})+(\lambda_{J+1}-\lambda_{J-1})) \tag{45}$$

$$U^{J,01}_{JJ} = -\frac{\sqrt{2J(J+1)}}{2J+1}(\sqrt{2}(\kappa_{J+1}-\kappa_{J-1})-(\lambda_{J+1}-\lambda_{J-1})). \tag{46}$$

As follows from these equations, even for very complicated and nonanalytic potentials $U(\boldsymbol{k}',\boldsymbol{k})$ the calculations of partial wave elements $U_{L'L}^{J,S'S}$ practically reduce to one-dimensional integrations for evaluation of partial elements α_l, β_l, etc. It is straightforward to check the consistency of representations of the $U_{L'L}^{J,S'S}$ elements given above and partial wave decompositions of scalar functions $(\alpha.\beta,\gamma,\ldots)$ employing the same elements. Substitution of expressions (39)–(46) into (11), (12), and (17)–(22) restores the original partial wave decompositions [as in (37)] of the functions α, β, γ, etc.

III. SUMMARY

The structure of the physically relevant operators must be compatible with general symmetry principles and frame transformations. Their application reduces the number of independent variables in an operator and restricts the allowed forms. In particular, the rotation invariance demands that an operator must be a scalar. Then the partial-wave decomposition of the operator can be reduced to the form where the angular dependence, combined into bipolar harmonics, intertwines in a complex way with the spin operators. The reduction formulas (13) of bipolar harmonics allows one to disentangle the spin and space degrees of freedom and splits the operator into parts where the spin tensors are multiplied by scalar functions with the known structure.

This method is applied to a direct derivation of the structure of the two-body operator for spin-$\frac{1}{2}$ particles within the framework of nonrelativistic dynamics. The two-body operator, compatible with invariance under translations, rotations and Galilean frame transformations, splits into two parts that are even and odd with respect to the space reflection. The time reversal (reciprocity) invariance constrains additionally the operator partial wave elements. At off-shell momenta the even part has eight terms with a different spin-tensor structure. At on-shell there are only six terms that reduce to five for identical particles. The momentum transfer representation was obtained and angular momentum decomposition for general spin-dependent potentials was developed in the analytical form.

The method used to construct a general structure of the two-body operator for spin-$\frac{1}{2}$ particles is straightforward and not confined to the cases considered above. It can also be implemented for particles with different spins.

APPENDIX

Assuming the translation, rotation, and Galilei invariance the general partial wave decomposition of the two-body operator $U(\boldsymbol{k}',\boldsymbol{k})$ for particles with spin 0 and $\frac{1}{2}$ can be written as

$$U(\boldsymbol{k}',\boldsymbol{k}) = \sum_{LL'JM} \imath^{L-L'}\Omega_{L'}^{JM}(\hat{\boldsymbol{k}}')U_{L'L}^{J}(k',k)\Omega_{L}^{JM}(\hat{\boldsymbol{k}})^{+}, \quad \text{(A1)}$$

where $J=|L\pm\frac{1}{2}|$ is the total angular momentum. The spinor spherical harmonics $\Omega_L^{JM}(\hat{\boldsymbol{k}})$ are defined as tensor spherical harmonics for spin $S=\frac{1}{2}$

$$\Omega_L^{JM}(\hat{\boldsymbol{k}}) = \sum_{M_L,\,m_s}\left(LM_L\frac{1}{2}m_s\middle|JM\right)Y_{LM_L}(\hat{\boldsymbol{k}})\left|\frac{1}{2}m_s\right\rangle.$$

By using expansion (6) for the product of spin functions the two-body operator $U(\boldsymbol{k}',\boldsymbol{k})$ can be expressed in the form

$$U(\boldsymbol{k}',\boldsymbol{k}) = \frac{1}{8\pi}\sum_{J,L}\hat{J}^2U_{LL}^{J}P_L(\mu) - \frac{1}{\sqrt{2}}\sum_{J,L,L'}\imath^{L-L'}(-1)^{J+L+L'+1/2}\hat{J}^2 \times\begin{Bmatrix} L & \frac{1}{2} & J \\ \frac{1}{2} & L' & 1\end{Bmatrix}U_{L'L}^{J}(\boldsymbol{\sigma}\cdot\Upsilon_1^{L,L'}(\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}')).$$

Substituting the bipolar harmonic reduction formulas (13) after some algebra we find the general decomposition of the two-body operator for particles with spin 0 and $\frac{1}{2}$

$$U(\boldsymbol{k}',\boldsymbol{k}) = f + ig(\boldsymbol{\sigma}\cdot\boldsymbol{n}) + i\kappa(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}) + i\lambda(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{k}}') \quad \text{(A2)}$$

where the coefficients f, g, etc., are functions of the scalars k, k', μ. These coefficients have the explicit representation

$$f = \frac{1}{4\pi}\sum_L\left\{ LU_{L\,L}^{L-1/2} + (L+1)U_{L\,L}^{L+1/2}\right\}P_L(\mu) \quad \text{(A3)}$$

$$g = \frac{1}{4\pi}\sum_L\left\{U_{L\,L}^{L-1/2} - U_{L\,L}^{L+1/2}\right\}P_L^1(\mu) \quad \text{(A4)}$$

$$\kappa = -\frac{1}{4\pi}\sum_L\left\{U_{L-1\,L}^{L-1/2} + U_{L+1\,L}^{L+1/2}\right\}P_L'(\mu) \quad \text{(A5)}$$

$$\lambda = \frac{1}{4\pi}\sum_L\left\{U_{L\,L-1}^{L-1/2} + U_{L\,L+1}^{L+1/2}\right\}P_L'(\mu) \quad \text{(A6)}$$

where $P_L^1(\mu)=-\sqrt{1-u^2}\,P_L'(\mu)$ is the associated Legendre function. The terms f and g (κ and λ) conserve (violate) parity. At on-shell momenta $k=k'$ these expressions coincide with the well-known decomposition of scattering amplitudes for particles with spin 0 and $1/2$ [2].

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