Boundary conditions of the hydrocascade model and relativistic kinetic equations for finite domains

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A detailed analysis of the coupled relativistic kinetic equations for two domains separated by a hypersurface having both space- and time-like parts is presented. Integrating the derived set of transport equations, we obtain the correct system of the hydro+cascade equations to model the relativistic nuclear collision process. Remarkably, the conservation laws on the boundary between domains conserve separately both the incoming and outgoing components of energy, momentum and baryonic charge. Thus, the relativistic kinetic theory generates twice the number of conservation laws compared to traditional hydrodynamics. Our analysis shows that these boundary conditions between domains, the *three flux discontinuity*, can be satisfied only by a special superposition of two *cutoff* distribution functions for the "out" domain. All these results are applied to the case of the phase transition between quark gluon plasma and hadronic matter. The possible consequences for an improved hydro+cascade description of the relativistic nuclear collisions are discussed. The unique properties of the *three flux discontinuity* and their effect on the space-time evolution of the transverse expansion are also analyzed. The possible modifications of both transversal radii from pion correlations generated by a correct hydro+cascade approach are discussed.

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I. INTRODUCTION

The modern history of relativistic hydrodynamics started more than 50 years ago when Landau suggested [1] its use to describe the expansion of the strongly interacting matter that is formed in high-energy hadronic collisions. Since that time there arose a fundamental problem of relativistic hydrodynamics known as the freeze-out problem. In other words, one has to know how to stop solving the hydrodynamical equations and convert the matter into free-streaming particles. There were several ways suggested to handle this, but only recently a new approach to solve the freeze-out problem in relativistic hydrodynamics has been discovered by Bass and Dumitru (BD model) [2] and further developed by Teaney, Lauret, and Shuryak (TLS model) [3]. These hydro +cascade models assume that the nucleus-nucleus collisions proceed in three stages: hydrodynamic expansion (hydro) of the quark gluon plasma (QGP), phase transition from the QGP to the hadron gas (HG), and the stage of hadronic rescattering and resonance decays (cascade). The switch from hydro- to cascade modeling takes place at the boundary between the mixed and hadronic phases. The spectrum of hadrons leaving this hypersurface of the QGP–HG transition is taken as input for the cascade.

This approach incorporates the best features of both the hydrodynamical and cascade descriptions. It allows for, on one hand, the calculation of the phase transition between the quark gluon plasma and hadron gas using hydrodynamics and, on the other hand, the freeze-out of hadron spectra using the cascade description. This approach allows one to overcome the usual difficulty of transport models in modeling phase transition phenomena. For this reason, this approach has been rather successful in explaining a variety of collective phenomena that has been observed at the CERN Super Proton Collider (SPS) and Brookhaven Relativistic Heavy Ion Collider (RHIC) energies. However, both the BD and TLS models face some fundamental difficulties which cannot be ignored (see a detailed discussion in Ref. [4]). Thus, within the BD approach the initial distribution for the cascade is found using the Cooper-Frye formula [5], which takes into account particles with all possible velocities, whereas in the TLS model the initial cascade distribution is given by the *cutoff* formula [6,7], which accounts for only those particles that can leave the phase boundary. As shown in Ref. [4], the Cooper-Frye formula leads to causal and mathematical problems in the present version of the BD model because the QGP-HG phase boundary inevitably has time-like parts. On the other hand, the TLS model does not conserve energy, momentum, and number of charges and this, as will be demonstrated later, is due to the fact that the equations of motion used in Ref. [3] are incomplete and, hence, should be modified.

These difficulties are likely in part responsible for the fact that the existing hydro+cascade models, like the more simplified ones, fail to explain the HBT *puzzle* [8], i.e., the fact that the experimental HBT radii at RHIC are very similar to those found at SPS, even though the center of mass energy is larger by an order of magnitude. Therefore, it turns out that the hydro+cascade approach successfully *parametrizes* the one-particle momentum spectra and their moments, but does not *describe* the space-time picture of the nuclear collision as probed by two-particle interferometry.

The main difficulty of the hydro+cascade approach looks similar to the traditional problem of freeze-out in relativistic hydrodynamics [6,7]. In both cases the domains (subsystems) have time-like boundaries through which the exchange of particles occurs and this fact should be taken into account. In relativistic hydrodynamics this problem was solved by the constraints which appear on the freeze-out hypersurface and provide the global energy-momentum and charge conservation [6,7,9]. A generalization of the usual Boltzmann equation, which accounts for the exchange of particles on the time-like boundary between domains in the relativistic kinetic theory, was given recently in Ref. [4]. It was shown that the kinetic equations describing the exchange of particles on the time-like boundary between subsystems should necessarily contain the δ-like source terms. From these kinetic equations the correct system of hydro+cascade equations to model the relativistic nuclear collision process was derived without specifying the properties of the separating hypersurface. However, both an explicit switch-off criterion from the hydro description to the cascade one and the boundary conditions between them were not considered in Ref. [4]. The present work is devoted to the analysis of the boundary conditions for the system of hydro+cascade equations. This is necessary to formulate the numerical algorithm for solving the hydro+cascade equations.

The paper is organized as follows. In Sec. II a brief derivation of the set of kinetic equations is given and source terms are obtained. In Sec. III the analog of the collision integrals is discussed and a fully covariant formulation of the system of coupled kinetic equations is found. The relation between the system obtained and the relativistic Boltzmann equation is also considered. The correct equations of motion for the hydro+cascade approach and their boundary conditions are analyzed in Sec. IV. There it is also shown that the existence of strong discontinuities across the space-like boundary, the time-like shocks, is in contradiction with the basic assumptions of a transport approach. The solutions of boundary conditions between the hydro and cascade domains for a single degree of freedom and for many degrees of freedom are discussed in Secs. V and VI, respectively. The conclusions are given in Sec. VII.

II. DRIFT TERM FOR SEMI-INFINITE DOMAIN

Let us consider two semi-infinite domains, "in" and "out," separated by the hypersurface Σ^* which, for the purpose of presenting the idea, we assume to be given in (3+1) dimensions by a single-valued function $t=t^*(\boldsymbol{x})=x_0^*(\boldsymbol{x})$. The latter is assumed to be a unique solution of the equation $\mathcal{F}^*(t,\boldsymbol{x})=0$ (a *switch-off criterion*) which has a positive time derivative $\partial_0\mathcal{F}^*(t^*,\boldsymbol{x})>0$ on the hypersurface Σ^*. Hereafter, all quantities defined at Σ^* will be marked with an asterisk. The distribution function $\phi_{in}(x,p)$ for $t\leqslant t^*(\boldsymbol{x})$ is assumed to belong to the in domain, whereas $\phi_{out}(x,p)$ denotes the distribution function of the "out" domain for $t\geqslant t^*(\boldsymbol{x})$ (see Fig. 1). In this work it is assumed that the initial conditions for $\phi_{in}(x,p)$ are given, whereas on Σ^* the function $\phi_{out}(x,p)$ is allowed to differ from $\phi_{in}(x,p)$ and this will modify the kinetic equations for both functions. For simplicity we consider a classical gas of point-like Boltzmann particles.

Similar to Ref. [10], we derive the kinetic equations for $\phi_{in}(x,p)$ and $\phi_{out}(x,p)$ from the requirement of particle number conservation. Therefore, the particles leaving one domain and crossing the hypersurface Σ^* should be subtracted from the corresponding distribution function and added to the other. Now, consider the closed hypersurface of the in domain, Δx^3 (shown as the contour *ABDE* in Fig. 1), which consists of two semiplanes σ_{t1} and σ_{t2} of constant time $t1$ and $t2$, respectively, that are connected from $t1$ to $t2>t1$ by the arc *BD* of the boundary $\Sigma^*(t1,t2)$ in Fig. 1. The original number of particles on the hypersurface σ_{t1} is given by the standard expression [10]

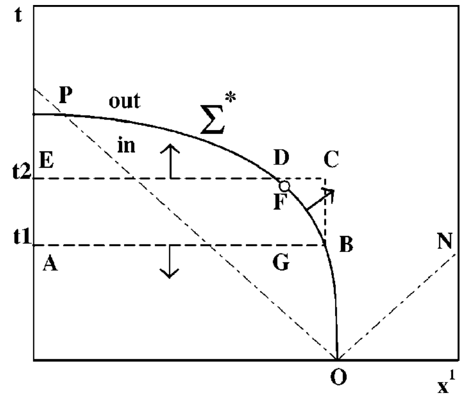

FIG. 1. Schematic two-dimensional picture of the boundary hypersurface Σ^* (solid curve). Arrows show the external normal vectors. The light cone *NOP* is shown by the dash-dotted line. The point *F* divides Σ^* into the time-like (*OF*) and space-like (*FP*) parts.

$$N_1 = -\int_{\sigma_{t1}} d\Sigma_\mu \frac{d^3p}{p^0} p^\mu \phi_{in}(x,p), \qquad (1)$$

where $d\Sigma_\mu$ is the external normal vector to σ_{t1} and, hence, the product $p^\mu d\Sigma_\mu \leqslant 0$ is non-positive. It is clear that these particles can cross either hypersurface σ_{t2} or $\Sigma^*(t1,t2)$. The corresponding numbers of particles are as follows:

$$N_2 = \int_{\sigma_{t2}} d\Sigma_\mu \frac{d^3p}{p^0} p^\mu \phi_{in}(x,p), \qquad (2)$$

$$N^*_{loss} = \int_{\Sigma^*(t1,t2)} d\Sigma_\mu \frac{d^3p}{p^0} p^\mu \phi_{in}(x,p)\Theta(p^\nu d\Sigma_\nu). \qquad (3)$$

The Θ function in the loss term (3) is very important because it accounts for the particles leaving the in domain (see also the discussion in Refs. [6,9]). For the space-like parts of the hypersurface $\Sigma^*(t1,t2)$ which are defined by negative sign $ds^2<0$ of the squared line element, $ds^2=dt^*(\boldsymbol{x})^2-d\boldsymbol{x}^2$, the product $p^\nu d\Sigma_\nu>0$ is always positive and, therefore, particles with all possible momenta can leave the in domain through the $\Sigma^*(t1,t2)$. For the time-like parts of $\Sigma^*(t1,t2)$ (with sign $ds^2>0$) the product $p^\nu d\Sigma_\nu$ can have either sign, and the Θ function *cuts off* those particles which return to the in domain.

Similarly one has to consider the particles coming to the in domain from outside. This is possible through the time-like parts of the hypersurface $\Sigma^*(t1,t2)$, if the particle momentum satisfies the inequality $-p^\nu d\Sigma_\nu>0$. In terms of the external normal $d\Sigma_\mu$ with respect to the in domain [this normal vector is shown as an arrow on the arc *BD* in Fig. 1 and will be used hereafter for all integrals over the hypersurface $\Sigma^*(t1,t2)$], the number of gained particles

$$N^*_{gain} = -\int_{\Sigma^*(t1,t2)} d\Sigma_\mu \frac{d^3p}{p^0} p^\mu \phi_{out}(x,p)\Theta(-p^\nu d\Sigma_\nu), \quad (4)$$

is, evidently, non-negative. Since the total number of particles is conserved, i.e., $N_2 = N_1 - N^*_{loss} + N^*_{gain}$, one can use the Gauss theorem to rewrite the obtained integral over the closed hypersurface Δx^3 as an integral over the 4-volume Δx^4 (area inside the contour *ABDE* in Fig. 1) surrounded by Δx^3

$$\int_{\Delta x^4} d^4x \frac{d^3p}{p^0} p^\mu \partial_\mu \phi_{in}(x,p) = \int_{\Sigma^*(t1,t2)} d\Sigma_\mu \frac{d^3p}{p^0} p^\mu \times [\phi_{in}(x,p) - \phi_{out}(x,p)] \times \Theta(-p^\nu d\Sigma_\nu). \quad (5)$$

Note that in contrast to the usual case [10], i.e., in the absence of a boundary Σ^*, the right-hand side (rhs) of Eq. (5) does not vanish identically.

The rhs of Eq. (5) can be transformed further to a 4-volume integral in the following sequence of steps. First we express the integration element $d\Sigma_\mu$ via the normal vector n^*_μ as follows ($dx^j > 0$, for $j=1,2,3$):

$$d\Sigma_\mu = n^*_\mu dx^1\, dx^2\, dx^3; \quad n^*_\mu \equiv \delta_{\mu 0} - \frac{\partial t^*(\boldsymbol{x})}{\partial x^\mu}(1-\delta_{\mu 0}), \quad (6)$$

where $\delta_{\mu\nu}$ denotes the Kronecker symbol. Then, using the identity

$$\int_{t1}^{t2} dt\, \delta(t - t3) = 1$$

for the Dirac δ function with $t1 \leq t3 \leq t2$, we rewrite the rhs integral in (5) as

$$\int_{\Sigma^*(t1,t2)} d\Sigma_\mu \cdots \equiv \int_{V^4_\Sigma} d^4x\, \delta(t - t^*(\boldsymbol{x})) n^*_\mu \ldots, \quad (7)$$

where short-hand notations are introduced for the 4-dimensional volume $V^4_\Sigma = (t2 - t1)\int_{\Sigma^*(t1,t2)} dx^1\, dx^2\, dx^3$ which is shown as the rectangle *GBCD* in Fig. 1 ($|GB| = |CD|$). Evidently, the Dirac δ function allows us to extend integration in (7) to the unified 4-volume $V^4_U = \Delta x^4 \cup V^4_\Sigma$ of Δx^4 and V^4_Σ (the volume V^4_U is shown as the area *ABCE* in Fig. 1). Finally, with the help of notations

$$\Theta_{out} \equiv \Theta(t - t^*(\boldsymbol{x})); \quad \Theta_{in} \equiv 1 - \Theta_{out}, \quad (8)$$

it is possible to extend the left-hand side (lhs) integral in Eq. (5) from Δx^4 to V^4_U. Collecting all the above results, from Eq. (5) one obtains

$$\int_{V^4_U} d^4x \frac{d^3p}{p^0} \Theta_{in} p^\mu \partial_\mu \phi_{in} = \int_{V^4_U} d^4x \frac{d^3p}{p^0} p^\mu n^*_\mu [\phi_{in} - \phi_{out}] \times \Theta(-p^\nu n^*_\nu)\delta(t - t^*(\boldsymbol{x})). \quad (9)$$

Since the volumes Δx^4 and V^4_U are arbitrary, one obtains the kinetic equation for the distribution function of the in domain

$$\Theta_{in} p^\mu \partial_\mu \phi_{in}(x,p) = C_{in}(x,p) + p^\mu n^*_\mu [\phi_{in}(x,p) - \phi_{out}(x,p)] \times \Theta(-p^\nu n^*_\nu)\delta(t - t^*(\boldsymbol{x})). \quad (10)$$

Note that the general solution of Eq. (9) contains an arbitrary function $C_{in}(x,p)$ [the first term in the rhs of (10)] which identically vanishes while being integrated over the invariant momentum measure d^3p/p_0. Such a property is typical for a collision integral [10], and we shall discuss its derivation in the subsequent section. To shorten the notation, the domain of each distribution function will be denoted as a subscripted italic capital letter A or B ($A,B \in \{in, out\}$) to avoid confusion with Greek 4-indices.

Similarly, one can obtain the equation for the distribution function of the out domain

$$\Theta_{out} p^\mu \partial_\mu \phi_{out}(x,p) = C_{out}(x,p) + p^\mu n^*_\mu [\phi_{in}(x,p) - \phi_{out}(x,p)] \times \Theta(p^\nu n^*_\nu)\delta(t - t^*(\boldsymbol{x})), \quad (11)$$

where the normal vector n^*_ν is given by (6). Note the asymmetry between the rhs of Eqs. (10) and (11): for the space-like parts of hypersurface Σ^* the source term with $\Theta(-p^\nu n^*_\nu)$ vanishes identically because $p^\nu n^*_\nu > 0$. This reflects the causal properties of the equations above: propagation of particles faster than light is forbidden, and hence no particle can (re)enter the in domain.

III. COLLISION TERM FOR SEMI-INFINITE DOMAIN

Since in the general case $\phi_{in}(x,p) \neq \phi_{out}(x,p)$ on Σ^*, the δ-like terms in the rhs of Eqs. (10) and (11) cannot vanish simultaneously on this hypersurface. Therefore, the functions $\Theta^*_{in} \equiv \Theta_{in}|_{\Sigma^*} \neq 0$ and $\Theta^*_{out} \equiv \Theta_{out}|_{\Sigma^*} \neq 0$ do not vanish simultaneously on Σ^* as well. The $\Theta(x)$ is not uniquely defined at $x=0$, and, therefore, there is some freedom to choose a convenient value at $x=0$. Since there is no preference between in and out domains, it is assumed that

$$\Theta^*_{in} = \Theta^*_{out} = \Theta(0) = \tfrac{1}{2}, \quad (12)$$

but the final results are independent of this choice. This result can be understood by considering the limit $a \to 0$ of the following definition: $\Theta(x) \equiv \frac{1}{2}\lim_{a\to 0}[\tanh(x/|a|) + 1]$.

Now, the collision terms for Eqs. (10) and (11) can be readily obtained. Adopting the usual assumptions for the distribution functions [10–12], one can repeat the standard derivation of the collision terms [10,12] and get the desired expressions. We shall not recapitulate this standard part, but only discuss how to modify the derivation for our purpose. First, one has to start the derivation in the Δx^4 volume of the in domain and then extend it to the unified 4-volume $V^4_U = \Delta x^4 \cup V^4_\Sigma$ similarly to the preceding section. Then, the first part of the collision term for Eq. (10) reads ($A,B \in \{in, out\}$)

$$C^I_{in}(x,p) = \Theta^2_{in}(I^G[\phi_{in},\phi_{in}] - I^L[\phi_{in},\phi_{in}]), \quad (13)$$

$$I^G[\phi_A,\phi_B] \equiv \tfrac{1}{2}\int D^9P\, \phi_A(p')\phi_B(p'_1) W_{pp_1|p'p'_1}, \quad (14)$$

$$I^L[\phi_A,\phi_B] \equiv \tfrac{1}{2}\int D^9P\,\phi_A(p)\phi_B(p_1)W_{pp_1|p'p_1'}, \quad (15)$$

where the invariant measure of integration is denoted by $D^9P \equiv (d^3p_1/p_1^0)(d^3p'/p'^0)(d^3p_1'/p_1'^0)$, and $W_{pp_1|p'p_1'}$ is the transition rate in the elementary reaction with energy-momentum conservation given in the form $p^\mu + p_1^\mu = p'^\mu + p_1'^\mu$. The rhs of (13) contains the standard gain and loss terms which are defined by Eqs. (14) and (15), respectively, weighted by the "probability" of collision between particles from the in domain given by the square of the Θ_{in} function. The value $\Theta_{in}^2 \equiv 1$ is found inside of the in domain, whereas $\Theta_{in}^2 = \Theta_{in}^{*2} = 1/4$ at the boundary Σ^* because, according to (12), for each value of the distribution function ϕ_{in} in the rhs of (13), only half of the boundary Σ^* belongs to the in domain. This can be better understood by considering, first, the above-mentioned tangent representation for the Θ function, and then taking the limit $a \to 0$ next.

It is easy to understand that on Σ^* the second part of the collision term [according to Eq. (12)] is defined by the collisions between particles of in and out domains

$$C_{in}^{II}(x,p) = \Theta_{in}\Theta_{out}(I^G[\phi_{in},\phi_{out}] - I^L[\phi_{in},\phi_{out}]). \quad (16)$$

Again, the product $\Theta_{in}\Theta_{out}=0$ everywhere, except at the hypersurface Σ^*, where it corresponds to the probability of collision at Σ^* for the particles coming from both domains. This can be easily seen from the hyperbolic tangent representation of the Θ function.

Combining (10), (13), and (16), one gets the kinetic equation for the in domain

$$\Theta_{in}p^\mu\partial_\mu\phi_{in}(x,p) = C_{in}^I(x,p) + C_{in}^{II}(x,p) + p^\mu n_\mu^* \times[\phi_{in}(x,p) - \phi_{out}(x,p)]\Theta(-p^\nu n_\nu^*) \times\delta(t - t^*(\boldsymbol{x})). \quad (17)$$

The kinetic equation for the out domain can be derived similarly; then, it can be represented in the form

$$\Theta_{out}p^\mu\partial_\mu\phi_{out}(x,p) = C_{out}^I(x,p) + C_{out}^{II}(x,p) + p^\mu n_\mu^* \times[\phi_{in}(x,p) - \phi_{out}(x,p)]\Theta(p^\nu n_\nu^*) \times\delta(t - t^*(\boldsymbol{x})), \quad (18)$$

where the evident notations for the collision terms $C_{out}^I \equiv \Theta_{out}^2(I^G[\phi_{out},\phi_{out}] - I^L[\phi_{out},\phi_{out}])$ and $C_{out}^{II} \equiv \Theta_{in}\Theta_{out} \times(I^G[\phi_{out},\phi_{in}] - I^L[\phi_{out},\phi_{in}])$ are used.

Equations (17) and (18) can be represented also in a covariant form with the help of the function $\mathcal{F}^*(t,\boldsymbol{x})$. Indeed, applying the definition of the derivative of the implicit function to $\partial_\mu t^*(\boldsymbol{x})$, one can rewrite the external normal vector (6) as $n_\mu^* \equiv \partial_\mu\mathcal{F}^*(t,\boldsymbol{x})/\partial_0\mathcal{F}^*(t,\boldsymbol{x})$. Now using the inequality $\partial_0\mathcal{F}^*(t^*,\boldsymbol{x})>0$ and the following identities $\delta(\mathcal{F}^*(t,\boldsymbol{x})) = \delta(t - t^*(\boldsymbol{x}))/\partial_0\mathcal{F}^*(t^*,\boldsymbol{x})$, $\Theta_A \equiv \Theta(S_A\mathcal{F}^*(t,\boldsymbol{x}))$, one can write Eqs. (17) and (18) in a fully covariant form

$$\Theta_A p^\mu\partial_\mu\phi_A(x,p) = C_A^I(x,p) + C_A^{II}(x,p) + p^\mu\partial_\mu\mathcal{F}^* \times[\phi_{in}(x,p) - \phi_{out}(x,p)]\Theta(S_A p^\nu\partial_\nu\mathcal{F}^*) \times\delta(\mathcal{F}^*(t,\boldsymbol{x})), \quad (19)$$

where the notations $A \in in$, $S_{in}=-1$ ($A \in out, S_{out}=1$) are introduced for the in (out) domain.

For the continuous distribution functions on Σ^*, i.e., $\phi_{out}|_{\Sigma^*} = \phi_{in}|_{\Sigma^*}$, the δ-like source terms on the rhs of Eqs. (17) and (18) vanish and one recovers the Boltzmann equations. Moreover, with the help of the evident relations

$$-\partial_\mu\Theta_{in} = \partial_\mu\Theta_{out} = \delta(\mathcal{F}^*(t,\boldsymbol{x}))\partial_\mu\mathcal{F}^*(t,\boldsymbol{x}), \quad (20)$$

$$C_{in}^I + C_{in}^{II} + C_{out}^I + C_{out}^{II} = I^G[\Phi,\Phi] - I^L[\Phi,\Phi], \quad (21)$$

where $\Phi(x,p) \equiv \Theta_{in}\phi_{in}(x,p) + \Theta_{out}\phi_{out}(x,p)$, one can get the following result summing up Eqs. (17) and (18):

$$p^\mu\partial_\mu\Phi(x,p) = I^G[\Phi,\Phi] - I^L[\Phi,\Phi]. \quad (22)$$

In other words, the usual Boltzmann equation follows from the system (19) automatically *without any assumption* about the behavior of ϕ_{in} and ϕ_{out} on the boundary hypersurface Σ^*. Also, Eq. (22) is valid not only under condition (12), but for *any choice* $0<\Theta_A^*<1$ obeying Eq. (8).

In fact, the system (19) generalizes the relativistic kinetic equation to the case of the strong temporal and spatial inhomogeneity, i.e., for $\phi_{in}(x,p) \neq \phi_{out}(x,p)$ on Σ^*. Of course, one has to be extremely careful while discussing the strong temporal inhomogeneity (or discontinuity on the space-like parts of Σ^*) such as the so-called *time-like shocks* [13,14] because, as shown in the subsequent section, their existence contradicts the usual assumptions [10–12] adopted for distribution functions.

From the system (19) it is possible to derive the macroscopic equations of motion for the energy-momentum tensor by multiplying the corresponding equation with p^ν and integrating it over the invariant measure. Thus, Eq. (19) generates the following expression $[T_A^{\mu\nu} \equiv \int(d^3p/p^0)p^\mu p^\nu\phi_A(x,p)]$:

$$\Theta_A\partial_\mu T_A^{\mu\nu} = \int\frac{d^3p}{p^0}p^\nu C_A^{II}(x,p) + \int\frac{d^3p}{p^0}p^\nu p^\mu\partial_\mu\mathcal{F}^* \times[\phi_{in}(x,p) - \phi_{out}(x,p)]\Theta(S_A p^\rho\partial_\rho\mathcal{F}^*)\delta(\mathcal{F}^*(t,\boldsymbol{x})). \quad (23)$$

Similar to the usual Boltzmann equation the momentum integral of the collision term C_{in}^I vanishes due to its symmetries [10], but it can be shown that the integral of the second collision term C_{in}^{II} does not vanish because it involves two different distribution functions.

The corresponding system of equations for the conserved current $N_A^\mu \equiv \int(d^3p/p^0)p^\mu\phi_A(x,p)$ can be obtained by direct integration of the system (19) with the invariant measure

$$\Theta_A\partial_\mu N_A^\mu = \int\frac{d^3p}{p^0}p^\mu\partial_\mu\mathcal{F}^*[\phi_{in}(t,\boldsymbol{x}) - \phi_{out}(t,\boldsymbol{x})] \times\Theta(S_A p^\rho\partial_\rho\mathcal{F}^*)\delta(\mathcal{F}^*(t,\boldsymbol{x})). \quad (24)$$

The above equation does not contain the contribution from antiparticles (just for simplicity), but the latter can be easily recovered. Note that in contrast to (23) the momentum integral of both collision terms vanishes in Eq. (24) due to symmetries.

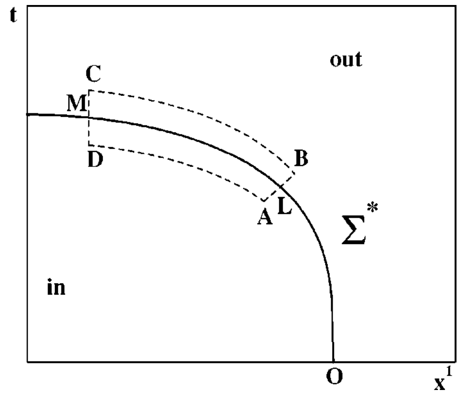

FIG. 2. Schematic two-dimensional picture of the integration contour to derive the boundary conditions (25)–(27) between the in and out domains. In the limit of a vanishing maximal distance $\Delta \to 0$ between the hypersurfaces AD and BC, both of these hypersurfaces are reduced to the part $\tilde{\Sigma}$ (an arc LM) of the boundary Σ^* between domains.

IV. CONSERVATION LAWS AT Σ^*

It is clear that Eqs. (19), (23), and (24) remain valid both for finite domains and for a multiple valued function $t = t^*(\boldsymbol{x})$ as well. To derive the whole system of these equations in the latter case, one has to divide the function $t^*(\boldsymbol{x})$ into the single-valued parts, but this discussion is beyond the scope of this paper. Using Eqs. (19), (23), and (24) we are ready to analyze the boundary conditions on the hypersurface Σ^*. The simplest way to get the boundary conditions is to integrate Eqs. (23) and (24). Indeed, integrating (23) over the 4-volume $V^4_{\tilde{\Sigma}}$ (shown as the area ABCD in Fig. 2) containing part $\tilde{\Sigma}$ of the hypersurface Σ^*, one obtains the energy-momentum conservation. Before applying the Gauss theorem to the lhs of (23), we note that the corresponding Θ_A function reduces the 4-volume $V^4_{\tilde{\Sigma}}$ to its part which belongs to the A domain. The latter is shown as area $ALMD$ ($BCML$) for $A \in in$ ($A \in out$) in Fig. 2. Then, in the limit of a vanishing maximal distance $\Delta \to 0$ between the hypersurfaces AD and BC in Fig. 2, the volume integral of the lhs of Eq. (23) can be rewritten as the two integrals $\int d\sigma_\mu T^{\mu\nu}_A$: the first integral is performed over the hypersurface $\tilde{\Sigma}$ shown as an arc LM in Fig. 2, and the second integral reduces to the same hypersurface but taken in the opposite direction, i.e., the ML arc in Fig. 2. Thus, the volume integral of the lhs of Eq. (23) vanishes in this limit for tensors $T^{\mu\nu}_A$ being continuous functions of coordinates, and we obtain

$$0 = \int_{V^4_{\tilde{\Sigma}}} d^4x \Theta_A \partial_\mu (T^{\mu\nu}_A(x,p))$$
$$\equiv \int_{V^4_{\tilde{\Sigma}}} d^4x \frac{d^3p}{p^0} p^\nu C^{II}_A(x,p) + \int_{V^4_{\tilde{\Sigma}}} d^4x \frac{d^3p}{p^0} \delta(\mathcal{F}^*(t,\boldsymbol{x})) p^\nu p^\mu \partial_\mu \mathcal{F}^*$$
$$\times [\phi_{in}(x,p) - \phi_{out}(x,p)] \Theta(S_A p^\rho \partial_\rho \mathcal{F}^*). \qquad (25)$$

Similarly to Sec. II, in the limit $\Delta \to 0$ the second integral on the rhs of (25) can be reexpressed as an integral over the closed hypersurface. Since the latter is arbitrary, then Eq. (25) can be satisfied if and only if the energy-momentum conservation occurs for every point of the hypersurface Σ^*

$$T^{\mu\nu}_{in\pm} \partial_\mu \mathcal{F}^*(t^*,\boldsymbol{x}) = T^{\mu\nu}_{out\pm} \partial_\mu \mathcal{F}^*(t^*,\boldsymbol{x}),$$

$$T^{\mu\nu}_{A\pm} \equiv \int \frac{d^3p}{p^0} p^\mu p^\nu \phi_A(x,p) \Theta(\pm p^\rho \partial_\rho \mathcal{F}^*). \qquad (26)$$

In deriving (26) from (25) we used the fact that the 4-volume integral of the second collision term C^{II}_A vanishes for finite values of distribution functions because of the Kronecker symbols. The results for the conserved current follow similarly from Eq. (24) after integrating it over the 4-volume $V^4_{\tilde{\Sigma}}$ and taking the limit $\Delta \to 0$

$$N^\mu_{in\pm} \partial_\mu \mathcal{F}^*(t^*,\boldsymbol{x}) = N^\mu_{out\pm} \partial_\mu \mathcal{F}^*(t^*,\boldsymbol{x}),$$

$$N^\mu_{A\pm} \equiv \int \frac{d^3p}{p^0} p^\mu \phi_A(x,p) \Theta(\pm p^\rho \partial_\rho \mathcal{F}^*). \qquad (27)$$

The fundamental difference between the conservation laws (26) and (27) and the ones of the usual hydrodynamics is that the systems (26) and (27) conserve the quantities of the outgoing from ($S_A = 1$) and incoming to ($S_A = -1$) in domain particles *separately*, whereas in the usual hydrodynamics only the sum of these contributions is conserved.

The trivial solution of Eqs. (26) and (27) corresponds to a continuous transition between in and out domains

$$\phi_{out}(x,p)|_{\Sigma^*} = \phi_{in}(x,p)|_{\Sigma^*}. \qquad (28)$$

This choice corresponds to the BD model [2]. The BD model gives a correct result for an oversimplified kinetics considered here. However, in the case of the first-order phase transition (or a strong crossover), which was a prime target of the hydro+cascade models [2,3], the situation is different. In the latter case the speed of sound either vanishes (or becomes very small) [15,16] and, hence, the rarefaction shock waves become possible [17–19]. The reason why the rarefaction shocks may exist lies in the anomalous thermodynamic properties [19] of the media near the phase transition region. In other words, on the boundary between the mixed and hadronic phases the rarefaction shocks are mechanically stable [19], whereas the compression shocks are mechanically unstable. This is also valid for the vicinity of the generalized mixed phase of a strong crossover.

One important consequence of the shock mechanical stability criterion is that the stable shocks necessarily are supersonic in the media where they propagate. The latter means that the continuous rarefaction flow in the region of phase transition is mechanically unstable as well, since a rarefaction shock, if it appears, propagates inside the fluid faster than the sound wave and, hence, it should change the fluid's state. Due to this reason the unstable hydrodynamic solutions simply do not appear [20,21].

Applying these arguments to the BD model, one concludes: for the first-order phase transition or strong crossover the sound wave in the (generalized) mixed phase may be unstable and the strong discontinuities of the thermodynamic quantities are possible [17–19]. The latter corresponds to the nontrivial solution of the conservation laws (26) and (27), which allows a discontinuity of the distribution function on two sides of the hypersurface Σ^*. Since there are twice the number of conservation laws compared to the usual hydrodynamics, it is impossible, as shown below, to build up the nontrivial solution of Eqs. (26) and (27) if the distribution functions on both sides of the hypersurface Σ^*, i.e., ϕ_{in} and ϕ_{out}, are taken to be the equilibrium ones.

Consider first the space-like parts of the hypersurface Σ^*. Then, Eqs. (26) and (27) for $S_A=-1$ vanish identically because of the inequality $p^\mu \partial_\mu \mathcal{F}^*(t^*,\boldsymbol{x})>0$, whereas for $S_A=1$ Eqs. (26) and (27) recover the usual hydrodynamical conservation laws at the discontinuity. However, it can be shown that the existence of strong discontinuities across the space-like hypersurfaces, the *time-like shocks* [13,14], is rather problematic because it leads to a contradiction of the basic assumptions adopted for the distribution function, even though the conservation laws (26) and (27) are formally fulfilled.

Indeed, according to the Bogolyubov's classification [11], a one-particle treatment can be established for a typical time Δt which, on one hand, should be much larger than the collision time τ_{coll}, and, on the other hand, should be much smaller than the relaxation time τ_{relax}

$$\tau_{coll} \ll \Delta t \ll \tau_{relax}. \qquad (29)$$

Similar to the usual Boltzmann equation (see also the discussions in Refs. [11,12]), in deriving the collision terms of Eq. (19) we implicitly adopted the requirement that the distribution function does not change substantially for times Δt less than the relaxation time τ_{relax}. However, at the discontinuities on the space-like parts of Σ^*, suggested in Refs. [13,14], the distribution function changes suddenly, i.e., $\Delta t=0$, and the left inequality (29) cannot be fulfilled at the *time-like shock*. Therefore, according to the Bogolyubov's classification [11], such a process, which is shorter than the typical collision time, belongs to a prekinetic or chaotic stage and, hence, cannot be studied at the level of a one-particle distribution function. It would instead require the analysis of a hierarchy of N-particle distribution functions, where N is the number of particles in the system. Thus, the existence of time-like shocks contradicts the adopted assumptions for a one-particle distribution. Their existence should be demonstrated first within the higher order distributions. This statement applies to several papers published by the Bergen group during the last few years where *time-like shocks* were attenuated in time using a phenomenological quasikinetic approach [22]. For the same reason, the use of equilibrium values for temperature and chemical potential in an attenuated time shock is rather problematic for time scales shorter than τ_{coll}. Note, however, that the discontinuities at the time-like parts of Σ^* (usual shocks) have no such restrictions and, hence, in what follows we shall analyze only these discontinuities.

V. BOUNDARY CONDITIONS AT Σ^* FOR A SINGLE DEGREE OF FREEDOM

Now, we have to find out whether it is possible to obtain the nontrivial solution of systems (26) and (27) using the parts of equilibrium distributions on the time-like segments of the hypersurface Σ^*. To simplify the presentation, first we consider the same kind of particles in both domains. It is convenient to transform the coordinate system $(t^*(\boldsymbol{x});\boldsymbol{x})$ into the special local frame $(t_L^*(\boldsymbol{x}_L);\boldsymbol{x}_L)$, which is the rest frame of discontinuity between the distributions ϕ_{in} and ϕ_{out}. This coordinate system will be indicated by the subscript L. The special local frame is defined as follows: the x axis should coincide with the local external normal vector to the hypersurface Σ^*, y- and z axes belong to the tangent hyperplane of Σ^*. In this case the external normal vector to the time-like parts of Σ^* is $n_\mu^*=(0;\partial_1\mathcal{F}_L^*;0;0)$, and one can readily check that the value of the derivative $\partial_1\mathcal{F}_L^*$ plays an important role in the conservation laws (26) and (27) only through the *cutoff* Θ function. Then, as in the theory of usual relativistic shocks [19–21], it can be shown that equations for the y- and z components of system (26) degenerate into the identities because of the symmetries of the energy-momentum tensor. Therefore, the number of independent equations at the discontinuity is 7: a switch-off criterion and six independent equations out of systems (26) and (27) [t- and x equations (26) and one equation (27) for two choices of $S_A=\{-1;+1\}$].

On the other hand the number of unknowns is 6 only: temperature T_{in}^* and baryonic chemical potential μ_{in}^* of the in domain, temperature T_{out}^* and baryonic chemical potential μ_{out}^* of the out domain, the collective velocity v_{in}^* of the in domain particles, and the collective velocity v_{out}^* of the particles of the out domain, which should be collinear to the normal vector n_μ^* in the rest frame of the discontinuity. A formal counting of equations and unknowns shows that it is impossible to satisfy the conservation laws (26) and (27) if the distribution functions on both sides are the equilibrium ones.

The last result means that instead of a traditional discontinuity we have to search for a principally new boundary condition on the hypersurface Σ^*. The analysis shows that there are two such possibilities with the equilibrium distribution function in the in domain and a special superposition of two *cutoff* equilibrium distributions for the out domain. The first possibility is to choose ϕ_{out} as follows:

$$\phi_{out}|_{\Sigma^*} = \phi_{in}(T_{in}^*,\mu_{in}^*,v_{in}^*)\Theta(p^1\partial_1\mathcal{F}_L^*) + \phi_{out}(T_{out}^*,\mu_{out}^*,v_{out}^*)\Theta(-p^1\partial_1\mathcal{F}_L^*), \qquad (30)$$

i.e., the distribution of outgoing particles from the in domain [the first term in the rhs of Eq. (30)] is continuous on the hypersurface Σ^*, whereas the distribution of the particles entering the in domain [the second term in the rhs of Eq. (30)] has a discontinuity on Σ^* which conserves the energy, momentum and baryonic charge because of the following boundary conditions $(\nu=\{0;1\})$:

$$T_{in-}^{1\nu}(T_{in}^*,\mu_{in}^*,v_{in}^*) = T_{out-}^{1\nu}(T_{out}^*,\mu_{out}^*,v_{out}^*), \qquad (31)$$

$$N^1_{in-}(T^*_{in},\mu^*_{in},v^*_{in}) = N^1_{out-}(T^*_{out},\mu^*_{out},v^*_{out}). \tag{32}$$

The above choice of boundary conditions at Σ^* reduces systems (26) and (27) for $S_A=1$ to the identities, and, hence, from the systems (26) and (27) there remain only three independent equations (31) and (32) for $S_A=-1$. Along with a switch-off criterion, these four equations can now be solved for six independent variables with the two variables chosen to be free for a moment. Thus, both the outgoing and incoming parts of the distribution function (30) can be chosen as the equilibrium ones, but with different temperatures, chemical potentials, and nonzero relative velocity $v^*_{rel} \equiv (v^*_{out} - v^*_{in})/(1-v^*_{out}v^*_{in})$ with respect to the distribution function ϕ_{in}.

Note a principal difference between this discontinuity and all the ones known in relativistic hydrodynamics: the out domain state consists, in general, of two different subsystems (fluxes) that have individual hydrodynamic parameters. It is clear that it is impossible to reduce three of those hydrodynamical parameters of one flux to those three of another flux because there are only two free variables out of six. Thus, together with the in domain flux there are in total three fluxes involved in this discontinuity. Therefore, it is appropriate to name it a *three flux discontinuity* in order to distinguish it from the ordinary shocks that are defined by maximum of two fluxes.

The outgoing component of the distribution (30) coincides with the choice of the boundary conditions suggested in the TLS model [3], whereas Eqs. (31) and (32) are missing in this model. For this reason, the TLS model fails to conserve energy, momentum, and charge. Note also that the lower values of the temperature $T^*_{out} \leqslant T^*_{in}$ and baryonic chemical potential $\mu^*_{out} \leqslant \mu^*_{in}$, which are typical for the rarefaction process considered in Ref. [3], should be compensated by an extra flow from the incoming particles to the in domain, i.e., v^*_{rel} should be opposite to the external normal vector n^*_μ in the rest frame of the *three flux discontinuity*. Therefore, such a discontinuity is analogous to the compression shock wave in relativistic hydrodynamics, and cannot appear in the rarefaction process for any of the hadronic species considered in Ref. [3].

Similarly, one can find another nontrivial solution of the systems (26) and (27) which corresponds to opposite choice of Eq. (30)

$$\phi_{out}|_{\Sigma^*} = \phi_{in}(T^*_{in},\mu^*_{in},v^*_{in})\Theta(-p^1\partial_1\mathcal{F}^*_L) + \phi_{out}(T^*_{out},\mu^*_{out},v^*_{out})\Theta(p^1\partial_1\mathcal{F}^*_L), \tag{33}$$

i.e., the incoming to the in domain component of the distribution above [the first term in the rhs of Eq. (33)] is continuous on hypersurface Σ^*, but the component leaving the in domain has a discontinuity on Σ^* which obeys the following conservation laws ($\nu=\{0;1\}$):

$$T^{1\nu}_{in+}(T^*_{in},\mu^*_{in},v^*_{in}) = T^{1\nu}_{out+}(T^*_{out},\mu^*_{out},v^*_{out}), \tag{34}$$

$$N^1_{in+}(T^*_{in},\mu^*_{in},v^*_{in}) = N^1_{out+}(T^*_{out},\mu^*_{out},v^*_{out}). \tag{35}$$

It is clear that both the outgoing and incoming components of the distribution (33) can be chosen as the equilibrium distribution functions. A simple analysis of the system (34) and (35) shows that for $T^*_{out} \leqslant T^*_{in}$ and $\mu^*_{out} \leqslant \mu^*_{in}$ the relative velocity v_{rel} in the local frame should be collinear to the external normal vector n^*_μ. Such a discontinuity is analogous to the rarefaction shock wave in the relativistic hydrodynamics. Thus, in contrast to the TLS choice, Eq. (33) should be used as the initial conditions for the out domain while studying the rarefaction process of matter with anomalous thermodynamic properties.

Now, we are ready to discuss how the nontrivial solutions (30) and (33) will modify the system of the hydro+cascade equations (19), (23), and (24). In what follows we assign the hydrodynamic equations to the in domain and the cascade ones to the out domain (the opposite case can also be considered). Inserting (30)–(32) into the in Eqs. (23) and (24), and into the out Eq. (19), one obtains the following system:

$$\Theta_{in}\partial_\mu T^{\mu\nu}_{in} = \int \frac{d^3p}{p^0} p^\nu C^{II}_{in}(x,p), \tag{36}$$

$$\Theta_{in}\partial_\mu N^\mu_{in} = 0, \tag{37}$$

$$\Theta_{out} p^\mu \partial_\mu \phi_{out}(x,p) = C^I_{out}(x,p) + C^{II}_{out}(x,p), \tag{38}$$

i.e., due to the boundary conditions (30)–(32) the δ-like terms have disappeared from the original system of equations. It is clear also that the source term in the rhs of Eq. (36) does not play any role because it is finite on the hypersurface Σ^* and vanishes everywhere outside Σ^*.

In order to obtain the system of hydro+cascade equations (36)–(38) for the nontrivial solution defined by Eqs. (33)–(35), the hydrodynamic description has to be extended to the outer ε-vicinity ($\varepsilon \to 0$) of the hypersurface Σ^*

$$\Theta_{out}\partial_\mu T^{\mu\nu}_{out} = \int \frac{d^3p}{p^0} p^\nu C^{II}_{out}(x,p), \tag{39}$$

$$\Theta_{out}\partial_\mu N^\mu_{out} = 0, \tag{40}$$

which in practice means that for Eqs. (33)–(35) one has to solve the cascade equation (38) a bit inside of the out domain infinitesimally close to Σ^* in order to remove the δ-like term in (38) and move this term to the discontinuity on the hypersurface Σ^*.

The remarkable feature of the system of hydro+cascade equations (36)–(40) is that each equation automatically vanishes outside the domain where it is specified. Also, by the construction, it is free of the principal difficulties of the BD and TLS models discussed above. The question how to conjugate the *three flux discontinuity* with the solution of the hydro equations (36), (37), (39), and (40) will be discussed in the next section.

VI. BOUNDARY CONDITIONS AT Σ^* FOR MANY DEGREES OF FREEDOM

In order to apply the above results to the description of the QGP-HG phase transition occurring in relativistic nuclear collisions, it is necessary to take into account the fact that the real situation differs from the previous consideration in two respects. The first one is that in the realistic case inside the in domain there should exist the QGP, whereas it should not appear in the out domain. Of course, the discussion of the QGP kinetic theory is a much more complicated problem and lies beyond the scope of this work. For our purpose it is sufficient to generalize the equations of motion (36)–(40) inside domains and the conservation laws (26) and (27) between these domains to the realistic case. Such a generalization can be made because in the case of the QGP-HG phase transition there will also be an exchange of particles between the in and out domains which must be accounted for by the δ-like source terms in the transport equations. The only important difference from the formalism developed in the preceding sections is that QGP must hadronize while entering the out domain, whereas the hadrons should melt while entering the in domain. Note, however, that in relativistic hydrodynamics one has to assume that all reactions, i.e., the QGP hadronization and melting of hadrons in this case, occur instantaneously. Under this assumption one can justify the validity of the equations of motion (36)–(40) and the conservation laws between QGP and HG on the boundary Σ^*.

The second important fact to be taken into account is that some hadrons have the large scattering cross sections with other particles and some hadrons have the small cross sections, because of this, different hadrons participate in the collective flow differently. A recent effort [23,24] to classify the inverse slopes of the hadrons at SPS lab energy 158 GeV·A led to the conclusion that the most abundant hadrons, e.g., pions, kaons, (anti)nucleons, Λ hyperons, etc., participate in the hadron rescattering and resonance decay till the very late time of expansion, whereas Ω hyperons, J/ψ, and ψ' mesons practically do not interact with the hadronic media and, hence, the freeze-out of their transverse momentum spectra (*kinetic freeze-out*) may occur just at hadronization temperature T_H. Therefore, the inverse slopes of the Ω, J/ψ, and ψ' particles are a combination of the thermal motion and the transversal expansion of the media from which these particles are formed.

These results for the Ω baryons and ϕ mesons were obtained within the BD and TLS models, whereas for the J/ψ and ψ' mesons it was suggested for the first time in Refs. [23,24]. Later these results were further refined in Ref. [25] by the simultaneous fit with only one free parameter (the maximal value of transversal velocity) of the measured Ω [26,27], J/ψ and ψ' [28] transverse momentum spectra in Pb+Pb collisions at 158 GeV·A that are frozen out at hadronization temperature T_H. The experimental situation with the ϕ mesons at SPS is, unfortunately, not clarified yet because the results of the NA49 [29] and NA50 [30] Collaborations disagree. The analysis of the transverse momentum spectra of Ω hyperons [31,32] and ϕ mesons [31] reported by the STAR Collaboration for energies $\sqrt{s}=130$ A·GeV in Refs. [33,34], respectively, and for $\sqrt{s}=200$ A·GeV in Ref. [32] shows that this picture remains valid for RHIC energies as well.

It is easy to find that for particles like ϕ, Ω, J/ψ, and ψ', which weakly interact with other hadrons, the distribution function ϕ_{out} should coincide with ϕ_{in}

$$\phi_{out}|_{\Sigma^*} = \phi_{in}(T_{in}^*, \mu_{in}^*, v_{in}^*)\Theta(p^1\partial_1\mathcal{F}_L^*), \tag{41}$$

where, in contrast to (30), there is no incoming component of the distribution because the noninteracting particles cannot rescatter and change their velocity. Note also that a small modification of the incoming part of J/ψ momentum distribution due to decay of heavier charmonia in the out domain can be safely neglected. Remarkably, the cascade initial condition (41) exactly coincides with the one used in the TLS model. Therefore, the main TLS conclusions [3] on the ϕ mesons and Ω hyperons remain unchanged, whereas for hadrons with large scattering cross sections the TLS conclusions may change significantly.

Omitting the contributions of weakly interacting hadrons from the components of the energy-momentum tensor and baryonic 4-current, one can generalize the boundary conditions (26) and (27) on the hypersurface Σ^* between the domains and formulate the energy-momentum and charge conservation laws in terms of the parts of the *cutoff* distribution functions. For definiteness we shall consider the first-order phase transition between QGP and hadronic matter throughout the rest of this work. The case of the second-order phase transition can be analyzed similarly. In terms of the local coordinates $(t_L^*(\boldsymbol{x}_L);\boldsymbol{x}_L)$, introduced in Sec. V, the conservation laws (26) and (27) can be generalized as follows ($\nu=\{0;1\}$):

$$\begin{aligned}\alpha_q \sum_{Q=q,\bar{q},\ldots} & T^{1\nu}_{Q\pm}(T_{in}^*, Z_Q\cdot\mu_{in}^*, v_{in}^*) \\ & + (1-\alpha_q)\sum_{H=\pi,K,\ldots} T^{1\nu}_{H\pm}(T_{in}^*, Z_H\cdot\mu_{in}^*, v_{in}^*) \\ & = \sum_{H=\pi,K,\ldots} T^{1\nu}_{H\pm}(T_{out}^{\pm}, Z_H\cdot\mu_{out}^{\pm}, v_{out}^{\pm}),\end{aligned} \tag{42}$$

$$\begin{aligned}\alpha_q \sum_{Q=q,\bar{q},\ldots} & N^{1}_{Q\pm}(T_{in}^*, Z_Q\cdot\mu_{in}^*, v_{in}^*) \\ & + (1-\alpha_q)\sum_{H=\pi,K,\ldots} N^{1}_{H\pm}(T_{in}^*, Z_H\cdot\mu_{in}^*, v_{in}^*) \\ & = \sum_{H=\pi,K,\ldots} N^{1}_{H\pm}(T_{out}^{\pm}, Z_H\cdot\mu_{out}^{\pm}, v_{out}^{\pm}),\end{aligned} \tag{43}$$

where α_q is the volume fraction of the QGP in a mixed phase, and the Q sums of the energy-momentum tensor and baryonic 4-current components, denoted as

$$T^{\mu\nu}_{Q\pm} \equiv \int \frac{d^3p}{p^0} p^\mu p^\nu \phi_Q(x,p)\Theta(\pm p^\rho\partial_\rho\mathcal{F}^*), \tag{44}$$

$$N^{\mu}_{Q\pm} \equiv \int \frac{d^3p}{p^0} p^\mu Z_Q \phi_Q(x,p)\Theta(\pm p^\rho\partial_\rho\mathcal{F}^*), \tag{45}$$

run over all corresponding degrees of freedom of QGP. The H sums also run over all hadronic degrees of freedom. In Eqs. (42) and (43) Z_Q and Z_H denote the baryonic charge of the corresponding particle species.

Now, from Eqs. (44) and (45) it is clearly seen that the correct hydro+cascade approach requires more detailed information about the microscopic properties of QGP than is usually provided by traditional equations of state. To proceed further we assume that those components are known. The general approach to calculate the angular and momentum integrals in Eqs. (44) and (45) was developed in Ref. [7] and was applied to the massive Boltzmann gas description in Refs. [7,35].

The important difference between conservation laws (42), (43) and (26), (27) is that in the out domain the temperature T^-_{out}, chemical potential μ^-_{out}, and relative velocity v^-_{out} of the incoming to Σ^* hadrons should differ from the corresponding quantities T^+_{out}, μ^+_{out}, and v^+_{out} of the outgoing from Σ^* particles, and both sets should differ from the quantities T^*_{in}, μ^*_{in}, and v^*_{in} of the in domain. In order to prove this statement, it is necessary to compare the number of equations and number of unknowns for the two distinct cases. Namely (i) if the initial state is in the mixed QGP-HG phase, and (ii) if the initial state belongs to the QGP.

In case (i) there are ten equations and ten unknowns.

(1) The equations are as follows: six conservation laws from Eqs. (42) and (43); value of the initial energy density; value of the initial baryonic density; the relation between initial temperature T^*_{in} and the baryonic chemical potential μ^*_{in} taken at the phase boundary; and the switch-off criterion.

(2) The unknowns are as follows: three temperatures T^*_{in}, T^-_{out}, T^+_{out}; three chemical potentials μ^*_{in}, μ^-_{out}, μ^+_{out}; three velocities v^*_{in}, v^-_{out}, v^+_{out} defined in the rest frame of a discontinuity; and the QGP fraction volume α_q.

Thus, in this case, one can find a desired solution of the system of ten transcendental equations, which is the most general form of the three flux discontinuity introduced by Eqs. (30)–(32).

To complete the solution of hydro equations (36), (37), (39), and (40), one must find the value of velocity v^*_{in} from the system of ten transcendental equations discussed above. This velocity then defines an ordinary differential equation $dx^1_L/dt^*_L=-v^*_{in}$ for the hypersurface Σ^* in the rest frame of matter of the in domain, which must be solved simultaneously with the hydro equations.

If initial state belongs to the interior of the QGP phase, case (ii), then the usual hydro solution will be valid till the system reaches the boundary with the mixed phase, from which the nontrivial discontinuity described by Eqs. (42) and (43) will begin. The differences from the previously considered case are now clear: in contrast to case (i), the volume fraction of QGP is fixed to unit $\alpha_q=1$; the energy and baryonic charge densities are no longer independent, but are completely defined by the temperature and baryonic chemical potential, which are connected by the entropy conservation for the continuous hydro solution in QGP.

Therefore, in case (ii) there are nine equations and nine unknowns, which are as follows.

(1) The equations are: six conservation laws from Eqs. (42) and (43); temperature dependence of the baryonic chemical potential $\mu^*_{in}=\mu^*_{in}(T^*_{in})$ due to the entropy conservation; the relation connecting temperature T^*_{in} and baryonic chemical potential μ^*_{in}, since they belong to the phase boundary; and the switch-off criterion.

(2) The unknowns, except for the fixed volume fraction $\alpha_q=1$, are the same as in case (i).

Again, the number of unknowns matches the number of equations, and the procedure to solve the system of hydro equations (36), (37), (39), and (40) simultaneously with the boundary conditions (42) and (43) is the same as in case (i).

Now, it is appropriate to discuss the switch-off criterion $\mathcal{F}^*(t,\boldsymbol{x})=0$ in more detail. By the construction of the hydro +cascade approach, the cascade treatment should be applied when hydrodynamics starts to lose its applicability: according to the original assumption the hydro equations (36), (37), (39), and (40) work well inside of the 4-volume surrounded by the hypersurface Σ^* and in the outer ε vicinity $(\varepsilon\to 0)$ of Σ^* [see also a discussion after Eq. (38)], whereas just outside of this domain the thermal equilibrium dismantles and one cannot use the *cutoff* equilibrium distributions interior of the out domain. Consequently, a switch-off criterion should be formulated solely for some quantity defined in the outer ε vicinity of hypersurface Σ^*, and it has to define the bounds of applicability of thermal equilibration and/or hydrodynamic description. Note that in the BD and TLS models this did not matter because both groups kept the cascade initial conditions as close as possible to the output of hydro. However, in the case of the *three flux discontinuity* on the time-like parts of hypersurface Σ^* the proper use of the switch-off criterion plays a decisive role in the construction of the mathematically correct hydro+cascade solution (see also a discussion of the freeze-out criterion in Refs. [6,7]). It is clear that, in contrast to the BD and TLS formulations, the switch-off criterion may generate a sizable effect while applied to interior of hadronic phase. This is so, because even a small difference (just a few MeV) between the temperature T^*_{in}, which belongs to the phase transition region, and temperatures T^-_{out} and T^+_{out} of the out domain may lead to a tremendous flow of outgoing hadrons because of the enormous latent heat of the QGP.

VII. CONCLUDING REMARKS

In the preceding sections we have derived a system of relativistic kinetic equations which describes the particle exchange between two domains separated by the hypersurface of arbitrary properties. We showed that the usual Boltzmann equation for the following sum of two distributions $\Phi(x,p) \equiv \Theta_{in}\phi_{in}(x,p)+\Theta_{out}\phi_{out}(x,p)$ automatically follows from the derived system, but not vice versa. Integrating the kinetic equations, we derived the system of the hydro+cascade equations for a single degree of freedom. Remarkably, the conservation laws on the boundary between two domains conserve the incoming and outgoing components of the energy, momentum, and baryonic charge separately, leading to twice the number of conservation laws on the separating hypersurface compared to the usual relativistic hydrodynamics. Then, we showed that for a single degree of freedom these boundary conditions between domains can be satisfied only by a special superposition of two *cutoff* equilibrium distributions for the out domain. Since the obtained discontinuity has three irreducible fluxes, it is named a *three flux discontinuity*, in contrast to usual shocks defined by two fluxes. It was also shown that the TLS-like choice of the boundary conditions, in contrast to the expectation of Ref. [3], corresponds to an analog of the compression shock in traditional hydrodynamics, and, therefore, cannot be used to model the rarefaction process.

Then, we showed that existence of the *time-like shocks* [13,14], formally rederived by this formalism, contradicts the usual assumptions adopted for the one-particle distributions and, hence, the solution of this problem requires the analysis of higher order distribution functions. Therefore, in the rest of the paper we concentrated on a detailed analysis of the discontinuities at the time-like hypersurfaces, i.e. the space-like shocks in terms of Refs. [13,14]. These results were then generalized to a more realistic case: when the mixed QGP-HG phase exists in the in domain and hadrons exist in the out domain. Such a generalization also required the exclusion of the hadrons with the small scattering cross section (like Ω, J/ψ, and ψ' particles) from the boundary conditions between domains. As we showed in the preceding section, the presence of the first-order phase transition makes the resulting system of transcendental equations more complicated than in the case of a single degree of freedom.

It turns out that a minimal number of variables in this discontinuity is either nine or ten, depending on the location of the initial state on the phase diagram. Therefore, on the hadronic side the *three flux discontinuity* should have two different flows with their own temperatures, chemical potentials, and collective velocities. This solution has a number of unique features in comparison with usual shocks.

(1) This discontinuity may generate a very strong, explosive-like flow of outgoing particles from the in domain, first because a huge latent heat of QGP is involved, and, second due to an extra momentum associated with the *cutoff* distribution. Indeed, considering the outgoing component of the distribution $\phi_{out}\Theta(p^1\partial_1\mathcal{F}^*_L)$ for massless pions in the frame where this function maximally resembles the noncut Boltzmann distribution, i.e., in the rest frame of the latter, one finds a nonvanishing collective velocity $v_\pi=(1+v_\sigma)/2$. Here, $v_\sigma \equiv dR_\perp/dt$ ($|v_\sigma| \leqslant 1$ for time-like parts of Σ^*) denotes the transversal radius velocity in this frame.

(2) The strong explosive flow of outgoing particles is localized at the time-like parts of the hypersurface Σ^*, whereas at the space-like parts of Σ^* there will be a continuous flow. It is even possible that for some choice of parameters the space-like boundary may be absent.

(3) The particle density of outgoing pions will strongly depend on the speed of the transversal radius expansion. Thus, for massless pions the particle density found according to the Eckart definition [10] is $\rho_\pi = [\rho_\pi(T^+_{out})]/4\sqrt{(1-v_\sigma)^3(3+v_\sigma)}$, i.e., it is smaller for all $v_\sigma > -1$ than the thermal particle density $\rho_\pi(T^+_{out})$. Therefore, the two particle correlations off the low particle density regions should be reduced. Since the situation $v_\sigma \gg -1$ is typical for the beginning of the transversal expansion [3], the main contribution to the transversal pion correlations will come from the later times of expansion. Thus, it is possible that the space-time region which defines the side and out pion correlation radii will be essentially more localized both in space and time than in traditional hydrodynamic solutions.

(4) Because there are two fluxes in the out domain, they will interact with each other. The resulting distribution should, of course, be found by the cascade simulations, but it is clear that the fastest of them will decelerate and the cold one will reheat. Beside the possibility to accelerate or decelerate the outgoing transversal flow more rapidly than in the BD and TLS models, the *three flux discontinuity* may naturally generate some turbulence patterns in the out domain.

Taking into account all these features, along with the fact that neither the BD nor TLS boundary conditions have such a strong discontinuity, we conclude that the *three flux discontinuity* opens a principally new possibility not only to resolve the HBT puzzle [8], but also to study some new phenomena, like a turbulence pattern, associated with a new kind of shock, a *three flux discontinuity*, in relativistic hydro +cascade approach.

Despite the reasonably good description of the one-particle spectra of the most abundant hadrons, even such a sophisticated model as the TLS one badly overestimates both of the transverse radii measured by pion interferometry like other hydrodynamic models. This is a strong indication that the hydro part of all existing hydro+cascade and hydrodynamic models requires an essential revision. How this revision will affect the present BD and TLS results is unclear at the moment, but the solution of the HBT puzzle [8] should serve as a good test for the correct picture of the space-time evolution during the posthadronization stage. The additional tests for the correct hydro+cascade equations should be the reproduction of three recently established signals of the deconfinement phase transition, i.e., the pion kink [36,37] seen at lab energy of $\sim$30 GeV$\cdot$A, the K^+/π^+ peak at the same lab energy [36] (the strangeness horn), and the plateau [38] in the inverse slope of the $K^{\pm}$ transverse momentum spectra at the whole range of the SPS energies (the step in caloric curves) measured by the NA49 Collaboration [39,40]. It is also necessary to check other predictions of the statistical model of the early stage [36], namely the anomalies in the entropy to energy fluctuations [41] (the "shark fin") and in strangeness to energy fluctuations [42] (the "tooth"), because both the shark fin and tooth may be sensitive to the turbulence behavior due to energy dissipation.

Note, however, that the completion of this task requires an additional research of the hydro+cascade approach. First, it is necessary to develop further the microscopic models of the QGP equation of state in order to find out the components of the *cutoff* energy-momentum tensor and baryonic 4-current required by Eqs. (42)–(45). This can be done, for example, within the phenomenological extensions [43–45] of the Hagedorn model. Second, a similar problem for hadrons should be solved as well, otherwise, as we discussed in the preceding section, the *switch-off* criterion from the hydro to cascade cannot be formulated correctly within the hydro +cascade approach. And, finally, for practical modeling it is necessary to formulate a mathematical algorithm to solve simultaneously the system of hydro+cascade equations (36)–(40) with the boundary conditions (42) and (43) between the hydro and cascade domains. These problems, however, should be considered elsewhere.

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