Role of fluctuations in the linear σ model with quarks

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We study the thermodynamics of the linear sigma model with constituent quarks beyond the mean-field approximation. By integrating out the quark degrees of freedom we derive an effective action for the meson fields which is then linearized around the ground state including field fluctuations. We propose a new method for performing exact averaging of complicated functions over the meson field fluctuations. Both thermal and zero-point fluctuations are considered. The chiral condensate and the effective meson masses are determined self-consistently in a rigorous thermodynamic framework. At zero chemical potential the model predicts a chiral crossover transition which separates two distinct regimes: heavy quarks and light pions at low temperatures, but light quarks and heavy mesons at high temperatures. The crossover becomes a first order phase transition if the vacuum pion mass is reduced from its physical value.

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I. INTRODUCTION

It is commonly accepted that quantum chromodynamics (QCD) is the true theory of strong interactions. Therefore, in principle, it should describe all phases of strongly interacting matter at all densities and temperatures. In practice QCD can be exactly solved only in some limiting cases: first, at very high densities and temperatures when the property of asymptotic freedom allows a perturbative expansion to be used, and second, at zero density and high enough temperatures when it can be discretized on a Euclidean lattice. Although some improvements at finite chemical potential have been achieved recently [1], the direct application of QCD at high baryon density, or for real-time processes, is still quite problematic at present. This makes it necessary to build effective models which respect only general properties of QCD, such as chiral symmetry, and operate with effective degrees of freedom, such as mesons and constituent quarks.

The linear sigma model (LσM) [2] is one of the most popular models of this kind which has been studied already for several decades (see, e.g., Refs. [3,4]). It incorporates correctly the phenomenology of low-energy strong interactions, including chiral symmetry. At low density and temperature the matter is assumed to be in a phase where chiral symmetry is spontaneously broken. A phase transition that restores chiral symmetry was predicted at high temperatures [5] or baryon densities [6]. According to Ref. [7] QCD with two massless flavors belongs to the same universality class as the O(4) LσM and therefore the phase transition is of the same order. In the case of nonzero quark masses chiral symmetry is explicitly broken and universality arguments cannot be applied. Then the character of the chiral transition depends on the detailed pattern of the symmetry breaking [8]. In recent years there have been many attempts to use different versions of the LσM to model QCD phase transitions at finite temperature [4,8,9], baryon density [10,11], and for nonequilibrium conditions [12–16].

Despite many studies the status of the LσM remains somewhat controversial. In most applications of this model only mesonic degrees of freedom are included explicitly [4,8,14]. Then the model shows a chiral phase transition, but the high-temperature phase is very different from what is expected for QCD, since there are no free quarks but only heavy mesons. On the other hand, if quark degrees of freedom are included from the beginning while fluctuations in the mesonic fields are ignored [11], the model also predicts a chiral transition at about the same temperature. Now, however, the low-temperature phase has the wrong structure since it is dominated by constituent quarks and, in the mean field approximation, pions play no role. Physically, of course, pions are expected to be the most relevant degrees of freedom at low temperatures. There have been several attempts to improve matters by including both constituent quarks and pions (see, e.g., Ref. [10]), but the calculations have been limited to the lowest order loop expansion. A more satisfactory approach has been pursued recently, namely the (approximate) solution of the renormalization group flow equations [17,18] which include effects due to field fluctuations. This appears to be particularly valuable in the neighborhood of critical points.

In the present paper we study a different, and possibly more transparent, approach to including field fluctuations. We deal with the full LσM including both constituent quarks and mesonic excitations and our goal is to proceed as far as possible without invoking any kind of mean-field approximation or perturbative expansion. We shall demonstrate below that one can indeed develop a practical computational scheme where the field fluctuations are incorporated in the thermodynamic potential to all orders in a self-consistent way. This formalism corresponds to summing up the infinite series of daisy and superdaisy diagrams. Of course, this is possible only within the Hartree approximation where the exchange diagrams are disregarded. The resulting physical picture appears to be close to QCD-based expectations, e.g. Ref. [19]. Namely, the model exhibits a smooth chiral cross-

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over transition at temperatures of 150–200 MeV from a low-temperature phase made of heavy constituent quarks and light pions to a high-temperature phase composed of light quarks and heavy mesonic excitations. We regard these results as quite satisfactory for modeling QCD, even though the model has no confinement. For effective theories that incorporate both chiral symmetry and confinement see Ref. [20].

The paper is organized as follows. In Sec. II a general LσM is formulated and its partition function is represented within the path integral formalism. We integrate out the quark degrees of freedom and approximately reduce the problem to a purely mesonic theory with a very nonlinear effective potential. This potential is then linearized around the correct ground state including equilibrium field fluctuations to all orders. Finally, the mesonic contribution to the thermodynamic potential is calculated in closed form. In Sec. III we describe a general method for evaluating averages of complicated functions over the fluctuations of the fields. This permits easy consistency checks and allows the equations to be put into a simple form. Results obtained by including or excluding zero-point contributions, in addition to the thermal fluctuations, are presented in Sec. IV. Here we also assess simplified approximations to the complete results. Our conclusions are presented in Sec. V. The relationship of the present approach to earlier work [21,22] is discussed in Appendix A, and the proof of a useful identity is given in Appendix B.

II. THEORY

A. LσM Lagrangian and partition function

In this paper we employ a standard version of the LσM model with $SU(2)_L \times SU(2)_R$ symmetry. The corresponding Lagrangian for quarks interacting with the σ and $\boldsymbol{\pi}$ meson fields is written as

$$\mathcal{L} = \mathcal{L}_{\overline{q}q} + \mathcal{L}_{Km} - U(\sigma, \boldsymbol{\pi}), \tag{1}$$

where the quark Lagrangian is

$$\mathcal{L}_{\overline{q}q} = \overline{q}[i\gamma_\mu \partial^\mu - g(\sigma + i\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi})]q, \tag{2}$$

the meson kinetic energy is

$$\mathcal{L}_{Km} = \tfrac{1}{2}(\partial_\mu \sigma \partial^\mu \sigma + \partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi}), \tag{3}$$

and the meson potential is given by

$$U(\sigma, \boldsymbol{\pi}) = \tfrac{1}{4}\lambda(\sigma^2 + \boldsymbol{\pi}^2 - \zeta)^2 - H\sigma. \tag{4}$$

Here we have included an explicit chiral symmetry breaking term of the conventional type, $H\sigma$. As is well known, the choice of the symmetry breaking term is not unique. The parameters of the model, g, λ, ζ, and H, will be specified later. The partition function of the grand canonical ensemble can be written as a functional integral over the quark and meson fields [23],

$$\begin{aligned} Z &= \mathrm{Tr}\, \exp\{-\beta(H - \mu\hat{N})\} \\ &= \int \mathcal{D}\overline{q}\mathcal{D}q\mathcal{D}\sigma\mathcal{D}\boldsymbol{\pi}\, \exp\left\{\int_0^\beta d\tau \int_V d^3x\, (\mathcal{L} + \mu\overline{q}\gamma^0 q)\right\}, \end{aligned} \tag{5}$$

where $\hat{N}$ is the quark number operator, $\beta = T^{-1}$ is the inverse temperature, μ is the quark chemical potential, V is the volume of the system, and $\tau = it$ denotes the imaginary time.

B. Integrating out the quarks

First we integrate out the quark degrees of freedom. The relevant part of the partition function is

$$Z_{\overline{q}q} = \int \mathcal{D}\overline{q}\mathcal{D}q\, \exp\left\{\int_0^\beta d\tau \int_V d^3x\, \overline{q}\hat{D}q\right\}, \tag{6}$$

where

$$\hat{D} = -\gamma^0 \frac{\partial}{\partial \tau} + i\boldsymbol{\gamma} \cdot \boldsymbol{\nabla} - g(\sigma + i\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\pi}) + \mu\gamma^0. \tag{7}$$

Formally one can integrate out the quark fields with the result

$$\ln Z_{\overline{q}q} = \int_0^\beta d\tau \int_V d^3x\, \ln \det \hat{D}. \tag{8}$$

The analysis of such an expression has been discussed in detail by Fraser [24]. Accordingly, one proceeds by moving the operators to the left and the fields which depend on τ and $\mathbf{x}$ to the right. This generates a series of commutators which involve the derivatives of the meson fields. Since our interest is in low energy properties we will discard these commutator terms, tacitly assuming that the meson mode amplitudes vary slowly in position and time. Then, evaluating the operator $\hat{D}$ in a frequency-three-momentum representation we get

$$\begin{aligned} \ln \det \hat{D} = \frac{\nu_q}{2\beta V} \sum_{\mathbf{p}n} \Big(&\ln\{\beta^2[\omega_n^2 + (E-\mu)^2]\} \\ &+ \ln\{\beta^2[\omega_n^2 + (E+\mu)^2]\}\Big), \end{aligned} \tag{9}$$

where the Matsubara frequency $\omega_n = (2n+1)\pi T$, the quark degeneracy ν_q is 12 for the two flavors employed here, and $E^2 = p^2 + m^2$. The quark effective mass is given by

$$m^2 = g^2(\sigma^2 + \boldsymbol{\pi}^2). \tag{10}$$

After performing the summation over n we get $\ln Z_{\overline{q}q} = -\int_0^\beta d\tau \int_V d^3x\, \Omega_{\overline{q}q}(m)$, where the quark-antiquark thermodynamic potential density is expressed as

$$\begin{aligned} \Omega_{\overline{q}q}(m) = -\frac{\nu_q T}{2\pi^2} \int dp\, p^2 \{&\beta E + \ln[1 + e^{-\beta(E-\mu)}] \\ &+ \ln[1 + e^{-\beta(E+\mu)}]\}. \end{aligned} \tag{11}$$

Note that this differs from the standard result in that the mass depends on the meson fields according to Eq. (10). Thus the partition function (5) can be written as

$$Z = \int \mathcal{D}\sigma\mathcal{D}\boldsymbol{\pi} \exp\left\{ \int_0^\beta d\tau \int_V d^3x \mathcal{L}_m \right\}, \tag{12}$$

where

$$\mathcal{L}_m = \mathcal{L}_{Km} - \widetilde{U}(\sigma, \boldsymbol{\pi}) \tag{13}$$

is the effective meson Lagrangian. Here the effective potential, including the contribution of quarks and antiquarks with effective mass m, is

$$\widetilde{U}(\sigma, \boldsymbol{\pi}) = U(\sigma, \boldsymbol{\pi}) + \Omega_{\overline{q}q}(m). \tag{14}$$

The Lagrangian (13) now contains only meson fields and it constitutes an effective meson theory with the very nonlinear interaction potential (14).

C. Linearization of the mesonic action

Let us consider the equations of motion for meson fields which follow from the effective meson Lagrangian (13):

$$\partial^\mu \partial_\mu \sigma + \frac{\partial \widetilde{U}}{\partial \sigma} = 0, \tag{15}$$

$$\partial^\mu \partial_\mu \boldsymbol{\pi} + \frac{\partial \widetilde{U}}{\partial \boldsymbol{\pi}} = 0. \tag{16}$$

We average these equations over the meson field fluctuations. Since the σ field is expected to develop a nonvanishing expectation value v, we decompose it as $\sigma = v + \Delta$ where Δ is the fluctuating part. By definition the average of Δ is zero and this is also true for any odd power of Δ, i.e., $\langle \Delta^n \rangle = 0$ for odd n. A similar remark applies to the pion field. Here and below angle brackets indicate averaging over the field fluctuations. A practical scheme for evaluating such averages is discussed in the following section.

As the thermal average of an odd number of fluctuating fields is zero, only the terms in $\widetilde{U}$ with odd powers of Δ will contribute to Eq. (15), yielding

$$\left\langle \frac{\partial \widetilde{U}(v + \Delta, \boldsymbol{\pi})}{\partial \Delta} \right\rangle = 0. \tag{17}$$

Since the pion field always occurs as $\boldsymbol{\pi}^2$, a single derivative with respect to a component π_i will always yield an odd number of fluctuating fields and thus the thermal average automatically vanishes. Notice that Eq. (17) includes all powers of the fluctuating fields, so the equation of motion contains fluctuations of the fields to all orders. This makes our approach different from numerous previous attempts to include fluctuations where an expansion is made around the mean field ground state. The second derivatives of the effective potential pick out the even powers of the fluctuating fields in $\widetilde{U}$ and these we identify with the meson masses:

$$m_\sigma^2 = \left\langle \frac{\partial^2 \widetilde{U}(v + \Delta, \boldsymbol{\pi})}{\partial \Delta^2} \right\rangle; \quad m_\pi^2 = \left\langle \frac{\partial^2 \widetilde{U}(v + \Delta, \boldsymbol{\pi})}{\partial \pi_i^2} \right\rangle. \tag{18}$$

We now linearize the complicated effective mesonic potential by setting

$$\widetilde{U}(v + \Delta, \boldsymbol{\pi}) \to \langle \widetilde{U}(v + \Delta, \boldsymbol{\pi}) \rangle + \tfrac{1}{2} m_\sigma^2 (\Delta^2 - \langle \Delta^2 \rangle) + \tfrac{1}{2} m_\pi^2 (\boldsymbol{\pi}^2 - \langle \boldsymbol{\pi}^2 \rangle). \tag{19}$$

Obviously, this becomes an identity if we average over the field fluctuations on both sides. The terms on the right containing the average quantities should be included directly in the total thermodynamic potential density. The remaining terms containing Δ^2 and $\boldsymbol{\pi}^2$ are combined with the kinetic energy to give the mesonic partition function:

$$Z_m = \int \mathcal{D}\Delta\mathcal{D}\boldsymbol{\pi} \exp\left\{ \int_0^\beta d\tau \int_V d^3x \left[\mathcal{L}_{Km} - \tfrac{1}{2} m_\sigma^2 \Delta^2 - \tfrac{1}{2} m_\pi^2 \boldsymbol{\pi}^2 \right] \right\}. \tag{20}$$

Following standard steps [23] one arrives at the thermodynamic potential density associated with the meson field fluctuations:

$$\Omega_m = -\frac{\ln Z_m}{\beta V} \equiv \Omega_\sigma + \Omega_\pi, \tag{21}$$

$$\Omega_\sigma = \frac{T}{2\pi^2} \int dp p^2 \left\{ \tfrac{1}{2} \beta E_\sigma + \ln(1 - e^{-\beta E_\sigma}) \right\}, \tag{22}$$

$$\Omega_\pi = \frac{3T}{2\pi^2} \int dp p^2 \left\{ \tfrac{1}{2} \beta E_\pi + \ln(1 - e^{-\beta E_\pi}) \right\}, \tag{23}$$

where $E_\sigma = \sqrt{p^2 + m_\sigma^2}$ and $E_\pi = \sqrt{p^2 + m_\pi^2}$. Two consistency relations for the meson masses follow directly from Eq. (20):

$$\langle \Delta^2 \rangle = 2 \frac{\partial \Omega_\sigma}{\partial m_\sigma^2}; \quad \langle \boldsymbol{\pi}^2 \rangle = 2 \frac{\partial \Omega_\pi}{\partial m_\pi^2}. \tag{24}$$

Finally, we can write the total thermodynamic potential density as

$$\Omega = \langle U(v + \Delta, \boldsymbol{\pi}) \rangle + \langle \Omega_{\overline{q}q}(m) \rangle - \tfrac{1}{2} m_\sigma^2 \langle \Delta^2 \rangle - \tfrac{1}{2} m_\pi^2 \langle \boldsymbol{\pi}^2 \rangle + \Omega_m(m_\sigma, m_\pi). \tag{25}$$

The subtraction of the third and fourth terms on the right is necessary to avoid double counting [25].

III. EVALUATION OF FIELD FLUCTUATIONS

A. Averaging procedure

Consider now the average of a complicated function $\mathcal{O}(v + \Delta, \boldsymbol{\pi}^2)$ over the fluctuating fields Δ and $\boldsymbol{\pi}$. The averaging can be carried out by generalizing the technique introduced in Ref. [26]. First, we expand the function in a Taylor series about the point $(v,0)$, then take the average term by term

$$\langle \mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)\rangle = \sum_{k,n} \mathcal{O}^{(k,n)}(v,0)\left\langle \frac{\Delta^k}{k!}\frac{\boldsymbol{\pi}^{2n}}{n!}\right\rangle, \tag{26}$$

where

$$\mathcal{O}^{(k,n)}(a,b) \equiv \frac{\partial^k}{\partial a^k}\frac{\partial^n}{\partial b^n}\mathcal{O}(a,b). \tag{27}$$

The next step is to reduce a general vertex $\langle \Delta^k \boldsymbol{\pi}^{2n}\rangle$ to powers of $\langle \Delta^2\rangle$ and $\langle \boldsymbol{\pi}^2\rangle$. The necessary counting factors for joining the meson fields at this vertex in all possible ways [21] are: $\langle \Delta^k\rangle = (k-1)!!\langle \Delta^2\rangle^{k/2}$ for k even, and zero for k odd. For pions, all species are equivalent so $\langle \pi_1^2\rangle = \langle \pi_2^2\rangle = \langle \pi_3^2\rangle = \frac{1}{3}\langle \boldsymbol{\pi}^2\rangle$ and therefore $\langle \boldsymbol{\pi}^{2n}\rangle = (2n+1)!!\langle \frac{1}{3}\boldsymbol{\pi}^2\rangle^n$. After substituting these factors in the series (26) we notice that the resulting averaging is equivalent to performing integrations with the following Gaussian weighting functions:

$$P_\sigma(z) = (2\pi\langle \Delta^2\rangle)^{-1/2}\exp\left(-\frac{z^2}{2\langle \Delta^2\rangle}\right), \tag{28}$$

$$P_\pi(y) = \sqrt{\frac{2}{\pi}}\left(\frac{3}{\langle \boldsymbol{\pi}^2\rangle}\right)^{3/2}\exp\left(-\frac{3y^2}{2\langle \boldsymbol{\pi}^2\rangle}\right). \tag{29}$$

Then, resumming the Taylor series, we obtain

$$\langle \mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)\rangle = \int_{-\infty}^{\infty} dz P_\sigma(z)\int_0^\infty dy y^2 P_\pi(y)\mathcal{O}(v+z,y^2). \tag{30}$$

This is a general result for any analytic function $\mathcal{O}$. Note that $\langle \mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)\rangle \to \mathcal{O}(v,0)$ in the limit when $\langle \Delta^2\rangle$ and $\langle \boldsymbol{\pi}^2\rangle \to 0$. The correspondence between Eq. (30) and the approximate expressions used in Refs. [21,22] is discussed in Appendix A. We also need the derivative of Eq. (30) with respect to some variable α. After two integrations by parts one obtains

$$\begin{aligned}\frac{\partial}{\partial\alpha}\langle \mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)\rangle = {}& \frac{\partial v}{\partial \alpha}\left\langle \frac{\partial \mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)}{\partial v}\right\rangle \\ &+ \frac{1}{2}\frac{\partial\langle \Delta^2\rangle}{\partial\alpha}\left\langle \frac{\partial^2\mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)}{\partial \Delta^2}\right\rangle \\ &+ \frac{1}{2}\frac{\partial\langle \boldsymbol{\pi}^2\rangle}{\partial\alpha}\left\langle \frac{\partial^2\mathcal{O}(v+\Delta,\boldsymbol{\pi}^2)}{\partial \pi_i^2}\right\rangle.\end{aligned} \tag{31}$$

Using this equation one can easily check that the derivative of the total thermodynamic potential density with respect to v is

$$\begin{aligned}\frac{\partial\Omega}{\partial v} = {}& \left\langle \frac{\partial \tilde{U}}{\partial v}\right\rangle + \frac{1}{2}\frac{\partial\langle \Delta^2\rangle}{\partial v}\left\{\left\langle \frac{\partial^2\tilde{U}}{\partial\Delta^2}\right\rangle - m_\sigma^2\right\} \\ &+ \frac{1}{2}\frac{\partial\langle \boldsymbol{\pi}^2\rangle}{\partial v}\left\{\left\langle \frac{\partial^2\tilde{U}}{\partial\pi_i^2}\right\rangle - m_\pi^2\right\} = 0.\end{aligned} \tag{32}$$

This vanishes due to the equation of motion (17) and the mass definitions (18). Thus we have the nontrivial result that Ω is a minimum with respect to variations in the scalar condensate v, as is required for sensible thermodynamics. This analysis amounts to the formal justification of the approach of Refs. [21,22] where the thermal averages were evaluated approximately in series form, and therefore thermodynamic consistency was obtained only approximately.

B. Zero-point fluctuations

The general formulas (11), (22), and (23) include both zero-point and thermal fluctuations. The former are divergent and thus require a proper renormalization procedure. The nontrivial issue of renormalization in self-consistent approximation schemes has been discussed for the LσM model by several authors [27–29]. We adopt their result for the regularization in four dimensions, namely

$$\begin{aligned}2\frac{d\Omega_\sigma^{\rm zpt}}{dm_\sigma^2} &= \frac{1}{4\pi^2}\int_0^\infty dp\frac{p^2}{E_\sigma} \to \frac{1}{16\pi^2}\left[m_\sigma^2\ln\frac{m_\sigma^2}{\Lambda_m^2} + \Lambda_m^2 - m_\sigma^2\right], \\ 2\frac{d\Omega_\pi^{\rm zpt}}{dm_\pi^2} &= \frac{3}{4\pi^2}\int_0^\infty dp\frac{p^2}{E_\pi} \to \frac{3}{16\pi^2}\left[m_\pi^2\ln\frac{m_\pi^2}{\Lambda_m^2} + \Lambda_m^2 - m_\pi^2\right].\end{aligned} \tag{33}$$

Note that the $+\Lambda_m^2$ term may be removed by redefinition of the constant ζ in the sigma model potential (4). The zero-point contributions to Ω then follow by integration [see Eqs. (35) and (36) below].

For the quark case it is natural to adopt a similar form, setting

$$2\frac{d\Omega_{qq}^{\rm zpt}}{dm^2} = -\frac{\nu_q}{2\pi^2}\int_0^\infty dp\frac{p^2}{E} \to -\frac{\nu_q}{8\pi^2}\left[m^2\ln\frac{m^2}{\Lambda_q^2} + \Lambda_q^2 - m^2\right]. \tag{34}$$

As is reasonable in a phenomenological model we have allowed the renormalization scale Λ_q in the fermion sector to differ from the scale Λ_m in the meson sector to allow for the different physics in the two sectors. Again, integration gives the result for $\Omega_{q\bar{q}}$ [see Eq. (39) below].

C. Practical evaluation

The thermodynamic potential densities including both zero-point and thermal fluctuations of the sigma and pion fields are, respectively,

$$\begin{aligned}\Omega_\sigma = {}& \frac{1}{64\pi^2}\left[m_\sigma^4\ln\frac{m_\sigma^2}{\Lambda_m^2} + \frac{1}{2}(3m_\sigma^2 - \Lambda_m^2)(\Lambda_m^2 - m_\sigma^2)\right] \\ &+ \frac{T}{2\pi^2}\int dp p^2\ln(1 - e^{-\beta E_\sigma}),\end{aligned} \tag{35}$$

$$\Omega_\pi = \frac{3}{64\pi^2}\left[m_\pi^4 \ln\frac{m_\pi^2}{\Lambda_m^2} + \frac{1}{2}(3m_\pi^2 - \Lambda_m^2)(\Lambda_m^2 - m_\pi^2)\right] + \frac{3T}{2\pi^2}\int dpp^2 \ln(1 - e^{-\beta E_\pi}). \tag{36}$$

The squared fluctuations of the meson fields are found by using Eq. (24):

$$\langle \Delta^2 \rangle = 2\frac{\partial \Omega_m}{\partial m_\sigma^2} = \frac{1}{16\pi^2}\left[m_\sigma^2 \ln\frac{m_\sigma^2}{\Lambda_m^2} + \Lambda_m^2 - m_\sigma^2\right] + \frac{1}{2\pi^2}\int dp\frac{p^2}{E_\sigma}n_B(E_\sigma), \tag{37}$$

$$\langle \boldsymbol{\pi}^2 \rangle = 2\frac{\partial \Omega_m}{\partial m_\pi^2} = \frac{3}{16\pi^2}\left[m_\pi^2 \ln\frac{m_\pi^2}{\Lambda_m^2} + \Lambda_m^2 - m_\pi^2\right] + \frac{3}{2\pi^2}\int dp\frac{p^2}{E_\pi}n_B(E_\pi), \tag{38}$$

where the Bose occupation number is $n_B(x) = (\exp\,\beta x - 1)^{-1}$. The thermodynamic potential density associated with quarks is

$$\langle \Omega_{\bar{q}q}(m)\rangle = -\frac{\nu_q}{32\pi^2}\left\langle m^4 \ln\frac{m^2}{\Lambda_q^2} + \frac{1}{2}(3m^2 - \Lambda_q^2)(\Lambda_q^2 - m^2)\right\rangle - \frac{\nu_q T}{2\pi^2}\int dpp^2\langle \ln[1 + e^{-\beta(E-\mu)}] + \ln[1 + e^{-\beta(E+\mu)}]\rangle, \tag{39}$$

where $E = \sqrt{p^2 + m^2}$.

Since the total thermodynamic potential (25) is a minimum with respect to variations in the scalar condensate v, the thermodynamic quantities can be found in standard fashion. The pressure $P = -\Omega$ and the energy density is

$$\begin{aligned}\mathcal{E} = {} & \langle U(v + \Delta, \boldsymbol{\pi})\rangle - \frac{1}{2}m_\sigma^2\langle\Delta^2\rangle - \frac{1}{2}m_\pi^2\langle\boldsymbol{\pi}^2\rangle + \frac{1}{64\pi^2}\left[m_\sigma^4 \ln\frac{m_\sigma^2}{\Lambda_m^2} + \frac{1}{2}(3m_\sigma^2 - \Lambda_m^2)(\Lambda_m^2 - m_\sigma^2)\right] + \frac{3}{64\pi^2}\left[m_\pi^4 \ln\frac{m_\pi^2}{\Lambda_m^2}\right. \\ & \left. + \frac{1}{2}(3m_\pi^2 - \Lambda_m^2)(\Lambda_m^2 - m_\pi^2)\right] + \frac{1}{2\pi^2}\int dpp^2\left[E_\sigma n_B(E_\sigma) + 3E_\pi n_B(E_\pi)\right] - \frac{\nu_q}{32\pi^2}\left\langle m^4 \ln\frac{m^2}{\Lambda_q^2} + \frac{1}{2}(3m^2 - \Lambda_q^2)\right. \\ & \left. \times(\Lambda_q^2 - m^2)\right\rangle + \frac{\nu_q}{2\pi^2}\int dpp^2\langle E[n_F(E,\mu) + n_F(E,-\mu)]\rangle,\end{aligned} \tag{40}$$

where the Fermi-Dirac occupation number is $n_F(x,y) = [\exp\,\beta(x-y) + 1]^{-1}$. Here we do not explicitly show a constant which must be subtracted from the pressure and added to the energy density in order to render P and $\mathcal{E}$ zero in the vacuum. The quark number density is

$$n = \frac{\nu_q}{2\pi^2}\int dpp^2\langle n_F(E,\mu) - n_F(E,-\mu)\rangle. \tag{41}$$

The entropy density can then be obtained from the standard thermodynamic relation $\mathcal{S} = \beta(\mathcal{E} + P - \mu n)$.

For the equation of motion (17) we need the first derivative of $\Omega_{\bar{q}q}$:

$$\left\langle \frac{\partial \Omega_{\bar{q}q}}{\partial \Delta}\right\rangle = g^2\langle \sigma A(m)\rangle, \tag{42}$$

where the function $A(m)$ is defined by

$$A(m) = 2\frac{\partial \Omega_{\bar{q}q}}{\partial m^2} = -\frac{\nu_q}{8\pi^2}\left[m^2 \ln\frac{m^2}{\Lambda_q^2} + \Lambda_q^2 - m^2\right] + \frac{\nu_q}{2\pi^2}\int dp\frac{p^2}{E}[n_F(E,\mu) + n_F(E,-\mu)]. \tag{43}$$

Within our approach the quark condensate can be easily expressed as

$$\langle \bar{q}q\rangle = \frac{1}{g}\left\langle \frac{\partial \Omega_{\bar{q}q}}{\partial \sigma}\right\rangle = g\langle \sigma A(m)\rangle. \tag{44}$$

For the meson masses in Eq. (18) we need second derivatives:

$$\left\langle \frac{\partial^2 \Omega_{\bar{q}q}}{\partial \Delta^2}\right\rangle = g^2\left\langle A(m) + 2g^2\sigma^2\frac{\partial A(m)}{\partial m^2}\right\rangle = \frac{g^2}{\langle\Delta^2\rangle}\langle \Delta(v + \Delta)A(m)\rangle, \tag{45}$$

$$\left\langle \frac{\partial^2 \Omega_{\bar{q}q}}{\partial \pi_i^2}\right\rangle = g^2\left\langle A(m) + 2g^2\pi_i^2\frac{\partial A(m)}{\partial m^2}\right\rangle = \frac{g^2}{\langle \pi_i^2\rangle}\langle \pi_i^2 A(m)\rangle, \tag{46}$$

where for the latter equalities we have used Eq. (30). Explicitly

$$\frac{\partial A(m)}{\partial m^2} = -\frac{\nu_q}{8\pi^2}\ln\frac{m^2}{\Lambda_q^2} - \frac{\nu_q}{4\pi^2}\int dp\frac{1}{E}[n_F(E,\mu) + n_F(E,-\mu)]. \tag{47}$$

When deriving Eqs. (43), (45), and (46) we have used the fact that $\Omega_{\bar{q}q}(m)$ and $A(m)$ are even functions of m. For simplicity we have suppressed the dependence of these functions upon T and μ.

Finally, we need to average the bare potential $U(v + \Delta, \boldsymbol{\pi})$ and its derivatives. Using the expression (30), or more elementary means, one obtains

$$\begin{aligned}\langle U(v + \Delta, \boldsymbol{\pi})\rangle &= \langle \tfrac{1}{4}\lambda(\sigma^2 + \boldsymbol{\pi}^2 - \zeta)^2 - H\sigma\rangle \\ &= \tfrac{1}{4}\lambda\{3(v^2 + \langle\Delta^2\rangle)^2 + (v^2 + \langle\boldsymbol{\pi}^2\rangle)^2 + \tfrac{2}{3}\langle\boldsymbol{\pi}^2\rangle^2 + 2\langle\Delta^2\rangle\langle\boldsymbol{\pi}^2\rangle - 2\zeta(v^2 + \langle\Delta^2\rangle \\ &\quad + \langle\boldsymbol{\pi}^2\rangle - \tfrac{1}{2}\zeta) - 3v^4\} - Hv.\end{aligned} \tag{48}$$

The first derivative of U needed in the equation of motion (17) is

$$\left\langle \frac{\partial U(v+\Delta,\boldsymbol{\pi})}{\partial \Delta} \right\rangle = \lambda v(v^2 + 3\langle\Delta^2\rangle + \langle\boldsymbol{\pi}^2\rangle - \zeta) - H. \tag{49}$$

The second derivatives of U needed for the meson masses in Eq. (18) are

$$\left\langle \frac{\partial^2 U(v+\Delta,\boldsymbol{\pi})}{\partial \Delta^2} \right\rangle = \lambda(3v^2 + 3\langle\Delta^2\rangle + \langle\boldsymbol{\pi}^2\rangle - \zeta),$$

$$\left\langle \frac{\partial^2 U(v+\Delta,\boldsymbol{\pi})}{\partial \pi_i^2} \right\rangle = \lambda\big(v^2 + \langle\Delta^2\rangle + \tfrac{5}{3}\langle\boldsymbol{\pi}^2\rangle - \zeta\big). \tag{50}$$

The equation of motion (17) involves the sum of Eqs. (42) and (49). It is interesting to note that the quark contribution of Eq. (42) vanishes if $v=0$. This is because $\langle\sigma A(m)\rangle$ becomes $\langle\Delta A(g\sqrt{\Delta^2+\boldsymbol{\pi}^2})\rangle$ which is an odd function of Δ. In the absence of explicit symmetry breaking, $H=0$, Eq. (49) permits a solution $v=0$ which is the true solution above the critical point.

The equation of motion and the equations for the meson masses have to be solved self-consistently. The integrals in Eq. (30) were evaluated numerically using 32-point Gaussian integration [30]. The necessary thermodynamic integrals were obtained by using the numerical approximation scheme of Ref. [31].

Assembling the contributions to the meson masses in Eq. (18) and using the equation of motion (17) for the case $v \neq 0$, we find

$$m_\sigma^2 = 2\lambda v^2 + g^2\left\langle \left(\frac{\Delta}{\langle\Delta^2\rangle} - \frac{1}{v}\right)(v+\Delta)A(m) \right\rangle + \frac{H}{v}, \tag{51}$$

$$m_\pi^2 = 2\lambda(\langle\pi_i^2\rangle - \langle\Delta^2\rangle) + g^2\left\langle \left(\frac{\pi_i^2}{\langle\pi_i^2\rangle} - \frac{v+\Delta}{v}\right)A(m) \right\rangle + \frac{H}{v}. \tag{52}$$

It is interesting to consider the high temperature limit when $\langle\Delta^2\rangle$ and $\langle\pi_i^2\rangle$ become equal. In this limit the first term on the right in Eq. (52) vanishes, but so does the second quark term (see Appendix B). Therefore we regain the usual high temperature result, $m_\pi \simeq H/v$. Since in this region v is very small, the quark contribution in Eq. (51) also vanishes approximately and the first term in the equation can be neglected, returning $m_\sigma \simeq m_\pi$ as expected.

IV. RESULTS

A. Choice of parameters

We need to choose the parameters λ, ζ, H, g, Λ_m, and Λ_q in the case where zero-point fluctuations are included (labeled ZPT). We also present calculations where zero-point fluctuations are excluded (labeled NOZPT) for two reasons. First they have often been excluded in the literature and second it is important to assess their influence in the ZPT case. Note that the parameters Λ_m and Λ_q are irrelevant to the NOZPT analysis.

Since chiral symmetry is spontaneously broken in the vacuum, the axial current requires that $v_{\text{vac}}=f_\pi$, where the pion decay constant is $f_\pi=92.4$ MeV. Goldstone's theorem states that in the absence of explicit symmetry breaking, $H=0$, the pion, as a Goldstone boson, should have zero mass in the phase with spontaneously broken symmetry. Then Eq. (52) gives in the vacuum

$$2\lambda(\langle\pi_i^2\rangle - \langle\Delta^2\rangle) + g^2\left\langle \left(\frac{\pi_i^2}{\langle\pi_i^2\rangle} - \frac{v+\Delta}{v}\right)A(m) \right\rangle = 0. \tag{53}$$

It is natural to assume that this equation is also valid in the case $m_\pi \neq 0$, or, stated differently, to take $H=m_\pi^2 f_\pi$. The solution of Eq. (53) at $T=0$ is $\langle\Delta^2\rangle=\langle\pi_i^2\rangle$ since, as shown in Appendix B, the quark contribution vanishes in this limit. This same condition was used by Lenaghan and Rischke [27] in the pure meson sector. For the NOZPT case $\langle\Delta^2\rangle=\langle\pi_i^2\rangle=0$ in the vacuum so the equation is automatically satisfied. In the ZPT case Eq. (53) requires Λ_m to be intermediate between the pion and sigma masses. Choosing vacuum masses $m_\sigma=700$ MeV and $m_\pi=138$ MeV gives $\Lambda_m=453$ MeV, as listed in Table I; values of similar order have been employed in Refs. [28,29,32]. This should not be directly viewed as a cutoff parameter. If the m_σ^4 term in Eq. (35) or the m_σ^2 term in Eq. (37) is compared with the result of using a four-dimensional cutoff to evaluate the integral, the appropriate cutoff parameter is more reasonable, namely $\Lambda_m e^{3/4}=960$ MeV or $\Lambda_m e^{1/2}=748$ MeV, respectively. The value of the coupling constant g is fixed by the requirement that the vacuum quark mass be approximately a third of the nucleon mass. We can define the mass by $M_n=\{\langle m^n\rangle\}^{1/n}$. It is not *a priori* obvious which power of the mass should be averaged. We examine the two lowest moments with $n=1$ and $n=2$ which are

$$M_1 = \langle m\rangle = g\langle\sqrt{\sigma^2+\boldsymbol{\pi}^2}\rangle, \tag{54}$$

$$M_2 = \langle m^2\rangle^{1/2} = g(v^2 + \langle\Delta^2\rangle + \langle\boldsymbol{\pi}^2\rangle)^{1/2}. \tag{55}$$

As we shall see below, the two definitions give masses which are closely similar, so it is of little consequence which choice is made. We actually used the second definition, setting $M_2=939/3$ MeV. To determine λ and Λ_q in the ZPT case we require the vacuum m_σ to be 700 MeV in Eq. (51) and constrain the vacuum quark condensate in Eq. (44). For the latter we choose the value $\langle\bar{q}q\rangle_{\text{vac}} \equiv \langle\bar{u}u+\bar{d}d\rangle = -2\times(225\text{ MeV})^3$ [33,34]. In the NOZPT case the vacuum quark condensate is zero and only the σ mass equation is required. The value of

TABLE I. Model parameters.

Parameter	ZPT	NOZPT
λ	7.114	27.58
$\zeta(\text{MeV}^2)$	-8.733×10^4	7.847×10^3
$H(\text{MeV}^3)$	1.760×10^6	1.760×10^6
g	2.844	3.387
$\Lambda_m(\text{MeV})$	453.4	
$\Lambda_q(\text{MeV})$	951.8	

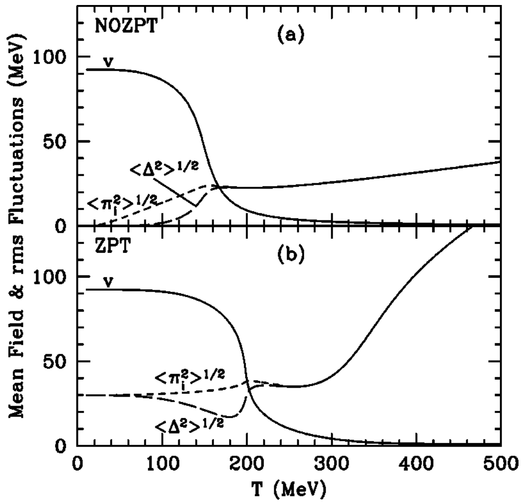

FIG. 1. The σ mean field v and the root mean square fluctuations as a function of temperature (a) without and (b) with zero-point fluctuations.

the scale Λ_q in Table I is about twice that in the meson sector, as seems intuitively reasonable. We remark that if Λ_q is chosen to be equal to Λ_m, and the quark condensate is predicted, rather than fitted, it turns out to be more than an order of magnitude smaller than the physical value used above. Furthermore, even when these renormalization scales differ, it is not possible to carry out the fit outlined above with m_σ=600 MeV. This is why we have chosen a somewhat larger value of 700 MeV for this poorly known mass. Thus the vacuum value of the quark condensate is a strong constraint. It is also responsible for the quite different values of ζ listed in columns two and three of Table I. These were obtained from the equation of motion (17).

Finally, we mention that we have examined results with a number of different parameter sets and qualitatively they are all very similar. Therefore the parameters of Table I will provide a representative set of results for this model.

We should further point out that while we require Goldstone's theorem to be satisfied at $T=0$, this does not guarantee that it will be fulfilled in general. This is because for $T,\mu\neq 0$ the right-hand side of Eq. (52) does not vanish even when the explicit symmetry-breaking term H is set to zero. The nonvanishing terms can be cancelled by including an additional (exchange) diagram [15,35] which is missing in our treatment. However, we focus on the more realistic case where the pion has its physical vacuum mass, m_π =138 MeV. Since we find that the contribution of these nonvanishing terms is relatively small, we shall keep the form (52) for the pion mass so as to preserve the structure of the theory.

B. Full model

In this paper we consider the case where the quark chemical potential μ is zero, corresponding to a net quark density n of zero. The behavior of the sigma mean field v is shown for the NOZPT case in Fig. 1(a) and for the ZPT case in Fig. 1(b). The value of v decreases smoothly to zero as the temperature increases so that chiral symmetry is approximately restored at high temperatures. Thus there is no sharp phase transition, but rather a crossover. The crossover temperature, determined from the maximum value of $|\partial v/\partial T|$, is 150 and 198 MeV in the NOZPT and ZPT cases, respectively. We remark that if quark contributions are omitted this temperature is about 70 MeV higher in accordance with Ref. [27], and in that case v falls off less rapidly with temperature. Lattice calculations suggest a smooth crossover, as we have found, and extrapolation to the chiral limit yields a phase transiiton with a critical temperature of 173±8 MeV [19]. The phase transition is thought to be of second order, although first order is a possibility [7]. Our model is unlikely to be accurate enough to distinguish the order, but it is of interest to examine the qualitative predictions as we approach the chiral limit. We do this in the simplest fashion by multiplying the physical vacuum pion mass by a factor <1 (the ZPT parameters are refitted with the remaining quantities unchanged, as specified in Sec. IV A). We find that when the pion mass reduction factor is $\frac{3}{4}$ the smooth crossover becomes a first order phase transition with a critical temperature of 197 MeV. This temperature is little changed by further reduction in the pion mass. For example, with a reduction factor of $\frac{1}{4}$ the temperature is 194 MeV. This appears to be an improvement on the prediction of the LσM without quarks [36] since it agrees with the lattice finding of a weak dependence of the critical temperature on the quark mass [19]. In comparison the renormalization group calculations [17] show a somewhat stronger dependence on the pion mass and further a phase transition (second order) is obtained only in the chiral limit. The transition temperature of 100–130 MeV found in Ref. [17] is substantially below the values given above, no doubt due to the rather small value of 430 MeV obtained for the σ mass.

Figure 1 also shows the mean square fluctuations, $\langle\Delta^2\rangle^{1/2}$ and $\langle\pi_i^2\rangle^{1/2}$. We required them to be equal at $T=0$ (see previous subsection), and they are equal again at high temperatures when v becomes small. As can be seen from Fig. 1(a) the thermal contribution in the intermediate region is larger for the pion since it has the smaller mass. We do not see the significant enhancement of the field fluctuations in the crossover region that would be expected were there to be a second order phase transition. The reason is that large thermal fluctuations are developed only at $m_\sigma \ll T$, which never holds in our calculations: in fact Fig. 2 shows that m_σ does not drop below 300 MeV. The shallow dip in $\langle\Delta^2\rangle^{1/2}$ in the 100–200-MeV temperature range which is seen in Fig. 1(b) arises from the decrease in m_σ. This causes the zero-point contribution to the fluctuations in Eq. (37) to decrease, becoming zero at $m_\sigma=\Lambda_m$. At higher temperatures m_σ rises with increasing T, causing the zero-point contribution to increase strongly and dominate over the thermal contribution for $T>300$ MeV. This is clearly seen by comparing Figs. 1(a) and 1(b).

The behavior of the meson and quark masses is illustrated in Fig. 2. For the meson masses the NOZPT and ZPT calculations give qualitatively similar results. As expected, the pion mass is an increasing function of T while the sigma mass develops a minimum in the transition region. Here the

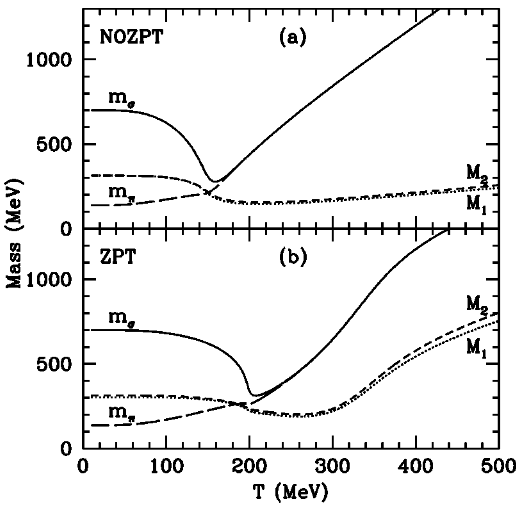

FIG. 2. The meson and quark masses as a function of temperature (a) without and (b) with zero-point fluctuations.

pion and sigma masses become nearly equal signaling the approximate restoration of chiral symmetry. At high temperatures, $T > 250$ MeV, the meson masses become large and grow according to $\sqrt{H/v}$. For the quark mass it is seen from Fig. 2 that the definitions (54) and (55) for M_1 (dotted curve) and M_2 (short-dashed curve) give closely similar results. At high temperatures the mass is small in comparison to the temperature in the NOZPT case [Fig. 2(a)], as it is in the renormalization group approach [17]. In contrast in the ZPT case the increase in the fluctuations seen in Fig. 1 causes an increase in the quark mass, such that it is of the same order as the temperature. Caldas *et al.* [37] report a similar result in their approach to including fluctuations. In this high-T regime the meson masses in Fig. 2(b) are approximately twice the quark mass, as would be expected for loosely bound states of quarks and antiquarks. It should be emphasized that this behavior is qualitatively different from the mean-field approximation where the constituent quark mass is defined to be gv, so that it vanishes at high temperatures.

The thermodynamic quantities, energy density, pressure, and entropy density, as functions of temperature are presented in Fig. 3. At high temperatures these are dominated by quarks. For a massless quark gas of degeneracy 12, $3P/T^4 = \mathcal{E}/T^4 = 6.91$ and $\mathcal{S}/T^3 = 9.21$, and in the NOZPT case in Fig. 3(a) these values are achieved at high temperatures. There is only a small contribution from mesons, since they are heavy, indicating that the system has effectively become a massless quark gas. The same cannot be said for the ZPT case in Fig. 3(b), where the asymptotic values are smaller than the massless limit. This is due the quark mass remaining comparable to the temperature. The quark condensate is displayed in Fig. 4 for the ZPT case since the NOZPT case has only a small thermal contribution. The condensate is dominated by the zero-point contribution. Even though one might expect the thermal part to increase with T, the condensate (44) is approximately $gv\langle A(m)\rangle$, so the decrease in v is sufficient to bring the thermal contribution to zero at high temperature. Thus $\langle\bar{q}q\rangle$ starts at $T=0$ with the chosen empirical value and rapidly becomes negligible when v becomes small. This is the physically expected behavior.

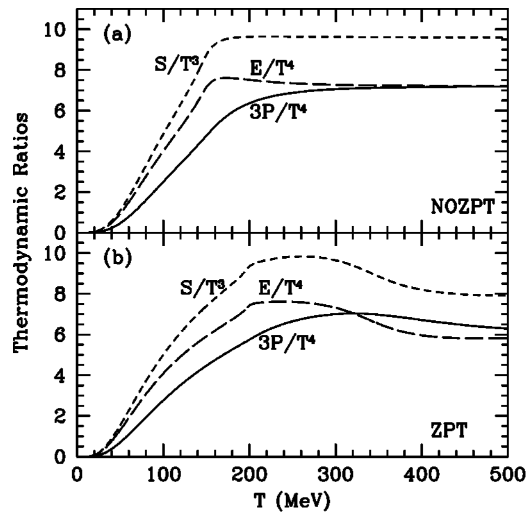

FIG. 3. The thermodynamic quantities $3P/T^4$, $\mathcal{E}/T^4$ and $\mathcal{S}/T^3$ as a function of temperature (a) without and (b) with zero-point fluctuations.

C. Approximations

The expressions above which involve $\Omega_{\bar{q}q}$ all require the calculation of two integrals over the Gaussian weighting functions as well as a momentum integration. Thus it is worthwhile to examine simpler procedures. The most natural simplification is to thermally average the quark mass and then insert it in Eqs. (42), (45), and (46). We first consider the M_1 mass definition in Eq. (54) which was the approximation adopted in Ref. [22]. Replacing m with M_1, the thermodynamic potential density is approximated $\langle\Omega_{\bar{q}q}(m)\rangle \to \Omega_{\bar{q}q}(M_1)$. In the equation of motion

$$\left\langle \frac{\partial\,\Omega_{\bar{q}q}}{\partial\,\Delta}\right\rangle = g\left\langle \frac{\sigma}{\sqrt{\sigma^2+\boldsymbol{\pi}^2}}\right\rangle M_1 A(M_1), \tag{56}$$

and for the meson masses

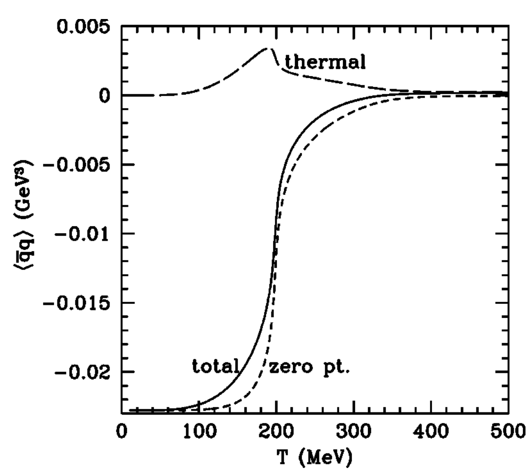

FIG. 4. Thermal, zero-point, and total values of the quark condensate as a function of temperature in the ZPT case.

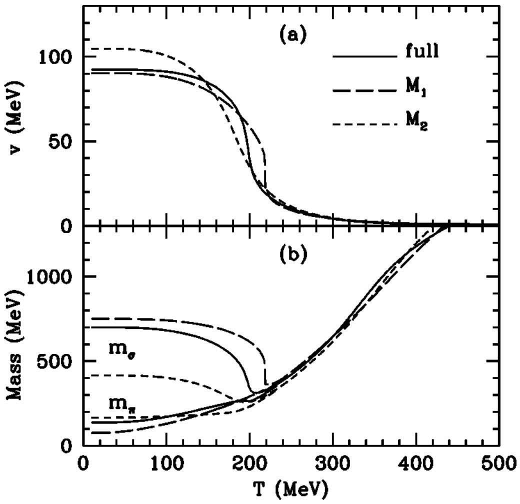

FIG. 5. Comparison of the full results with the M_1 and M_2 approximations: (a) σ mean field v and (b) meson masses.

$$\left\langle \frac{\partial^2 \Omega_{\overline{q}q}}{\partial \Delta^2} \right\rangle = g \left\langle \frac{\boldsymbol{\pi}^2}{(\sigma^2 + \boldsymbol{\pi}^2)^{3/2}} \right\rangle M_1 A(M_1),$$

$$\left\langle \frac{\partial^2 \Omega_{\overline{q}q}}{\partial \pi_i^2} \right\rangle = g \left\langle \frac{\sigma^2 + \frac{2}{3}\boldsymbol{\pi}^2}{(\sigma^2 + \boldsymbol{\pi}^2)^{3/2}} \right\rangle M_1 A(M_1). \tag{57}$$

The M_2 case defined in Eq. (55) is even simpler. Here the thermodynamic potential density is approximated $\langle \Omega_{\overline{q}q}(m) \rangle \to \Omega_{\overline{q}q}(M_2)$, and in the equation of motion

$$\left\langle \frac{\partial \Omega_{\overline{q}q}}{\partial \Delta} \right\rangle = g^2 \langle \sigma \rangle A(M_2) = g^2 v A(M_2). \tag{58}$$

The second derivatives are $g^2 A(M_2)$ for both sigma and pion. In this case the quark contributions to the meson masses in Eqs. (51) and (52) vanish.

Note that ambiguity in the evaluation of Eqs. (42), (45), and (46) when using a thermally averaged quark mass is resolved by using Eq. (32) to require that $\partial\Omega/\partial v$ be identically zero. Then the thermodynamic potential density is a minimum with respect to variations in the scalar condensate v, as it should be.

In Figs. 5 and 6 we compare the full ZPT results with those obtained by using the M_1 and M_2 approximations for various quantities of interest. The parameters in the left column of Table I are used throughout. At temperatures above 220 MeV both approximations are able to reproduce the exact results reasonably well with, perhaps, a slight preference for the M_1 case. Below that temperature there are larger deviations between the exact and the approximate results. The crossover occurs at a 15-MeV higher temperature in the M_1 approximation. At lower temperatures the M_1 approximation gives a reasonable, although not extremely accurate account of the exact results. The M_2 approximation, on the other hand, shows marked deviations for the sigma mean field and mass, v and m_σ, and the quark condensate. Of course, this can be much improved by refitting the parameters at $T=0$. However, then one finds that there are no physical solutions

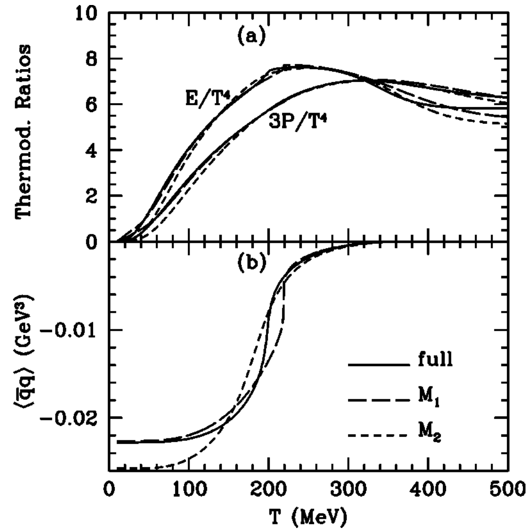

FIG. 6. Comparison of the full results with the M_1 and M_2 approximations: (a) thermodynamic quantities $3P/T^4$ and $\mathcal{E}/T^4$ and (b) quark condensate.

at temperatures just above 300 MeV. We conclude that the M_1 approximation is to be preferred and that it describes the trends of the exact results quite well, being most accurate at high temperature.

V. CONCLUSIONS

Our main goal in this paper was to understand the role played by field fluctuations in effective chiral models. We have shown how to average field functions of arbitrary complexity over the field fluctuations. This allowed a set of self-consistent equations for the average field and the masses to be formulated which led to consistent thermodynamics. We have applied this approach to the linear sigma model, including both mesonic and quark degrees of freedom. The quark degrees of freedom were integrated out and the effective action was linearized according to Eq. (19). Thus we described the σ and π mesons as quasiparticles and their properties were properly taken into account in the thermodynamic potential.

We have considered two versions of the model: ZPT where both zero-point and thermal fluctuations are included, and NOZPT where only thermal fluctuations are present. The former was able to describe the quark condensate and the vacuum value was chosen according to QCD-based estimates. To accommodate this value a sigma mass of 700 MeV was employed which, while within the broad range set by the Particle Data Group [38], is 100 MeV larger than the traditional value. We also needed to employ separate renormalization scales for the mesonic and quark zero-point contributions of 450 and 950 MeV, respectively. For both versions of the model we required the constituent quark mass in vacuum to be 1/3 of the free nucleon mass.

Numerical results were presented only for the case of zero net quark density, i.e., $\mu=0$. The calculations revealed an interesting and consistent picture. With increasing temperature we saw a gradual decrease of the condensate and an increase of the sigma and pion fluctuations. The model gave a crossover type of chiral transition at a temperature of about 198 MeV (150 MeV) in the ZPT (NOZPT) case. In the transition region the rms field fluctuations became comparable in magnitude to the condensate. On the other hand we did not observe the strong increase in the sigma field fluctuations which would appear in the vicinity of a true critical point. The restoration of chiral symmetry was seen in the behavior of the sigma and pion masses, which became degenerate above the transition region, and in the behavior of the quark condensate which decreased to zero.

The effective quark mass first showed a modest decrease with temperature, but above the transition region it started to increase and this trend was quite strong for the ZPT case. Since the condensate was already nearly zero, this growth was induced entirely by the meson field fluctuations. In condensed matter physics this phenomenon is known as pseudogap formation. At high temperatures the meson masses increased much more rapidly than the quark mass, so that they effectively decoupled from the system and we had a nearly ideal gas of quarks. In the ZPT case the quark effective mass was comparable to the temperature, however, in the NOZPT case the mass was much smaller so that the thermodynamics closely resembled a massless quark gas. On the other hand, at low temperatures quarks were heavy in comparison to the temperature while pions were relatively light so that we had an ideal gas of pions with mass close to the physical mass. Certainly, a qualitatively similar behavior is expected on the basis of QCD. The transition between these two regimes occurs at a temperature which is surprisingly close to that found in lattice QCD simulations [19]. In addition we considered simplified approaches in which the quark mass was treated as a number rather than a function of the meson fields; these were found to be quite successful in the high-temperature, chiral-restored regime, but less accurate at low temperatures.

We also briefly discussed the dependence of the chiral transition on the vacuum pion mass. As the chiral limit was approached by reducing the pion mass to a fraction $\leqslant 0.75$ of its true vacuum value the crossover transition became a first order phase transition. The critical temperature was found to be quite insensitive to the value of the pion mass, as in lattice analyses [19].

In the future it would be interesting to perform calculations for a nonzero quark chemical potential. This would give the possibility of exploring the phase diagram of the model in the T-μ plane and comparing it with the predictions of other QCD-motivated models. It would also be interesting to optimize the model in order to achieve better correspondence with QCD. Some obvious omissions from the present approach include vector mesons, strange quarks, the gluon condensate and gluonlike excitations [26,39]. The latter would provide the correct degrees of freedom at high temperature. Also in the future, more realistic patterns of symmetry breaking, including for instance the $\mathrm{U}(1)_A$ anomaly, should be considered.

ACKNOWLEDGMENTS

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APPENDIX A: CONNECTION TO PREVIOUS WORK

The strategy followed in Refs. [21,22] was to assume for the purposes of counting that

$$\langle\Delta^2\rangle = \langle\pi_i^2\rangle = \tfrac{1}{4}\langle\Delta^2 + \boldsymbol{\pi}^2\rangle \equiv \tfrac{1}{4}\langle\boldsymbol{\psi}^2\rangle, \tag{A1}$$

although in the numerical evaluation of the final expressions the actual values of the thermal averages were used. Equation (A1) is strictly true only in the high temperature limit, where the sigma and pion masses become degenerate, however, at low temperatures one can also show that the correct expressions are obtained. Now, writing $\sigma^2+\boldsymbol{\pi}^2=v^2+\boldsymbol{\psi}^2+2v\Delta$, the troublesome cross term $2v\Delta$ was expanded out. Using Eq. (A1) the expressions needed could be cast in terms of a function $f(v^2+\boldsymbol{\psi}^2)$.

Equation (A1) allows the general expression (30) to be written as a four-dimensional integral for the thermal average of the function in question:

$$\langle f(v^2+\boldsymbol{\psi}^2)\rangle = \frac{4}{\pi^2\langle\boldsymbol{\psi}^2\rangle^2}\int d^4\ell\ \exp\left(-\frac{2\ell^2}{\langle\boldsymbol{\psi}^2\rangle}\right)f(v^2+\ell^2). \tag{A2}$$

Since only the magnitude $\ell^2=\ell_0^2+\ell_1^2+\ell_2^2+\ell_3^2$ occurs in the integrand, the angular integration may be carried out giving

$$\langle f(v^2+\boldsymbol{\psi}^2)\rangle = \frac{8}{\langle\boldsymbol{\psi}^2\rangle^2}\int_0^\infty d\ell\,\ell^3\ \exp\left(-\frac{2\ell^2}{\langle\boldsymbol{\psi}^2\rangle}\right)f(v^2+\ell^2). \tag{A3}$$

Equation (A3) then allows an easy evaluation of the principal expressions used in Refs. [21,22]. Defining a new integration variable according to $x^2=1+\ell^2/v^2$, then setting $z^2=2v^2/\langle\boldsymbol{\psi}^2\rangle$, and integrating by parts twice one obtains

$$\left\langle \ln\left(\frac{v^2+\boldsymbol{\psi}^2}{v_{\rm vac}^2}\right)\right\rangle = \ln\left(\frac{v^2}{v_{\rm vac}^2}\right) + 1 + (1-z^2)e^{z^2}E_1(z^2), \tag{A4}$$

as given in Refs. [21,22], with $v_{\rm vac}$ denoting the vacuum value of v. The exponential integral is defined [30] by $E_1(y)=\int_1^\infty dt\,t^{-1}e^{-yt}$. A similar procedure yields

$$\langle\sqrt{v^2+\boldsymbol{\psi}^2}\rangle = \frac{3v}{4z}\left[2z + \left(1-\frac{2}{3}z^2\right)\sqrt{\pi}e^{z^2}\,\mathrm{erfc}(z)\right], \tag{A5}$$

as given in Ref. [22], with the complementary error function defined [30] by $\mathrm{erfc}(z)=1-2\pi^{-1/2}\int_0^z e^{-t^2}dt$.

APPENDIX B: AN IDENTITY FOR THE CASE $\langle\Delta^2\rangle=\langle\pi_i^2\rangle$

Here we show that the quark contribution to Eq. (52) vanishes in the case $\langle\Delta^2\rangle=\langle\pi_i^2\rangle$. First we prove by induction that in this case

$$\langle \Delta^{2n+2} \boldsymbol{\psi}^{2m} \rangle = (2n+1)\langle \Delta^{2n} \pi_i^2 \boldsymbol{\psi}^{2m} \rangle, \tag{B1}$$

where n and m are integers. It is simple to verify this equation in the cases $m=0$ and $m=1$. After integrating by parts the following relations are obtained:

$$\langle \Delta^{2n+2} \boldsymbol{\psi}^{2j} \rangle = \langle \Delta^2 \rangle [(2n+1)\langle \Delta^{2n} \boldsymbol{\psi}^{2j} \rangle + 2j \langle \Delta^{2n+2} \boldsymbol{\psi}^{2j-2} \rangle],$$

$$\langle \Delta^{2n} \pi_i^2 \boldsymbol{\psi}^{2j} \rangle = \langle \pi_i^2 \rangle [\langle \Delta^{2n} \boldsymbol{\psi}^{2j} \rangle + 2j \langle \Delta^{2n} \pi_i^2 \boldsymbol{\psi}^{2j-2} \rangle]. \tag{B2}$$

Since $\langle \Delta^2 \rangle = \langle \pi_i^2 \rangle$ these relations show that if Eq. (B1) holds for $m=j-1$ then it holds for $m=j$ and, since it holds for $m=0$ and 1, it therefore holds in general. The relation can also be proved by using combinatorial arguments.

Now consider a function $f(m^2)$, where $m^2 = g^2[(v+\Delta)^2 + \boldsymbol{\pi}^2]$. Expanding $2g^2 v\Delta$ in a Taylor series

$$\langle \Delta f(m^2) \rangle = \sum_{j=0}^{\infty} \left\langle \frac{f^{\{2j+1\}}(g^2(v^2+\boldsymbol{\psi}^2))}{(2j+1)!} (2g^2 v)^{2j+1} \Delta^{2j+2} \right\rangle, \tag{B3}$$

where we have used the fact that odd powers of Δ give a vanishing contribution, and $f^{\{n\}}$ denotes the nth derivative of f. Utilizing Eq. (B1)

$$\begin{aligned}\langle \Delta f(m^2) \rangle &= 2g^2 v \sum_{j=0}^{\infty} \left\langle \frac{f^{\{2j+1\}}(g^2(v^2+\boldsymbol{\psi}^2))}{(2j)!} \pi_i^2 (2g^2 v\Delta)^{2j} \right\rangle \\ &= 2g^2 v \langle \pi_i^2 f^{\{1\}}(m^2) \rangle, \end{aligned} \tag{B4}$$

again using the fact that odd powers of Δ give zero contribution. Thus in the case $\langle \Delta^2 \rangle = \langle \pi_i^2 \rangle$ we have the relation

$$\begin{aligned}\left\langle -\frac{\Delta f(m^2)}{v} + 2g^2 \pi_i^2 \frac{\partial f(m^2)}{\partial m^2} \right\rangle &= \left\langle \left(\frac{\pi_i^2}{\langle \pi_i^2 \rangle} - \frac{v+\Delta}{v} \right) f(m^2) \right\rangle \\ &= 0, \end{aligned} \tag{B5}$$

where the second equality is obtained using the relation (46). Equation (B5) was used in the text.

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