Goldstone bosons in the pairing Hamiltonian: The path integral approach

M. B. Barbaro and A. Molinari
Dipartimento di Fisica Teorica, Università di Torino and Istituto Nazionale de Fisica Nucleare, Sezione di Torino, Torino, Italy

F. Palumbo
INFN, Laboratori Nazionali di Frascati, Frascati, Italy

M. R. Quaglia
Dipartimento di Fisica, Università di Genova and Istituto Nazionale di Fisica Nucleare, Sezione di Genova, Genova, Italy
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As a first step to derive the interacting boson model from a microscopic nuclear Hamiltonian, we bosonize the pairing Hamiltonian in the framework of the path integral formalism, respecting both the particle number conservation and the Pauli principle. Special attention is paid to the role of the Goldstone bosons. We construct the saddle point expansion which reproduces the sector of the spectrum associated with the addition or removal of nucleon pairs.

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I. INTRODUCTION

The problem of relating the interacting *boson* model (IBM), so successful in interpreting the low-energy nuclear phenomenology, to an underlying *fermionic* nuclear Hamiltonian has been attacked in a number of papers (see in particular [1–3]).

In principle, a possible procedure to deal with it could be (1) to derive an effective interaction in a chosen model space, (2) to express it in terms of pairing, quadrupole and other forces, and (3) to bosonize the model Hamiltonian thus obtained.

Less ambitiously one could start by assuming from the outset a multipole expansion of the effective interaction in the chosen model space.

Following these steps one should be able to relate the parameters appearing in the IBM to those occurring in the fundamental nucleon-nucleon interaction and the bosonic operators to the original fermionic ones.

A vast literature exists concerning the problem of mapping the fermionic model space into the bosonic one [4]. However, it appears to us that the actual realization of these mappings still deserves futher analyses, especially in connection to the nature of the interacting bosons, namely, whether they are Goldstone bosons or not. Let us comment on why the Goldstone bosons should play a role in nuclear structure and how they relate to some of the bosons of the IBM.

In this connection, we recall that underlying the IBM is the recognition that the nuclear interaction is attractive enough to form pairs of nucleons. In low-energy phenomena these conserve their identity inside nuclei, thus becoming the relevant degrees of freedom for the collective nuclear levels. The analogy between these pairs of nucleons and the Cooper pairs of superconductivity is clearly suggestive and indeed the simplified version of the BCS theory of superconductivity represented by the so-called pairing Hamiltonian was applied to atomic nuclei long ago and was dealt with by the Bogoliubov-Valatin (BV) transformation.

The pairing Hamiltonian, in its simplest form, governs the dynamics of pairs of nucleons moving in a mean field and coupled to zero angular momentum (referred to as *s* bosons in the IBM). The great success of the IBM stemmed from the introduction of pairs of nucleons coupled to angular momentum 2 as well, the *d* bosons, which can also be treated through a generalized BV transformation. In all cases the BV transformation does not conserve the number of particles. Although elaborated methods have lately been devised to preserve the number of particles [4], it is of importance to realize that this feature of the BV transformation is connected to the central concept lying at the core of superconductivity, namely, the spontaneous breaking of the global gauge invariance related to the particle number conservation. Thus in a superconducting system there must exist an associated Goldstone boson. Now in the IBM nuclei are indeed viewed as superconducting systems, even if only approximately, since they are finite systems. As a consequence, the IBM should embody at least one Goldstone boson. Other Goldstone bosons should of course appear if other symmetries, like the rotational one, are (approximately) spontaneously broken.

In infinite systems the Goldstone fields carry specific signatures: indeed they live in the coset space of the spontaneously broken group U(1) of the global gauge transformation $e^{iq\Lambda/\hbar}$, q being the electric charge, with respect to the unbroken subgroup Z_2 consisting of the gauge transformations associated with $\Lambda=0$ and $\Lambda=\pi\hbar/q$ and display only derivative interactions. If these distinctive features survive in finite systems, the identification of Goldstone bosons becomes of relevance not only for a deeper understanding of the bosonization mechanism, but also for a convenient choice of the variables. One could indeed choose bosonic fields not living in the coset space, but at the price of rendering the formalism quite cumbersome, as exemplified by the chiral physics [5].

Motivated by the above heuristic considerations, in this paper we present an investigation of the program outlined at the beginning of this Introduction, limiting ourselves to confronting the third point, namely, to considering only the pairing interaction. Yet, since, in a model space, the pairing interaction is an important component of any realistic effective

interaction [6], then our work may also be viewed as a first step in the derivation of the IBM.

We will use the path integral formalism where the problem of superconductivity is readily solved independently of which quantum numbers the Cooper pairs carry [5]. The basic question we address is to what extent the features of superconductivity in an infinite system, in particular the signatures of the Goldstone fields, survive in a finite system. Our main result is that indeed these signatures are exactly preserved in the pairing model. Accordingly we argue that the extension of the latter to include higher multipolarities of the force should be easily feasible, as confirmed by recent work [7].

The plan of the present paper is as follows.

In Sec. II we discuss from the point of view illustrated above the well known spectrum of the pairing Hamiltonian, which is characterized by the two quantum numbers n (the number of pairs) and s (the number of broken pairs), naturally related to two types of excitations. Those associated with n relate to the addition or removal of a correlated pair of nucleons and can be viewed as Goldstone bosons stemming from the (almost) broken conservation of particle number. Those related to s, namely, to the breaking of pairs (seniority excitations), we view as corresponding to the Higgs particles.

In Sec. III we derive the Euclidean path integral formulation and introduce auxiliary bosonic fields via the Hubbard-Stratonovitch transformation.

In Secs. IV and V we set up a saddle point expansion of the effective action of the auxiliary fields. The expansion parameter turns out to be the inverse of the energy

$$M = \frac{g\Omega}{2}, \tag{1}$$

g being the strength of the pairing force and $\Omega = j + 1/2$ the pair degeneracy of a single-particle level with angular momentum j (for the sake of simplicity we shall consider the case of one level only).

Within such an expansion we succeed in reproducing the excitation energies associated with the addition or removal of pairs of nucleons. We do not explore, however, the seniority excitations, which are not easily accommodated in the framework of our expansion since their energies are of the order of M. A reasonable estimate, based on the available nuclear phenomenology [8], yields $M \geqslant 3$ MeV in the region of the Sn isotopes. As anticipated, our analysis shows that the pairing model encodes in a striking way the basic features of spontaneous symmetry breaking in an infinite system: *indeed the field which describes the Goldstone excitations lives in the coset of the* $U(1)$ *symmetry related to particle number conservation with respect to the conserved* Z_2 *subgroup and displays only derivative couplings*.

Then we investigate two paths for selecting a sector of a given number of nucleons: one based on the chemical potential (Sec. VI) and the other on a projection operator (Sec. VII). Since at zero temperature, even in the presence of the chemical potential, the number of particles does not fluctuate, the two formalisms lead exactly to the same result.

In Sec. VIII we derive the Hamiltonian of the s bosons. In the present case, since the spectrum of the pairing model is known, this Hamiltonian has been determined in a simpler, direct way by Talmi [16]. This work, however, ignores the Pauli principle, which, therefore, must be added *a posteriori* as an *ad hoc* prescription. Moreover our approach has the merit that it can be applied as well when forces of higher multipolarity are active, thus opening the way to the microscopical derivation of the IBM. In Sec. IX we present our conclusions and outlook.

For the sake of completeness, in concluding this Introduction, we mention two approaches to the problem of bosonization attempted in the past.

The first, also limited to the pairing interaction, was based on the use of even Grassmann variables in the generating functional [10,11], but substantial difficulties were met in its extension to include the quadrupole interaction.

The second was carried out by Mukherjee and Nambu [12] who explored in depth the connection between the BCS theory of superconductivity and the IBM. These authors, linearizing in the frame of the mean field approach a nuclear BCS Hamiltonian embodying a contact four-nucleon interaction, accounted for second order corrections. They were thus able to derive a bosonic Hamiltonian expressed as a sum of Casimir operators and hence qualitatively of the IBM type. However, these authors actually explored an infinite homogeneous system and dealt with finite nuclei only in some approximate schemes, thus failing to enforce the particle number conservation. Moreover their approach appears hardly suitable for a realistic derivation of the IBM model, in particular as far as the fermion-boson mapping is concerned.

II. THE EXCITATIONS OF THE PAIRING HAMILTONIAN AS GOLDSTONE AND HIGGS BOSONS

According to the framework outlined in the Introduction we consider the schematic pairing Hamiltonian, which accounts for much of the physics of the low-energy spectra of nuclei. Our aim is to establish a connection between its well known spectrum and the (almost) spontaneous breaking of the particle number conservation, which entails the existence of Goldstone and Higgs modes.

In its simplest version the pairing Hamiltonian describes a system of interacting bounded identical nucleons living in one single-particle level of angular momentum j and reads

$$\hat{H} = \sum_{m=-j}^{j} e_m \hat{\lambda}_m^\dagger \hat{\lambda}_m - g\hat{A}^\dagger \hat{A} \tag{2}$$

where

$$\hat{A} = \sum_{m>0} (-1)^{j-m} \hat{\lambda}_{-m} \hat{\lambda}_m, \tag{3}$$

$\hat{\lambda}^\dagger$, $\hat{\lambda}$ are the usual creation and annihilation nucleon operators, m is the third component of the angular momentum j, the e_m are the (negative) single-particle energies, and g is the strength of the pairing force. For the sake of simplicity we set $e_m = e$ independent of m. In the Conclusions we will mention how and when the level dispersion can be accounted for [13].

In this paper we shall consider an even number of identical nucleons only. In such a case the energy spectrum is given by the well-known formula [14]

$$E_{n,s} = 2en - gn(\Omega - n + 1) + gs(\Omega - s + 1), \quad n \geqslant s, \quad (4)$$

where n is the number of pairs and s the pair seniority.[1] Clearly Eq. (4) holds valid for

$$n \leqslant \Omega \quad (5)$$

and not only for $s \leqslant n$, but for

$$s \leqslant \Omega - n \quad (6)$$

as well, as a consequence of the Pauli principle.

In an infinite system the energy of the Goldstone bosons vanishes with the associated quantum number. This does not occur in a finite system, but, to the extent that the energy spectrum of the latter displays a pattern similar to that of an infinite system, it should exhibit two quite different energy scales. Actually the excitation energies associated with both the quantum numbers n and s appear to be of order $g\Omega$. However, the Goldstone nature of the energy spectrum associated with the quantum number n is clearly apparent when one considers the excitations with respect to the minimum. This, if n is viewed as a continuous variable, occurs for

$$\nu_0 = \frac{1}{2}(\Omega + 1) - \frac{e}{g}. \quad (7)$$

In fact, since n assumes only integer values, the minimum of Eq. (4) takes place for

$$n_0 = [\nu_0], \quad (8)$$

$[\cdots]$ meaning the integral part. Introducing then the shifted quantum number

$$\nu = n - n_0 \quad (9)$$

Eq. (4) becomes

$$E_{n_0+\nu,s} = g\nu^2 + 2g\nu(n_0 - \nu_0) - gn_0(2\nu_0 - n_0) + gs(\Omega - s + 1). \quad (10)$$

Now we see that the addition (or removal) of one pair of nucleons with respect to the ground state requires an energy of order g: this is the energy of the Goldstone boson. Instead, the energy required to break a pair, the seniority energy, is of order $g\Omega$: this is the energy of a Higgs boson. Such correspondence can be made more strict by taking the limits $g \to 0$, $\Omega \to \infty$, with $g\Omega$ constant: then the Higgs energy would stay finite, while the energy of the Goldstone boson would vanish.

We should now point out that using the physical values for Ω, g, and e appropriate, for example, to the Sn nucleus (these can be taken from [8,9]), one would obtain a value for ν_0 corresponding to an unphysical nucleus. However, the excitation energies of both modes, measured with respect to the minimum [see Eq. (10)] are *essentially independent* of the single-particle energy e. Hence our argument, although heuristic, remains qualitatively correct. The real justification of its validity will be given in the next section.

Finally, since in the pairing Hamiltonian the degeneracy Ω is fixed by the model space, the excitation energies related to the quantum numbers ν and s can be predicted once two conditions are chosen in order to fix the parameters e and g.

III. THE GENERATING FUNCTIONAL

As is well known [15] the path integral must be evaluated in its discretized form. The discretized Euclidean action of our system is

$$S = \tau \sum_{t=-N_0/2}^{N_0/2-1} \left\{ -g\bar{A}(t)A(t-1) + \sum_{m=-j}^{j} [\bar{\lambda}_m(t)(\nabla_t^+ + e)\lambda_m(t-1)] \right\}, \quad (11)$$

where τ is the time spacing, N_0 the number of points on the time lattice,

$$(\nabla_t^{\pm} f)(t) = \pm \frac{1}{\tau}[f(t \pm 1) - f(t)], \quad (12)$$

and

$$Z = \int [d\bar{\lambda} d\lambda] e^{-S} \quad (13)$$

the generating functional. Moreover λ, $\bar{\lambda}$,

$$A = \sum_{m>0} (-1)^{j-m} \lambda_{-m} \lambda_m,$$

and

$$\bar{A} = \sum_{m>0} (-1)^{j-m} \bar{\lambda}_m \bar{\lambda}_{-m}$$

are Grassmann variables. We recall that the fermion fields must satisfy antiperiodic boundary conditions in time.

Now, to cast the action in a form convenient for the saddle point expansion, we perform a number of manipulations: their role will be illustrated when appropriate. First we shift the time label in the variables λ (but not $\bar{\lambda}$) according to

$$\lambda(t-1) \to \lambda(t) \quad (14)$$

in order to have $\bar{A}$ and A with the same time argument. This yields for the action the expression

[1] According to our definition the pair seniority quantum number s is half the usual seniority v and corresponds to the number of pairs not coupled to angular momentum $J=0$, and as such, blind to the action of the pairing force.

$$S^I = \tau \sum_{t=-N_0/2}^{N_0/2-1} \left\{ -g\overline{A}(t)A(t) + \sum_{m=-j}^{j} [\overline{\lambda}_m(t)(\nabla_t^+ + e)\lambda_m(t)] \right\}. \tag{15}$$

Carrying out next the Hubbard-Stratonovitch transformation, we get the new action

$$S^{II} = \tau \sum_{t=-N_0/2}^{N_0/2-1} \left\{ g\overline{\eta}(t)\eta(t) + g\overline{\eta}(t)A(t) + g\eta(t)\overline{A}(t) + \sum_{m=-j}^{j} [\overline{\lambda}_m(t)(\nabla_t^+ + e)\lambda_m(t)] \right\}. \tag{16}$$

Clearly the auxiliary fields $\overline{\eta}$ and η should satisfy periodic boundary conditions.

Finally we introduce the Goldstone field θ through the polar representation [5] for the auxiliary fields

$$\eta = \sqrt{\rho}e^{2i\theta}, \quad \overline{\eta} = \sqrt{\rho}e^{-2i\theta}. \tag{17}$$

The field ρ has been placed under a square root to avoid the Jacobian which would otherwise appear. Notice that for this change of variable to be one to one (with the only exception of the point $\rho=0$), θ must vary in the range $0 \leqslant \theta < \pi$. Hence the field θ lives in the coset space of the (almost) broken symmetry group U(1) of particle conservation with respect to the unbroken subgroup Z_2, as appropriate to a Goldstone field [5]. From the periodic boundary conditions for the η field periodic boundary conditions for ρ and θ follow as well.

We note now that, after the transformation (17), the θ field appears in the action (16) with nonderivative couplings whereas the Goldstone field should display only derivative couplings. However, the former can be eliminated introducing the following transformation on the nucleon fields:

$$\lambda_m = e^{i\theta}\psi_m, \quad \overline{\lambda}_m = e^{-i\theta}\overline{\psi}_m. \tag{18}$$

As a consequence of the above transformation the following operators:

$$q^{\pm} = \exp(\mp i\theta)\nabla_t^{\pm}\exp(\pm i\theta) \pm e, \tag{19}$$

whose matrix elements read

$$(q^+)_{t_1t_2} = \frac{1}{\tau}[\exp\{i\tau(\nabla_t^+\theta)_{t_1}\}\delta_{t_2,t_1+1} - \delta_{t_1t_2}] + e\delta_{t_1t_2}, \tag{20}$$

$$(q^-)_{t_1t_2} = \frac{1}{\tau}[\delta_{t_1t_2} - \exp\{i\tau(\nabla_t^+\theta)_{t_2}\}\delta_{t_2,t_1-1}] - e\delta_{t_1t_2}, \tag{21}$$

will appear in the action. The θ field appears only in these operators and therefore under a derivative, as appropriate to a Goldstone field.

Hence Eq. (16) can be recast in the form

$$S^{III} = \tau \sum_{t=-N_0/2}^{N_0/2-1} \left\{ g\rho + \sum_{m>0} [\overline{\psi}_m q^+ \psi_m + \overline{\psi}_{-m} q^+ \psi_{-m} + g\sqrt{\rho}(-1)^{j-m}(\overline{\psi}_m\overline{\psi}_{-m} + \psi_{-m}\psi_m)] \right\} = \tau \sum_{t=-N_0/2}^{N_0/2-1} \left\{ g\rho + \sum_{m>0} \{[\overline{\psi}_m + g\sqrt{\rho}(-1)^{j-m}\psi_{-m}(q^+)^{-1}] \times q^+(\psi_m + g\sqrt{\rho}(-1)^{j-m}(q^+)^{-1}\overline{\psi}_{-m}) - g^2\rho\psi_{-m}(q^+)^{-1}\overline{\psi}_{-m} + \overline{\psi}_{-m}q^+\psi_{-m}\} \right\}. \tag{22}$$

Now we first integrate over the fermionic fields $\overline{\psi}_m$ and ψ_m for a given positive m: this yields $\mathrm{Det}(q^+)$, independent of m. Likewise, performing the integration over $\overline{\psi}_m$ and ψ_m for a given negative m, we get $\mathrm{Det}\{q^+ + g^2\rho[(q^+)^{-1}]^T\}$, T meaning the transpose operation, again m independent. Lumping the two results together and exploiting the relation

$$(q^+)^T = -q^-, \tag{23}$$

we find for the fermionic functional integration the result

$$[\mathrm{Det}(-q^-q^+ + g^2\rho)]^{\Omega}. \tag{24}$$

Thus, disregarding here and in the following all the field independent factors, we get for the generating functional the following expression:

$$Z = \int_0^{\infty} [d\rho] \int_0^{\pi} [d\theta] \exp(-S_{eff}), \tag{25}$$

with

$$S_{eff} = \tau \sum_t g\rho - \mathrm{Tr}\,\ln(-q^-q^+ + g^2\rho). \tag{26}$$

Note that the argument of the logarithm is symmetric, but not Hermitian. The trace must be taken over the quantum number $m>0$ and the time. The U(1) symmetry is now nonlinearly realized in the invariance of S_{eff} under the substitution

$$\theta \to \theta + \alpha, \tag{27}$$

with α time independent.

IV. THE SADDLE POINT

In this section we look for a minimum of S_{eff} at constant fields: hence only the time independent component of the ρ field, to be referred to as $\overline{\rho}$, will enter into the effective action. We start by defining

$$M = \sqrt{e^2 + g^2\overline{\rho}} \tag{28}$$

and

$$P^{-1} = -\nabla_t^+\nabla_t^- + e(\nabla_t^+ - \nabla_t^-) + M^2 = -(1 - e\tau)\Box + M^2, \tag{29}$$

where $\Box = \nabla_t^+\nabla_t^-$. Notice that $e\tau$ cannot be neglected with respect to 1. Indeed, in our calculations we will first perform

the limit $N_0 \to \infty$ and then we shall let $\tau \to 0$. The effective action at constant fields reads then

$$\overline{S_{eff}} = \tau \sum_t g\overline{\rho} - \mathrm{Tr}\ \ln\ P^{-1}. \tag{30}$$

The trace is conveniently evaluated in the Fourier representation, yielding

$$\overline{S_{eff}} = \tau N_0 g\overline{\rho} - \Omega \sum_{n_0=-N_0/2}^{N_0/2-1} \ln\left[4(1-e\tau)\sin^2\frac{\pi}{N_0}(n_0+1/2) + \tau^2 M^2 \right], \tag{31}$$

where the antiperiodic boundary conditions of the nucleon fields have been taken into account. Converting the sum into an integral we get

$$\overline{S_{eff}} = N_0\tau\left\{ g\overline{\rho} - \frac{2\Omega}{\tau}\ln\left[\frac{1}{2}(\tau M + \sqrt{4(1-e\tau)+\tau^2 M^2}) \right] \right\} = N_0\tau(g\overline{\rho} + \Omega e - \Omega M) + \mathcal{O}(\tau^2). \tag{32}$$

Notice that the piece Ωe stems from the term τe in Eq. (31). The minimum of $\overline{S_{eff}}$ occurs for

$$M = \frac{g\Omega}{2}, \tag{33}$$

which is independent of e and $\overline{\rho}$. Inserting the above into Eq. (28) one gets for the value $\overline{\rho_0}$ of $\overline{\rho}$ at the minimum the expression

$$\overline{\rho_0} = \frac{\Omega^2}{4} - \frac{e^2}{g^2}, \tag{34}$$

so that $\overline{S_{eff}}$ at the minimum is

$$S_0 = N_0\tau\left(-\frac{M\Omega}{2} - \frac{e^2}{g} + \Omega e \right). \tag{35}$$

Although the values for Ω, g, and e appropriate for the Sn isotopes would lead to a negative $\overline{\rho_0}$, actually in selecting a given nucleus (see Sec. VI) the replacement $e \to \epsilon = e - \mu$ (μ being the chemical potential) should be performed. When this is done, as will be shown in Sec. VI, $\overline{\rho_0}$ turns out indeed to be positive, as it should be, in the physical range $1 \leqslant n \leqslant \Omega$, attaining its maximum value for $n \simeq \Omega/2$, namely, for a half-filled level.

V. THE SADDLE POINT EXPANSION

To perform this expansion we start by defining the fluctuation of the ρ field according to

$$\rho = \overline{\rho_0} + r = \overline{\rho_0}\left(1 + \frac{r}{\overline{\rho_0}} \right) \tag{36}$$

and by noticing that the generating functional (25) now reads

$$Z = \int_{-\overline{\rho_0}}^{\infty} [dr] \int_0^{\pi} [d\theta] \exp(-S_{eff}). \tag{37}$$

Now two cases should be considered.

(a) $\overline{\rho_0}$ is sufficiently large: then the functional integral defining Z becomes Gaussian and an expansion in $r/\overline{\rho_0}$ can clearly be performed. Actually, as will be later shown [see formula (57) below], $\overline{\rho_0}$ is indeed large when $n \sim \Omega/2$, namely, when the level where the pairs live is far from being fully occupied or empty.

(b) $\overline{\rho_0}$ is small, which occurs for $n \simeq 1$ or $n \simeq \Omega$ [see again formula (57) below]. In this case the shift in Eq. (36) is absent and the ρ field acts only through its fluctuations, which are small, thus assuring the validity of the expansion.

To proceed further we rewrite S_{eff} in the form

$$S_{eff} = \tau \sum_t g(\overline{\rho_0} + r) + \mathrm{Tr}\ \ln\ P - \mathrm{Tr}\ \ln[\mathbb{1} + P(R_1 + R_2)], \tag{38}$$

where

$$R_1 = -q^- q^+ + (\nabla_t^+ + e)(\nabla_t^- - e) \tag{39}$$

and

$$R_2 = g^2 r. \tag{40}$$

We set then

$$S_{eff} = \sum_{r=0}^{\infty} S_r, \tag{41}$$

the term S_0 being the saddle point contribution, given by Eq. (35). This grows like Ω^2; however, it contains also a term of order Ω and a term of order 1, which should be kept if an expansion in powers of $1/\Omega$ is sought for. It seems to us, however, more convenient to stick to the definition (29) for the operator P and to compute the further contributions to the expansion (41) (the quantum fluctuations) by developing the logarithm (38): the terms thus obtained are naturally organized in powers of M^{-1}. It is worth recalling that this expansion does not break the U(1) invariance.

In the following we shall confine ourselves to evaluating, in addition to the first order terms, those quadratic in $\nabla_t\theta$ and r.

A. First order contributions

These contributions stem from the term linear in r and from the first term in the expansion of the logarithm; hence

$$S_1 = \tau g \sum_t r_t - \mathrm{Tr}[P(R_1 + R_2)]. \tag{42}$$

The explicit computation of the second term on the right-hand side (RHS) of the above yields

$$-\mathrm{Tr}(PR_1) = \frac{\Omega}{\tau^2}\sum_t P_{tt}\left\{ [1 - e^{2i(\theta_{t+1}-\theta_t)}] - [1 - e^{i(\theta_{t+1}-\theta_t)}] \times \left[\tau^2 M^2 - \frac{\tau^2}{P_{tt}} + 2(1-e\tau) \right] \right\}, \tag{43}$$

where P_{tt} is found to read

$$P_{tt} = \frac{\tau}{2M}\left(1 - e\tau + \frac{\tau^2 M^2}{4}\right)^{-1/2}. \tag{44}$$

By expanding the exponentials up to second order in θ we get

$$-\mathrm{Tr}(PR_1) = \frac{\Omega}{2M}\left[1 + \left(M + \frac{3}{2}e\right)\tau\right]\tau\sum_t \frac{(\theta_{t+1} - \theta_t)^2}{\tau^2} + \mathcal{O}(\tau^3). \tag{45}$$

The contribution arising from the third term on the RHS of Eq. (42) turns out to be

$$-\mathrm{Tr}(PR_2) = -\Omega g^2 \sum_t P_{tt} r_t. \tag{46}$$

Notably this contribution, linear in r, is canceled by the first term in S_1, owing to the equation for the minimum of the action $\bar{S}_{eff}$. The cancellation holds to the order $\mathcal{O}(\tau^2)$, which is the approximation we keep in our analysis and in obtaining the equation for the action minimum.

In conclusion, for the first order contribution to the action we get

$$S_1 = -\mathrm{Tr}(PR_1) = \frac{1}{g}\left[1 + \left(M + \frac{3}{2}e\right)\tau\right]\tau\sum_{t=-\infty}^{\infty} \theta(-\Box)\theta, \tag{47}$$

where Σ_t runs from $-\infty$ to ∞ since we let $N_0 \to \infty$ in evaluating P. We note that these contributions are of order Ω and 1, namely, are $\mathcal{O}(1/\Omega)$ with respect to those of the saddle point.

B. Second order contributions

We have seen in the previous subsection that all the terms linear in r cancel out: hence the r integration remains undefined. Our aim now is to ascertain whether the surviving terms in r stabilize the action.

Among these we consider the contributions arising from the second term in the expansion of the logarithm. They read

$$S_2 = \frac{1}{2}\mathrm{Tr}(PR_1)^2 + \mathrm{Tr}(PR_1PR_2) + \frac{1}{2}\mathrm{Tr}(PR_2)^2. \tag{48}$$

In the above the first term is r independent, the second is linear in r, and the third one is quadratic. Therefore, for the present purpose, it is sufficient to evaluate the latter. For this we have found

$$\frac{1}{2}\mathrm{Tr}(PR_2)^2 = \frac{g^4\Omega}{2}\sum_{tt_1} r_t [P_{tt_1}]^2 r_{t_1}. \tag{49}$$

Hence the integral over r is well defined.

In conclusion we remark that, as will be seen in the following sections, in order to obtain the Goldstone boson energies we must find out how they depend upon the single-particle energy e. For this purpose we have to perform in the integral expressing the generating functional Z_1 (associated with the action S_1) the θ integration, which appears to be Gaussian, but actually it is not, because θ is compact. Yet we can choose $\nabla_t\theta$ as a new integration variable, thus rendering the integral Gaussian. We then get

$$-\frac{1}{N_0\tau}\ln Z_1 = \frac{3}{4}e + \frac{M}{2} = \frac{3e + g\Omega}{4}. \tag{50}$$

We notice that this contribution stems from the term $(M + 3e/2)\tau$ in S_1, which is irrelevant because it vanishes in the formal continuum limit.

VI. FIXING THE PARTICLE NUMBER BY THE CHEMICAL POTENTIAL

In this section we apply the saddle point expansion to a specific nucleus using the method of the chemical potential. For this purpose we replace e with

$$\epsilon = e - \mu, \tag{51}$$

μ being the chemical potential. Its value is fixed according to

$$\langle\hat{N}\rangle = \frac{1}{N_0\tau}\frac{\partial}{\partial\mu}\ln Z = -\frac{1}{N_0\tau}\frac{\partial}{\partial\epsilon}\ln Z, \tag{52}$$

where

$$\hat{N} = \sum_{m=-j}^{j} \hat{\lambda}_m^\dagger \hat{\lambda}_m \tag{53}$$

is the particle number operator. Since, however, we shall let $N_0 \to \infty$ with τ constant, which corresponds to the limit of vanishing temperature $T = 1/(N_0\tau)$, we are allowed to replace $\langle\hat{N}\rangle$ with $2n$. We also notice that because M does not depend upon e [see Eq. (1)], it does not depend on μ either. So Eq. (52) becomes

$$n = \frac{1}{2N_0\tau}\frac{\partial}{\partial\epsilon}(S_0 - \ln Z_1) = -\frac{\epsilon}{g} + \frac{\Omega + 3/4}{2}, \tag{54}$$

which gives

$$\mu = g(n - \Omega/2 - 3/8) + e \tag{55}$$

for the chemical potential. Hence, in the presence of the chemical potential, the energy of the system becomes

$$E_{n,0} = \frac{1}{N_0\tau}(S_0 - \ln Z_1) + 2\,\mu n = 2en - gn(\Omega - n + 3/4) + \frac{g}{8}\left(5\Omega + \frac{9}{8}\right). \tag{56}$$

We thus see from the above that in our approach the excitation spectrum of the pairing Hamiltonian is reproduced with good accuracy. On the other hand the ground state energy differs from the exact value $-g[(\Omega+1)/2 - e/g]^2$ by the quantity $(3g\Omega + g - e)/4$, which corresponds to a relative error of order $1/\Omega$.

We conclude this section by further examining the issue, already addressed in the beginning of Sec. V, of the validity of our expansion. For this purpose it is of importance to assess the size of $\bar{\rho}_0$. With this aim we replace in Eq. (34) e

by ϵ and use Eq. (55), dropping the term $-3/8$ in the parentheses on the RHS, thus getting

$$\overline{\rho_0} = n(\Omega - n). \tag{57}$$

Now, when $n \sim \Omega/2$, then the single-particle energy e almost coincides with the chemical potential μ. In such a situation ϵ is almost vanishing and, from Eq. (57), $\overline{\rho_0} \simeq \Omega^2/4$. This large value corresponds to the situation when the level where the pairs live is neither fully filled nor almost empty. On the other hand $\overline{\rho_0}$ attains its lowest value when $n=1$ or $n=\Omega$. It is remarkable that even in these cases, where an expansion in $1/\overline{\rho_0}$ cannot be performed, our approach still yields the correct excitation spectrum of the system.

VII. FIXING THE PARTICLE NUMBER BY THE PROJECTION OPERATOR

Owing to the importance of properly fixing the particle number n, in this section we address the problem through an alternative procedure, namely, by introducing in the path integral the projection operator

$$\mathcal{P}_n = \int_{-\pi}^{+\pi} \frac{d\alpha}{2\pi} e^{-i(\hat{N}-2n)\alpha}. \tag{58}$$

Using then the variables (18) and performing the Hubbard-Stratonovitch transformation as previously done, we get for the generating functional the expression

$$Z^{(n)} = \int_{-\pi}^{+\pi} \frac{d\alpha}{2\pi} \int [d\lambda\, d\overline{\lambda}\, d\psi\, d\overline{\psi}\, d\eta\, d\overline{\eta}] e^{-S^{(n)}} \tag{59}$$

where

$$S^{(n)} = \tau \sum_{t=-N_0/2}^{N_0/2-1} \left\{ g\rho + \sum_{m>0} [\overline{\psi}_m q_\sigma^+ \psi_m + \overline{\psi}_{-m} q_\sigma^+ \psi_{-m} + g\sqrt{\rho}(\overline{\psi}_m \overline{\psi}_{-m} + \psi_{-m}\psi_m)] \right\} - 2N_0\tau(\sigma - e)n \tag{60}$$

(remember that the label n indicates the number of pairs), having defined

$$\sigma = e + \frac{i\alpha}{N_0\tau} \tag{61}$$

and

$$q_\sigma^\pm = \exp(\mp i\theta)\nabla_t^\pm \exp(\pm i\theta) \pm \sigma. \tag{62}$$

Next we carry out the integration over the fermionic degrees of freedom, getting for the partition function the expression

$$Z^{(n)} \propto \int_{e-i\pi/N_0\tau}^{e+i\pi/N_0\tau} d\sigma \int [d\rho\, d\theta] \exp[-S_{eff}^{(n)}(\sigma,\rho,\theta)], \tag{63}$$

where

$$S_{eff}^{(n)} = -2N_0\tau(\sigma - e)n + \tau \sum_t g\rho - \mathrm{Tr}\,\ln[-q_\sigma^- q_\sigma^+ + g^2\rho]. \tag{64}$$

At constant fields Eq. (64) simplifies to

$$\overline{S^{(n)}}(\sigma,\overline{\rho}) = N_0\tau[g\overline{\rho} + 2ne + (\Omega - 2n)\sigma - \Omega\sqrt{\sigma^2 + g^2\overline{\rho}}], \tag{65}$$

which is stationary when

$$\frac{\partial \overline{S^{(n)}}(\sigma,\overline{\rho})}{\partial \overline{\rho}} = N_0\tau g\left(1 - \frac{\Omega g}{2\sqrt{\sigma^2 + g^2\overline{\rho}}}\right) = 0 \tag{66}$$

and

$$\frac{\partial \overline{S^{(n)}}(\sigma,\overline{\rho})}{\partial \overline{\rho}} = N_0\tau\left(-\frac{\Omega g}{M} + \Omega - 2n\right) = 0. \tag{67}$$

The solutions of the above equations read

$$\overline{\rho_0} = n(\Omega - n) \tag{68}$$

and

$$\sigma_0 = \frac{g}{2}(\Omega - 2n). \tag{69}$$

Finally the effective action (65) at the minimum σ_0 turns out to be

$$\overline{S^{(n)}}(\sigma_0,\overline{\rho_0}) = N_0\tau E_0^{(n)} = N_0\tau[2ne - gn(\Omega - n)], \tag{70}$$

which differs from the zero seniority spectrum of the pairing Hamiltonian. Indeed the latter has $\Omega+1$, rather than Ω, inside the parentheses on the RHS of Eq. (70).

Note that the action (65) is an analytic function of σ inside an integration path deformed to encompass the saddle point σ_0. This path goes from $e - i\pi/N_0\tau$ to $\sigma_0 - i\pi/N_0\tau$ along a straight line parallel to the real axis, then from $\sigma_0 - i\pi/N_0\tau$ to $\sigma_0 + i\pi/N_0\tau$ along a straight line parallel to the imaginary axis, and finally it goes back from $\sigma_0 + i\pi/N_0\tau$ to $e + i\pi/N_0\tau$. When $N_0 \to \infty$, the contributions coming from the paths parallel to the real axis cancel each other, while the one parallel to the imaginary axis vanishes: hence there are no corrections to the saddle point contribution.

Next, with the aim of checking the results obtained in the framework of the chemical potential method, we evaluate, using the projection operator, the first order correction in the saddle point expansion in ρ and θ. For this scope we set, as in Eq. (36),

$$\rho = \overline{\rho_0} + r. \tag{71}$$

The action (64) can then be recast as follows:

$$S_{eff}^{(n)} \simeq \overline{S_0^{(n)}}(\sigma_0,\overline{\rho_0}) + S_1^{(n)} \tag{72}$$

where

$$S_1^{(n)} = N_0\tau g r - \mathrm{Tr}[P(\sigma_0,\overline{\rho_0})(R_1^{(n)} + R_2^{(n)})] \tag{73}$$

with

$$R_1^{(n)} = -[q_\sigma^- + (\sigma - \sigma_0)][q_\sigma^+ - (\sigma - \sigma_0)] + (\nabla_t^+ + \sigma_0)(\nabla_t^- - \sigma_0) \tag{74}$$

and

$$R_2^{(n)} = g^2 r. \tag{75}$$

The contribution

$$\mathrm{Tr}[P(\sigma_0,\overline{\rho_0})R_2^{(n)}] = g^2 \mathrm{Tr}\ P(\sigma_0,\overline{\rho_0})r = g^2\frac{\tau}{g}\sum_t r \tag{76}$$

cancels the first term in Eq. (73). Hence the latter simply becomes

$$S_1^{(n)} = -\mathrm{Tr}[P(\sigma_0,\overline{\rho_0})R_1^{(n)}]. \tag{77}$$

A calculation, similar to the one carried out in Sec. V A, yields then

$$-\mathrm{Tr}[PR_1^{(n)}] = \frac{\Omega}{2M}\left[1+\left(M+\frac{3}{2}\sigma_0\right)\tau\right]\tau\sum_t \frac{(\theta_{t+1}-\theta_t)^2}{\tau^2}. \tag{78}$$

Using now the expression (69) for σ_0 we get

$$S_1^{(n)} = \frac{1}{g}\left[1+g\left(\frac{5}{4}\Omega-\frac{3}{2}n\right)\tau\right]\tau\sum_t \theta(-\Box)\theta \tag{79}$$

and by performing the θ integration [again using Eq. (69)] we finally obtain the first order energy

$$E_0^{(n)} + E_1^{(n)} = 2ne - gn\left(\Omega - n + \frac{3}{4}\right) + \frac{5}{8}g\Omega, \tag{80}$$

which coincides with the excitation spectrum obtained with the chemical potential.

VIII. THE HAMILTONIAN OF THE *s* BOSONS

To complete our program (restricted, we recall, to the pairing potential) we derive below the bosonic Hamiltonian corresponding to our effective action. The most general, particle conserving, quartic Hamiltonian for a system of bosons, confined to live in one single-particle level, reads in normal form

$$H(\hat{b}^\dagger,\hat{b}) = h\hat{b}^\dagger\hat{b} + v\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}, \tag{81}$$

$\hat{b}^\dagger,\hat{b}$ being bosonic creation-annihilation operators acting in a Fock space. They satisfy canonical commutation relations. The values of the parameters h,v were obtained in Ref. [16] in such a way to yield the pairing Hamiltonian spectrum. Of course the Hamiltonian thus obtained, being intrinsically bosonic, patently violates the Pauli principle and therefore the condition $n<\Omega$ should be added *a posteriori*, when using Eq. (81) in describing a system of fermions. Here, to show how this condition naturally emerges instead in our framework, we obtain the parameters h and v with our methods. To this purpose we write the path integral associated with Eq. (81), namely,

$$Z = \int [db^* db]\exp(-S), \tag{82}$$

where

$$S = \tau\sum_{t=1}^{N_0}\{b_{t+1}^*\nabla_t b_t + H(b_{t+1}^*, b_t)\} \tag{83}$$

and the b^*,b are holomorphic variables satisfying periodic boundary conditions in time. We now compare the above to our effective action.

To this end we introduce the polar representation

$$b = \sqrt{\rho}\,\exp(i\theta),\quad b^* = \sqrt{\rho}\,\exp(-i\theta) \tag{84}$$

in terms of which the generating functional and the action read

$$Z = \int_0^\infty [d\rho]\int_{-\pi}^{\pi}[d\theta]\exp(-S), \tag{85}$$

$$S = \tau\sum_t\{\sqrt{\rho_{t+1}}\exp(-i\theta_{t+1})\nabla_t[\sqrt{\rho_t}\exp(i\theta_t)] + H[\sqrt{\rho_{t+1}}\exp(-i\theta_{t+1}),\sqrt{\rho_t}\exp(i\theta_t)]\}. \tag{86}$$

Now we again look for a minimum of S at constant fields and perform a saddle point expansion. The calculation is basically the same as the one previously developed and hence will not be reported here. We only quote the result of the comparison with our effective action: it yields

$$h = 2e - g\Omega - g/4,\quad v = g,\quad n<\Omega. \tag{87}$$

The inequality in the above equation is necessary for the two path integrals to coincide and follows from the positivity of $\overline{\rho_0}$, given in Eq. (57): thus in our approach such a condition, far from being artificial, is necessarily implied by the formalism itself. Obviously the considerations following Eq. (41) hold valid as well here. Thus h will be affected by an error of order $1/M$. Indeed, for the Hamiltonian (81) to reproduce exactly the pairing spectrum it must be

$$h = 2e - g\Omega,\quad v = g. \tag{88}$$

Finally, it is of importance to stress once more that the above discussion can be generalized to include other types of bosons, as, for example, those appearing in the quartic Hamiltonian of the IBM model, respecting basic symmetries like particle number conservation, rotational invariance, etc. It is then clear that our approach opens the way to microscopically deduce the Arima-Iachello model: in this case to fix the coupling constants one should again write the corresponding path integral and compare the resulting bosonic action with the effective one found extending the procedure developed in this paper to a fermionic Hamiltonian including forces of higher multipolarity.

IX. CONCLUSIONS

In this paper we have carried out an investigation concerning the possibility of a systematic bosonization of a realistic nuclear Hamiltonian for the description of the low-lying sector of the nuclear spectrum. Our study is admittedly preliminary since it is limited to the pairing interaction, which is of course only a component (although an important

one) of an effective interaction pretending to be realistic. However, having overcome the main difficulty we expected, namely, the one of going from an infinite to a finite system, we hope to be able to solve the bosonization problem in the presence of other types of interaction [7].

Our approach is based on the concept of symmetry breaking and on the related properties of the interaction among the bosonic fields: indeed this framework is the most suitable for deriving, rather than assuming, a model like the IBM.

To develop our scheme we have used the path integral formalism because of the large flexibility it allows both in choosing and in dealing with the variables appropriate to the problem. This has led us to deduce an asymptotic expansion for the system's spectrum in the parameter $M^{-1}=2/g\Omega$.

The Euclidean path integral clearly encompasses the whole Fock space of the system. To deal with a specific nucleus a given number of pairs must be selected (a procedure not to be confused, of course, with the projection of the particle number when this is violated). In our approach a definite particle sector has been chosen using both the chemical potential and the projection operator methods: the two turn out to be completely equivalent.

Notably our expansion to first order reproduces with good accuracy the energy of the pair addition and removal modes (or, in the language of the IBM, of the *s* bosons). Moreover the requirement that $\overline{\rho_0}$ should be positive entails the inequality $n<\Omega$ [see again Eq. (57)], thus implementing the action of the Pauli principle. This is conceptually important for the consistency of our scheme, which respects both the Pauli principle and the particle number conservation. Worth noticing is that this crucial feature is absent in the framework developed in [16,17], but correctly dealt with in many treatments based on the mapping procedure [1,4].

Our expansion cannot account for the seniority excitations, whose energies, in the framework of the pairing Hamiltonian (2), are larger than M: hence they should be separately treated. For this scope clearly an analysis of the excitations associated with the field ρ should be performed. In this connection it is worth recalling that in the scheme of the BV transformation these modes are described in terms of quasiparticles whose energies, as well known, are expressed in terms of the gap Δ. In an infinite system Δ is associated with the order parameter by setting a non zero vacuum expectation value for the pair (Cooper) field and it signals the onset of the superconducting phase; in a finite system it measures the strength of the mean field (referred to as the "pair field") which, in the BV scheme, linearizes the pair interaction.

Finally we should observe that in our analysis the impact on the excitation spectrum stemming from the removal of the degeneracy of the levels where the pairs are sitting and of the higher order terms in the saddle point expansion has not been explored. Concerning the first issue, if the spacing between the single-particle energy levels is small with respect to M, it can easily be accounted for within the present perturbative scheme. Indeed this approach has already been pursued in Ref. [13]. Actually the Sn isotopes are well suited for such perturbative treatment since here the distance between the single-particle energies of concern appears to be much smaller than M.

Concerning the second point, we can only say that it would certainly be both interesting and important to examine in more depth our asymptotic expansion.

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[1] J. N. Ginocchio, Phys. Lett. **79B**, 173 (1978); Nucl. Phys. **A376**, 438 (1982); H. B. Geyer, Phys. Rev. C **34**, 2373 (1986); C. W. Johnson and J. N. Ginocchio, *ibid.* **50**, R571 (1994).

[2] *Algebraic Approaches to Nuclear Structure*, edited by R. F. Casten (Harwood-Academic, Chur, Switzerland, 1993).

[3] A. Arima, T. Otsuka, F. Iachello, and I. Talmi, Phys. Lett. **66B**, 201 (1977); T. Otsuka, A. Arima, F. Iachello, and I. Talmi, *ibid.* **76B**, 139 (1978).

[4] A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63**, 375 (1991).

[5] S. Weinberg, *The Quantum Theory of Fields* (Cambridge University Press, Cambridge, 1996), Vol. 2, Chap. 21.

[6] D. J. Dean and M. Hjort-Jensen, Rev. Mod. Phys. **75**, 607 (2003).

[7] F. Palumbo, nucl-th/0405045; N. Lo Iudice and F. Palumbo (in progress).

[8] A. Bohr and B. Mottelson, *Nuclear Structure* (Benjamin, New York, 1969), Vols. 1 and 2.

[9] D. R. Bes and R. A. Broglia, in *Elementary Modes of Excitations in Nuclei*, Proceedings of the International School of Physics "Enrico Fermi," Course LXIX, Varenna 1977, edited by A. Bohr and R. A. Broglia (North Holland, Amsterdam, 1977), p. 59.

[10] M. B. Barbaro, A. Molinari, and F. Palumbo, Nucl. Phys. **B487**, 492 (1997).

[11] M. B. Barbaro, A. Molinari, F. Palumbo, and M. R. Quaglia, Phys. Lett. B **476**, 477 (2000).

[12] M. Mukherjee and Y. Nambu, Ann. Phys. (N.Y.) **191**, 143 (1991); Y. Nambu and M. Mukherjee, Phys. Lett. B **209**, 1 (1988).

[13] M. B. Barbaro, R. Cenni, A. Molinari, and M. R. Quaglia, Phys. Rev. C **66**, 034310 (2002).

[14] D. J. Rowe, *Nuclear Collective Motion* (Methuen, London, 1970).

[15] J. W. Negele and H. Orland, *Quantum Many-Particle Systems* (Addison-Wesley, Reading, MA, 1988).

[16] I. Talmi, in [9], p. 352.

[17] I. Talmi, in *Unfolding the Matter of Nuclei*, Proceedings of the International School of Physics "Enrico Fermi," Course CXXXVIII, Varenna, 1998, edited by A. Molinari and R. A. Ricci (North Holland, Amsterdam, (1998), 335.