E(5) and X(5) critical point symmetries obtained from Davidson potentials through a variational procedure

Dennis Bonatsos,[1,*] D. Lenis,[1,†] N. Minkov,[2,‡] D. Petrellis,[1,§] P. P. Raychev,[2,‖] and P. A. Terziev[2,¶]
[1]*Institute of Nuclear Physics, NCSR "Demokritos", GR-15310 Aghia Paraskevi, Attiki, Greece*
[2]*Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 72 Tzarigrad Road, BG-1784 Sofia, Bulgaria*
(Received 24 February 2004; published 16 August 2004)

Davidson potentials of the form $\beta^2+\beta_0^4/\beta^2$, when used in the E(5) framework bridge the U(5) and O(6) symmetries, while they bridge the U(5) and SU(3) symmetries when used in the X(5) framework. Using a variational procedure, we determine for each value of angular momentum L the value of β_0 at which the rate of change of various physical quantities [energy ratios, intraband B(E2) ratios, quadrupole moment ratios] has a maximum, the collection of the values of the physical quantity formed in this way being a candidate for describing its behavior at the relevant critical point. Energy ratios lead to the E(5) and X(5) results, while intraband B(E2) ratios and quadrupole moments lead to the E(5)-β^4 and X(5)-β^4 models. A new derivation of the Holmberg-Lipas formula for nuclear energy spectra is obtained as a by-product.

DOI: 10.1103/PhysRevC.70.024305 PACS number(s): 21.60.Ev, 21.60.Fw

I. INTRODUCTION

The recently introduced E(5) [1] and X(5) [2] models are supposed to describe shape phase transitions in atomic nuclei, the former being related to the transition from U(5) (vibrational) to O(6) (γ-unstable) nuclei, and the latter corresponding to the transition from U(5) to SU(3) (rotational) nuclei. In both cases the original Bohr collective Hamiltonian [3] is used, with an infinite well potential in the collective β variable. Separation of variables is achieved in the E(5) case by assuming that the potential is independent of the collective γ variable, while in the X(5) case the potential is assumed to be of the form $u(\beta)+u(\gamma)$. For brevity we are going to refer to these two cases of separation of variables as "the E(5) framework" and "the X(5) framework" respectively, although the former case has been known for a long time [4]. The selection of an infinite well potential in the β variable in both cases is justified by the fact that the potential is expected to be flat around the point at which a shape phase transition occurs. Experimental evidence for the occurence of the E(5) and X(5) symmetries in some appropriate nuclei is growing (Refs. [5–8], respectively).

In the present work we examine if the choice of the infinite well potential is the optimum one for the description of shape phase transitions. For this purpose, we need one-parameter potentials which can span the U(5)-O(6) region in the E(5) framework, as well as the U(5)-SU(3) region in the X(5) framework. It turns out that the exactly soluble [9,10] Davidson potentials [11]

$$u(\beta)=\beta^2+\frac{\beta_0^4}{\beta^2}, \tag{1}$$

where β_0 is the position of the minimum of the potential, do possess this property.

Taking into account the fact that various physical quantities should change most rapidly at the point of the shape phase transition [12], we locate for each value of the angular momentum L the value of β_0 for which the rate of change of the physical quantity is maximized. The collection of the values of the physical quantity formed in this way should then correspond to the behavior of this physical quantity at the critical point. As appropriate physical quantities we have used energy ratios within the ground state band, as well as intraband B(E2) ratios and quadrupole moment ratios within the ground state band, and within excited bands.

The energy ratios within the ground state band lead to results very similar to these provided by the infinite well potential in both the E(5) and the X(5) frameworks, thus indicating that the choice of the infinite well potential in both cases is the optimum one. Intraband B(E2) ratios and quadrupole moment ratios lead to results close to the ones provided by the E(5)-β^4 [13,14] and X(5)-β^4 [15] models, which use a $u(\beta)=\beta^4/2$ potential in the E(5) and X(5) framework, respectively. (It should be noticed at this point that a quartic anharmonicity has been used earlier in the description of even nuclei in Refs. [16,17].) Further discussion of these results is deferred to Sec. VIII.

The variational procedure used here is analogous to the one used in the framework of the variable moment of inertia (VMI) model [18], where the energy is minimized with respect to the (angular momentum dependent) moment of inertia for each value of the angular momentum L separately. The variational procedure used here also resembles the standard Ritz variational method of quantum mechanics [19], since the former utilizes a trial potential containing a free parameter, while in the latter a trial wave function involving a free parameter is involved. The main difference between

*Email address: bonat@inp.demokritos.gr
†Email address: lenis@inp.demokritos.gr
‡Email address: nminkov@inrne.bas.bg
§Email address: petrellis@inp.demokritos.gr
‖Deceased.
¶Email address: terziev@inrne.bas.bg

 ©2004 The American Physical Society

the two methods lies in the way in which the value of the parameter is determined. In the Ritz approach this is achieved by minimizing the energy, while in the present case it is done by maximizing the rate of change of the relevant physical quantity.

L-dependent potentials are also not unheard of in nuclear physics. They are known to occur in the framework of the optical model potential [20–22], as well as in the case of quasimolecular resonances, such as $^{12}C+^{12}C$ [23].

In the framework of the Interacting Boson Model (IBM) [24], there have been attempts to consider the U(5) to O(6) transition [13], as well as the U(5) to SU(3) transition [25] in terms of one-parameter schematic Hamiltonians. In the present approach, in contrast, Davidson potentials are used directly in the original Bohr Hamiltonian, without the intervention af any IBM approximations.

On the other hand, exactly soluble models have been constructed by using the Coulomb and Kratzer [26] potentials in the original Bohr Hamiltonian in the E(5) [27] and X(5) [28] frameworks, while a two-parameter quasiexactly soluble model [29–31] has been constructed by using [32] the sextic oscillator with a centrifugal barrier [33] in the E(5) framework. Some of these potentials will be further commented below.

Spectra and B(E2) transition rates for Davidson potentials in the E(5) and X(5) frameworks are considered in Sec. II, while the relevant limiting symmetries are discussed in Sec. III. The variational procedure is then applied to energy ratios (Sec. IV), excitation energies (Sec. V), B(E2) ratios (Sec. VI), and quadrupole moment ratios (Sec. VII), while finally Sec. VIII contains a discussion of the present results and plans for further work. A preliminary version of this work, limited to the variational study of the spectra of ground state bands, has been reported in Ref. [34].

II. SPECTRA AND B(E2) TRANSITION RATES

A. The E(5) framework

In the E(5) model it is assumed that the potential appearing in the original Bohr Hamiltonian [3] (in which the collective variables β and γ appear) depends only on the variable β, i.e., $V(\beta,\gamma)=U(\beta)$. One can then proceed to separation of variables in the standard way [3,4], seeking wave functions of the form $\Psi(\beta,\gamma,\theta_i)=f(\beta)\Phi(\gamma,\theta_i)$, where θ_i $(i=1,2,3)$ are the Euler angles describing the orientation of the deformed nucleus in space. Reduced energies $\epsilon=(2B/\hbar^2)E$ and reduced potentials $u=(2B/\hbar^2)U$ [1], where B is the mass parameter appearing in the original Bohr Hamiltonian, are introduced for simplicity.

In the equation involving the angles (i.e., γ and the Euler angles), the eigenvalues of the second order Casimir operator of SO(5) occur, having the form $\Lambda=\tau(\tau+3)$, where $\tau=0$, 1, 2, …, is the quantum number characterizing the irreducible representations (irreps) of SO(5), called the "seniority" [35]. This equation has been solved by Bès [36].

As described in more detail in Ref. [34], the "radial" equation (involving the variable β) can be solved exactly for the Davidson potentials of Eq. (1), the eigenfunctions being Laguerre polynomials, and the energy eigenvalues having the form [9,10] (in $\hbar\omega=1$ units)

$$E_{n,\tau}=2n+1+\left[\left(\tau+\frac{3}{2}\right)^2+\beta_0^4\right]^{1/2}. \tag{2}$$

For $\beta_0=0$ the original solution of Bohr [3,37], which corresponds to a five-dimensional (5D) harmonic oscillator characterized by the symmetry $U(5)\supset SO(5)\supset SO(3)\supset SO(2)$ [38], is obtained. The values of angular momentum L contained in each irrep of SO(5) (i.e., for each value of τ) are given by the algorithm [24] $\tau=3\nu_\Delta+\lambda$, where $\nu_\Delta=0,1,\ldots$, is the missing quantum number in the reduction $SO(5)\supset SO(3)$, and $L=\lambda,\lambda+1,\ldots,2\lambda-2,2\lambda$ (with $2\lambda-1$ missing).

The levels of the ground state band are characterized by $L=2\tau$ and $n=0$. Then the excitation energies of the levels of the ground state band relative to the ground state are

$$E_{0,L,\mathrm{exc}}=E_{0,L}-E_{0,0}=\tfrac{1}{2}([(L+3)^2+4\beta_0^4]^{1/2}-[9+4\beta_0^4]^{1/2}). \tag{3}$$

The "radial" equation is also exactly soluble for $u(\beta)$ being a 5D infinite well [$u(\beta)=0$ if $\beta\leqslant\beta_W$, $u(\beta)=\infty$ for $\beta>\beta_W$], in which case one obtains the E(5) model of Iachello [1], the eigenfunctions being Bessel functions $J_{\tau+3/2}(z)$ (with $z=\beta k$, $k=\sqrt{2BE/\hbar^2}$), while the spectrum is determined by the zeros of the Bessel functions, having the form $E_{\xi,\tau}=(\hbar^2/2B)k_{\xi,\tau}^2$ with $k_{\xi,\tau}=x_{\xi,\tau}/\beta_W$, where $x_{\xi,\tau}$ is the ξ-th zero of the Bessel function $J_{\tau+3/2}(z)$. The spectra of the E(5) and Davidson cases become directly comparable by establishing the formal correspondence $n=\xi-1$.

In what follows the ratios

$$R_{n,L}=\frac{E_{n,L}-E_{0,0}}{E_{0,2}-E_{0,0}},\quad \underline{R}_{n,L}=\frac{E_{n,L}-E_{n,0}}{E_{n,2}-E_{n,0}} \tag{4}$$

with the notation $E_{n,L}$, will be used. In the former case energies in all bands are measured relative to the ground state and normalized to the excitation energy of the $L=2$ state of the ground state band (as in Ref. [1]), while in the latter case energies in each band are measured relative to the bandhead $(L=0)$ of this band and normalized to the excitation energy of the $L=2$ state of this band. From Eq. (2) it is clear that $\underline{R}_{n,L}$ obtains identical values for all bands with $L=2\tau$, irrespectively of n. For the ground state band $(n=0)$ the simplified notation $R_L\equiv R_{0,L}$ will also be used.

The quadrupole operator has the form [4]

$$T^{(E2)}_\mu=t\alpha_\mu=t\beta\left[\mathcal{D}^{(2)}_{\mu,0}(\theta_i)\cos\gamma+\frac{1}{\sqrt{2}}[\mathcal{D}^{(2)}_{\mu,2}(\theta_i)+\mathcal{D}^{(2)}_{\mu,-2}(\theta_i)]\sin\gamma\right], \tag{5}$$

where t is a scale factor and $\mathcal{D}(\theta_i)$ denote Wigner functions of the Euler angles. The calculation of B(E2) rates proceeds as in Refs. [1,14].

In what follows, the intraband ratios

$$R_{n,L}^{B(E2)} = \frac{B[E2;(L+2)_n \to L_n]}{B(E2;2_0 \to 0_0)}, \quad \underline{R}_{n,L}^{B(E2)} = \frac{B[E2;(L+2)_n \to L_n]}{B(E2;2_n \to 0_n)} \tag{6}$$

will be used. In the former case the B(E2) intraband transition rates of all bands are normalized to the B(E2) transition rate between the two lowest states of the ground state band (as in Ref. [1]), while in the latter case the B(E2) intraband transition rates within each band are normalized to the B(E2) transition rate between the two lowest states of this band.

It should be noted at this point that quadrupole moments in this framework vanish, if one is limited to the quadrupole operator of Eq. (5), because of a $\Delta\tau = \pm 1$ selection rule. This case is reminiscent of the vanishing (to lowest order) of the quadrupole moments in the O(6) limit of IBM [24]. Nonvanishing quadrupole moments can be obtained by including the next order terms in the quadrupole operator of Eq. (5).

B. The X(5) framework

Starting again from the original Bohr Hamiltonian [3], one seeks solutions of the relevant Schrödinger equation having the form $\Psi(\beta,\gamma,\theta_i) = \phi_K^L(\beta,\gamma) D_{M,K}^L(\theta_i)$, where θ_i $(i=1,2,3)$, are the Euler angles, $\mathcal{D}(\theta_i)$ denote Wigner functions of them, L are the eigenvalues of angular momentum, while M and K are the eigenvalues of the projections of angular momentum on the laboratory-fixed z axis and the body-fixed z' axis, respectively. As pointed out in Refs. [2,39], an approximate separation of variables can be achieved by assuming that the potential has a minimum around $\gamma=0$, as well as that it can be separated into two terms, one depending on β and the other depending on γ, i.e., $u(\beta,\gamma) = u(\beta) + u(\gamma)$.

As described in more detail in Ref. [34], the "radial" equation occuring in this case can be solved exactly in the case of the Davidson potentials of Eq. (1), the energy eigenvalues being (in $\hbar\omega = 1$ units)

$$E_{n,L}^{(K)} = 2n + 1 + \left[\tfrac{1}{3}(L(L+1) - K^2) + \tfrac{9}{4} + \beta_0^4\right]^{1/2}. \tag{7}$$

The levels of the ground state band are characterized by $n=0$ and $K=0$. Then the excitation energies relative to the ground state are given by

$$E_{0,L,\mathrm{exc}}^{(0)} = E_{0,L}^{(0)} - E_{0,0}^{(0)} = \left[\tfrac{1}{3}L(L+1) + \tfrac{9}{4} + \beta_0^4\right]^{1/2} - \left[\frac{9}{4} + \beta_0^4\right]^{1/2}, \tag{8}$$

which can easily be put into the form

$$E'_{0,L,\mathrm{exc}} = \frac{E_{0,L,\mathrm{exc}}^{(0)}}{\left[\frac{9}{4} + \beta_0^4\right]^{1/2}} = \left[1 + \frac{L(L+1)}{3\left(\frac{9}{4} + \beta_0^4\right)}\right]^{1/2} - 1, \tag{9}$$

which is the same as the Holmberg-Lipas formula [40]

$$E_H(L) = a_H\left[\sqrt{1 + b_H L(L+1)} - 1\right], \tag{10}$$

with $a_H = 1$ and $b_H = 1/(27/4 + 3\beta_0^4)$. The Holmberg-Lipas formula has built in the concept of the variable moment of inertia (VMI) model [18] (i.e., the increase of the moment of inertia as a function of angular momentum), as pointed out in Ref. [41].

The "radial" equation is also exactly soluble for $u(\beta)$ being a 5D infinite well potential, similar to the one mentioned below Eq. (3). In this case the X(5) model of Iachello [2] is obtained, in which the eigenfunctions are Bessel functions $J_\nu(k_{s,L}\beta)$ with [2,39] $\nu = \{[L(L+1) - K^2]/3 + 9/4\}^{1/2}$, while the spectrum is determined by the zeros of the Bessel functions, the relevant eigenvalues being $E_{\beta;s,L} = (\hbar^2/2B)(k_{s,L})^2$ with $k_{s,L} = x_{s,L}/\beta_W$, where $x_{s,L}$ is the sth zero of the Bessel function $J_\nu(k_{s,L}\beta)$.

The spectra of the X(5) and Davidson cases become directly comparable by establishing the formal correspondence $n = s-1$. In addition to the energy ratios $R_{n,L}$ and $\underline{R}_{n,L}$, defined in Eq. (4), which will be used for $K=0$ bands, the ratios

$$\underline{R}'_{n,L} = \frac{E_{n,L}^{(2)} - E_{n,2}^{(2)}}{E_{n,3}^{(2)} - E_{n,2}^{(2)}}, \tag{11}$$

defined within the $K=2$ band, will be used below. The quadrupole operator is again given by Eq. (5), while the B(E2) transition rates are calculated in the way described in Refs. [2,15].

In addition to the intraband B(E2) ratios defined in Eq. (6), interband B(E2) transition rate ratios

$$R_{n,L,n',L'}^{B(E2)} = \frac{B(E2;L_n \to L'_{n'})}{B(E2;2_0 \to 0_0)} \tag{12}$$

will be used. Quadrupole moments are defined by [24]

$$Q_{s,L} = \frac{4\sqrt{\pi}}{5}(L_s L_s 2 | L_s - L_s 0)\langle L_s \| T^{(E2)} \| L_s \rangle. \tag{13}$$

In what follows, the ratios

$$R_{n,L}^{Q} = \frac{Q_{n,L}}{Q_{0,2}}, \quad \underline{R}_{n,L}^{Q} = \frac{Q_{n,L}}{Q_{n,2}} \tag{14}$$

will be used.

III. LIMITING SYMMETRIES

A. The U(5) and O(6) limits in the E(5) framework

For $\beta_0 = 0$ it is clear that in the E(5) framework the original vibrational model of Bohr [3,37] (with $R_4 = 2$) is obtained, while for large β_0 the O(6) limit of the interacting boson model (IBM) [24] for large boson numbers, which coincides with the γ-unstable rotator (with $R_4 = 2.5$) is approached [9].

The gradual evolution from the U(5) to the O(6) limit, as β_0 is increased, is depicted in Fig. 1, where several energy ratios R_L within the ground state band are depicted, and in Fig. 2, where the intraband B(E2) ratios $R_{0,L}^{B(E2)}$ (within the ground state band) and $R_{1,L}^{B(E2)}$ (within the $n=1$ band) are shown. The limiting values at the right-hand side are in agreement with the O(6) predictions for large boson numbers, given by [24,41] $E(L) = AL(L+6)$, $R_L = L(L+6)/16$,

$$B(E2;L+2 \to L) = a\frac{L+2}{L+5}, \quad \underline{R}_{n,L}^{B(E2)} = \frac{5}{2}\frac{L+2}{L+5}, \tag{15}$$

where A and a are constants and Eq. (15) is valid for all values of n, taking into account the difference between the

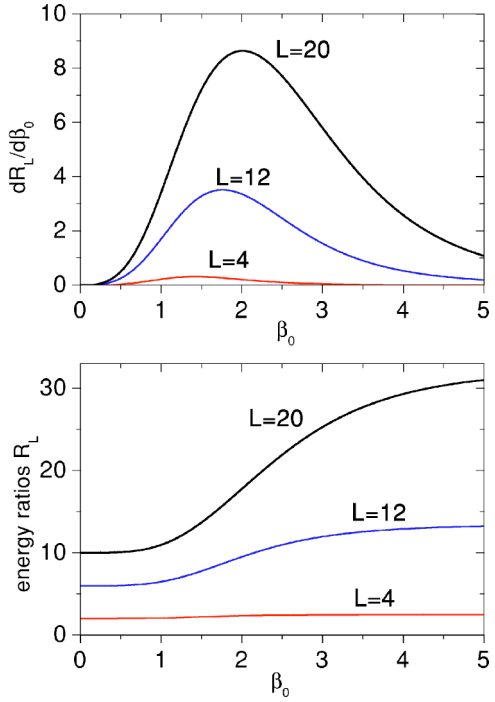

FIG. 1. (Color online) R_L energy ratios for the ground state band (for $L=4$, 12, 20) and their derivatives $dR_L/d\beta_0$ vs the parameter β_0, calculated using Davidson potentials [Eq. (1)] in the E(5) framework. The R_L curves demonstrate the evolution from the U(5) symmetry (on the left) to the O(6) limit of IBM with large boson numbers (on the right). See Secs. III A and IV A for further details.

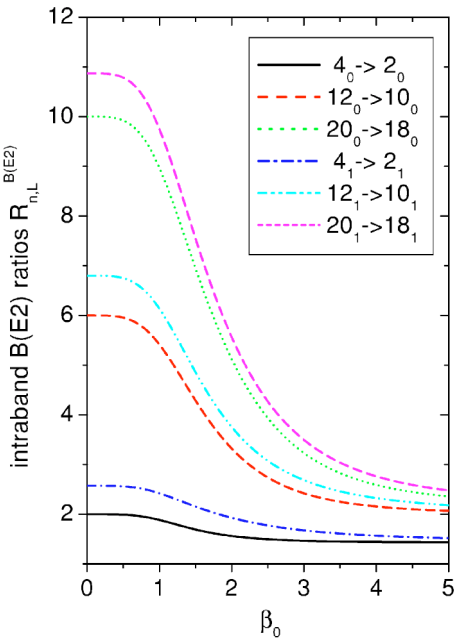

FIG. 2. (Color online) Intraband B(E2) ratios [Eq. (6)] $R_{0,L}^{B(E2)}$ (for the ground state band) and $R_{1,L}^{B(E2)}$ (for the $n=1$ band) for $L=2$, 10, 18, vs the parameter β_0. The curves show the evolution from the U(5) symmetry (on the left) to the O(6) limit of IBM with large boson numbers (on the right). See Secs. III A and VI A for further discussion.

$R_{n,L}^{B(E2)}$ and $\underline{R}_{n,L}^{B(E2)}$ ratios, defined in Eq. (6), while at the left-hand side the U(5) values, corresponding to [14,24] $E(L)=AL$, $R_L=L/2$,

$$B[E2;(L+2)_0 \to L_0] = a(L+2), \quad R_{0,L}^{B(E2)} = \frac{L+2}{2}, \tag{16}$$

$$B[E2;(L+2)_1 \to L_1] = a'\frac{(L+2)(L+7)}{L+5},$$
$$\underline{R}_{1,L}^{B(E2)} = \frac{5}{14}\frac{(L+2)(L+7)}{L+5}, \tag{17}$$

where A, a, a' constants, are obtained.

It should be noticed that the O(6) limit of IBM (with large boson numbers) is also obtained [27] by using in the E(5) framework the exactly soluble Kratzer potential [26], which has the form

$$u(\beta) = -2\mathcal{D}\left(\frac{\beta_0}{\beta} - \frac{1}{2}\frac{\beta_0^2}{\beta^2}\right) = -\frac{A}{\beta} + \frac{B}{\beta^2}, \tag{18}$$

where $\mathcal{D}$ is the depth of the minimum of the potential, which is located at β_0, while $A=2\beta_0\mathcal{D}$ and $B=\beta_0^2\mathcal{D}$. The O(6) limit is obtained for large values of $\mathcal{D}$ or large values of β_0. In the case of the Kratzer potential, however, it is clear that small values of $\mathcal{D}$ or small values of β_0 lead [27] to the Coulomb potential.

B. The X(5)-β^2 and SU(3) limits in the X(5) framework

In the X(5) framework for $\beta_0=0$ the exactly soluble X(5)-β^2 model (with $R_4=2.646$) is obtained, the details of which can be found in Ref. [15], while for large β_0 the SU(3) limit of IBM with large boson numbers, which coincides with the rigid rotator (with $R_4=3.333$) is obtained. In what follows the occurence of the SU(3) limit will be discussed in more detail.

It is clear that the Holmberg-Lipas formula gives rotational spectra for small values of b_H, at which one can keep only the first L-dependent term in the Taylor expansion of the square root appearing in Eq. (10), leading to energies proportional to $L(L+1)$. It is then clear that rotational spectra are expected for large values of β_0, for which small values of b_H occur. One can easily verify that the R_L ratios for large β_0 approach the predictions of the SU(3) limit of IBM at large boson numbers, which correspond to the rigid rotator with $E(L)=AL(L+1)$ and $R_L=L(L+1)/6$, where A is constant [24], the agreement to the SU(3) results being quite good already at $\beta_0=8$. The same can be seen for the ratios $\underline{R}'_{0,L}$, regarding the $n=0$, $K=2$ band, which in the rigid rotator case correspond to the limiting values $\underline{R}'_{0,L}=L(L+1)/6-1$.

One can also consider the intraband B(E2) ratios $\underline{R}_{0,L}^{B(E2)}$ (within the ground state band, which has $n=0$) and $\underline{R}_{1,L}^{B(E2)}$ (within the next band, which is characterized by $n=1$). In the SU(3) limit (for infinite number of bosons) one has for all $K=0$ bands [24]

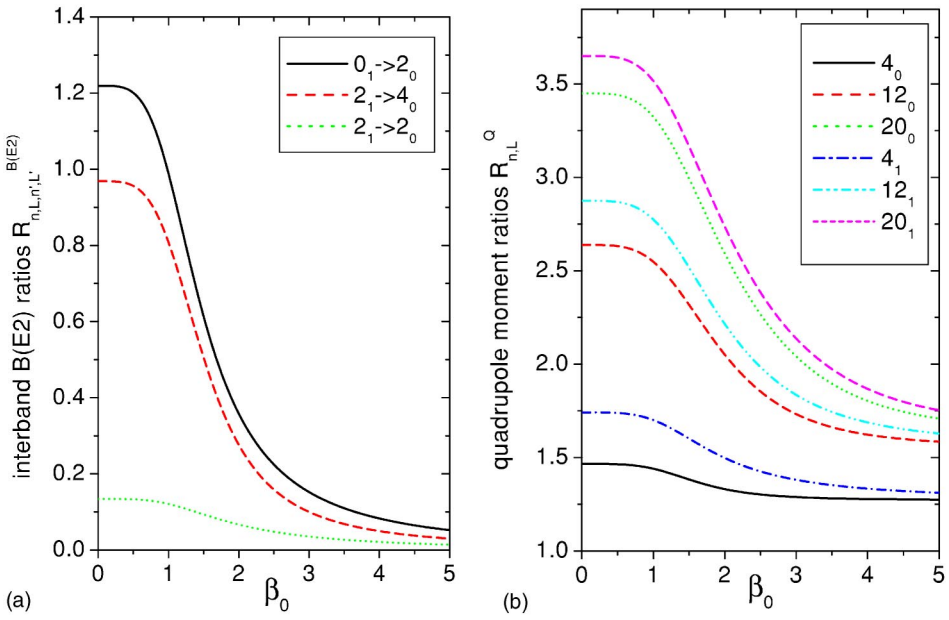

FIG. 3. (Color online) (a) Interband B(E2) ratios $R^{B(E2)}_{1,L,0,L'}$ [Eq. (12)] from the $n=1$, $K=0$ band to the ground state band vs β_0. (b) Quadrupole moment ratios [Eq. (14)] $R^{Q}_{0,L}$ (for the ground state band) and $R^{Q}_{1,L}$ (for the $n=1$, $K=0$ band) for $L=4$, 12, 20, vs β_0. In all cases the curves show the evolution from the X(5)-β^2 model (on the left) to the SU(3) limit of IBM with large boson numbers (on the right). See Secs. III B, VI B, and VII for further discussion.

$$B(E2;L+2\to L)=a\frac{(L+2)(L+1)}{(2L+3)(2L+5)},$$
$$\underline{R}^{B(E2)}_{n,L}=\frac{15}{2}\frac{(L+2)(L+1)}{(2L+3)(2L+5)}, \tag{19}$$

where a is constant, for all values of n. One can easily verify that the $n=0$ and $n=1$ results still differ a little at $\beta_0=4$, becoming almost identical to the SU(3) behavior at $\beta_0=8$.

Interband B(E2) transition rates from the $n=1$ band to the $n=0$ band [defined in Eq. (12)], which are forbidden in the SU(3) framework [24], are shown in Fig. 3(a). The rapid fall of these transitions towards zero with increasing β_0 can be seen in Fig. 3(a), where the X(5)-β^2 limiting values on the left are in agreement with the ones given in Ref. [15].

Ratios of quadrupole moments within the $n=0$ and $n=1$ bands are shown in Fig. 3(b). In the SU(3) limit (for infinite number of bosons) one has [24] (for all values of n) $Q_{n,L}=aL/(2L+3)$, where a is constant, corresponding to $\underline{R}^{Q}_{n,L}=7L/(4L+6)$. One can easily verify that at $\beta_0=4$ some differences between the $n=0$ and $n=1$ cases are still visible, while at $\beta_0=8$ both cases become almost identical to the SU(3) values.

The evolution of quadrupole moments from the X(5)-β^2 case to the SU(3) limiting values is clearly seen in Fig. 3(b). The SU(3) limiting values for $n=0$ and $n=1$ on the right do converge with increasing β_0 [taking into account the difference between the ratios $R^{Q}_{n,L}$ and $\underline{R}^{Q}_{n,L}$, defined in Eq. (14)]. Since no results for quadrupole moments for the X(5)-β^2 case are given in Ref. [15], they are reported here in Table I. Quadrupole moments for the X(5) model, as well as for the X(5)-β^4, X(5)-β^6, and X(5)-β^8 models (defined in Ref. [15]) are also listed in Table I as a by-product.

It should be noted that the SU(3) limit of IBM (at large boson numbers) is also obtained [28] by using in the X(5) framework the exactly soluble Kratzer potential [26] of Eq. (18). The SU(3) limit is obtained for large values of B. In the case of the Kratzer potential, however, it is clear that small values of B lead [28] to the Coulomb potential.

IV. VARIATIONAL PROCEDURE APPLIED TO ENERGY RATIOS

A. E(5) framework

We first consider the ratios R_L within the ground state band, since these ratios are well-known indicators of collectivity in nuclear structure [42]. As seen in Fig. 1, these ratios increase with β_0, the increase becoming very steep at some value $\beta_{0,m}(L)$ of β_0, where the first derivative $dR_L/d\beta_0$ reaches a maximum value, while the second derivative $d^2R_L/d\beta_0^2$ vanishes. Numerical results for $\beta_{0,m}$ are shown in Table II, together with the values of R_L occuring at these points, which are compared to the R_L ratios occuring in the ground state band of the E(5) model [1]. Very close agreement of the values determined by the procedure described above with the E(5) values is observed in Table II, as well as in Fig. 4(a), where these ratios are also shown, together with the corresponding ratios of the U(5) and O(6) limits. Finally, the potentials obtained for different angular momenta L are depicted in Fig. 5.

It is worth mentioning at this point that the consequences of replacing in the E(5) framework the infinite well potential by a well of finite depth have been studied in detail [43], the main conclusion being that many key features of E(5) remain essentially unchanged, even if the depth of the potential is radically changed. This observation implies that the E(5) predictions, reassured above through the variational procedure, are stable and do not depend sensitively on any parameter like the depth of the potential.

The success of the variational procedure when applied to energy ratios within the ground state band encourages its use for excited bands as well. Since it is reasonable to treat each band as a separate entity, the ratios $\underline{R}_{n,L}$ [defined in Eq. (4)] should be used for this purpose. From Eq. (2) it is clear,

TABLE I. Quadrupole moments [Eq. (13)] of the X(5)-β^4, X(5)-β^6, and X(5)-β^8 models, compared to the predictions of the X(5) and X(5)-β^2 models for some $K=0$ bands. See Sec. II B for details.

band	L	X(5)-β^2	X(5)-β^4	X(5)-β^6	X(5)-β^8	X(5)
$s=1$						
	0	0.000	0.000	0.000	0.000	0.000
	2	1.000	1.000	1.000	1.000	1.000
	4	1.466	1.413	1.391	1.380	1.358
	6	1.823	1.699	1.648	1.622	1.572
	8	2.127	1.925	1.842	1.800	1.719
	10	2.395	2.114	2.000	1.941	1.828
	12	2.638	2.277	2.132	2.058	1.913
	14	2.862	2.422	2.247	2.158	1.982
	16	3.070	2.552	2.348	2.245	2.038
	18	3.266	2.671	2.439	2.322	2.086
	20	3.450	2.781	2.522	2.391	2.127
	22	3.626	2.884	2.598	2.454	2.162
	24	3.794	2.980	2.669	2.512	2.193
	26	3.954	3.070	2.734	2.565	2.221
	28	4.109	3.155	2.795	2.615	2.245
	30	4.258	3.236	2.853	2.661	2.268
$s=2$						
	0	0.000	0.000	0.000	0.000	0.000
	2	1.245	1.107	1.034	0.991	0.894
	4	1.742	1.521	1.411	1.346	1.197
	6	2.095	1.796	1.655	1.573	1.379
	8	2.387	2.012	1.840	1.743	1.508
	10	2.643	2.191	1.992	1.880	1.608
	12	2.875	2.347	2.120	1.995	1.689
	14	3.088	2.486	2.233	2.095	1.756
	16	3.287	2.612	2.333	2.182	1.813
	18	3.473	2.727	2.423	2.260	1.863
	20	3.651	2.833	2.505	2.331	1.906

however, that in the case of the Davidson potentials and for bands with $L=2\tau$ the $\underline{R}_{n,L}$ ratios will be identical to R_L for all values of n. This is a special feature of the Davidson potentials, due to their oscillatorlike spectrum. This feature is lifted when one considers generalized Davidson potentials of the form

$$u(\beta) = \beta^{2n} + \frac{\beta_0^{4n}}{\beta^{2n}}, \; n = 2,3,4,\ldots, \tag{20}$$

which will be briefly discussed in Sec. VIII.

B. X(5) framework

The steps of the previous subsection have been repeated in the X(5) case, shown in Table III. Very close agreement with the ground state band of the X(5) model [2] is observed, while the relevant potentials form a figure similar to Fig. 5.

TABLE II. Parameter values $\beta_{0,m}$, where the absolute value of the first derivative of a physical quantity has a maximum, while the second derivative vanishes, for the R_L energy ratios and the $E_{0,L,\mathrm{exc}}$ excitation energies [Eq. (3)] of the ground state band of the Davidson potentials in the E(5) framework, together with the corresponding values of each physical quantity (labeled by "var"), which are compared to appropriate E(5) or U(5) results. In the case of $E_{0,L,\mathrm{exc}}$, the corresponding R_L ratios are also shown. See Secs. IV A and V A for further details.

L	$\beta_{0,m}$	R_L var	R_L E(5)	$\beta_{0,m}$	$E_{0,L,\mathrm{exc}}$ var	R_L var	R_L U(5)
2				1.399	0.709	1.000	1.000
4	1.421	2.185	2.199	1.538	1.423	2.008	2.000
6	1.522	3.549	3.590	1.658	2.142	3.021	3.000
8	1.609	5.086	5.169	1.766	2.862	4.037	4.000
10	1.687	6.793	6.934	1.866	3.583	5.054	5.000
12	1.759	8.667	8.881	1.960	4.303	6.070	6.000
14	1.825	10.705	11.009	2.049	5.022	7.084	7.000
16	1.888	12.906	13.316	2.135	5.739	8.096	8.000
18	1.947	15.269	15.799	2.217	6.455	9.105	9.000
20	2.004	17.793	18.459	2.297	7.168	10.111	10.000

For excited $K=0$ bands the ratios $\underline{R}_{n,L}$ should be considered. However, because of Eq. (7), it is clear that all these ratios for all $K=0$ bands are identical to R_L for all values of n, due to the oscillatorlike form of the spectrum of Davidson potentials. This is lifted when using the generalized Davidson potentials of Eq. (20), which will be further discussed in Sec. VIII.

For $K=2$ bands the ratios $\underline{R}'_{n,L}$ [defined in Eq. (11)] should be used. The relevant results for the $n=0$, $K=2$ band are reported in Table III and Fig. 6(a). Not only the energy ratios obtained through the variational procedure are very close to the X(5) results [15,39], but in addition the $\beta_{0,m}$ values obtained for the even values of L are very close to the corresponding ones obtained from the application of the variational procedure to the ground state band, discussed above.

V. VARIATIONAL PROCEDURE APPLIED TO EXCITATION ENERGIES

A. E(5) framework

It is instructive to apply the variational procedure developed in the previous section to isolated energy levels instead of energy ratios. For this purpose the excitation energies $E_{0,L,\mathrm{exc}}$ of the levels of the ground state band, given by Eq. (3), will be used. The values $\beta_{0,m}$ where the absolute value of the first derivative (since the first derivative is negative in this case) becomes maximum are reported in Table II, together with the corresponding $E_{0,L,\mathrm{exc}}$ values. Using the $E_{0,L,\mathrm{exc}}$ values obtained in this way, one can calculate the relevant R_L ratios. As seen in Table II, the results obtained in this way are very close to the U(5) results.

This result is easy to explain: From the experimental data (see Ref. [44], for example) it is known that excitation ener-

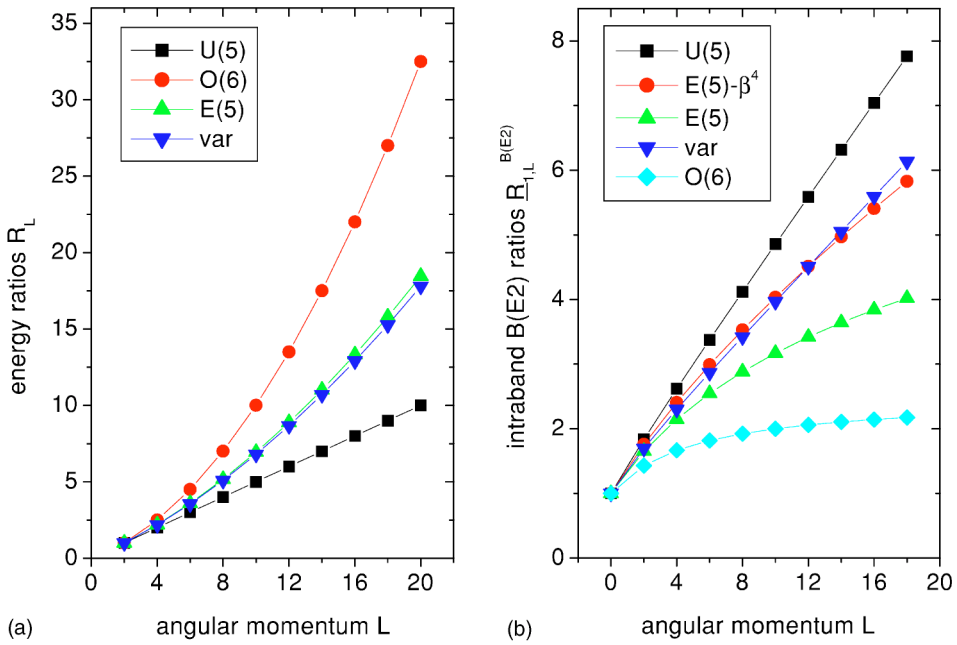

FIG. 4. (Color online) Energy ratios R_L for the ground state band (a), and intraband B(E2) ratios $R^{B(E2)}_{1,L}$ [Eq. (6)] (b) for the $n=1$ band, obtained through the variational procedure (labeled by "var") using Davidson potentials in the E(5) framework, compared to the values provided by the U(5), O(6), E(5), and E(5)-β^4 models. See Secs. IV A and VI A for further details.

gies within a series of isotopes drop very rapidly in the region of the vibrational limit, as one moves away from the pure vibrational behavior (see, for example, the chains of the Sm, Gd, Dy isotopes), while the changes near the rotational limit as one moves from one isotope to the next are minimal (see, for example, the Th, U, Pu isotopes). Trying then to identify a series of energy levels corresponding to the most rapid changes in the excitation energies, one naturally ends up with the vibrational limit. Therefore the application of the variational procedure to isolated energy levels just demonstrates the effectiveness of the method, leading to results physically expected.

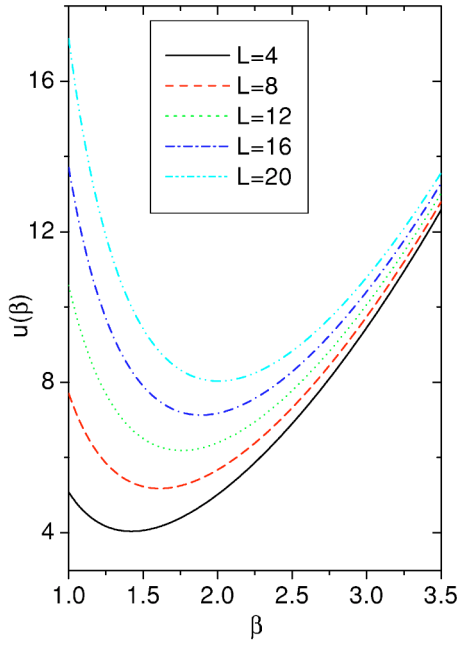

FIG. 5. (Color online) Davidson potentials [Eq. (1)] obtained for different angular momenta L through the application of the variational procedure to energy ratios within the ground state band in the E(5) framework. The β_0 values corresponding to these potentials are listed in Table II. See Sec. IV A for further discussion.

B. X(5) framework

The steps of the previous subsection are repeated using the excitation energies of Eq. (8). Numerical results are shown in Table III, and are seen to be very close to the results provided by the X(5)-β^2 model [15], which corresponds to a Davidson potential with $\beta_0=0$ and represents the most vibrational behavior possible within the realm of the model used, which spans the X(5)-β^2 to SU(3) region.

VI. VARIATIONAL PROCEDURE APPLIED TO B(E2) RATIOS

A. E(5) framework

The success of the variational procedure when applied to the energy ratios R_L also encourages its use for intraband B(E2) ratios. Considering each band as a separate entity, it is reasonable to use the ratios $R^{B(E2)}_{n,L}$, defined in Eq. (6). It should be emphasized that the different choice of the denominator in the ratios $R^{B(E2)}_{n,L}$ and $R^{B(E2)}_{n,L}$ is not a trivial matter of normalization, since one divides in each case by a different function of β_0, being led in this way to different results when the variational procedure is applied.

As we have seen in Fig. 2, the B(E2) ratios go down from their U(5) values at $\beta_0=0$ to the O(6) limiting values at large β_0. Therefore in this case we are going to determine the values $\beta_{0,m}$ at which the absolute value of the first derivative, $|dR^{B(E2)}_{n,L}/d\beta_0|$, has a maximum, while the second derivative

TABLE III. Parameter values $\beta_{0,m}$ where the absolute value of the first derivative of a physical quantity has a maximum, while the second derivative vanishes, for the R_L energy ratios and the $E^{(0)}_{0,L,\mathrm{exc}}$ excitation energies [Eq. (8)] of the ground state band, as well as for the $R'_{0,L}$ energy ratios [Eq. (11)] of the $n=0$, $K=2$ band of the Davidson potentials in the X(5) framework, together with the corresponding values of each physical quantity (labeled by "var"), which are compared to appropriate X(5) or X(5)-β^2 results. In the case of $E_{0,L,\mathrm{exc}}$, the corresponding R_L ratios are also shown. See Secs. IV B and V B for further details.

L	$\beta_{0,m}$	R_L var	R_L X(5)	$\beta_{0,m}$	$E^{(0)}_{0,L,\mathrm{exc}}$ var	R_L var	R_L X(5)-β^2	L	$\beta_{0,m}$	$R'_{0,L}$ var	$R'_{0,L}$ X(5)
2				1.329	0.397	1.000	1.000				
4	1.334	2.901	2.904	1.470	1.055	2.655	2.646	4	1.339	2.157	2.163
6	1.445	5.419	5.430	1.604	1.804	4.540	4.507	5	1.404	3.454	3.472
8	1.543	8.454	8.483	1.726	2.591	6.517	6.453	6	1.463	4.881	4.919
10	1.631	11.964	12.027	1.840	3.394	8.539	8.438	7	1.517	6.431	6.497
12	1.711	15.926	16.041	1.947	4.206	10.582	10.445	8	1.567	8.101	8.205
14	1.785	20.330	20.514	2.049	5.022	12.634	12.465	9	1.614	9.886	10.037
16	1.855	25.170	25.437	2.146	5.839	14.690	14.494	10	1.658	11.786	11.994
18	1.922	30.442	30.804	2.240	6.656	16.745	16.529				
20	1.985	36.146	36.611	2.330	7.472	18.798	18.568				

vanishes. Results for the $n=1$ band are depicted in Fig. 4(b), where the U(5) and O(6) results, calculated from Eqs. (17) and (15) , are shown for comparison. In addition, the results given by the original E(5) model [1], as well as by the E(5)-β^4 model [13,14], which uses a $u(\beta)=\beta^4/2$ potential in the E(5) framework instead of an infinite well potential, are exhibited. It is clear that the variational procedure leads to results which are quite close to the E(5)-β^4 case. One can easily verify that the same fact is seen for the ground state band (with $n=0$). We defer further discussions of these results to Sec. VIII.

It should be mentioned at this point that the first derivative of the $B(E2:L+2\rightarrow L)$ values with respect to β_0 does not exhibit a maximum, while the second derivative does not vanish at any value other than $\beta_0=0$. Therefore the variational procedure cannot be applied to isolated B(E2) values, being applicable to B(E2) ratios only.

B. X(5) framework

The steps of the previous subsection are repeated in the X(5) case. It is found that for both the ground state band ($n=0$, $K=0$) and the $n=1$, $K=0$ band the variational procedure leads to results which are quite close to the X(5)-β^4 case [15], which uses a $u(\beta)=\beta^4/2$ potential in the X(5) framework instead of an infinite well one. Further discussion of these results is deferred to Sec. VIII. As in the E(5) case, the variational procedure cannot be applied to isolated B(E2) values.

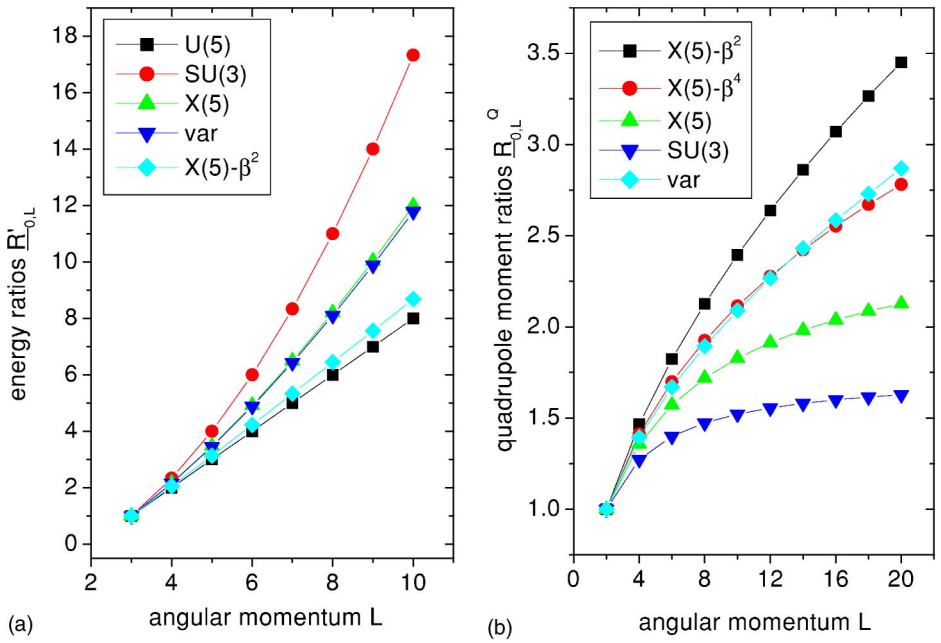

FIG. 6. (Color online) Energy ratios $R^{0,L}_{,}$ [Eq. (11)] (a) for the $n=0$, $K=2$ band, and quadrupole moment ratios $R^{Q}_{0,L}$ [Eq. (14)] (b) for the ground state band, obtained through the variational procedure (labeled by "var") using Davidson potentials in the X(5) framework, compared to the values provided by the U(5), SU(3), X(5), X(5)-β^2, and X(5)-β^4 models. See Secs. IV B and VII for further details.

VII. VARIATIONAL PROCEDURE APPLIED TO QUADRUPOLE MOMENTS

The variational procedure can also be applied to quadrupole moments. Since quadrupole moments in the E(5) framework vanish to lowest order, as mentioned in Sec. II A, we are limited here to the X(5) case. Treating each band as a separate entity, one should use the ratios $R^Q_{n,L}$, defined in Eq. (14), which involve quadrupole moments of only one band, in contrast to the ratios $R^Q_{n,L}$, defined also in Eq. (14), which involve quadrupole moments from two different bands, except in the case of the ground state band. Results for the ground state band ($n=0$) are plotted in Fig. 6(b), where the SU(3) results (given in Sec. III B) are shown for comparison. In addition, the results provided by the X(5), X(5)-β^2 and X(5)-β^4 models, given in Table I, are shown. It is clear that the results of the variational procedure are close to the X(5)-β^4 values. One can easily see that the same holds for the $n=1$ band. These results will be further discussed in the next section.

VIII. DISCUSSION

The main results obtained in the present work are summarized here.

(1) A variational procedure for determining the values of physical quantities at the point of shape phase transitions in nuclei has been suggested. Using one-parameter potentials spanning the region between the two limiting symmetries of interest, the parameter values at which the rate of change of the physical quantity becomes maximum are determined for each value of the angular momentum separately and the corresponding values of the physical quantity at these parameter values are calculated. The values of the physical quantity collected in this way represent its behavior at the critical point.

(2) The method has been applied in the shape phase transition from U(5) to O(6), using one-parameter Davidson potentials [11] and considering the energy ratios $R_L=E(L)/E(2)$ within the ground state band as the relevant physical quantity, leading to a band which practically coincides with the ground state band of the E(5) model [1]. It has also been applied in the same way in the shape phase transition from U(5) to SU(3), leading to a band which practically coincides with the ground state band of the X(5) model [2]. Energy ratios within the lowest $K=2$ band in the latter case also lead to the relevant X(5) results [15,39].

(3) The method has also been applied to intraband B(E2) ratios of the ground state band and the first excited band in the U(5)-O(6) transition region, leading to results very close to the ones provided by the E(5)-β^4 model [13,14], which uses a $u(\beta)=\beta^4/2$ potential instead of an infinite well potential in the E(5) framework. It has also been applied to intraband B(E2) ratios and ratios of quadrupole moments of the ground state band and the first excited band in the U(5)-SU(3) transition region, leading to results very similar to the ones provided by the X(5)-β^4 model [15], which uses a $u(\beta)=\beta^4/2$ potential instead of an infinite well potential in the X(5) framework.

(4) The method has also been applied to isolated excitation energies of the ground state band, leading to a U(5) band in the E(5) framework and to a X(5)-β^2 [15] band in the X(5) framework. [The U(5) and X(5)-β^2 models correspond to the use of a harmonic oscillator potential $u(\beta)=\beta^2/2$ in the E(5) and X(5) frameworks, respectively.] These results are expected, since it is known [44] that the most rapid change (drop) of the excitation energies in a series of isotopes occurs as one starts moving away from the vibrational limit towards the rotational limit.

(5) It should be emphasized that the application of the method was possible because the Davidson potentials correctly reproduce the U(5) and O(6) symmetries of IBM (with large boson numbers) in the E(5) framework (for small and large parameter values, respectively), as well as the relevant X(5)-β^2 [15] and SU(3) symmetries in the X(5) framework (for small and large parameter values, respectively). The occurence of SU(3) (with large boson numbers) in the X(5) framework is a new result, which has been proved by considering energy, intraband and interband B(E2), and quadrupole moment ratios, while the occurence of O(6) in the E(5) framework has essentially been observed earlier [9] and has been corroborated here by considering energy and intraband B(E2) ratios.

(6) As a by-product, a derivation of the Holmberg-Lipas formula [40] has been achieved using Davidson potentials in the X(5) framework.

(7) As another by-product, quadrupole moments for the X(5) model and the X(5)-β^{2n} models [15] for $n=1$, 2, 3, 4 have been calculated.

The following comments are now appropriate.

(i) The fact that the application of the variational procedure to energy ratios within the ground state band in the E(5) and X(5) frameworks leads to results very close to the ground state bands of the E(5) and X(5) models suggests that the selection of the infinite well potential is the optimum one in both cases.

(ii) The fact that the application of the variational procedure to intraband B(E2) transition ratios and to quadrupole moment ratios leads to results close to the E(5)-β^4 [13,14] and X(5)-β^4 [15] models in the E(5) and X(5) frameworks, respectively, suggests that further studies are needed. In particular, it is of interest to apply the variational procedure using the generalized Davidson potentials of Eq. (20) as "trial potentials" in both the E(5) and X(5) frameworks. These potentials with $\beta_0=0$ are known to smoothly approach the E(5) and X(5) models from the U(5) direction [14,15], as the power of β in the β^{2n} term increases. It is expected that these potentials with $\beta_0\neq 0$ will be smoothly approaching the E(5) and X(5) models from the O(6) and SU(3) directions, respectively. Then the results of the variational procedure could converge towards the E(5) and X(5) results with increasing n.

(iii) The application of the variational procedure to energy ratios involving levels of excited bands with $n>0$ is trivial in the case of the Davidson potentials, because of their harmonic oscillator features, but it becomes nontrivial in the case of the generalized Davidson potentials, and therefore this task should be undertaken.

(iv) The generalized Davidson potentials are known to possess the appropriate limiting behavior for $\beta_0=0$ [14,15]

and are expected to approach the appropriate limits [near O(6) and SU(3) in the E(5) and X(5) frameworks, respectively] for large values of β_0. Any other potential/Hamiltonian bridging the relevant pairs of symmetries [U(5)-O(6) and U(5)-SU(3)] could be equally appropriate.

(v) It is interesting that the most general (up to two-body terms) IBM Hamiltonian appropriate for the U(5) to O(6) transition leads [13] to the E(5)-β^4 model, in agreement to the results mentioned in (ii). It will also be interesting to examine if appropriate symmetry-conserving higher order terms [45–48], when added to this Hamiltonian, modify this conclusion.

ACKNOWLEDGMENTS

Partial support through the NATO Collaborative Linkage Grant No. PST.CLG 978799 is gratefully acknowledged.

[1] F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000).
[2] F. Iachello, Phys. Rev. Lett. **87**, 052502 (2001).
[3] A. Bohr, Mat. Fys. Medd. K. Dan. Vidensk. Selsk. **26**, No. 14 (1952).
[4] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).
[5] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **85**, 3584 (2000).
[6] N. V. Zamfir, M. A. Caprio, R. F. Casten, C. J. Barton, C. W. Beausang, Z. Berant, D. S. Brenner, W. T. Chou, J. R. Cooper, A. A. Hecht, R. Krücken, H. Newman, J. R. Novak, N. Pietralla, A. Wolf, and K. E. Zyromski, Phys. Rev. C **65**, 044325 (2002).
[7] R. F. Casten and N. V. Zamfir, Phys. Rev. Lett. **87**, 052503 (2001).
[8] R. Krücken, B. Albanna, C. Bialik, R. F. Casten, J. R. Cooper, A. Dewald, N. V. Zamfir, C. J. Barton, C. W. Beausang, M. A. Caprio, A. A. Hecht, T. Klug, J. R. Novak, N. Pietralla, and P. von Brentano, Phys. Rev. Lett. **88**, 232501 (2002).
[9] J. P. Elliott, J. A. Evans, and P. Park, Phys. Lett. **169B**, 309 (1986).
[10] D. J. Rowe and C. Bahri, J. Phys. A **31**, 4947 (1998).
[11] P. M. Davidson, Proc. R. Soc. London, Ser. A **135**, 459 (1932).
[12] V. Werner, P. von Brentano, R. F. Casten, and J. Jolie, Phys. Lett. B **527**, 55 (2002).
[13] J. M. Arias, C. E. Alonso, A. Vitturi, J. E. Garcia-Ramos, J. Dukelsky, and A. Frank, Phys. Rev. C **68**, 041302 (2003).
[14] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev, and P. A. Terziev, Phys. Rev. C **69**, 044316 (2004).
[15] D. Bonatsos, D. Lenis, N. Minkov, P. P. Raychev, and P. A. Terziev, Phys. Rev. C **69**, 014302 (2004).
[16] O. K. Vorov and V. G. Zelevinsky, Yad. Fiz. **37**, 1392 (1983) [Sov. J. Nucl. Phys. **37**, 830 (1983)].
[17] O. K. Vorov and V. G. Zelevinsky, Nucl. Phys. **A439**, 207 (1985).
[18] M. A. J. Mariscotti, G. Scharff-Goldhaber, and B. Buck, Phys. Rev. **178**, 1864 (1969).
[19] W. Greiner, *Quantum Mechanics–An Introduction* (Springer, Berlin 1989).
[20] H. Fiedelday and S. A. Sofianos, Z. Phys. A **311**, 339 (1983).
[21] R. S. Mackintosh, S. G. Cooper, and A. A. Ioannides, Nucl. Phys. **A472**, 85 (1987); **A476**, 287 (1988).
[22] D. Bonatsos and H. Müther, Nucl. Phys. **A469**, 23 (1989).
[23] W. Greiner, J. Y. Park, and W. Scheid, *Nuclear Molecules* (World Scientific, Singapore 1995).
[24] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
[25] J. P. Draayer, V. G. Gueorguiev, F. Pan, and Y. Luo, nucl-th/0311094.
[26] A. Kratzer, Z. Phys. **3**, 289 (1920).
[27] L. Fortunato and A. Vitturi, J. Phys. G **29**, 1341 (2003).
[28] L. Fortunato, and A. Vitturi, J. Phys. G **30**, 627 (2004).
[29] A. V. Turbiner and A. G. Ushveridze, Phys. Lett. A **126**, 181 (1987).
[30] A. V. Turbiner Commun. Math. Phys. **118**, 467 (1988).
[31] A. V. Turbiner, Zh. Eksp. Teor. Fiz. **94**, 33 (1988) [Sov. Phys. JETP **67**, 230 (1988)].
[32] G. Lévai and J. M. Arias, Phys. Rev. C **69**, 014304 (2004).
[33] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics* (IOP Publishing, Bristol 1994).
[34] D. Bonatsos, D. Lenis, N. Minkov, D. Petrellis, P. P. Raychev, and P. A. Terziev, Phys. Lett. B **584**, 40 (2004).
[35] G. Rakavy, Nucl. Phys. **4**, 289 (1957).
[36] D. R. Bès, Nucl. Phys. **10**, 373 (1959).
[37] G. G. Dussel and D. R. Bès, Nucl. Phys. **A143**, 623 (1970).
[38] E. Chacón and M. Moshinsky, J. Math. Phys. **18**, 870 (1977).
[39] R. Bijker, R. F. Casten, N. V. Zamfir, and E. A. McCutchan, Phys. Rev. C **68**, 064304 (2003).
[40] P. Holmberg and P. O. Lipas, Nucl. Phys. **A117**, 552 (1968).
[41] R. F. Casten, *Nuclear Structure from a Simple Perspective* (Oxford University Press, Oxford 1990).
[42] C. A. Mallmann, Phys. Rev. Lett. **2**, 507 (1959).
[43] M. A. Caprio, Phys. Rev. C **65**, 031304 (2002).
[44] M. Sakai, At. Data Nucl. Data Tables **31**, 399 (1984).
[45] P. Van Isacker and J. Q. Chen, Phys. Rev. C **24**, 684 (1981).
[46] K. Heyde, P. Van Isacker, M. Waroquier, and J. Moreau, Phys. Rev. C **29**, 1420 (1984).
[47] G. Vanden Berghe, H. E. De Meyer, and P. Van Isacker, Phys. Rev. C **32**, 1049 (1985).
[48] J. Vanthournout, H. De Meyer, and G. Vanden Berghe, Phys. Rev. C **38**, 414 (1988).