PHYSICAL REVIEW C VOLUME 7, NUMBER 3 MARCH 1973

Thresholds and Resonances in a Three-Body Model

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(Received 1 September 1972)

An exactly soluble three-body model is constructed which consists of two light particles and an infinitely heavy one having internal degrees of freedom. This model is designed to simulate various phenomena associated with isobaric-analog states, showing up in scattering and transfer reactions. The formulation and solution are done in the framework of Faddeev's theory using nonlocal separable potentials. The cross sections are calculated in the neighborhood of the channel thresholds. The effect of a resonance on the threshold is studied in detail and threshold anomalies are searched.

INTRODUCTION

The anomalous behavior of the nuclear cross sections at the channel thresholds was predicted many years ago by Wigner.[1,2] Such a threshold anomaly, a so-called Wigner cusp was observed experimentally by Malmberg[3] in the ^{7}Li$(p, p)^7$Li reaction in the neighborhood of the (p, n) threshold. A similar cusp was found also in the ^{7}Li-$(p, p'\gamma)^7$Li reaction.[4] As far as we know, since that time no other clear-cut example of the threshold anomaly has been reported. In the case of the ^{7}Li$(p, p)^7$Li reaction the (p, n) threshold lies in the vicinity of a strong resonance.

It was pointed out by Moore *et al.*[5] that a pronounced dip can be observed in the excitation function of the ^{90}Zr$(d, p)^{91}$Zr at $E_d = 7$ MeV and this energy just corresponds to the threshold energy of the ^{90}Zr$(d, n)^{91}$Nb* process where the residual nucleus remains in an isobaric analog state associated with the ground state of the ^{91}Zr. To explain this dip as a threshold anomaly seemed to be the most appealing interpretation,[6-8] but, if this simple explanation is correct, then similar anomalies are expected to appear in the (d, p) cross sections on other nuclei too. The experimental efforts, however, to observe threshold effects in other areas of the Periodic Table were unsuccessful up till now. If one assumes a coincidence of the threshold with a resonance just as it happens in the case of the ^{7}Li then in principle it is possible to understand the peculiar behavior of the ^{90}Zr.[9]

The strength function for p-wave neutrons has its maximum in the vicinity of $A = 90$; therefore it seems reasonable to expect such a coincidence.

To clarify the situation it would be desirable to perform such an analysis in which the couplings of the various channels and the resonance phenomena are treated in a consistent way. At the moment to attack this problem in its full complexity is beyond our capabilities, but to construct an exactly soluble model which incorporates all the essential features of the problem is possible. In this paper we define such a model and studying its solution we try to draw some tentative conclusions for the real nuclei. The model is essentially a generalized version of the Lane model. The target and the deuteron are considered as a three-body system and a charge exchange interaction with the target is assumed. Since most of the technical details of the model were published earlier[10-12] we restrict ourselves only to the most important points.

MODEL

In our previous paper we have constructed an exactly soluble three-body model for the study of resonance phenomena in nuclear reactions, assuming all three constituents of the model to be elementary. Now we drop this assumption and we generalize our model allowing internal degrees of freedom and internal excitations for the particles.

The general form for the Hamiltonian of the three-body system is given by

$$H = h_1 + h_2 + h_3 + V_1 + V_2 + V_3, \tag{1}$$

where h_α stands for the Hamiltonian of the particle α and the interaction between the particles β and γ is denoted by $V_\alpha(\beta \neq \gamma \neq \alpha)$. If a particle does not have internal degrees of freedom or if the internal states are degenerate, then the Hamiltonian h_α is just the kinetic energy K_α.

In the three-body system there are four channels defined by the channel Hamiltonians:

$$H_\alpha = H_0 + V_\alpha \quad (\alpha = 0, 1, 2, 3), \tag{2}$$

where

$$H_0 = h_1 + h_2 + h_3, \quad V_0 = 0. \tag{3}$$

The possible initial and final states $|\Psi_{\alpha a}\rangle$ are the solutions of the channel Hamiltonians:

$$H_\alpha |\Psi_{\alpha a}\rangle = E_{\alpha a} |\Psi_{\alpha a}\rangle. \tag{4}$$

The transition amplitude between the $|\Psi_{\alpha a}\rangle$ and the $|\Psi_{\beta b}\rangle$ states is obtained as the "on-shell" matrix element of the transition operator

$$\langle \Psi_{\beta b} | U_{\beta\alpha}(z) | \Psi_{\alpha a}\rangle |_{E_{\alpha a} = E_{\beta b} = z - i\epsilon}. \tag{5}$$

The transition operators $U_{\beta\alpha}(z)$ satisfy the Faddeev-type[13] equations introduced by Alt, Grassberger, and Sandhas[14]:

$$U_{\beta\alpha}(z) = (1 - \delta_{\beta\alpha})(z - H_0) + \sum_\gamma (1 - \delta_{\beta\gamma}) T_\gamma(z) G_0(z) U_{\gamma\alpha}(z), \tag{6}$$

where $T_\gamma(z)$ are the two-body T operators satisfying the Lippmann-Schqinger equations

$$T_\gamma(z) = V_\gamma + V_\gamma G_0(z) T_\gamma(z) \tag{7}$$

and $G_0(z)$ is the Green function of the noninteracting system

$$G_0(z) = (z - H_0)^{-1}. \tag{8}$$

In the construction of the model system, we introduce the following simplifying assumptions.[12]

The first particle is a "neutron" (n'). The second particle is a "nucleon" which can be either in "proton" (p) or "neutron" (n) state. The masses of the first and second particles are equal ($m_1 = m_2 = \frac{1}{2}$) and we forget about the spin variables. The third particle is a "heavy nucleus" ($m_3 = \infty$) which can be either in "core" state characterized by the isospin quantum numbers T, M_T and by the energy E_c or it can be in the "analog" state having the same isospin T, but with the projection $M_T - 1$ and energy E_A. As a consqeuence of these assumptions the Hamiltonian can be represented as a two by two matrix in the space of the third parti-

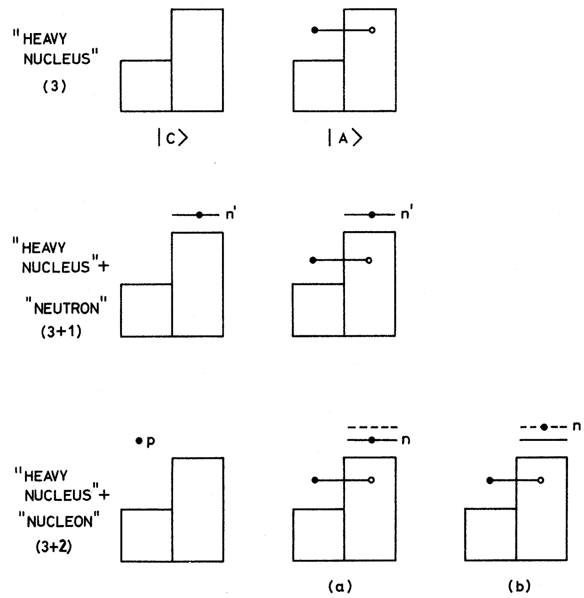

FIG. 1. The characteristic features of the two-body subsystems.

TABLE I. The parameters of the potentials defined by Eq. (11) and the coordinates on the complex momentum plane of the bound-state and resonance poles of the two-body T matrices. In the case of $\alpha = 1$, 2 the poles are obtained by taking $\Lambda^{(1)}_{CA} = \Lambda^{(1)}_{AC} = 0$ and $E_A = 0$.

	S	1	2	1	2	1	2
$\lambda_{\alpha s}$ (MeV$^{3/4}$)		11.00	18.25	−1.50	0	1.00	0
$\delta_{\alpha s}$ (MeV$^{3/4}$)		0	−79.00	81.64	0	0	0
$\beta_{\alpha s}$ (MeV$^{1/2}$)		3.80	3.16	5.00	⋯	6.00	⋯
$\gamma_{\alpha s}$ (MeV$^{1/2}$)		⋯	9.48	3.80	⋯	⋯	⋯
		C	A	C	A	C	A
$\Lambda^{(\alpha)}_{ii'}$	C	0.040	0.012	1.000	0	27.599	0
	A	0.012	1.000	0	1.000	0	5.520
Bound-state poles (MeV$^{1/2}$)		⋯	$1.912i$	$2.906i$	$2.906i$	$1.061i$	⋯
Resonance poles (MeV$^{1/2}$)		⋯	$\pm 0.945 - 0.043i$	⋯	⋯	⋯	⋯
E_C (MeV)	0.00	E_A (MeV)	7.10				

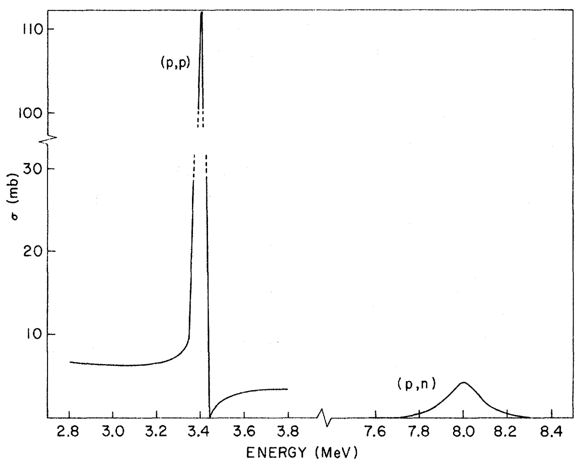

FIG. 2. The (p,p) and (p,n) cross sections in the $\alpha=1$ subsystem.

cle states

$$\langle i|H|i'\rangle = (K_1+K_2+E_i)\delta_{ii'} + (\langle i|V_2|i\rangle + \langle i|V_3|i\rangle)\delta_{ii'} + \langle i|V_1|i'\rangle , \qquad (9)$$

where E_i is the eigenvalue of h_3

$$h_3|i\rangle = E_i|i\rangle \quad (i=C,A).$$

It is a well-known fact that if one represents the interactions by local potentials, then the Faddeev equations lead to a system of coupled integral equations of two variables. The solution of such a task is more than tedious. In order to circumvent this difficulty we will use separable, nonlocal potentials. In this case the two-body T matrices are also separable and can be obtained very easily in an algebraic way. As a consequence of the separability of the T matrices the Faddeev equations are reduced to a set of coupled integral equations of one variable. In addition to this practical reason there are some further arguments in favor of the separable nonlocal potentials. First, there exists a number of such two-nucleon potentials which are capable of reproducing fairly well the scattering phases and the properties of the deuteron.[15] Second, detailed calculations show that both the bound states and the scattering states of the three-nucleon system can be interpreted reasonably well by means of very simple separable potentials.[16] It was pointed out many times that the success of these calculations can be found in the fact that even the simplest separable potential is able to generate the proper pole structure of the T matrix and the nucleon systems are only moderately sensitive for the further details. Furthermore it is worthwhile to emphasize that the separable potentials are very useful and very convenient tools for model calculations and for the investigation of specific problems of nuclear reactions. By properly choosing the form factors of the potentials one can easily generate the desired bound state and resonance poles of the T matrices and one can automatically get rid of unnecessary complications, which would mask the essential features of the model problem to be studied.

Exactly because of these circumstances we define the model Hamiltonian in terms of nonlocal separable potentials which in momentum representation read as follows:

$$\langle \vec{k}_1\vec{k}_2 i|H_0|\vec{k}_1'\vec{k}_2' i'\rangle = (k_1^2+k_2^2+E_i)\delta(\vec{k}_1-\vec{k}_1')\delta(\vec{k}_2-\vec{k}_2')\delta_{ii'} \quad (\hbar=1), \qquad (10)$$

$$\langle \vec{k}_1\vec{k}_2 i|V_\alpha|\vec{k}_1'\vec{k}_2' i'\rangle = -\delta(\vec{p}_\alpha-\vec{p}_\alpha')\Lambda^{(\alpha)}_{ii'}\sum_{s=1}^{2} g_{\alpha s}(q_\alpha)g_{\alpha s}(q_\alpha'), \qquad (11)$$

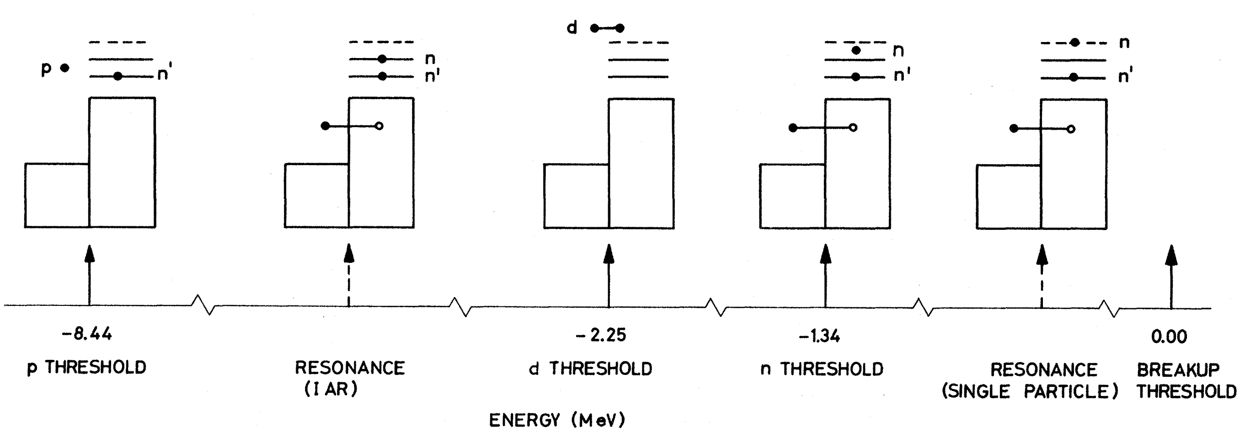

FIG. 3. The possible configurations of the three-body system below the breakup threshold.

where $\vec{q}_1=\vec{k}_2$, $\vec{p}_1=\vec{k}_1$, $\vec{q}_2=\vec{k}_1$, $\vec{p}_2=\vec{k}_2$, and finally $\vec{q}_3=\frac{1}{2}(\vec{k}_1-\vec{k}_2)$ and $\vec{p}_3=\vec{k}_1+\vec{k}_2$ are the relative and total momenta of the two light particles, respectively. The form factors are chosen as follows:

$$g_{\alpha s}(q_\alpha)=\frac{\lambda_{\alpha s}}{q_\alpha^2+\beta_{\alpha s}^2}+\frac{\delta_{\alpha s}}{q_\alpha^2+\gamma_{\alpha s}^2}\ . \tag{12}$$

The coupling constants $\Lambda_{ii'}^{(\alpha)}$ form diagonal matrices except for $\alpha=1$. The nondiagonal element $\Lambda_{CA}^{(1)}$ $(=\Lambda_{AC}^{(1)})$ plays the role of the strength parameter of the charge exchange between the "nucleon" and the "heavy nucleus." The T matrix has the simple form

$$\langle\vec{k}_1\vec{k}_2 i\,|\,T_\alpha(z)\,|\,\vec{k}_1'\vec{k}_2' i\rangle=-\delta(\vec{p}_\alpha-\vec{p}_\alpha')\sum_{s,s'=1}^{2}g_{\alpha s}(q_\alpha)\tau_{is,i's'}^{(\alpha)}(z-p_\alpha^2)g_{\alpha s'}(q_\alpha')\,. \tag{13}$$

Substituting into the Lippmann-Schwinger equation, an explicit expression can be obtained for the inverse of τ:

$$[\tau^{(\alpha)}(z)]_{is,i's'}^{-1}=[\Lambda^{(\alpha)}]_{ii'}^{-1}\delta_{ss'}+\delta_{ii'}\int_0^\infty\frac{g_{\alpha s}(q_\alpha)g_{\alpha s'}(q_\alpha)}{z-q_2^2-E_i}\,4\pi q_2^2dq_\alpha\,. \tag{14}$$

PROPERTIES OF THE TWO-BODY SUBSYSTEMS

Studying the pole trajectories of the T matrices, one can get appropriate sets of potential parameters providing the desired bound states, resonances, and scattering properties of the two-body subsystems. The potential parameters and the "coordinates" of the bound-state and resonance poles are listed in Table I. Using self-explanatory notations, the characteristic features of the subsystems are summarized in Fig. 1.

In order to simulate the Coulomb effect, the potential acting on p was chosen to be considerably weaker than the potentials acting on n and n'.[12] As a consequence of this choice there is no bound state for p.

To see the behavior of the "heavy nucleus + nucleon" system, we have calculated the (p,p) and (p,n) cross sections using the on-shell elements of the two-body T matrix. Actually this is equivalent with the solution of the Lane model.

The cross sections exhibited in Fig. 2 show two resonances. The first resonance appearing in the (p,p) cross section corresponds to the isobaric-analog state [see the configuration (a) in Fig. 1];

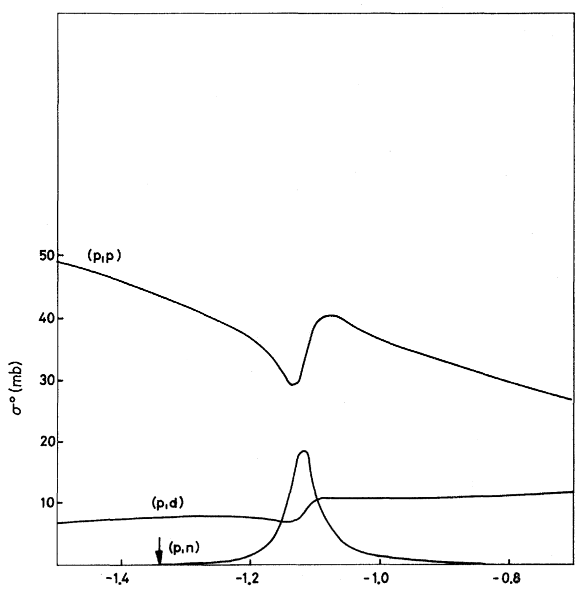

FIG. 4. The energy dependence of the (p,p), (p,n), and (p,d) cross sections for $l=0$, in the case when the resonance is far from the n threshold indicated by the arrow.

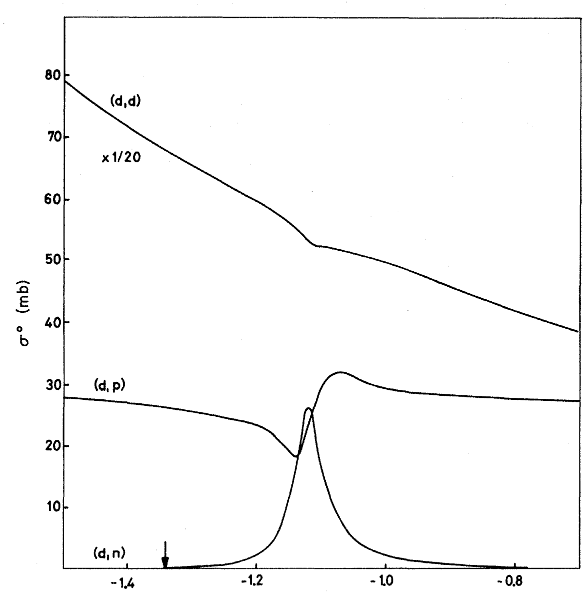

FIG. 5. The energy dependence of the (d,d), and (d,p), and (d,n) cross sections for $l=0$, in the case when the resonance is far from the n threshold indicated by the arrow.

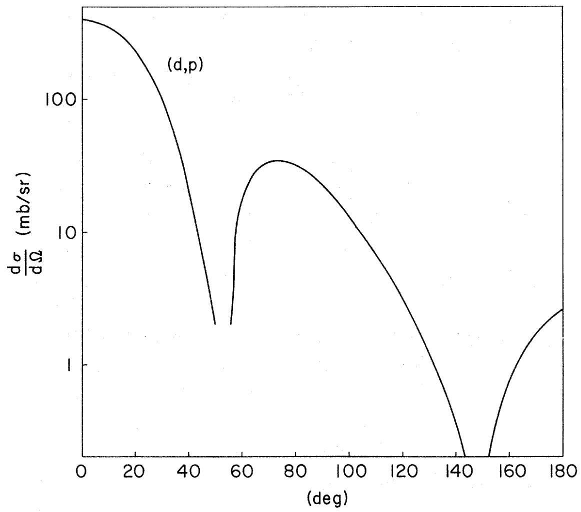

FIG. 6. A typical differential cross section for the (d,p) reaction summed for the first five partial waves at $E=-0.95$ MeV.

the other one is a single-particle resonance in the n channel [see the configuration (b)].

CROSS SECTIONS

The possible configurations of the three-body system below the breakup threshold ($E<0$) are depicted in Fig. 3 together with the main features of the spectrum. After having fixed the properties of the one- and two-body subsystems, the solution of the Faddeev equations is quite straightforward (see Ref. 10). In our particular case, for each value of the total angular momentum the Faddeev equations can be reduced to a coupled system of integral equations of one variable for the eight complex-valued amplitudes corresponding to the eight different form factors figuring in the two-body interactions.

Below the breakup threshold where the singularities of the kernels can be handled very easily, the solution was obtained by Gaussian quadrature. To get numerically reliable solutions it was enough to use 16 Gaussian points. We have performed the calculations for the (p,p), (p,n), (p,d) and for the (d,d), (d,n), (d,p) processes. The correctness of the solutions and the numerical precision were checked by the optical theorem and by the detailed balance theorem. The energy dependence of the cross sections for the dominant $l=0$ case is exhibited in Figs. 4 and 5. The differential cross section for the (d,p) process is shown in Fig. 6. Since the particles have no spin and the interactions act only in relative s states, the total angular momentum is just the same as the orbital momentum of the bombarding particle. The contribution of the higher partial waves above $l=4$ was found to be negligible.

DISCUSSION

First of all, we must emphasize that the results obtained for the various cross sections are exact in the framework of the given model. This

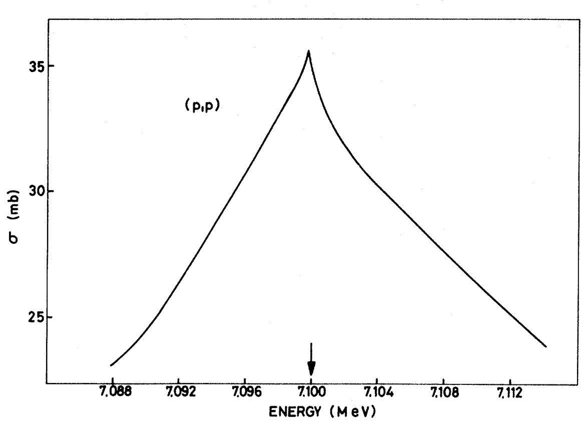

FIG. 7. The cusp in (p,p) cross section in the $\alpha=1$ subsystem, when the resonance is nearly on the threshold. The parameters are the same as in Table I, except for $\lambda_{11}=11.215$ MeV$^{1/2}$, $\Lambda^{(1)}_{CC}=0.0425$, $\Lambda^{(1)}_{AA}=1.0025$, and $\Lambda^{(1)}_{CA}=\Lambda^{(1)}_{AC}=0.06$.

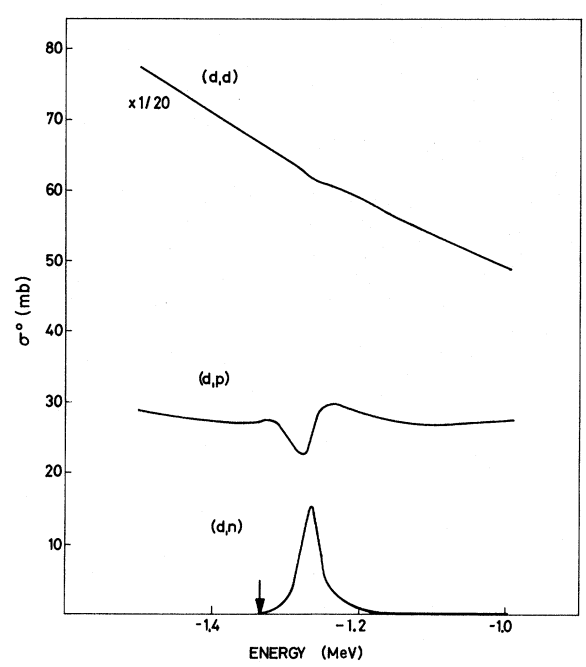

FIG. 8. The energy dependence of the (d,d), (d,p), and (d,n) cross sections for $l=0$, when the resonance is near to the n threshold. The potential parameters are the same as in Table I, except for $\Lambda^{(1)}_{CA}=0.006$ and $\Lambda^{(3)}_{AA}=6.072$. The threshold is indicated by the arrow.

means that after having defined the Hamiltonian we have no freedom to neglect or overemphasize any aspect of the reaction process by means of approximations or additional assumptions. On the other hand, we are able to relate uniquely the structure of the cross sections to the properties of the two-body subsystems as it is demonstrated in Fig. 3. Analyzing the results obtained so far, we have to stress that threshold anomaly appears neither in the "heavy nucleus + nucleon" system (Fig. 2) nor in the three-body system (Figs. 4 and 5). The *punctum saliens* of this paper can be formulated by the following question: What will happen if the resonance approaches the threshold? To get the answer we have repeated the calculations for various sets of parameters.

In the case of the "heavy nucleus + nucleon" system we have shifted the single-particle resonance to the threshold. The result is shown in Fig. 7. The conclusion is quite clear, namely if the resonance overlap with the threshold and the coupling between the two channels is strong enough, a pronounced Wigner cusp arises. This corresponds just to the case of the $^7\mathrm{Li}+p$ reaction. In the case of the full three-body system we have shifted the resonance again step by step into the neighborhood of the threshold by increasing the n-n' interaction strength. A typical result is to be seen on Fig. 8. In addition to the resonance effect no distinguishable threshold anomaly was found.

More precisely, if there exists any threshold effect at all then it is smeared out by the computational errors due to the Gaussian quadrature. Obviously, it is impossible to draw firm conclusions from such a simplified model for real nuclear processes. Therefore, we are forced to formulate our conclusion in the following precautionary way. The results obtained from our model support the assumption according to which the necessary but not sufficient condition of the observability of threshold anomaly is the overlap of a resonance with the threshold.

ACKNOWLEDGMENT

We are indebted very much to Dr. J. Zimanyi for many illuminating discussions and for critical remarks. One of us (I.L.) wants to thank Professor C. Mayer-Böricke for the hospitality extended to him in the Institut für Kernphysik and for putting at his disposal the facilities of the KFA, where the work was completed.

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