

B. Zeidman, *Phys. Rev.* **164**, 1274 (1967).

<sup>50</sup>D. G. Fleming, J. Cerny, C. C. Maples, and N. K. Glendenning, *Phys. Rev.* **166**, 1012 (1968).

<sup>51</sup>B. Margolis and N. de Takacsy, *Can. J. Phys.* **44**, 1431 (1966).

<sup>52</sup>G. E. Walker and D. Schlobohm, *Nucl. Phys.* **A140**, 49 (1970).

<sup>53</sup>H. R. Hiddleston, J. Aymar, and S. E. Darden, *Bull. Am. Phys. Soc.* **16**, 489 (1971); and H. R. Hiddleston, private communication.

<sup>54</sup>N. Dombey and P. K. Kabir, *Phys. Rev. Letters* **17**, 730 (1966).

<sup>55</sup>S. L. Adler, *Phys. Rev. Letters* **18**, 519 (1967).

<sup>56</sup>E. G. Adelberger and D. P. Balamuth, *Phys. Rev. Letters* **27**, 1597 (1971).

<sup>57</sup>A. I. Sanda and G. Shaw, *Phys. Rev. Letters* **26**, 1087 (1971).

<sup>58</sup>A. J. Ferguson, *Angular Correlation Methods in Gamma Ray Spectroscopy* (North-Holland, Amsterdam, 1965).

PHYSICAL REVIEW C

VOLUME 7, NUMBER 3

MARCH 1973

## Spin and Spin-Isospin Symmetry Energy of Nuclear Matter

J. Dabrowski

*Institute for Nuclear Research, Warsaw, Poland*

and

P. Haensel

*Institute for Theoretical Physics, Warsaw University, Warsaw, Poland*

(Received 13 September 1972)

The expression,  $E/A = \epsilon_{\text{vol}} + \frac{1}{2}\epsilon_{\tau}\alpha_{\tau}^2 + \frac{1}{2}\epsilon_{\sigma}(\alpha_n + \alpha_p)^2 + \frac{1}{2}\epsilon_{\sigma\tau}(\alpha_n - \alpha_p)^2$ , for the ground-state energy of nuclear matter with an excess of neutrons, of spin-up neutrons, and of spin-up protons (characterized by the corresponding parameters,  $\alpha_{\tau} = (N - Z)/A$ ,  $\alpha_n = (N_{\uparrow} - N_{\downarrow})/A$ , and  $\alpha_p = (Z_{\uparrow} - Z_{\downarrow})/A$ ), contains three symmetry energies: the isospin symmetry energy  $\epsilon_{\tau}$ , the spin symmetry energy  $\epsilon_{\sigma}$ , and the spin-isospin symmetry energy  $\epsilon_{\sigma\tau}$ . General expressions for  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$  are obtained in terms of the  $K$  matrix which depends on four different Fermi momenta. With suitable approximations, numerical values of  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$  (and also of  $\epsilon_{\tau}$ ) are derived using the Brueckner-Gammel-Thaler, the Hamada-Johnston, and the Reid soft-core nucleon-nucleon potentials. The most reliable results, obtained with the Reid soft-core potential, are:  $\epsilon_{\tau} = 61$  MeV,  $\epsilon_{\sigma} = 74$  MeV, and  $\epsilon_{\sigma\tau} = 73$  MeV. The possibility of estimating the energies of the spin and spin-isospin modes of collective nuclear excitations is discussed.

### I. INTRODUCTION

Let us consider nuclear matter composed of  $N_{\uparrow}$  neutrons with spin up,  $N_{\downarrow}$  neutrons with spin down,  $Z_{\uparrow}$  protons with spin up, and  $Z_{\downarrow}$  protons with spin down. All the nucleons are contained in a periodicity box of volume  $\Omega$ . The composition of the system may be characterized by  $A = N_{\uparrow} + N_{\downarrow} + Z_{\uparrow} + Z_{\downarrow}$ , the neutron-excess parameter

$$\alpha_{\tau} = (N_{\uparrow} + N_{\downarrow} - Z_{\uparrow} - Z_{\downarrow})/A = (N - Z)/A,$$

the neutron-spin-up-excess parameter  $\alpha_n = (N_{\uparrow} - N_{\downarrow})/A$ , and the proton-spin-up-excess parameter  $\alpha_p = (Z_{\uparrow} - Z_{\downarrow})/A$ .

The ground-state energy  $E$  of the system is subject to two conditions:

$$E(A, \alpha_{\tau}, \alpha_n, \alpha_p) = E(A, -\alpha_{\tau}, \alpha_p, \alpha_n), \quad (1)$$

$$E(A, \alpha_{\tau}, \alpha_n, \alpha_p) = E(A, \alpha_{\tau}, -\alpha_n, -\alpha_p). \quad (2)$$

The first one results from the charge independence of nuclear forces, and the second one from

the time-reversal invariance of the interaction (or, in other words, from the requirement that  $E$  does not change when the spins of all the nucleons are reversed).

Because of these two conditions the expansion of  $E$  in powers of  $\alpha_{\tau}$ ,  $\alpha_n$ , and  $\alpha_p$  takes the form

$$E/A = \epsilon_{\text{vol}} + \frac{1}{2}\epsilon_{\tau}\alpha_{\tau}^2 + \frac{1}{2}\epsilon_{\sigma}(\alpha_n + \alpha_p)^2 + \frac{1}{2}\epsilon_{\sigma\tau}(\alpha_n - \alpha_p)^2, \quad (3)$$

where powers higher than quadratic are neglected.

In expression (3), apart from the volume energy,  $\epsilon_{\text{vol}}$ , and the usual (isospin) symmetry energy, denoted here by  $\epsilon_{\tau}$ , we have two more quantities:  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$ . Quantity  $\epsilon_{\sigma}$  is the measure of additional energy necessary to maintain a spin excess in the system, characterized by the spin-excess parameter,  $\alpha_{\sigma} = \alpha_n + \alpha_p = (N_{\uparrow} + Z_{\uparrow} - N_{\downarrow} - Z_{\downarrow})/A$ . Quantity  $\epsilon_{\sigma}$  is referred to further as the spin symmetry energy. Quantity  $\epsilon_{\sigma\tau}$  is the measure of additional energy necessary to maintain in the system an excess of spin-up neutrons and spin-

down protons, characterized by the spin-isospin-excess parameter,  $\alpha_{\sigma\tau} = \alpha_n - \alpha_p = (N_{\uparrow} + Z_{\downarrow} - N_{\downarrow} - Z_{\uparrow})/A$ . Quantity  $\epsilon_{\sigma\tau}$  is referred to further as spin-isospin-symmetry energy.

The purpose of our work is to calculate  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$ . Having this in view we apply the method of calculating the isospin symmetry energy,  $\epsilon_{\tau}$ , presented in Ref. 1 (hereafter referred to as BD). As a matter of fact, the method requires a slight modification. The isospin (crucial as regards  $\epsilon_{\tau}$ ) is a vector in the abstract isospin space, whereas the spin (crucial as regards  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$ ) is a vector in the real space. Thus, one has to take into account the relative orientation of the spin with regard to other vectors in the real space, e.g., momenta. This becomes essential in the case of a two-body tensor interaction.

Notice should also be drawn to the fact that four Fermi momenta appear in our considerations, viz.,  $\kappa_n$  for neutrons with spin up,  $\lambda_n$  for neutrons with spin down,  $\kappa_p$  for protons with spin up, and  $\lambda_p$  for protons with spin down. These Fermi momenta are related to the excess parameters by the relations:

$$\left. \begin{matrix} \kappa_n \\ \lambda_n \end{matrix} \right\} = k_F^3 (1 + \alpha_{\tau} \pm \alpha_{\sigma} \pm \alpha_{\sigma\tau}), \quad (4a)$$

$$\left. \begin{matrix} \kappa_p \\ \lambda_p \end{matrix} \right\} = k_F^3 (1 - \alpha_{\tau} \pm \alpha_{\sigma} \mp \alpha_{\sigma\tau}), \quad (4b)$$

## II. GENERAL FORMULAS FOR $\epsilon_{\sigma}$ AND $\epsilon_{\sigma\tau}$

If we divide the total energy  $E$  into the kinetic and potential parts,  $E = E_{\text{kin}} + E_{\text{pot}}$ , then expansion (3) of  $E_{\text{kin}}$  leads to the following values of the kinetic parts of  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$ :

$$\epsilon_{\sigma}^{\text{kin}} = \epsilon_{\sigma\tau}^{\text{kin}} = \epsilon^{\text{kin}} = \frac{2}{3} \epsilon_F, \quad (6)$$

where the Fermi energy  $\epsilon_F = \hbar^2 k_F^2 / 2M$ . Thus, we have

$$\epsilon_{\sigma} = \epsilon^{\text{kin}} + \epsilon_{\sigma}^{\text{pot}}, \quad \epsilon_{\sigma\tau} = \epsilon^{\text{kin}} + \epsilon_{\sigma\tau}^{\text{pot}}, \quad (7)$$

where the potential parts of the symmetry energies are:

$$\epsilon_{\sigma}^{\text{pot}} = \left( \frac{\partial^2 E_{\text{pot}}/A}{\partial \alpha_{\sigma}^2} \right)_0, \quad \epsilon_{\sigma\tau}^{\text{pot}} = \left( \frac{\partial^2 E_{\text{pot}}/A}{\partial \alpha_{\sigma\tau}^2} \right)_0. \quad (8)$$

Subscript 0 refers to the value of the derivative at point  $\alpha_{\tau} = \alpha_{\sigma} = \alpha_{\sigma\tau} = 0$ .

According to the Brueckner theory the potential energy is given by the expression

$$E_{\text{pot}} = \frac{1}{2} \sum_{s_3} \sum_{t_3} \sum_{\vec{m}}^{(t_3 s_3)} V(\vec{m} s_3 t_3), \quad (9)$$

where

$$V(\vec{m} s_3 t_3) = \sum_{s'_3} \sum_{t'_3} \sum_{\vec{m}'}^{(t'_3 s'_3)} (\vec{m} s_3 t_3 \vec{m}' s'_3 t'_3 | K(\kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m} s_3 t_3 \vec{m}' s'_3 t'_3) - \text{exchange}, \quad (10)$$

and  $t_3$  denotes the third component of the isospin of the nucleons. We use the convention  $t_3 = \frac{1}{2} (-\frac{1}{2})$  for neu-

where  $k_F$  is the Fermi momentum for nuclear matter with  $N_{\uparrow} = N_{\downarrow} = Z_{\uparrow} = Z_{\downarrow} = \frac{1}{4} A$ ,

$$k_F^3 = 3\pi^2 \rho / 2 = \frac{3\pi^2}{2} \frac{A}{\Omega}. \quad (5)$$

The effective interaction between nucleons in nuclear matter will depend intrinsically on all the four Fermi momenta. This results in additional complications.

Our calculation will be carried out in terms of the Brueckner theory of nuclear matter.<sup>2</sup> It should be emphasized, however, that our general expressions for  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$  do not depend on how the effective nucleon-nucleon interaction in nuclear matter (the  $K$  matrix) is derived from nuclear forces.

In Sec. II, we derive the general expression for  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$  in terms of the effective nucleon-nucleon interaction in nuclear matter. Similar approximations to those which have been used in BD are applied to Sec. III to simplify the expression for rearrangement contributions to  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$ . In Sec. IV we describe the procedure of calculating the  $K$  matrix starting with three forms of nucleon-nucleon forces (the Brueckner-Gammel-Thaler,<sup>3</sup> the Hamada-Johnston,<sup>4</sup> and the Reid soft-core<sup>5</sup> potential). In Sec. V, numerical results are presented and discussed. In particular, the possibility of estimating the energies of the spin and spin-isospin modes of collective nuclear excitations is pointed out.

trons (protons). The third component of the spin of the nucleons is denoted by  $s_3$ . The sum  $\sum_{\vec{m}}^{(t_3 s_3)}$  denotes summation over all momenta states occupied by nucleons with the third component of isospin and spin equal to  $t_3$  and  $s_3$ , respectively. We thus have, e.g.,

$$\sum_{\vec{m}}^{(\frac{1}{2} \frac{1}{2})} = \sum_{\vec{m}}^{m < \kappa_n} . \quad (11)$$

In Eq. (10) the intrinsic dependence of the  $K$  matrix on the four Fermi momenta has been explicitly shown. The order in which the Fermi momenta appear as arguments of  $K$  will be kept the same throughout this paper, and so, for instance, the third argument always stands for the Fermi momentum of protons with spin up.

Now, we shall pass to the representation of the total spin and isospin of the two nucleons,  $s$  and  $T$ , with the corresponding third components,  $m_s$  and  $T_3$ . Using the notation

$$(\vec{m}\vec{m}' s m_s T T_3 | K(\kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m}\vec{m}' s m_s T T_3) = (\vec{m}\vec{m}' | K(s m_s, T T_3; \kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m}\vec{m}') , \quad (12)$$

we may write, for instance,  $V(\vec{m}\uparrow n) = V(\vec{m} s_3 = \frac{1}{2} t_3 = \frac{1}{2})$  in the form:

$$\begin{aligned} V(\vec{m}\uparrow n) = & \sum_{\vec{m}'}^{m' < \kappa_n} (\vec{m}\vec{m}' | 2K(11, 11; \kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m}\vec{m}') + \sum_{\vec{m}'}^{m' < \lambda_n} (\vec{m}\vec{m}' | \sum_s K(s 0, 11; \kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m}\vec{m}') \\ & + \sum_{\vec{m}'}^{m' < \kappa_p} (\vec{m}\vec{m}' | \sum_T K(11, T 0; \kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m}\vec{m}') + \sum_{\vec{m}'}^{m' < \lambda_p} (\vec{m}\vec{m}' | \frac{1}{2} \sum_s \sum_T K(s 0, T 0; \kappa_n \lambda_n \kappa_p \lambda_p) | \vec{m}\vec{m}') . \end{aligned} \quad (13)$$

We introduce expression (13), together with similar expressions for  $V(\vec{m}\uparrow n)$ ,  $V(\vec{m}\uparrow p)$ , and  $V(\vec{m}\downarrow p)$ , into Eq. (9), and obtain an expression for  $E_{\text{pot}}$  which depends on  $\alpha_\sigma$  and  $\alpha_{\sigma\tau}$  in two ways: first, through the upper limits of the sums over  $\vec{m}$  and  $\vec{m}'$ , and secondly, through the intrinsic dependence of  $K$  on the four Fermi momenta, related to  $\alpha_\sigma$  and  $\alpha_{\sigma\tau}$  by Eqs. (4). Thus, when we calculate the second derivatives of  $E_{\text{pot}}$  indicated in Eqs. (8), we get two parts of  $\epsilon_\sigma^{\text{pot}}$  and  $\epsilon_{\sigma\tau}^{\text{pot}}$ ,

$$\epsilon_\sigma^{\text{pot}} = \epsilon_\sigma^{(0)\text{pot}} + \Delta\epsilon_\sigma, \quad \epsilon_{\sigma\tau}^{\text{pot}} = \epsilon_{\sigma\tau}^{(0)\text{pot}} + \Delta\epsilon_{\sigma\tau}, \quad (14)$$

the first part,  $\epsilon_\sigma^{(0)\text{pot}}$  and  $\epsilon_{\sigma\tau}^{(0)\text{pot}}$ , resulting from the first type of the dependence, and the second part,  $\Delta\epsilon_\sigma$  and  $\Delta\epsilon_{\sigma\tau}$ , resulting from the second type of the dependence of  $E_{\text{pot}}$  on  $\alpha_\sigma$  and  $\alpha_{\sigma\tau}$ . The terms  $\Delta\epsilon_\sigma$  and  $\Delta\epsilon_{\sigma\tau}$  will be referred to as the rearrangement part of  $\epsilon_\sigma$  and  $\epsilon_{\sigma\tau}$ , respectively, and the nonrearrangement part of  $\epsilon_\sigma$  ( $\epsilon_{\sigma\tau}$ ) will be denoted by

$$\epsilon_\sigma^{(0)} = \epsilon_\sigma^{\text{kin}} + \epsilon_\sigma^{(0)\text{pot}}, \quad \epsilon_{\sigma\tau}^{(0)} = \epsilon_{\sigma\tau}^{\text{kin}} + \epsilon_{\sigma\tau}^{(0)\text{pot}} . \quad (14')$$

Analogous notation will be used in the discussion of the corresponding parts of  $\epsilon_\tau$ . Here, we do not write the formulas for  $\epsilon_\tau$ , since they have been presented in BD.

The calculation of  $\epsilon_\sigma^{\text{pot}}$  and  $\epsilon_{\sigma\tau}^{\text{pot}}$ , i.e., the calculation of the second derivatives in Eqs. (8), is lengthy but straightforward. So let us mention only that in obtaining our final expressions for  $\epsilon_\sigma^{\text{pot}}$  and  $\epsilon_{\sigma\tau}^{\text{pot}}$  we applied two relations

$$K(s m_s, T - T_3; \kappa_n \lambda_n \kappa_p \lambda_p) = K(s m_s, T T_3; \kappa_p \lambda_p \kappa_n \lambda_n), \quad (15a)$$

$$K(s - m_s, T T_3; \kappa_n \lambda_n \kappa_p \lambda_p) = K(s m_s, T T_3; \lambda_n \kappa_n \lambda_p \kappa_p), \quad (15b)$$

which reflect the charge independence and the time-reversal invariance of nuclear forces.

Our final results for  $\epsilon_\sigma^{(0)\text{pot}}$  and  $\epsilon_{\sigma\tau}^{(0)\text{pot}}$  are:

$$\epsilon_\sigma^{(0)\text{pot}} = 2 \int \frac{d\vec{k}_F}{4\pi} S^\sigma(\vec{k}_F) + \frac{1}{3} k_F \left( \frac{\partial V_0(m)}{\partial m} \right)_{m=k_F}, \quad (16)$$

$$\epsilon_{\sigma\tau}^{(0)\text{pot}} = 2 \int \frac{d\vec{k}_F}{4\pi} S^{\sigma\tau}(\vec{k}_F) + \frac{1}{3} k_F \left( \frac{\partial V_0(m)}{\partial m} \right)_{m=k_F}, \quad (17)$$

where  $V_0(m)$  is the single-nucleon potential in the case of  $N_\uparrow = N_\downarrow = Z_\uparrow = Z_\downarrow = A/4$ . The quantities  $S^\sigma$  and  $S^{\sigma\tau}$  are expressed by  $K(s m_s, T; k_F)$  which is the effective interaction (independent of  $T_3$ ) in the case of

$N_{\uparrow} = N_{\downarrow} = Z_{\uparrow} = Z_{\downarrow} = A/4$ . We have:

$$\left. \begin{array}{l} S^{\sigma}(\vec{k}) \\ S^{\sigma\tau}(\vec{k}) \end{array} \right\} = \frac{A}{8} \int \frac{d\vec{k}_F}{4\pi} (\vec{k} \vec{k}_F | \sum_T (2T \pm 1) [K(11, T; k_F) - \frac{1}{2}K(10, T; k_F) - \frac{1}{2}K(00, T; k_F)] | \vec{k} \vec{k}_F). \quad (18)$$

Our final results for  $\Delta\epsilon_{\sigma}$  and  $\Delta\epsilon_{\sigma\tau}$  are:

$$\Delta\epsilon_{\sigma} = \Delta_0\epsilon + \Delta_1\epsilon_{\sigma}, \quad \Delta\epsilon_{\sigma\tau} = \Delta_0\epsilon + \Delta_1\epsilon_{\sigma\tau}, \quad (19)$$

where  $\Delta_0\epsilon$  denotes the same expression as in BD, and is related to the rearrangement potential  $V_{OR}(k_F)$  in the case of  $N_{\uparrow} = N_{\downarrow} = Z_{\uparrow} = Z_{\downarrow} = A/4$ ,

$$\Delta_0\epsilon = -\frac{2}{3}V_{OR}(k_F). \quad (20)$$

The expressions for  $\Delta_1\epsilon_{\sigma}$  and  $\Delta_1\epsilon_{\sigma\tau}$  are:

$$\begin{aligned} \Delta_1\epsilon_{\sigma} = \frac{A}{2} \left[ \int_m \int \frac{d\vec{k}_F}{4\pi} (\vec{k}_F \vec{m} | k_F \left( \frac{\partial}{\partial\kappa} - \frac{\partial}{\partial\lambda} \right) \sum_T (2T+1) K(11, T; \kappa\lambda\kappa\lambda) | \vec{k}_F \vec{m}) \right. \\ \left. + \frac{1}{4} \int_m \int_{m'} (\vec{m} \vec{m}' | k_F^2 \left( \frac{\partial}{\partial\kappa} - \frac{\partial}{\partial\lambda} \right)^2 \sum_T (2T+1) [K(11, T; \kappa\lambda\kappa\lambda) + \frac{1}{2} \sum_s K(s0, T; \kappa\lambda\kappa\lambda)] | \vec{m} \vec{m}') \right] \quad (21) \end{aligned}$$

and

$$\begin{aligned} \Delta_1\epsilon_{\sigma\tau} = \frac{A}{2} \left\{ \int_m \int \frac{d\vec{k}_F}{4\pi} (\vec{k}_F \vec{m} | k_F \left( \frac{\partial}{\partial\kappa} - \frac{\partial}{\partial\lambda} \right) 2K(11, 11; \kappa\lambda\lambda\kappa) | \vec{k}_F \vec{m}) \right. \\ \left. + \frac{1}{4} \int_m \int_{m'} (\vec{m} \vec{m}' | k_F^2 \left( \frac{\partial}{\partial\kappa} - \frac{\partial}{\partial\lambda} \right)^2 \left( 2K(11, 11; \kappa\lambda\lambda\kappa) + \sum_s K(s0, 11; \kappa\lambda\lambda\kappa) \right) \right. \\ \left. + \sum_T \left[ K(11, T0; \kappa\lambda\lambda\kappa) + \frac{1}{2} \sum_s K(s0, T0; \kappa\lambda\lambda\kappa) \right] | \vec{m} \vec{m}') \right\}, \quad (22) \end{aligned}$$

where all the derivatives are to be calculated at the point  $\kappa_n = \lambda_n = \kappa_p = \lambda_p = k_F$ , and where  $\int_m = (4\pi k_F^3)^{-1} \int_{m < k_F}^{d\vec{m}}$ .

It should be noticed that the effective interaction  $K$  required for calculating  $\Delta_1\epsilon_{\sigma}$  and  $\Delta_1\epsilon_{\sigma\tau}$  depends only on two different Fermi momenta. For the case of  $\Delta_1\epsilon_{\sigma}$  [Eq. (21)] we have  $\kappa_n = \kappa_p = \kappa$  and  $\lambda_n = \lambda_p = \lambda$ , i.e.,  $N_{\uparrow} = Z_{\uparrow}$  and  $N_{\downarrow} = Z_{\downarrow}$ . In this case  $K$  does not depend on  $T_3$  which has been omitted in our notation. For the case of  $\Delta_1\epsilon_{\sigma\tau}$  [Eq. (22)] we have  $\kappa_n = \lambda_p = \kappa$  and  $\lambda_n = \kappa_p = \lambda$ , i.e.,  $N_{\uparrow} = Z_{\downarrow}$  and  $N_{\downarrow} = Z_{\uparrow}$ .

### III. APPROXIMATE EXPRESSIONS FOR $\Delta_1\epsilon_{\sigma}$ AND $\Delta_1\epsilon_{\sigma\tau}$

Although it is possible to calculate the  $K$  matrices which depend on two different Fermi momenta, this calculation is very tedious. For instance, such a calculation has been performed in Ref. 6 for  $K(\kappa\kappa\lambda\lambda)$ , which is the  $K$  matrix necessary to calculate  $\epsilon_{\tau}$ . On the other hand, an approximation has been applied in BD which enables calculation of  $\epsilon_{\tau}$  using  $K(k_F)$ , which is the  $K$  matrix when all the four Fermi momenta are equal. The same type of approximation will be applied in calculating  $\Delta_1\epsilon_{\sigma}$  and  $\Delta_1\epsilon_{\sigma\tau}$  in the present work.

The following approximations are introduced for calculating  $\Delta_1\epsilon_{\sigma}$ :

$$K(11, T; \kappa\lambda\kappa\lambda) \approx K(11, T; \kappa), \quad (23a)$$

$$K(s0, T; \kappa\lambda\kappa\lambda) \approx K(s0, T; k'_F), \quad (23b)$$

and the following approximations for calculating  $\Delta_1\epsilon_{\sigma\tau}$ :

$$K(11, 11; \kappa\lambda\lambda\kappa) \approx K(11, 1; \kappa), \quad (24a)$$

$$K(s0, 11; \kappa\lambda\lambda\kappa) \approx K(s0, 1; k'_F), \quad (24b)$$

$$K(11, T0; \kappa\lambda\lambda\kappa) \approx K(11, T; k'_F), \quad (24c)$$

$$K(s0, T0; \kappa\lambda\lambda\kappa) \approx \frac{1}{2} [K(s0, T; \kappa) + K(s0, T; \lambda)], \quad (24d)$$

where

$$k'_F = 2^{-1/2} [\kappa^2 + \lambda^2]^{1/2}, \quad (25)$$

and where the  $K$  matrices on the right-hand side of Eqs. (23) and (24) are calculated for the case of  $N_1 = N_2 = Z_1 = Z_2 = \frac{1}{2}A$  with the indicated common value of the Fermi momentum. These  $K$  matrices are independent of  $T_3$ .

The motivation for the above approximation is the same as for the analogous approximations in BD. Let us explain, for instance, approximations (24a, b, d). Approximation (24a) states that the effect of the spin-isospin excess on the scattering of two spin-up neutrons is determined by the shift of the Fermi momentum of the spin-up neutrons. Approximation (24b) states that the same effect on the scattering of two neutrons with opposite spin directions is determined approximately by the shift of an average value [Eq. (25)] of the Fermi momenta of the spin-up and spin-down neutrons. To explain approximation (24d) let us consider, for instance, the case  $s=1$ ,  $T=1$ , in which

$$|s0, T0\rangle = \frac{1}{2}\{|\uparrow n \uparrow p\rangle + |\uparrow p \uparrow n\rangle + |\uparrow p \downarrow n\rangle + |\downarrow n \uparrow p\rangle\}. \quad (26)$$

The former two states on the right-hand side of Eq. (26) are states of nucleons whose Fermi momenta equal  $\kappa$ , and the latter two states on the right-hand side of Eq. (26) are states whose Fermi momenta equal  $\lambda$  (in the spin-isospin excess nuclear matter with  $\kappa_n = \kappa$ ,  $\lambda_n = \lambda$ ,  $\kappa_p = \lambda$ ,  $\lambda_p = \kappa$ ). If the spin-flip amplitudes are neglected, we see that half of  $K(s0, T0; \kappa\lambda\lambda\kappa)$  represents the scattering of two nucleons whose Fermi momenta equal  $\kappa$ , and the other half the scattering of two nucleons whose Fermi momenta equal  $\lambda$ . This leads us to the approximation (24d).

When the approximate expressions (23) and (24) are introduced into Eqs. (21) and (22) we get the following approximate formulas:

$$\begin{aligned} \Delta_1 \epsilon_\sigma = \frac{A}{2} \left\{ \int \frac{d\hat{k}_F}{4\pi} \int_m (\vec{k}_F \vec{m} | k_F \frac{d}{dk_F} \sum_T (2T+1) K(11, T; k_F) | \vec{k}_F \vec{m}) \right. \\ \left. + \frac{1}{4} \int_m \int_{m'} (\vec{m} \vec{m}' | [k_F^2 \frac{d^2}{dk_F^2} \sum_T (2T+1) K(11, T; k_F) + \frac{1}{2} k_F \frac{d}{dk_F} \sum_T \sum_s (2T+1) K(s0, T; k_F)] | \vec{m} \vec{m}') \right\} \end{aligned} \quad (27)$$

and

$$\begin{aligned} \Delta_1 \epsilon_{\sigma\tau} = \frac{A}{2} \left\{ \int \frac{d\hat{k}_F}{4\pi} \int_m (\vec{k}_F \vec{m} | k_F \frac{d}{dk_F} 2K(11, 1; k_F) | \vec{k}_F \vec{m}) \right. \\ \left. + \frac{1}{4} \int_m \int_{m'} (\vec{m} \vec{m}' | [k_F^2 \frac{d^2}{dk_F^2} (2K(11, 1; k_F) + \frac{1}{2} \sum_s \sum_T K(s0, T; k_F)) \right. \\ \left. + k_F \frac{d}{dk_F} (\sum_s K(s0, 1; k_F) + \sum_T K(11, T; k_F))] | \vec{m} \vec{m}') \right\}. \end{aligned} \quad (28)$$

#### IV. CALCULATIONAL PROCEDURE

The main problem in applying the general formulas of Sec. II for the calculation of  $\epsilon_\sigma$  and  $\epsilon_{\sigma\tau}$  consists in determining the effective nucleon-nucleon interaction  $K$ . Since we shall apply the approximate expressions of Sec. III for  $\Delta\epsilon_\sigma$  and  $\Delta\epsilon_{\sigma\tau}$ , we avoid the tedious problem of determining the  $K$  matrix for the case of two different Fermi momenta, and calculate the  $K$  matrix which depends only on one Fermi momentum  $k_F$ . In the present work this is performed in terms of the Brueckner theory, where  $K$  is determined from the nucleon-nucleon potential  $v$  by solving the equation

$$(\vec{k}_1 \vec{k}'_2 | K | \vec{k}_F \vec{k}'_F) = (\vec{k}_1 \vec{k}_2 | v | \vec{k}_F \vec{k}'_F) + \sum_{\vec{k}'_1 \vec{k}'_2} (\vec{k}_1 \vec{k}_2 | v | \vec{k}'_1 \vec{k}'_2) \frac{Q}{z - \epsilon(k'_1) - \epsilon(k'_2)} (\vec{k}'_1 \vec{k}'_2 | K | \vec{k}_F \vec{k}'_F), \quad (29)$$

in which pure kinetic energies,  $\epsilon(k) = \hbar^2 k^2 / 2M$ , in the intermediate states are used, and where the exclusion principle operator is approximated by its angle average value, i.e., averaged over the directions of center-of-mass momentum of the two nucleons. By  $z$  we denote the sum of the self-consistent single-particle energies of the states  $\vec{k}_F$  and  $\vec{k}'_F$ .

Equation (29) refers to the case when the momenta of the two interacting nucleons are equal to  $\hbar\vec{k}_F$  and  $\hbar\vec{k}'_F$ , i.e., when the two nucleons are on the Fermi surface. The effective interaction  $K$  between two such nucleons is only needed for calculating  $S^\sigma(\vec{k}_F)$  and  $S^{\sigma\tau}(\vec{k}_F)$  according to Eq. (18).

It should be noted that in this case

$$z = 2[\epsilon_F + V_0(k_F)]. \quad (30)$$

The use of pure kinetic energies in the intermediate states in Eq. (29), which in recent years has become a standard procedure, deserves some comment. This procedure simplifies radically the calculation of the  $K$  matrix. Essential for its justification is the smallness of the contribution to the binding energy of nuclear matter from three-body clusters (see, for example, Ref. 2 and also Ref. 7, where the relevant references are given). The existing estimates seem to indicate that this contribution is small. The estimates are, however, only approximate. In our opinion the problem of proper choice of the particle energies in the intermediate states has, so far, not been solved satisfactorily. It should also be mentioned that the choice of the free-particle energies has been criticized recently by Baker and Gammel<sup>8</sup> and also by Petry and Schütte<sup>9</sup> who have shown that this choice is incorrect for certain classes of two-body interaction.

Fortunately, the consequences of the ambiguity in the choice of the intermediate-state energies are much less important for the spin and spin-isospin symmetry energies than for the volume energy, and the equilibrium density. The volume energy is a small difference between big kinetic and potential energy parts, whereas  $\epsilon_0$  and  $\epsilon_{\sigma\tau}$  are sums of a positive kinetic and positive potential part [Eq. (7)]. Furthermore, the contributions to  $\epsilon_0$  and  $\epsilon_{\sigma\tau}$  which contain  $S^\sigma$  and  $S^{\sigma\tau}$  [Eqs. (16) and (17)], appear to be insensitive to wide changes of the gap in the single-particle spectrum at the Fermi surface, and by reducing the gap we may simulate the effect of adding an attractive single-particle potential to the energies of the intermediate states. So, only less than half of the value of  $\epsilon_0$  and  $\epsilon_{\sigma\tau}$  is directly affected by the choice of the intermediate-state energies.

Three different nucleon-nucleon potentials were used as input: the Brueckner-Gammel-Thaler (BGT) potential,<sup>3</sup> the Hamada-Johnston (HJ) potential,<sup>4</sup> and the Reid soft-core (RSC) potential.<sup>5</sup> All these potentials have been applied before, in the Brueckner theory of nuclear matter. In particular, the equilibrium density and the self-consistent single-particle energy spectrum of the occupied states have been determined by Brueckner and Gammel<sup>3</sup> for the BGT potential, and by Banerjee and Sprung<sup>7</sup> for the HJ and RSC potentials. In our calculation we use the values of  $k_F$  and  $z$  [Eq. (30)], and also the value of  $[m \partial V_0(m) / \partial m]_{m=k_F}$  determined in Refs. 3 and 7.

In this way we avoid the necessity of solving the

whole self-consistency problem, and to determine  $\epsilon_0^{(0)\text{pot}}$  and  $\epsilon_{\sigma\tau}^{(0)\text{pot}}$  from Eqs. (16) and (17), we have to solve Eq. (29) for the  $K$  matrix elements on the Fermi surface only. We do this in the standard way by using the integral equation for the wave function of the relative motion of the two nucleons in configuration space, with the hard core of the two-body potential being replaced by a hard shell. Like in Ref. 7, the RSC potential is treated, for computational convenience, as having a very small hard core (0.04 fm). We apply a partial-wave expansion, and confine ourselves to partial waves with the orbital angular momentum  $L \leq 2$ . Since the RSC potential of Ref. 5 has not been determined in the  ${}^3D_3$  state, this state is neglected in our calculation for the RSC potential. The integral equations for the radial dependence of the partial waves are approximated by linear algebraic equations, and solved by the Gaussian method. Otherwise the whole procedure, including meshes, is the same as in Ref. 10.

The value of  $\Delta_0\epsilon$  may be obtained from Eq. (20) and from the known relation<sup>11</sup>

$$\epsilon_F + V_0(k_F) + V_{0R}(k_F) = \epsilon_{\text{vol}}, \quad (31)$$

valid at the equilibrium density. All quantities except  $V_{0R}$  have been determined in Refs. 3 and 7 for the BGT, and HJ and RSC potentials, respectively.

To calculate  $\Delta_1\epsilon_0$  and  $\Delta_1\epsilon_{\sigma\tau}$  from the approximate expressions (27) and (28), the density or Fermi momentum dependence of the  $K$  matrix must be known. In the case of the BGT potential this density dependence has been determined by Brueckner, Gammel, and Weitzner<sup>12</sup> (cf. also Ref. 13), who have shown that the variation of  $K$  with density could be entirely included in the  $s$  wave repulsive part of the  $K$  matrix. Using the very simple form of this part of the  $K$  matrix given in Ref. 12, we have calculated  $\Delta_1\epsilon_0$  and  $\Delta_1\epsilon_{\sigma\tau}$  for the BGT potential in exactly the same way as the analogous term,  $\Delta_1\epsilon_\tau$ , of  $\epsilon_\tau$  was calculated in BD.

For the case of the RSC potential, Sprung and Banerjee<sup>14</sup> have constructed a local effective interaction,

$$V(r) = \sum_{ST} \sum_i [a_i(ST) + b_i(ST)k_F^{1/2}] e^{-r/\lambda_i} \Lambda_{ST}, \quad (32)$$

that reproduces the  $K$  matrix elements appearing in their nuclear-matter calculation<sup>7</sup> with the RSC potential. By  $\Lambda_{ST}$  we denote the projection operator for two-nucleon states of total spin  $S$  and total isospin  $T$ . The values of constants  $a_i(ST)$ ,  $b_i(ST)$ , and  $\lambda_i$  are given in Ref. 14. Here  $V(r)$  is con-

sidered in the form (32) referred to in Ref. 14 as G-O force, since this form proves to reproduce better the  $K$  matrix elements than other forms considered in Ref. 14 (the G-1 and G-3) do. The representation of the  $K$  matrix in the form (32) has been used in the present work to calculate  $\Delta_1 \epsilon_\sigma$  and  $\Delta_1 \epsilon_{\sigma\tau}$  [Eqs. (27) and (28)] for the case of RSC potential. In this calculation only contributions of the  $S$ ,  $P$ , and  $D$  waves have been included. Let us notice that tensor and spin-orbit components of the effective interaction, also determined by Sprung,<sup>15</sup> do not contribute to expressions (27) and (28) for  $\Delta_1 \epsilon_\sigma$  and  $\Delta_1 \epsilon_{\sigma\tau}$ .

In the case of HJ potential, we do not have any simple representation of the density dependence of the  $K$  matrix elements. Because of this, we did not calculate  $\Delta_1 \epsilon_\sigma$  and  $\Delta_1 \epsilon_{\sigma\tau}$  for the HJ potential.

#### V. NUMERICAL RESULTS AND DISCUSSION

The values of  $k_F$ ,  $z$ ,  $\frac{1}{3}k_F[\partial V_0/\partial m]_{m=k_F}$ , and  $\Delta_0 \epsilon$  have been adopted from Refs. 3 and 7 in our calculation, and are shown in the upper part of Table I. The value of  $z$  for the BGT potential deserves a comment. In our calculation, pure kinetic energies are used in the intermediate states, whereas a single-particle potential has been included in the intermediate-states single-particle energies of Ref. 3, and thus the gap between the single-particle energy spectrum of the occupied and unoccupied states has been lowered. The value of  $z$ , suitable for our calculation, has been fixed by the requirement that it leads in our calculations to the value of  $\epsilon_\tau^{(0)\text{pot}}$ , found in Ref. 3. Actually, the values of  $\epsilon_\sigma^{(0)\text{pot}}$  and  $\epsilon_{\sigma\tau}^{(0)\text{pot}}$  would change only slightly, if in the case of the BGT potential a value of

TABLE I. Calculated values (in MeV) of the three symmetry energies, and values of  $k_F$  (in  $\text{fm}^{-1}$ ) and other quantities (in MeV), used in the calculation.

Potential	BGT	HJ	RSC
$k_F$	1.49 <sup>a</sup>	1.27 <sup>b</sup>	1.43 <sup>b</sup>
$z$	-8.1 <sup>a</sup>	-33.4 <sup>b</sup>	-42.8 <sup>b</sup>
$\frac{1}{3}k_F(\partial V_0/\partial m)k_F$	11.4 <sup>a</sup>	9.6 <sup>b</sup>	16.6 <sup>b</sup>
$\Delta_0 \epsilon = -\frac{2}{3}V_{0R}(k_F)$	-8.2 <sup>a</sup>	-6.0 <sup>b</sup>	-7.4 <sup>b</sup>
$\epsilon_\tau^{(0)}$	51.8	43.8	58.9
$\epsilon_\sigma^{(0)}$	51.7	42.3	58.0
$\epsilon_{\sigma\tau}^{(0)}$	79.8	54.7	76.1
$\epsilon_\tau$	64.1		60.5
$\epsilon_\sigma$	64.9		74.1
$\epsilon_{\sigma\tau}$	76.6		73.0

<sup>a</sup> Determined in Ref. 3.

<sup>b</sup> Determined in Ref. 7.

$z$  of a magnitude similar to that for HJ and RSC potentials were used.

The lower part of Table I shows the results of the present calculation. Values of  $\epsilon_\tau$  calculated in terms of the theory given in BD are also included in Table I.

The three potentials considered lead to different equilibrium densities, i.e., different values of  $k_F$ . To compare the symmetry energies produced by the three potentials, it is therefore necessary to provide for the differences in the  $k_F$  values. This has been done by calculating the symmetry energies for the HJ and RSC potentials for the BGT value of  $k_F = 1.49 \text{ fm}^{-1}$  in the following way. The parts containing  $S(\vec{k}_F)$  [Eqs. (16) and (17)] have been obtained from our calculated values by assuming a linear dependence on  $\rho$ . The term  $\frac{1}{3}k_F[\partial V_0/\partial m]_{m=k_F}$  has been obtained from the values of the effective mass as a function of  $k_F$ , given in Ref. 7. The rearrangement part of the symmetry energy has been calculated in terms of the effective interaction [Eq. (32)]. The resulting values of the symmetry energies are shown in Table II, together with the corresponding values for the BGT potential.

As is seen from Table II, the two potentials HJ and RSC lead to very similar values of all the three symmetry energies. This reflects the similarity of the two potentials. Although the HJ potential contains a quadratic spin-orbit part, this is balanced by the fact that the RSC potential is different in each  $STJ$  channel. The fact that the RSC potential has a soft core, whereas the HJ potential has a hard core, does not affect the values of the symmetry energies calculated at the same density. On the other hand, the RSC potential leads to a higher equilibrium density than the HJ potential.<sup>7</sup> Because of this, the symmetry energies calculated at the equilibrium densities of the two potentials are different, as is seen in Table I.

On the other hand, the old BGT potential, adjusted to earlier nucleon-nucleon scattering data, produces lower values of the symmetry energies,

TABLE II. The three symmetry energies (in MeV), corrected for the same density, corresponding to  $k_F = 1.49 \text{ fm}^{-1}$ .

Potential	BGT	HJ	RSC
$\epsilon_\tau^{(0)}$	51.8	68.0	66.1
$\epsilon_\sigma^{(0)}$	51.7	65.5	65.1
$\epsilon_{\sigma\tau}^{(0)}$	79.8	85.5	85.6
$\epsilon_\tau$	64.1		67.7
$\epsilon_\sigma$	64.9		84.0
$\epsilon_{\sigma\tau}$	76.6		81.7

calculated at the same density, as is shown in Table II. However, in one respect the three potentials are similar, viz., the nonrearrangement parts of the symmetry energies, calculated with any of the three potentials, satisfy the relation

$$\epsilon_{\tau}^{(0)} \simeq \epsilon_{\sigma}^{(0)} < \epsilon_{\sigma\tau}^{(0)}. \quad (33)$$

The situation changes when we add the rearrangement parts. As for the BGT potential, relation (33) holds also for full values of the symmetry energies; for the RSC potential we have

$$\epsilon_{\tau} < \epsilon_{\sigma} \simeq \epsilon_{\sigma\tau}. \quad (34)$$

Thus the relative values of the three symmetry energies depend on the values of the rearrangement corrections.

Let us note that the differences between  $\epsilon_{\tau}$ ,  $\epsilon_{\sigma}$ , and  $\epsilon_{\sigma\tau}$  are due to the spin dependence of nuclear forces. Hence if the forces were spin independent we would have  $\epsilon_{\tau} = \epsilon_{\sigma} = \epsilon_{\sigma\tau}$ .

It seems to us that our results, obtained with the more up to date RSC potential, are more reliable than those obtained using the old BGT potential. Also in the case of the RSC potential the rearrangement contributions to the symmetry energies have been calculated in terms of the carefully adjusted local effective interaction of Ref. 14, whereas in the case of BGT potential a simplified representation of the density dependence of the  $K$  matrix, restricted to the  $S$  state,<sup>12</sup> has been applied. Consequently, we consider the following RSC values of the three symmetry energies to be our most reliable results:

$$\epsilon_{\tau} = 61 \text{ MeV}, \quad \epsilon_{\sigma} = 74 \text{ MeV}, \quad \epsilon_{\sigma\tau} = 73 \text{ MeV}. \quad (35)$$

The isospin symmetry energy,  $\epsilon_{\tau}$ , is obviously of greater practical importance than the other two symmetry energies, since medium and heavy nuclei have an appreciable neutron excess resulting from the Coulomb forces. Several empirical estimates of  $\epsilon_{\tau}$  are available in the literature. For instance, by assuming that the isospin symmetry energy contains a volume and a surface part, Green<sup>16</sup> finds  $\epsilon_{\tau} = 61$  MeV, and Cameron<sup>17</sup> finds  $\epsilon_{\tau} = 63$  MeV. Both these values agree very well with our RSC value, Eq. (35), and also with the BGT value,  $\epsilon_{\tau} = 64$  MeV. Of the theoretical calculations of  $\epsilon_{\tau}$  let us mention Ref. 6 in which the value  $\epsilon_{\tau} = 56$  MeV has been obtained for the BGT potential, and the work by Siemens,<sup>18</sup> who has obtained the value  $\epsilon_{\tau} = 62$  MeV for the RSC potential. Since in Refs. 6 and 18 two different Fermi momenta for protons and neutrons have been used explicitly in the calculation of the  $K$  matrix, the good agreement between the value of  $\epsilon_{\tau}$  obtained

in Ref. 18 and the value given in Eq. (35) shows that probably the approximations of Sec. III, together with the local effective interaction, Eq. (32), are sufficiently accurate in the case of the RSC potential. On the other hand, the difference between our BGT value,  $\epsilon_{\tau} = 64$  MeV, and the value of Ref. 6,  $\epsilon_{\tau} = 56$  MeV, seems to indicate that the representation of the density dependence of the  $K$  matrix (Ref. 12) is not accurate enough for the symmetry-energy problem.

The spin and spin-isospin symmetry energies,  $\epsilon_{\sigma}$  and  $\epsilon_{\sigma\tau}$ , are not related directly to the properties of nuclear ground states (the spin of even-even nuclei is zero in their ground state). They are, however, similar to  $\epsilon_{\tau}$  related to the properties of certain collective nuclear excited states. To explain this point let us recall the Goldhaber-Teller<sup>19</sup> model which has been very successful in a qualitative explanation of the position of the giant dipole resonance that appears in the nuclear photoeffect. In this model, as developed by Steinwedel and Jensen,<sup>20</sup> the giant dipole state consists of a dipole vibration of the proton fluid against the neutron fluid, with the restoring force being determined by the isospin symmetry energy,  $\epsilon_{\tau}$ . The total nucleon density, i.e., the sum of proton and neutron densities, is assumed to be constant over the nuclear volume which is a sphere of radius  $R$ . Generally, for a vibration of a given multipolarity  $\lambda$  and of the order  $n$ , the energy of this charge vibration (isospin or  $\tau$  vibration) is given by

$$E(\tau; \lambda, n) = \hbar(\epsilon_{\tau}/M)^{1/2} R^{-1} z_n^{(\lambda)}, \quad (36)$$

where  $z_n^{(\lambda)}$  are discrete numbers, tabulated as, e.g., in Ref. 21. For the giant-dipole case ( $\lambda=1$ ,  $n=1$ ) we have  $z_1^{(1)} = 2.08$ . In Eq. (36) we have assumed  $N = Z = \frac{1}{2}A$ .

It has been pointed out a long time ago<sup>22</sup> that beside this isospin mode of vibration, there exist others in which protons with spin up and neutrons with spin down move against protons with spin down and neutrons with spin up (spin-isospin or  $\sigma\tau$  mode), or in which nucleons with spin up move against nucleons with spin down (spin or  $\sigma$  mode). Extensive studies of the properties of these modes have been performed by Überall, Walecka, and others (cf. Refs. 23–28). Apart from the  $\tau$  mode, the  $\sigma\tau$  mode, which plays an essential role in the muon capture and in the inelastic electron scattering, is best established. As shown in Refs. 26 and 27 both these processes may be described in terms of a generalized hydrodynamical model which includes all the three  $\tau$ ,  $\sigma$ , and  $\sigma\tau$  modes of vibration. In this generalized hydrodynamical model of Überall the frequencies of the different



vibration modes, i.e., the energies of the collective excited states, are phenomenological parameters. The values of  $\epsilon_\sigma$  and  $\epsilon_{\sigma\tau}$  calculated in the present paper enable us to relate these frequencies or energies to nuclear forces. The simplest way of doing this, for instance in the case of spin vibrations, is to repeat the derivation of Steinwedel and Jensen<sup>20</sup> with the neutron (proton) fluid replaced by the fluid of nucleons with spin up (down). The restoring force for the spin vibrations is then determined by the spin symmetry energy,  $\epsilon_\sigma$ . Note that within the quadratic approximation of Eq. (3) the  $\tau$ ,  $\sigma$ , and  $\sigma\tau$  vibrations are not coupled. In this way for the energy of the  $\sigma$  vibration (and similarly for the  $\sigma\tau$  vibration) of a given multipolarity  $\lambda$  and of the order  $n$  we have

$$\begin{aligned} E(\sigma; \lambda, n) &= \hbar(\epsilon_\sigma/M)^{1/2} R^{-1} z_n^{(\lambda)}, \\ E(\sigma\tau; \lambda, n) &= \hbar(\epsilon_{\sigma\tau}/M)^{1/2} R^{-1} z_n^{(\lambda)}. \end{aligned} \quad (37)$$

The derivation of Eqs. (36) and (37) involves, however, several approximations, for instance, neglect of the surface diffuseness. As this effect is expected to play a similar role in the  $\tau$  and  $\sigma\tau$  modes, it seems to be more reasonable to discuss the ratio

$$E(\sigma\tau; \lambda, n)/E(\tau; \lambda, n) = (\epsilon_{\sigma\tau}/\epsilon_\tau)^{1/2}, \quad (38)$$

for which one gets the value 1.1 both with our RSC [Eq. (35)] and BGT results, as well as with our HJ results for  $\epsilon_{\sigma\tau}^{(0)}$  and  $\epsilon_\tau^{(0)}$ .

The location of the  $\tau$  and  $\sigma\tau$  giant-dipole levels seems to be best established in the  $^{16}\text{O}$  nucleus, where both the muon capture<sup>29,30</sup> and inelastic scattering measurements can be interpreted<sup>25,26</sup> in terms of definite  $\tau$  and  $\sigma\tau$  levels. According to

Raphael, Überall, and Werntz<sup>26</sup> the energies of the  $\tau$  and  $\sigma\tau$  dipole  $1^-$  levels in  $^{16}\text{O}$  are 22.0 and 24.5 MeV, and thus their ratio 1.1 coincides with our estimate.

Although the above discussion shows that our results for the symmetry energies are relevant for the problem of the  $\tau$ ,  $\sigma$ , and  $\sigma\tau$  modes of nuclear excitations, one should add the following remark. In real nuclei the orbital angular momentum  $L$  is coupled with spin  $S$ . For instance in the case of the  $\sigma\tau$  giant dipole resonance ( $S=1$ ,  $L=1$ ) we have, in fact, three  $J^\pi$  states:  $0^-$ ,  $1^-$ ,  $2^-$ . To get the splitting of these three states, and thus to be able to make any more detailed comparison with the levels in finite nuclei, the coupling between the motion of the two fluids and the spin should be built into the generalized Steinwedel-Jensen model.

As it is well known, the isospin-symmetry energy,  $\epsilon_\tau$ , is closely related to the isospin-dependent part of the nuclear single-particle potential (cf. BD). Similarly, the symmetry energies,  $\epsilon_\sigma$  and  $\epsilon_{\sigma\tau}$ , are related to the spin and spin-isospin dependent parts of the nuclear single-particle potential, which recently have been studied extensively, both experimentally and theoretically (see, for example, Satchler<sup>31</sup>). This problem will, however, be presented in a separate publication.

#### ACKNOWLEDGMENTS

The authors wish to thank the staff of the Computer Center of the State Power Dispatch Station in Warsaw for their assistance in performing the numerical calculations on the CDC 3170 computer.

<sup>1</sup>K. A. Brueckner and J. Dabrowski, Phys. Rev. **134**, B722 (1964).

<sup>2</sup>For a review of the Brueckner theory on nuclear matter, see, e.g., J. Dabrowski, in *Proceedings of the International Course on Nuclear Theory, Trieste, 1969* (International Atomic Energy Agency, Vienna, Austria, 1970), p. 131.

<sup>3</sup>K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958); the parameters of the BGT potential are given in their Table III.

<sup>4</sup>T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 383 (1963).

<sup>5</sup>R. V. Reid, Ann. Phys. (N.Y.) **50**, 411 (1968).

<sup>6</sup>K. A. Brueckner, S. A. Coon, and J. Dabrowski, Phys. Rev. **168**, 1184 (1968).

<sup>7</sup>P. K. Banerjee and D. W. L. Sprung, Can. J. Phys. **49**, 1899 (1971).

<sup>8</sup>G. A. Baker, Jr., and J. L. Gammel, Phys. Rev. C **6**, 403 (1972).

<sup>9</sup>H. R. Petry and D. Schütte, Phys. Letters **38B**, 159 (1972).

<sup>10</sup>S. A. Coon and J. Dabrowski, Phys. Rev. **140**, B287 (1965).

<sup>11</sup>K. A. Brueckner, Phys. Rev. **100**, 597 (1958); N. M. Hugenholtz and L. Van Hove, Physica **24**, 363 (1958).

<sup>12</sup>K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).

<sup>13</sup>K. A. Brueckner and D. T. Goldman, Phys. Rev. **116**, 424 (1959).

<sup>14</sup>D. W. L. Sprung and P. K. Banerjee, Nucl. Phys. **A168**, 273 (1971).

<sup>15</sup>D. W. L. Sprung, Nucl. Phys. **A182**, 97 (1972).

<sup>16</sup>A. E. S. Green, Rev. Mod. Phys. **30**, 569 (1958); Phys. Rev. **95**, 1006 (1954).

<sup>17</sup>A. G. W. Cameron, Can. J. Phys. **35**, 1021 (1957).

<sup>18</sup>P. J. Siemens, Nucl. Phys. **A141**, 225 (1970).

<sup>19</sup>A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **15**, 81 (1945), in Russian; M. Goldhaber and E. Teller, Phys. Rev. **74**, 1046 (1948).

<sup>20</sup>H. Steinwedel and H. Jensen, Z. Naturforsch. **5a**, 413 (1950).

<sup>21</sup>M. Huber, Am. J. Phys. **35**, 685 (1967).

- <sup>22</sup>W. Wild, Bayer. Akad. Wiss., Math.-Naturwiss. Klasse, 18, 371 (1956); A. E. Glassgold, W. Heckrotte, and K. M. Watson, Ann. Phys. (N.Y.) 6, 1 (1959); S. Hatanoto, Progr. Theoret. Phys. (Kyoto) 24, 418 (1960).  
<sup>23</sup>H. Überall, Phys. Rev. 137, B502 (1965).  
<sup>24</sup>R. Raphael and H. Überall, Phys. Rev. 152, 899 (1966).  
<sup>25</sup>H. Überall, Nuovo Cimento 41B, 25 (1966).  
<sup>26</sup>R. Raphael, H. Überall, and C. Werntz, Phys. Letters 24B, 15 (1967).

- <sup>27</sup>H. Überall, Acta Phys. Austriaca 30, 89 (1969).  
<sup>28</sup>L. L. Foldy and J. D. Walecka, Nuovo Cimento 34, 1026 (1964).  
<sup>29</sup>V. Evseev, T. Kozłowski, V. Roganov, and J. Wojtkowska, in *High-Energy Physics and Nuclear Structure*, edited by S. Devons (Plenum, New York, 1970), p. 157.  
<sup>30</sup>M. E. Plett and S. E. Sobottka, Phys. Rev. C 3, 1003 (1971).  
<sup>31</sup>G. R. Satchler, Particles Nuclei 1, 197 (1971).

## Center-of-Mass Motion in Many-Particle Systems\*

D. J. Ernst, C. M. Shakin, and R. M. Thaler

*Department of Physics, Case Western Reserve University, Cleveland, Ohio 44106*

(Received 10 October 1972)

We have reexamined the problem of constructing a nuclear ground-state wave function consistent with the requirement of translational invariance, beginning with a wave function which does not have this property. A class of transformations is constructed which transforms the initial-state vector into an eigenstate of total momentum. The particular transformation of this class which leads to the lowest intrinsic energy is given.

A wave function generated by a shell model, Hartree-Fock, Brueckner-Hartree-Fock, or other such calculation, is centered about a point in space (generally taken as the origin). Such a wave function violates the fundamental requirement of translational invariance. There have been, over the years, many different methods<sup>1-4</sup> proposed to generate a wave function consistent with the requirement of translational invariance from one which is not consistent with this symmetry. That these methods can yield very different results has recently been noted.<sup>4,5</sup> Perhaps the most intuitively attractive prescription is that which reinterprets the variables  $\vec{r}_i$  (which locate the particles with respect to some arbitrary origin) as the variables  $\vec{r}_i - 1/A \sum_j \vec{r}_j$  (which choose that origin to be the center of mass of the system). This prescription implies the substitution

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_A) \\ - \Psi_0 \left( \vec{r}_1 - \frac{1}{A} \sum_j \vec{r}_j, \vec{r}_2 - \frac{1}{A} \sum_j \vec{r}_j, \dots, \vec{r}_A - \frac{1}{A} \sum_j \vec{r}_j \right). \quad (1)$$

It is clear that the wave function on the right is translationally invariant, as a translation of all the coordinates,  $\vec{r}_i \rightarrow \vec{r}_i + \vec{\Delta}$ , leaves the wave function unchanged. Such a wave function may be thought of as an eigenfunction of total momentum with momentum eigenvalue zero. In this paper we

will rewrite this prescription in operator form, generalize this operator so that it contains a continuous set of nonequivalent prescriptions, and finally discuss the properties and physical meanings of these operators.

We define the center-of-mass position operator and its canonical conjugate, the total momentum operator, in second-quantized form as

$$\vec{\mathfrak{X}}^{(A)} = \frac{1}{A} \int d^3 \vec{r} a^\dagger(\vec{r}) \vec{r} a(\vec{r}) \quad (2)$$

and

$$\vec{\mathfrak{P}} = \int d^3 \vec{k} a^\dagger_{\vec{k}} \vec{k} a_{\vec{k}}, \quad (3)$$

where  $a^\dagger$ ,  $a$  are fermion operators. The relation between  $|\Psi\rangle$  and  $|\Psi_{\vec{P}=0}\rangle$  is then given by

$$|\Psi_{\vec{P}=0}\rangle \equiv \mathcal{O}_{\vec{P}=0} |\Psi\rangle = \int \delta(\vec{R} - \vec{\mathfrak{X}}^{(A)}) e^{-i\vec{R} \cdot \vec{\mathfrak{P}}} d^3 \vec{R} |\Psi\rangle. \quad (4)$$

The operator  $\mathcal{O}_{\vec{P}=0}$  first translates each variable in  $|\Psi\rangle$  by an amount  $-\vec{R}$  and then the  $\delta$  function fixes  $\vec{R}$  to be the center of mass of the system, thus reproducing the relation prescribed by Eq. (1). The  $\vec{R}$  integral cannot be done explicitly merely by replacing the variable  $\vec{R}$  in the exponent by the operator  $\vec{\mathfrak{X}}^{(A)}$ , because  $\vec{\mathfrak{X}}^{(A)}$  does not commute with  $\vec{\mathfrak{P}}$ .

The operator  $\mathcal{O}_{\vec{P}=0}$  can be expressed in an instruc-