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Off-Shell Effects in Triton Binding-Energy and Form-Factor Calculations

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We calculate the triton binding energy and charge form factor for a set of phase-shift-equivalent, S-wave, spin-independent potentials. The Faddeev equations are solved by the method of Padé approximants. We present a partial wave expansion for the triton form factor for a $J=0$ spatially symmetric wave function. A 6-MeV variation in triton binding energy results occurs for the potentials studied. All but 1 MeV of this off-shell variation is attributable to changes in the deuteron wave function. The relation between the triton binding energy and deuteron wave function is discussed. We find that the triton binding energy is sensitive to the relative amount of scale distortion in the momentum-space deuteron wave function. For the potentials studied, less binding in the triton is associated with deuteron wave functions that are more "compressed," i.e., fall off more rapidly in momentum space for $k<2$ fm^{-1}. We also find that potentials that yield less binding in the triton (in worse agreement with experiment) yield a diffraction minimum in the form factor at smaller momentum transfer (in better agreement with experiment). The disparity between the binding-energy results and form-factor results with respect to experiment may indicate a role for three-body forces in the three-nucleon system.

I. INTRODUCTION

One of the basic unresolved problems of present day nuclear physics is the detailed nature of the two-nucleon (N-N) interaction. Accurate phase-shift analyses of the experimental N-N elastic scattering data for low partial waves and for energies where pion production is not important ($E_{\mathrm{LAB}}<350$ MeV) are available[1]; however, the N-N data only determine the asymptotic form of the N-N wave function, or alternatively, the on-shell properties of the N-N transition (T) matrix. Elastic N-N scattering experiments tell nothing about the off-energy-shell T matrix. To predict the properties of the few- or many-nucleon system, one must have a knowledge of the off-shell T matrix.[2,3] Moreover, recent calculations[4-7] indicate that off-shell behavior plays a major role in predictions of the binding energies of nuclear matter and ^{16}O.

This paper investigates the consequences of the off-shell uncertainty in the N-N interaction in the three-nucleon bound-state problem. The three-nucleon bound-state problem provides an excellent opportunity to learn something about the N-N interaction because, unlike most nuclear many-body problems, an exact theory is available through the Faddeev equations formalism.[2] Recent numerical advances[8-10] render the three-nucleon bound-state problem manageable for potentials that act in a small number of partial waves.

Previous studies[11-13] of off-shell effects in the triton indicate that triton-binding-energy results are moderately sensitive to off-shell properties. In this paper we attempt to explain how off-shell variations in the triton binding energy and form-factor results arise from changes in the momentum distributions of the deuteron wave function and off-shell T matrix. To investigate off-shell effects, we calculate the triton binding energies and charge form factors for a set of exactly phase-shift-equivalent potentials. Section II briefly reviews the unitary-transformation method of generating these potentials. All potentials investigated are unitarily equivalent, in the two-body sense, to a two-term Yukawa potential acting in relative S waves. The Yukawa potential is spin-independent and corresponds very closely to the V potential of Malfliet and Tjon.[9] We thus do not attempt to examine the roles of spin dependence, tensor forces, or higher partial waves in the triton problem. We employ a simplified model of the N-N interaction to isolate the role of off-shell properties. Since the triton is believed to be in a predominantly spatially symmetric state, the averaging of the spin dependence of the N-N interaction is not an unreasonable approximation.[14]

Section III of this paper describes the solution of the Faddeev equations for the triton problem. We apply a modification of the methods of Malfliet and Tjon[9,10] to extract the triton binding energy and wave function. We present partial-wave expansions for the completely antisymmetrized three-nucleon wave function and for the triton (or ^{3}He) form factor for a spatially symmetric $J=0$ wave function. The actual calculations of the form factor include angular momentum states up to $l=8$ in

the partial-wave expansion. Previous Faddeev calculations of the form factor[15] only include the $l=0$ contribution.

Results of the calculations with phase-shift-equivalent potentials appear in Sec. IV. Large variations (greater than 1 MeV) in the triton-binding-energy results occur only for those potentials that yield significantly different deuteron wave functions. Potentials that predict less binding in the triton also predict the diffraction minimum in the triton (or ^{3}He) charge form factor at lower values of the momentum transfer. In fact, the form-factor calculations seem to suggest that off-shell changes in the form factor are governed to a high degree by the off-shell changes in the binding energy.

In Sec. V we discuss the off-shell dependence of the results of Sec. IV. The discussion centers mainly on the dependence of the triton binding energy on the deuteron wave function. We employ results obtained from the unitary-pole approximation (UPA)[16] to isolate the role of the deuteron wave function. The main conclusion is that the triton-binding-energy results are sensitive to the relative amounts of distortion present in the deuteron wave function (with respect to some fixed deuteron wave function) in momentum space. Potentials that yield steeper momentum (k) dependence in the deuteron wave function for $k<2$ fm^{-1} yield less binding in the triton. In cases where potentials yield virtually the same deuteron wave function, changes in the low and intermediate momentum components ($k<2$ fm^{-1}) of the low-energy half-shell T matrix seem to govern the off-shell changes in triton results.

Section VI explores the implications of the observed off-shell dependence of the triton results with respect to current experimental and theoretical constraints on the off-shell N-N interaction. We conclude that the off-shell uncertainty in triton-binding-energy predictions could be several MeV even for "reasonably realistic" potentials. This uncertainty follows mainly from the lack of experimental constraints on the low-energy $^1S_0$ interaction. This conclusion is especially important in view of the fact that Faddeev calculations[9] for the Reid potential[17] yield an incorrect triton binding energy (6.5 MeV vs 8.5 MeV experiment). The discrepancy between the ^{3}He charge form factor as calculated with the Reid potential[15] with experiment is more mysterious. By the results of this paper, potentials that differ off shell that give better agreement with experiment for the position of the diffraction minimum in the form factor give worse agreement for the triton binding energy. This trend may imply an important role for three-body forces.

II. UNITARILY EQUIVALENT POTENTIALS

Details of the unitary-transformation method of generating phase-shift-equivalent potentials appear elsewhere.[4,6] A brief review follows below.

We start with a two-body "untransformed" Hamiltonian $H=H_0+V$, where H_0 is the relative kinetic energy and V is the two-body potential. We introduce a two-body operator U that is short ranged

$$\langle\vec{r}\,|\,U\,|\,\vec{r}\,'\rangle \xrightarrow[r \text{ or } r'\to\infty]{} \delta(\vec{r}-\vec{r}\,')$$

and unitary ($U^\dagger U = UU^\dagger = 1$). If the Hamiltonian H has a spectrum E_n and eigenstates Ψ_n (i.e., $H\Psi_n = E_n\Psi_n$), then $\tilde{H}\tilde{\Psi}_n = E_n\tilde{\Psi}_n$ and $\tilde{\Psi}_n(\vec{r})_{r\to\infty} \Psi_n(\vec{r})$, where $\tilde{\Psi}_n = U\Psi_n$ and $\tilde{H} = UHU^\dagger$. Therefore, the two-body "transformed" Hamiltonian yields the same spectrum as H and, in particular, will leave the two-body bound-state energies invariant. By unitarity the eigenstates of $\tilde{H}$ preserve the orthonormality of states ($\langle\tilde{\Psi}_n|\Psi_m\rangle = \langle\Psi_n|\Psi_m\rangle$). Also, since $\tilde{\Psi}_n$ and Ψ_n are asymptotically equal, $\tilde{H}$ preserves the phase shifts of H. While the two-body potentials V and $\tilde{V}=\tilde{H}-H_0$ are unitarily equivalent in the two-body problem, these two-body potentials do not yield unitarily equivalent wave functions in the many-body problem. Therefore, the two-body potentials V and $\tilde{V}$, when substituted as the interaction between pairs in a many-body Hamiltonian, do not yield the same many-body spectrum.

In this paper we employ the rank-one unitary transformation

$$U = 1 - 2\Lambda\,, \tag{1}$$

where

$$\langle rl\gamma\,|\,\Lambda\,|\,r'l'\gamma\rangle = g_l^\gamma(r)\,g_{l'}^\gamma(r')\,\delta_{ll'}\,\delta_{l0}\,.$$

The variable r is the relative displacement. The quantum numbers l and l' stand for relative orbital angular momentum and γ stands for the "good" quantum numbers J, M (total angular momentum and projection), S (spin), and II_3 (isospin and projection). We choose $g_l^\gamma(r)$ to have the form

$$g_l^\gamma(r) = Ce^{-\alpha_0 r}(\beta_0+\beta_1 r+\beta_2 r^2)\,, \tag{2}$$

where C is determined by the unitarity condition $\int_0^\infty r^2 dr\,[g_l^\gamma(r)]^2 = 1$. In this study the "untransformed" potential V is given by

$$\langle rl\gamma\,|\,V\,|\,r'l'\gamma\rangle = V(r)\delta(r-r')/r^2\delta_{ll'}\,\delta_{l0}\,, \tag{3}$$

where $V(r) = -574.32e^{-1.55r}/r + 1448.44e^{-3.11r}/r$. The potential (3) is very close to the V potential of Malfliet and Tjon.[9] It represents the average of singlet and triplet S-wave potentials that individually give good fits to the $^1S_0$ and $^3S_1$ N-N scattering data up to 300 MeV (LAB). This potential yields a two-body bound state at -0.416 MeV. The potential (3),

while clearly a simplification of the true N-N force, is a suitable starting point for the purpose of studying off-shell effects in the triton.

We do not introduce the transformation of Eqs. (1), (2) on any special physical grounds. The principal reason for its use is computational convenience. With the choices of Eqs. (1)–(3), analytic expressions are available for the matrix elements of $\tilde{V}$ in momentum space.[18] We do not attempt to theoretically justify the transformation we employ. The transformation merely provides a tool to vary the unknown part of the interaction, i.e., the off-shell T matrix elements. The main goal is to determine how changes in the off-shell N-N interaction affect predictions of triton properties. Note, however, that by the proper choice of the parameters α_0, β_i in Eq. (2), one can limit the range of the transformation. Therefore, one may choose the parameters of the transformation such as not to affect the two-nucleon wave function (or potential) in the one-pion-exchange (OPE) tail region ($r > 2$ fm). Thus, with the appropriate choice of parameters, the transformation (1), (2) is not inconsistent with our knowledge from meson theory of the N-N interaction at large distances.

The matrix elements of $\tilde{V}$ determine the two-body T matrix $\tilde{t}(\omega)$ through the Lippmann-Schwinger equation

$$\langle kl\gamma|\,\tilde{t}(\omega)\,|k'l'\gamma\rangle = \langle kl\gamma|\,\tilde{V}\,|k'l'\gamma\rangle + \sum_{l_1}\int_0^\infty \frac{q^2dq\,\langle kl\gamma|\,\tilde{V}\,|ql_1\gamma\rangle\langle ql_1\gamma|\,\tilde{t}(\omega)\,|k'l'\gamma\rangle}{\omega - q^2}\,. \tag{4}$$

The matrix elements of $\tilde{t}(\omega)$ provide the input for the solution of the Faddeev equations for the three-body problem.

III. SOLUTION OF THE FADDEEV EQUATIONS FOR THE TRITON

We now describe our method of solution of the Faddeev equations. We adhere to the notation of Maffliet and Tjon.[9] If the three nucleons have momenta $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$, respectively, the ket notation $|\vec{p}\,\vec{q}\rangle_1$ indicates a state where particles 2 and 3 have relative momentum $\vec{p}$, i.e., $\vec{p} = (\vec{k}_2 - \vec{k}_3)/2$ (we assume equal masses). The spectator particle (here particle 1) has a momentum relative to the center of mass of 2 and 3 proportional to $\vec{q}$, i.e.,

$$\vec{q} = \frac{1}{2\sqrt{3}}(\vec{k}_2 + \vec{k}_3 - 2\vec{k}_1)\,.$$

Clearly, the state $|\vec{p}\,\vec{q}\rangle_1$ can be expressed in terms of states where 2 or 3 is the spectator, e.g., in the three-body center-of-mass frame

$$|\vec{p}\,\vec{q}\rangle_1 = |-\vec{p}/2 + \sqrt{3}\,\vec{q}/2\,,\ -\sqrt{3}\,\vec{p}/2 - \vec{q}/2\rangle_2 \text{ etc.}$$

In partial-wave notation the ket $|pq\alpha\beta\rangle_1$ indicates a state where

$$p = |\vec{k}_2 - \vec{k}_3|/2\,, \quad q = \frac{1}{2\sqrt{3}}\,|\vec{k}_2 + \vec{k}_3 - 2\vec{k}_1|$$

and the quantum numbers α are l (relative orbital angular momentum of 2 and 3), L (spectator orbital angular momentum), and J, M (total orbital angular momentum and projection). We indicate the spin-isospin quantum numbers by the label β. Since we deal with spin-independent potentials, we will usually omit the spin-isospin label. The state $|pq\alpha\beta\rangle_1$ is antisymmetrized with respect to interchange of particles 2 and 3.

The three-body T matrix satisfies a three-body Lippmann-Schwinger equation

$$T(\omega) = V + V\frac{1}{\omega - H_0}\,T(\omega)\,, \tag{5}$$

where V includes the interaction of all the particles and H_0 is the total kinetic energy. Standard numerical techniques of solving integral equations do not apply to Eq. (5) because the kernel of Eq. (5) is noncompact (e.g., δ functions appear in the kernel). Faddeev[2] overcame this problem by recasting Eq. (5) into the form

$$T(\omega) = T^{(1)}(\omega) + T^{(2)}(\omega) + T^{(3)}(\omega)\,,$$

where

$$T^{(i)}(\omega) = T_i(\omega) + T_i(\omega)\frac{1}{\omega - H_0}\sum_{j\neq i} T^{(j)}(\omega)\,. \tag{6}$$

In Eq. (6) the operator $T_i(\omega)$ is the three-body T matrix for the case where only particles $j, k \neq i$ interact. In the three-body center-of-mass frame $T_1(\omega)$ is related to the two-body T matrix by ${}_1\langle\vec{p}\,\vec{q}|\,T_1(\omega)\,|\vec{p}\,'\vec{q}\,'\rangle_1 = \langle\vec{p}|\,t(\omega - q^2)\,|\vec{p}\,'\rangle\,\delta(\vec{q} - \vec{q}\,')$.

For S-wave spin-independent forces, the three-body bound-state problem in the triton state [$J = 0$,

S (spin) $=\frac{1}{2}$, I (isospin) $=\frac{1}{2}$, we need not consider coupling of J and S] reduces to that of three identical bosons. After partial-wave decomposition[19] the Faddeev equations reduce to the two-dimensional integral equation

$$\Psi(p,q)=\varphi(p,q)+\frac{4}{\sqrt{3}\,q}\int_0^\infty q'\,dq'\int_{L(q,q')}^{U(q,q')}p'\,dp'\,\frac{t(p,p_1';\omega-q^2)\psi(p',q')}{\omega-q'^2-p'^2}\,, \tag{7}$$

where, for any spatially symmetric state Ψ

$$\psi(p,q)\equiv {}_1\langle pq\alpha|\,T^{(1)}(\omega)\,|\Psi\rangle={}_i\langle pq\alpha|\,T^{(i)}(\omega)\,|\Psi\rangle \quad \text{for } \alpha\text{: } l=L=J=0$$

$$[\,{}_1\langle pq\alpha|\,T^{(1)}(\omega)\,|\Psi\rangle=0 \quad \text{for } \alpha\text{: } l=L\neq 0, J=0]\,,$$

$$\varphi(p,q)\equiv {}_1\langle pq\alpha\text{: } L=l=J=0|\,T_1(\omega)\,|\Psi\rangle\,,$$

$$L(q,q')=|\,2q-q'\,|\sqrt{3}\,;\ U(q,q')=(2q+q')/\sqrt{3}\,,$$

and

$$p_1'=(p'^2+q'^2-q^2)^{1/2}\,.$$

The S-wave two-body T matrix is denoted by $t(k,k';s)\equiv\langle k0\alpha|\,t(s)\,|k'0\alpha\rangle$ with the normalization $t(k,k;k^2+i\epsilon)=-2\pi^{-1}e^{i\delta_0(k)}\sin\delta_0(k)/k$ where $\delta_0(k)$ is the S-wave phase shift. Implicit in the suppression of spin-isospin quantum numbers in Eq. (7) is that the three nucleons are interacting in a spatially symmetric spin-isospin antisymmetric state.

To solve Eq. (7) we apply the method of Padé approximants.[10,20] Formally, one can write Eq. (7) as

$$\psi(\lambda)=\Phi+\lambda\kappa\psi(\lambda)\,, \tag{8}$$

where $\lambda=1$ for the physical problem. The Neumann series, generated by the iteration of Eq. (8) is simply a power series in λ, i.e.,

$$\psi(\lambda)(\mathrm{NS})=\Phi+\lambda\kappa\Phi+\lambda^2\kappa^2\Phi+\cdots=\sum_{i=0}^{\infty}a_i\lambda^i\,, \tag{9}$$

where, in reality, the a_i are functions of momentum. The Nth Padé approximant to ψ is given by

$$\psi_{[N,N]}(\lambda)=\sum_{i=0}^{N}A_i\lambda^i\Big/\Big(1+\sum_{i=1}^{N}B_i\lambda^i\Big)\,, \tag{10}$$

where one determines the coefficients A_i, B_i by matching the first $2N+1$ coefficients of the power series for $\psi_{[N,N]}$ with the corresponding a_i of Eq. (9). We note from Eqs. (9) and (10) that the Padé approximants to $\psi(\lambda)$ may have poles whereas the Newmann series cannot predict poles. The physical solution $\psi(\lambda=1)$ has a pole when the energy is equal to the triton binding energy, i.e., $\omega=-ME_T/\hbar^2$. We therefore search for the value of ω such that the Nth Padé approximant to $\psi(1)$ indicates a pole. Of course, N must be large enough that an essentially converged result (with increasing N) is obtained. For all potentials of this paper, convergence occurs by $N=12$ even when the Newmann series is very divergent.

The solution of Eq. (8) by Padé approximants requires the evaluation of a large number of double integrals. We evaluated the integrals by employing a 16-point Gaussian quadrature formula in the q' variable of Eq. (7) with the interval $(0,1)$ mapped to $(0,\infty)$. In the p' variable we employed a variable Gaussian mesh of up to 20 points. The pole position was stable, to within 2%, with varying number of mesh points in the p' and q' variables for all potentials studied.

To extract the bound-state wave function ${}_1\langle pq\alpha|\,\Psi_B\rangle$ we exploit the Low equation

$$T(\omega)=V+V\frac{1}{\omega-H}\,V\,. \tag{11}$$

For $\omega\approx\omega_B$, where ω_B is the bound-state eigenvalue, Eq. (11) implies

$${}_1\langle pq\alpha|\,\Psi_B\rangle\sim {}_1\langle pq\alpha|\,T(\omega)\,|\Psi\rangle/(p^2+q^2-\omega_B)\,, \tag{12}$$

where $\omega_B=-M|\,E_T\,|/\hbar^2$. By employing Eq. (6), the definition of $\psi(p,q)$ in Eq. (7), and the fact that for identical particles[19] ${}_j\langle pq\alpha|\,T^{(i)}(\omega)\,|\Psi\rangle={}_i\langle pq\alpha|\,T^{(j)}(\omega)\,|\Psi\rangle$ we obtain, for S-wave interactions and $J=0$,

$${}_1\langle pq\alpha|\,\Psi_B\rangle\sim\frac{\delta_{J0}}{p^2+q^2-\omega_B}\Big[\psi(p,q)\delta_{l0}\delta_{L0}+\int_0^\infty p'^2dp'\int_0^\infty q'^2dq'\sum_{n=2}^{3}{}_n\langle pq\alpha|\,p'q'\alpha'\text{: } l'=L'=J'=0\rangle_1\psi(p',q')\Big]\,. \tag{13}$$

With the recoupling coefficients of Refs. 9 and 19, Eq. (13) becomes

$$
\begin{aligned}
{}_1\langle pq\alpha|\Psi_B\rangle \sim \frac{\delta_{J0}}{p^2+q^2-\omega_B}\Big[&\psi(p,q)\delta_{l0}\delta_{L0}+\delta_{Ll}(2l+1)^{1/2}\\
&\times\int_{-1}^{1}dx\,\psi([\,p^2/4+3q^2/4+\sqrt{3}\,pqx/2]^{1/2},\ [\,3p^2/4+q^2/4-\sqrt{3}\,pqx/2]^{1/2})P_l(x)\Big]\ \text{for even } l \qquad (14)\\
&=0\quad \text{for odd } l\,.
\end{aligned}
$$

The spin-isospin dependence, which is suppressed throughout our treatment, is completely antisymmetric, i.e., the spin-singlet–isospin-triplet and spin-triplet–isospin-singlet components (with respect to particles 2 and 3) are of equal magnitude but opposite sign.

The body form factor for the trinucleon is given by

$$F(q^2)=\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}\langle\Psi_B|\rho(\vec{x})|\Psi_B\rangle, \qquad (15)$$

where $\vec{x}$ is the displacement from the center of mass and $\rho(\vec{x})$ is the matter-density operator. Replacement of the matter-density operator by the charge-density operator yields the charge form factor of the triton (or ^{3}He). In terms of the bound-state wave functions in momentum space, the expression for the body form factor becomes, for a spatially symmetric $J=0$ wave function derived from an S-wave spin-independent force

$$F(q^2)=\frac{1}{2}\sum_{\substack{\alpha:\ l=L\,(\text{even})\\ J=0}}\int_0^\infty p'^2dp'\int_0^\infty q'^2dq'\int_{-1}^{1}dx\,{}_1\langle p'q'\alpha|\Psi_B\rangle\langle\Psi_B|\,p',q_0(x),\alpha\rangle_1 P_l\{[\,q'-qx/\sqrt{3}\,]/q_0(x)\}\,, \qquad (16)$$

where $q_0(x)=(q'^2+q^2/3-2qq'x/\sqrt{3})^{1/2}$. Eq. (16) is equivalent to Eq. (1) of Harper, Kim, and Tubis[21] for a $J=0$ spatially symmetric wave function.

Notice that in Eq. (14) for the wave function and in Eq. (16) for the body form factor that contributions from $l=L>0$ partial waves occur although the N-N interaction is only in relative S waves. In our calculations of the triton form factor described later in Sec. IV we include contributions up to $l=8$ in Eq. (16). Table I indicates the convergence of the square of the form factor for the potential of Eq. (3) (henceforth referred to as potential $U0$) as higher angular momentum states are included. The convergence of the form factors for the potentials studied in the next section is almost as good as that indicated in Table I for potential $U0$. The values of q^2 listed in Table I take into account the relativistic correction

$$q^2[\text{of Eq. (16)}]\to q^2\left(1+\frac{\hbar^2q^2}{36M^2c^2}\right)\ \text{(of Table I)}\,. \qquad (17)$$

Having thus described the method of calculation of the triton binding energy and form factor, we now present the results of our calculation.

IV. RESULTS

Table II indicates the transformation parameters, the triton binding energies (E_T), and the nuclear matter properties predicted by the two-term Yukawa potential of Eq. (3), $U0$, and by seven phase-shift-equivalent potentials ($U1$–$U7$). Table II also indicates the values of E_T predicted by the UPA for later discussion in Sec. V. The overlap integral $\langle g|\,\psi_d\rangle$, where $g(r)$ is the transformation function and ψ_d is the deuteron wave function predicted by $U0$, is a measure of how much the "transformed" deuteron wave function, $\tilde{\psi}_d$, differs from ψ_d. This can be seen from the relation

$$\langle\tilde{\psi}_d-\psi_d|\tilde{\psi}_d-\psi_d\rangle^{1/2}=2|\langle g|\,\psi_d\rangle|\,.$$

First consider those potentials in Table I for which the deuteron wave functions are virtually unchanged (i.e., $U0$, $U1$, and $U2$). Despite very large changes in nuclear matter results (26.6 MeV/A in the binding energy), these potentials yield a maximum variation of only about one MeV in E_T. The variations in E_T are in the same direction as those in nuclear matter but much smaller. In fact, the variations in E_T are comparable to the variations in nuclear-matter binding energy calculated at densities of $k_F\approx 0.6$ to 1.0 fm^{-1}, which are well below the empirical nuclear-matter density of $k_F=1.36$ fm^{-1}. The comparison of the triton and nuclear matter results for $U0$–2 may merely reflect the fact that the triton is a less dense system than nuclear matter. Hadjimichael and Jackson[13] report

TABLE I. The triton form factor $F^2(q^2)$ as a function of the maximum angular momentum state (l_{max}) included in the partial wave expansion (16).

q^2(fm^{-2})	l_{max} =	$F^2(q^2)$ 0	2	4	6	8
1.0		4.0289×10^{-1}	4.0235×10^{-1}	4.0254×10^{-1}	4.0258×10^{-1}	4.0248×10^{-1}
4.0		4.3713×10^{-2}	4.3662×10^{-2}	4.3758×10^{-2}	4.3766×10^{-2}	4.3762×10^{-2}
8.0		3.3056×10^{-3}	3.3697×10^{-3}	3.3829×10^{-3}	3.3808×10^{-3}	3.3794×10^{-3}
12.5		1.5186×10^{-4}	1.6918×10^{-4}	1.6892×10^{-4}	1.6813×10^{-4}	1.6791×10^{-4}
16.0		3.8828×10^{-6}	6.3523×10^{-6}	6.0395×10^{-6}	5.9089×10^{-6}	5.8854×10^{-6}
20.0		2.4794×10^{-6}	1.5372×10^{-6}	1.7599×10^{-6}	1.8114×10^{-6}	1.8161×10^{-6}

somewhat larger variations (over 3 MeV) in E_T for potentials that give similar deuteron properties; however, the overlaps $\langle g|\psi_d\rangle$ for the potentials of Ref. 13 become as large as −0.04.

If we lift the restriction of nearly identical deuteron wave functions, much larger variations in E_T (up to 6 MeV) occur. The direction of the variations in E_T are usually, but not always, in the same direction as the variations in nuclear-matter binding energy. Comparisons of the triton and nuclear-matter binding energies of $U0$ with $U3$ and $U2$ with $U5$ show variations in opposite directions. Interestingly, there seems to be a correlation between the overlap $\langle g|\psi_d\rangle$ and E_T, with potentials giving smaller positive or greater negative overlaps yielding more binding in the triton. Evidently, the triton binding energy is mainly sensitive to some property of the deuteron wave function.

Figure 1 illustrates the deuteron wave functions, in momentum space, of potentials $U0$, $U3$, $U5$, and $U7$. Up to $k\approx0.5$ fm^{-1} the wave functions are relatively close. This closeness probably reflects the short-rangedness of the transformations and the fact that all potentials have the same deuteron binding energy (0.416 MeV) and scattering length. We observe that potentials that yield deuteron wave functions that decrease more rapidly with momentum at moderate values of k give less binding in the triton. Figure 2 shows a similar pattern between the triton binding energy and the deuteron form factor[22] $|f(q^2)|$ as a function of the momentum transfer (q). Since we do not deal with potentials with tensor forces, the deuteron form factors have only monopole terms and may give diffraction minima. Potentials that give the first diffraction minimum at smaller q^2 yield less binding in the triton (Fig. 2). We include Fig. 2 because the form factor is more closely related to experiment (i.e., elastic electron-deuteron scattering experiments) than is the wave function.

The results presented in Table II and Figs. 1 and 2 imply that a consistent relation exists between the triton binding energy and deuteron wave function (or form factor). Just which part of the deuteron wave function in momentum space is important in determining the triton binding energy is not yet clear. Differences in the deuteron wave functions that distinguish potentials in the triton problem persist over large regions of momentum space. The later discussion of Sec. V attacks the relation between E_T and $\psi_d(k)$ more quantitatively. Next we consider the variations in the triton-charge–form-factor results for phase-shift-equivalent potentials.

Figure 3 illustrates the triton-charge form factor for potentials $U0$, $U2$, and $U7$. With a spatially symmetric $J=0$ wave function and the assumption of a zero electric form factor for the neutron, there is no distinction between the charge form

TABLE II. Triton and nuclear-matter properties of the transformed potentials. Binding energies are indicated by positive values.

Potential	$U0$	$U1$	$U2$	$U3$	$U4$	$U5$	$U6$	$U7$
α_0 (fm^{-1})	···	2.40	1.80	2.00	2.40	2.40	2.80	2.80
β_0	···	1.00	1.00	0.00	1.00	1.00	0.00	0.00
β_1 (fm^{-1})	···	−0.8392	−0.681	1.00	−0.82	−0.80	1.00	1.00
β_2 (fm^{-2})	···	0.00	0.00	−0.64	0.00	0.00	−0.68	−0.64
E/A nuclear matter (MeV)	33.0	15.5	6.4	~0.0	15.3	13.9	6.7	3.4
Saturation k_F (fm^{-1})	2.00	1.35	1.00	~0.0	1.33	1.20	0.95	0.75
$\langle g\|\psi_d\rangle$	···	0.000 28	0.000 38	−0.117	0.0098	0.0194	0.0666	0.0986
E_T (MeV)	7.65	7.03	6.59	9.68	6.65	6.25	4.57	3.66
E_T^{UPA} (MeV)	7.55	7.55	7.55	50	7.22	6.86	5.19	3.61

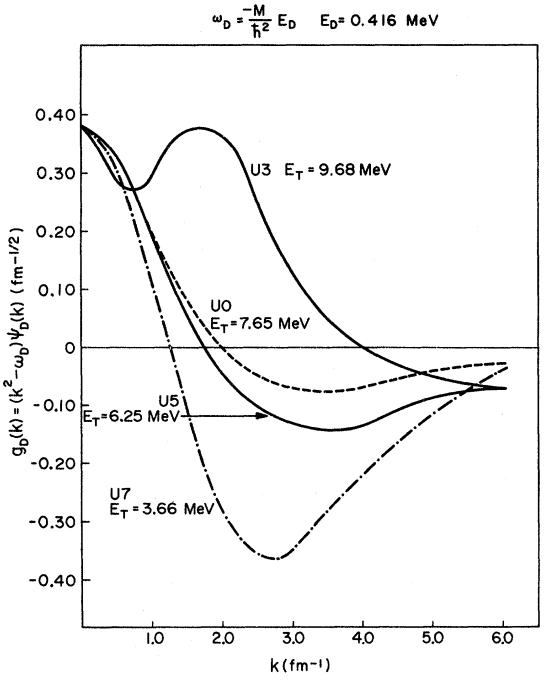

FIG. 1. The deuteron momentum-space wave functions and triton binding energies for potentials $U0$, $U3$, $U5$, and $U7$.

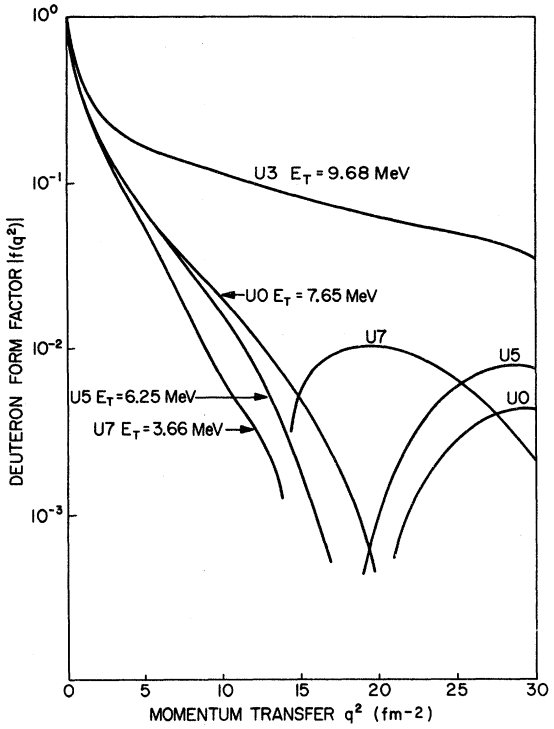

FIG. 2. The deuteron-body form factor $|f(q^2)|$ for the potentials of Fig. 1.

factors of the triton and ^{3}He. (We ignore Coulomb effects or possible charge-dependent nuclear forces.) For all potentials the integration meshes in the p' and q' variables in Eq. (16) consist of 16 Gaussian points mapped to the interval $(0, \infty)$. For potentials $U0$ and $U2$ we employ a 16-point Gaussian quadrature in the x variable of Eqs. (14) and (16). We employ a 32-point quadrature for $U7$. The dotted error bars represent estimates of the computational errors. These computational errors are mainly attributable to interpolation uncertainties. The numerical solution of Eq. (7) provides $\psi(p, q)$ as an array on a fixed mesh. The evaluation of the wave function [Eq. (14)] and form factor [Eq. (16)] requires values of $\psi(p, q)$ both on and off the fixed mesh. To determine the values of $\psi(p, q)$ off the fixed mesh, we employed a modified form of linear interpolation.[23] Once the body form factor of Eq. (16) is evaluated, multiplication by the proton electric form factor, which we take from the analytic expressions of Janssens *et al.*,[24] yields the triton-charge form factor. The values of q^2 in Fig. 3 in-

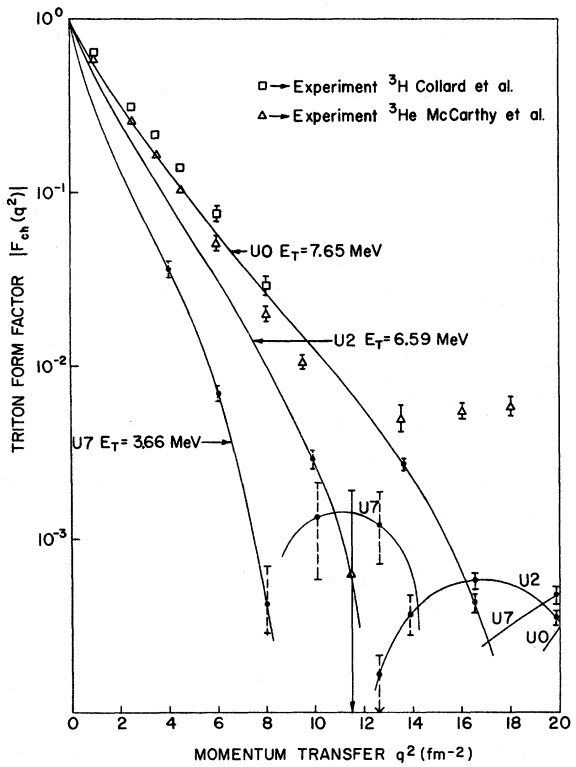

FIG. 3. The triton charge form factor $|F_{CH}(q^2)|$ and triton binding energy for potentials $U0$, $U2$, and $U7$. The experimental points are from Ref. 24. The dotted error bars represent estimates on the computational error. In the approximations of this paper there is no distinction between the form factor of the triton and that of ^{3}He.

clude the same relativistic connection as those of Table I [see Eq. (17)]. The neutron form factor is taken to be zero.

The form factors predicted by the phase-shift-equivalent potentials of Fig. 3 have qualitatively similar shapes up to the first diffraction minimum; however, potentials that yield less binding in the triton give form factors that are more "compressed" in q^2 and predict diffraction minima at lower momentum transfers. The observed relation between E_T and the position of the diffraction minimum is similar to the trend between E_T and the diffraction minimum of the deuteron form factor. At low momentum transfers ($q^2 \lesssim 6$ fm^{-2}) the binding energy alone seems to determine the form factor. Note, for example, the excellent agreement between the form factor of potential $U0$ and the experimental form factor of ^{3}He for $q^2 \lesssim 6$ fm^{-2}. The binding energy of $U0$ is, in fact, very close to the experimental binding energy of ^{3}He (7.65 MeV vs 7.8 MeV). At larger momentum transfers the qualitative features of the triton form factor resemble those of the deuteron form factor (e.g. compare the form factors of $U0$ and $U7$ in Figs. 2 and 3). The above results suggest that off-shell variations in the N-N interaction affect the deuteron-form-factor, triton-binding-energy, and triton-form-factor results in a correlated manner. We do not gain much off-shell information from calculating the triton form factor that we cannot obtain from the triton binding energy and the deuteron form factor.

Potential $U0$ gives a triton form factor similar to that obtained from the more realistic Reid soft-core potential.[15] Like with the Reid potential, the diffraction minimum occurs at too large a momentum transfer ($q^2 \approx 18$ fm^{-2}). The calculation presented here raises the question whether any two-nucleon potential that fits the two-body data can yield agreement with experiment for the triton (and ^{3}He). The cost of matching the experimental diffraction minimum[25] at $q^2 \approx 11.5$ fm^{-2} seems to be to lower the binding energy. Realistic potentials already predict too little binding in the triton[9] (e.g., 6.5 MeV for the Reid potential vs 8.5 MeV experiment). Furthermore, all of the potentials fail, by at least an order of magnitude, to predict the experimental charge form factor for $q^2 \approx 20$ fm^{-2}. We shall return to the significance of our form factor results with respect to experiments and other calculations[13, 15, 26] in Sec. VI.

Due to computational difficulties, we have presented the calculations of the triton form factor for only three phase-shift-equivalent potentials. These three cases provide, numerically, the most accurate results. Preliminary calculations with some of the other phase-shift-equivalent potentials, although less accurate numerically, indicate the same observed trends between E_T, $|F_{CH}(q^2)|$, and $|f(q^2)|$. In the next section we discuss the relation between the triton observables, the deuteron wave function, and the off-shell T matrix more quantitatively.

V. DISCUSSION OF RESULTS

From the results of Sec. IV, the dominant influence in the off-shell variations of triton-binding-energy results are changes in the deuteron wave function. To isolate the role of the deuteron wave function, we first study the dependence of the triton binding energy, as predicted by the UPA,[16] on the deuteron wave function. The discussion of this dependence, which appears later in this section, involves essentially the effects of scale transformations and scale distortions of the two-body potentials and wave functions. First, however, we review the relevant UPA equations.

Suppose a potential V predicts a deuteron wave function ψ_d and two-body binding energy E_d. The UPA employs the simple fact that one can write a one-term separable potential, $V^{\rm UPA}$, that predicts the same wave function and bound-state energy as V. The UPA potential becomes for S waves

$$V^{\rm UPA}(k,k') = -\lambda_d g_d(k) g_d(k') , \tag{18}$$

where

$$g_d(k) = (k^2 - \omega_d)\psi_d(k)$$

and

$$\omega_d = -|ME_d/\hbar^2| ,$$

$$\lambda_D = \left[\int q^2 dq \, g_d{}^2/(q^2 - \omega_d) \right]^{-1} .$$

The UPA T matrix is given by

$$t^{\rm UPA}(k,k';\omega) = -\lambda_d g_d(k) g_d(k')/D(\omega) , \tag{19}$$

where $D(\omega) = 1 - \lambda_d \int q^2 dq \, g_d{}^2(q)/(q^2 - \omega)$. Since the UPA T matrix preserves the bound-state pole and residue of the full T matrix, t, the approximation $t \approx t^{\rm UPA}$ should be good for $\omega \approx \omega_d$. For $|\omega - \omega_d| \gg 0$, the UPA may not be a good approximation. For example, the UPA to potential $U0$ gives phase shifts at intermediate energies (20 MeV $\lesssim E_{\rm LAB} \lesssim$ 150 MeV) that are only in fair agreement with those of the full potential (see Table III). The usual justification for applying the UPA to the triton problem is that one expects triton results to be sensitive to only the low-energy properties of the two-body T matrix. The chief advantage of the UPA is that the separability of $t^{\rm UPA}$ reduces the two-dimensional integral equation (7) to a one dimensional integral equation.

The triton binding energies predicted by the UPA appear on the bottom line of Table I. In all cases except $U3$ the UPA is a fairly good approximation in predicting E_T. We shall see later that potential $U3$ is an extreme case that corresponds to a transformation of range 4.5 fm. For the moment we concentrate on those potentials for which $E_T \approx E_T^{\mathrm{UPA}}$.

To relate changes in E_T (or E_T^{UPA}) and $F(q^2)$ to changes in the deuteron wave function, we first show that the off-shell effects described in Sec. IV are similar to the effects produced by scale transformations. Suppose a two-nucleon potential with matrix elements $V(k,k')$ yields a many-body (A-body) wave function $\psi(k_1, k_2 \ldots k_A)$ with eigenvalue E_A. By making a scale transformation $k_i \to \lambda k_i$ in the many-body Schrödinger equation in momentum space one can convince oneself that the "scaled" potential $V_\lambda(k,k') \equiv \lambda V(\lambda k, \lambda k')$ yields a wave function

$$\psi_\lambda(k_1, k_2 \ldots k_A) \sim \psi(\lambda k_1, \lambda k_2 \ldots \lambda k_A) \tag{20}$$

with eigenvalue

$$E_{A\lambda} \equiv E_A(V_\lambda) = E_A/\lambda^2 . \tag{21}$$

Some other useful relations obtained from the scaled potentials are

$$\begin{aligned} &E_{d\lambda} = E_d/\lambda^2 , \\ &\psi_{d\lambda}(k) = \lambda^{3/2}\psi_d(\lambda k) , \\ &f_\lambda(q^2) = f(\lambda^2 q^2) , \\ &t_\lambda(k,k';\omega) = \lambda t(\lambda k, \lambda k'; \lambda^2\omega) , \\ &F_{A\lambda}(q^2) = F_A(\lambda^2 q^2) , \end{aligned} \tag{22}$$

where

$$F_A(q^2) = \int d\tau\, e^{-i\vec{q}\cdot\vec{x}} \rho_A(\vec{x}) \quad (\rho_A = \text{matter density}) .$$

The first three equations of Eq. (22) refer to deuteron properties, the fourth to the two-body T matrix, and the last to any A-body (including three-body) form factor. By changing the momentum variables of Eq. (7) to the scaled momenta λp, λq, $\lambda p'$, $\lambda q'$, one can verify that $\psi_\lambda(p,q) \equiv \psi(\lambda p, \lambda q)$ is a solution with respect to the spatially symmetric state $\Psi_\lambda(\vec{p},\vec{q}) \equiv \lambda^2 \Psi(\lambda\vec{p}, \lambda\vec{q})$ with the T matrix $t_\lambda(k,k';\omega)$ at energy ω/λ^2 if $\psi(p,q)$ is a solution of Eq. (7) with respect to $\Psi(\vec{p},\vec{q})$ at energy ω. Therefore, if the T matrix $t(k,k';\omega)$ yields a pole in $\psi(p,q)$ at energy ω_B, the T matrix $t_\lambda(k,k';\omega)$ yields a pole at energy ω_B/λ^2, which verifies Eq. (21) for $A=3$. Equations (20)–(22) indicate that if two potentials (or two sets of two-body wave functions) are related by the scale transformation $k \to \lambda k$, the binding energies and form factors scale in a correlated manner. Specifically, scaled potentials (or scaled deuteron wave functions in the UPA) that give less binding in the three-body system yield three-body form factors that fall off more rapidly with q^2 with diffraction minima at lower q^2. This relation between the binding energy and form factor under a scale transformation is qualitatively similar to the off-shell variations of E_T and $|F(q^2)|$ observed in Sec. IV. To some degree, therefore, the triton predictions respond to off-shell changes in the potential *as if* the two-body potentials (or deuteron wave functions in the UPA) were related by scale transformations.

The following situation exists. In the deuteron problem the unitarily equivalent potentials V and $\tilde{V}$ yield the same deuteron binding energy and thus do not yield deuteron wave functions that are connected by a simple scale transformation. If they were so connected, the two-body binding energies would not be invariant under the scale transformation, except for the trivial case $\lambda = 1$ [see Eq. (21)]. In the three-body case, the binding energy and form factor results suggest that, under the potential $\tilde{V}$, the triton feels an "effective scaled potential" V_{λ_T}, where $V_{\lambda_T}(k,k') = \lambda_T V(\lambda_T k, \lambda_T k')$ for some value of λ_T. Equivalently, in the UPA, the triton feels an "effective deuteron wave function" $\psi_{\lambda_T d}(k) = \lambda_T^{3/2}\psi_d(\lambda_T k)$. Empirically, the distortion parameter λ_T can be determined by Eq. (21) with $E_A \to E_T$, $E_{A\lambda} \to \tilde{E}_T$ and $V_\lambda \to \tilde{V}$. The question arises: What properties of the deuteron wave function determine λ_T?

Suppose triton observables were only sensitive to the Fourier components of the deuteron wave function in a narrow range of momenta about $k \approx k_0$. In this case if a potential $\tilde{V}$ yields a deuteron wave function $\psi_d(k)$, such that

$$\tilde{\psi}_d(k) \approx \lambda^{3/2}(k_0)\psi_d[k\lambda(k_0)] \quad \text{for } k \approx k_0 \tag{23}$$

the triton would effectively "feel" a deuteron wave

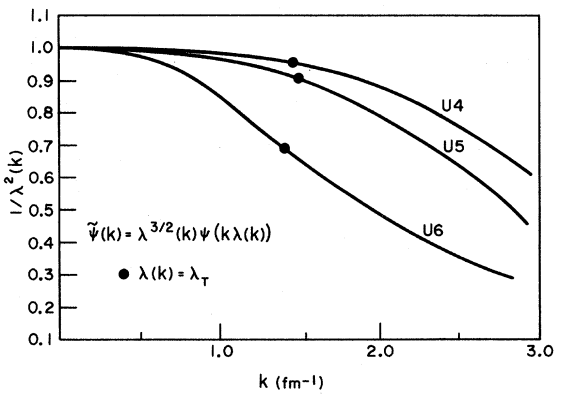

FIG. 4. The distortion function $1/\lambda^2(k)$, as determined from the deuteron wave functions of potentials $U4$, $U5$, and $U6$ [$\lambda(k) \equiv 1$ for $U0$]. The heavy dot indicates where $1/\lambda^2(k) = 1/\lambda_T^2 \equiv \tilde{E}_T^{\mathrm{UPA}}/E_T^{\mathrm{UPA}}$.

function $\psi_{\lambda_T d}(k) = \lambda_T^{3/2}\psi_d(\lambda_T k)$, where $\lambda_T = \lambda(k_0)$. Therefore the triton observables of the potentials V and $\tilde{V}$ would be related through Eqs. (21) and (22) with $V_\lambda \to \tilde{V}$, $E_A \to E_T$, $E_A \to \tilde{E}_T$, $F_\lambda \to \tilde{F}$, and $\lambda \to \lambda(k_0)$.

Figure 4 illustrates the values of $1/\lambda^2(k)$ as functions of the momentum k as obtained from the deuteron wave functions of potentials $U4$–6 through the relation $\tilde{\psi}(k) = \lambda^{3/2}(k)\psi[k\lambda(k)]$. The reference potential $V[\lambda(k) \equiv 1]$ is $U0$. The heavy dots in Fig. 4 indicate the points on the curves at which $1/\lambda^2(k) = 1/\lambda_T{}^2 \equiv \tilde{E}_T^{\rm UPA}/E_T^{\rm UPA}$ for each potential. We refer to the points so described as "centroids." We have the remarkable fact that the centroids fall near $k \approx 1.45$ fm^{-1} relatively independent of the potential.[27] The centroid falls close to $k = 1.45$ fm^{-1} even for potential $U6$, which is a fairly radical potential for the triton as indicated by its triton binding energy in Table II. In other words, for the transformed potentials considered, the triton binding energy results are varying *as if* the triton were only influenced by Fourier components in the deuteron wave function for $k \approx 1.45$ fm^{-1}.

An important question arises concerning the generality of the results of Fig. 4, i.e., $\lambda_T \approx \lambda$ ($k = 1.45$ fm^{-1}). Does this relation somehow depend on the types of deuteron wave functions we consider? We would expect the triton results to be sensitive to a spread of Fourier components in the deuteron wave function rather than just the behavior over a very narrow range of momenta. Remember that all the deuteron wave functions we treat (except that of $U3$) are appreciably different only inside a range (in r space) of 1.5 fm or less. In momentum space, all of the deuteron wave functions are approximately equal for $k \lesssim 0.5$ fm^{-1}. Also, all of the deuteron wave functions (except that of $U3$) have qualitatively similar features (see Fig. 1).

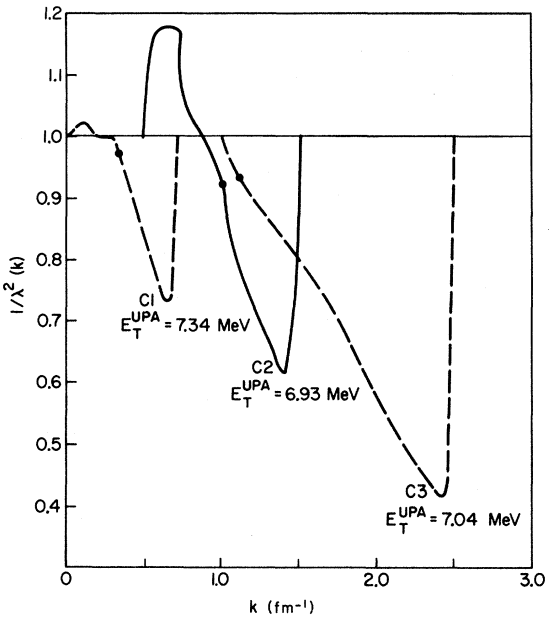

FIG. 5. The distortion functions $1/\lambda^2(k)$ and UPA triton binding energies for three hypothetical deuteron wave functions. For purposes of calculation, the deuteron binding energy was fixed at $E_d = 0.416$ MeV. The heavy dots have the same meaning as in Fig. 4. Again, $\lambda(k) \equiv 1$ for potential $U0$.

To test the sensitivity of the triton binding energy to different Fourier components in $\psi_d(k)$, we computed E_T in the UPA from deuteron wave functions that differ from that of $U0$ over finite regions in momentum space. The resulting binding energies and distortion functions $1/\lambda^2(k)$ appear as curves C 1–3 in Fig. 5. A fourth deuteron wave function (not shown in Fig. 5) was considered that altered the Fourier components for $k \geq 2$ fm^{-1}. The binding energy obtained with this wave function was virtually the same as that of $U0$, which indicates the insensitivity of E_T to momentum components for $k > 2$ fm^{-1}. We note that for curves C 1–3 the positions of the centroids occur at different values of k; however, in each case the empirically determined value of λ_T (i.e., $1/\lambda_T{}^2 = \tilde{E}_T^{\rm UPA}/E_T^{\rm UPA}$) is not inconsistent with λ_T being some kind of weighted average of the distortion function $\lambda(k)$. The wave functions and binding energies represented in Fig. 5 indicate that the triton is sensitive to a spread of momenta in the deuteron wave function rather than just to a narrow range of momenta. There is no guarantee, however, that the deuteron wave functions in Fig. 5 correspond to transformations of suitably short range in r space; curve C 1, in fact, corresponds to a very long-range transformation.

An interpretation of the three-body results of this paper might be as follows: The empirically determined value of λ_T (i.e., $1/\lambda_T{}^2 = \tilde{E}_T^{\rm UPA}/E_T^{\rm UPA}$) represents some "average" value of the distortion function $\lambda(k)$ over a distribution of momenta relevant to the triton problem. For the short-range transformations considered, the differences in the triton binding energy results are likely attributable to different distortions present in $\tilde{\psi}_d(k)$ over a region of momentum space centered about $k \approx 1.45$ fm^{-1}. This does not mean that E_T is more sensitive to changes in momentum components for $k \approx 1.45$ fm^{-1} than it is to lower momenta. This interpretation only means that the low Fourier components of the wave functions do not differ sufficiently under short-range transformations to account for the observed changes in E_T. The situation is analagous to the sensitivity of the wound integral κ of nuclear matter to off-shell T matrix elements. According to Haftel and Tabakin[6] the

peak of sensitivity of the wound integral to half-off-shell T matrix elements occurs when the T matrix element is 1 to 2 fm^{-1} off energy shell (Fig. 12 of Ref. 6). However, most of the *changes* in κ between potentials are mainly attributable to differences in T matrix elements that are 3 fm^{-1} or more off energy shell (Fig. 14 of Ref. 6). In the triton the peak of sensitivity may come at fairly low k (this seems to be indicated by Fig. 5); however, for potentials whose deuteron wave functions differ only at short distances, the changes in E_T seem to be coming from differences in distortions in $\tilde{\psi}_d(k)$ at moderate (1.0 fm$^{-1} \lesssim k < 2.0$ fm^{-1}) values of k.

Admittedly, the above discussion is intuitive. The results of Figs. 4 and 5 are at least consistent with the arguments presented. A rigorous treatment involves considering the effects of scale distortions rather than constant scale transformations, on the energies and wave functions of the three-body system. Even in the two-body case the effects of scale distortion are nontrivial. In the Appendix we show that under a slowly varying scale distortion of the potential the two-body eigenvalue is approximately determined by the expectation value of the distortion function between the two-body wave functions. This result makes plausible,

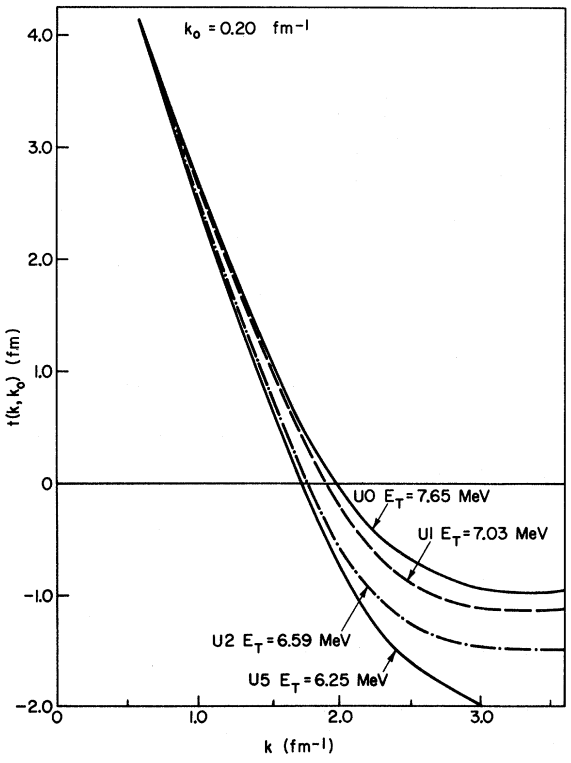

FIG. 6. The half-shell T matrix $t(k, k_0 = 0.2$ fm$^{-1})$ and triton binding energies for potentials $U0$, $U2$, and $U5$.

but does not prove, that λ_T is the average value of $\lambda(k)$ over the two-body relative momentum distribution of the three-body wave function. Clearly, the role of scale distortion in the two- and three-body problems requires further study. A question of utmost importance is whether the connection between the triton results and distortion in the deuteron wave function is a general result of a peculiarity of the types of off-shell changes studied here.

What about the cases where the deuteron wave function does not change? The results of Sec. IV indicate a residual (about 1-MeV) dependence on the off-shell properties of the T matrix not dominated by the deuteron pole. In Fig. 6 we plot half-shell T matrix elements[28] $t(k, k_0)$ as functions of k for $k = 0.2$ fm^{-1} for potentials $U0$, $U1$, $U2$, and $U5$. We include $U5$ for comparison even though its deuteron wave function differs somewhat from that of $U0$. Like in the case of the dependence of E_T on $\psi_d(k)$, potentials whose half-shell T matrix elements vary more steeply with k (for $k < 2$ fm^{-1}) generally yield less binding in the triton. The low-energy half-shell T matrix, as a function of k, seems to play a role analagous to the deuteron wave function. This role of the half-shell T matrix is not surprising in light of the similar ways the T matrix and deuteron wave function scale in Eq. (22). The main point here is that although changes do not appear in the deuteron wave function, variations in E_T occur if changes appear in the low-energy half-shell T matrix. Interestingly further calculations of half-shell T matrix elements for $k_0 > 0.4$ fm^{-1} (not presented here) indicate a breakdown between the observed relation between E_T and $t(k, k_0)$. This breakdown perhaps indicates that E_T is sensitive to only the low-energy properties of the off-shell T matrix.

So far we have ignored the unusual properties of potential $U3$, i.e., the large disagreement between E_T and E_T^{UPA} (Table II). Two properties of potential $U3$ account for the failure of the UPA in the triton problem. First, the phase shifts as predicted by the UPA for $U3$ differ radically from those predicted by the full potential, as shown in Table III. Even at low energies (e.g., $k_0 = 0.4$ fm^{-1}, $E_{\mathrm{LAB}} \approx 13$ MeV) the UPA phase shifts for potential $U3$ are much more attractive than the true phase shifts. Potential $U3$ is the only potential in this paper for which the UPA destroys approximate phase-shift equivalence at such low energies. The second unusual property of $U3$ is the extreme energy dependence of the half-shell T matrix at very low energies. Figure 7 depicts $t(k, k_0)$ obtained from potentials $U0$ and $U3$ for $k_0 = 0.0$ and 0.4 fm^{-1}. For the potential $U3$ the k dependence of $t(k, k_0)$ is strikingly different at $k_0 = 0.4$ fm^{-1} than at $k_0 = 0.0$

TABLE III. Phase-shifts in the unitary-pole approximation $E_{LAB}=2\hbar^2 k_0^2/M$.

k_0 (fm^{-1})	δ (rad)	$U0$	$U3$	$U5$
0.2	1.7654	1.7620	1.7800	1.7616
0.4	1.3159	1.2993	1.4235	1.2970
0.6	1.0271	0.9852	1.3967	0.9778
0.8	0.8025	0.7278	1.6019	0.7117
1.0	0.6170	0.5100	1.7652	0.4817
1.2	0.4593	0.3287	1.7440	0.2863
1.4	0.3224	0.1848	1.5953	0.1317
1.6	0.2019	0.0806	1.4037	0.0286
1.8	0.0945	0.0187	1.1889	0.0026
2.0	−0.0019	0.0001	0.9776	0.0912
2.4	−0.1684	0.0719	0.6007	0.5990
2.6	−0.2408	0.1349	0.4442	0.7955
2.8	−0.3070	0.1935	0.3122	0.8825

fm^{-1}. None of the other potentials studied (e.g. $U0$) exhibit this extreme energy dependence in the low-energy half-shell T matrix. The large disagreement between δ and δ_{UPA} (Table III) for $U3$ is no doubt a result of the rapid energy dependence of the T matrix. For potential $U3$ the deuteron pole simply does not dominate the two-nucleon T matrix at low energies. Evidently, the potential $U3$ produces strong two-body virtual states or antibound states near zero energy. The lack of pole dominance at low energies is indicative of a potential of longer range than what we would believe

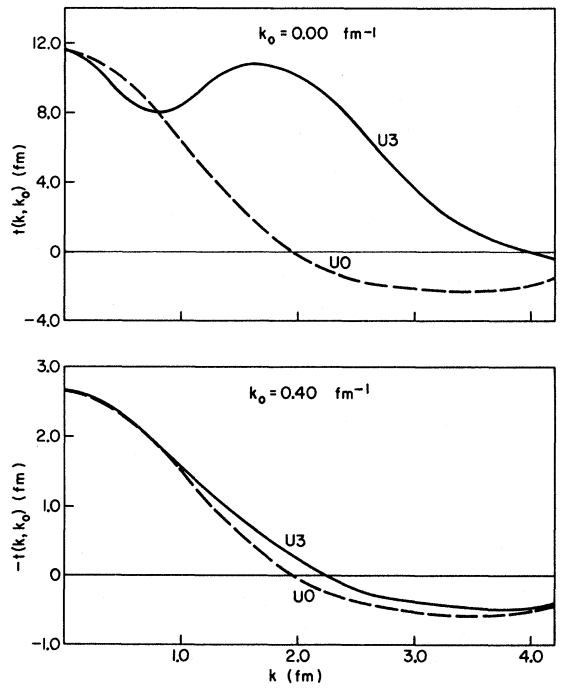

FIG. 7. The half-shell T matrix $t(k,k_0)$ for potentials $U0$ and $U3$ for $k_0=0.0$ and 0.4 fm^{-1}.

reasonable for the N-N interaction.[29] In fact, the range of the transformation that produces potential $U3$ is about 4.5 fm, which is well into the OPE region.

While the pathological properties of the potential $U3$ are interesting from a theoretical standpoint, they are of little practical consequence. Potentials like $U3$ may be rejected on both theoretical and experimental grounds. A more practical question emerges from the discussion of this section. Namely, how do the off-shell results observed here affect the current status of the three-nucleon problem with respect to experiment and with respect to current phenomenological models of the N-N interaction? We devote the next section to this question.

VI. CONSEQUENCES OF RESULTS

The results and discussion of Secs. IV and V indicate that the predictions of triton observables for phase-shift-equivalent potentials critically depend on the relative distortion of the deuteron wave function. The question arises – how may experimental uncertainties in the deuteron wave function affect predictions of the triton binding energy? The best measurements available of the deuteron wave function come from electron-deuteron elastic scattering experiments. These experiments essentially measure the electromagnetic form factors of the deuteron.[30]

From Eq. (22) it becomes apparent that one could employ the deuteron form factor, $f(q^2)$, as the basis of a scale distortion analysis instead of using the deuteron wave function. Plots like Fig. 4 that employ $\tilde{f}(q^2)$ instead of $\tilde{\psi}_d(k)$ suggest that the observed changes in E_T are attributable to changes in the deuteron form factors over a range of momentum transfers (q) centered about $q^2 \approx 12$ fm^{-2}. For $q^2 \approx 12$ fm^{-2}, the experimental error in $f(q^2)$ is about $\pm 5\%$.[30] This experimental error implies an uncertainty in the relevant distortion function $\lambda(q^2)$, as deduced from experimental plots $f(q^2)$ with error bars, of about 4.5%. Extrapolated to the Reid soft-core potential, which gives a good fit to the experimental deuteron-electric form factor, the experimental error in $f(q^2)$ implies about a ± 0.3-MeV uncertainty in E_T. This uncertainty is only a small fraction of the 2-MeV disagreement between E_T as predicted by the Reid potential (6.5 MeV) and experiment ($E_T = 8.5$ MeV). The estimate of ± 0.3 MeV presented here should not be taken too literally since we have ignored possible off-shell effects due to the tensor force or spin-dependent aspects of the N-N interaction. However, this estimate does indicate that one may have to look

beyond the N-N interaction in the deuteron (triplet-even) state to attain agreement with experiment.

In realistic N-N interactions the $^1S_0$ interaction is different from the $^3S_1$ interaction. The $^1S_0$ interaction, while almost as strong as the $^3S_1$ interaction, does not support a bound state. Instead, the virtual bound-state "singlet-deuteron" resonance dominates the $^1S_0$ T matrix at low energies. Presumably, triton results should be sensitive to the "singlet-deuteron" form factor as well as to the deuteron wave function. Unfortunately, direct experimental measurements of the singlet-deuteron form factor (or low-energy half-shell $^1S_0$ T matrix elements) are nonexistent. One may ask, however, what effects do "reasonable" variations of the low-energy off-shell T matrix in the $^1S_0$ state may have on the predictions of triton observables.

To examine the question of "reasonable" off-shell variations in the $^1S_0$ state, we employ the model of Picker, Redish, and Stephenson[31] for the half-shell T matrix. The model of Picker, Redish, and Stephenson[31] applies certain smoothness constraints to the two-nucleon scattering wave functions at distances of less than 1.4 fm. The solid curves in Fig. 8 indicate the maximum variation in the $^1S_0$ zero-energy half-shell T matrix in the model

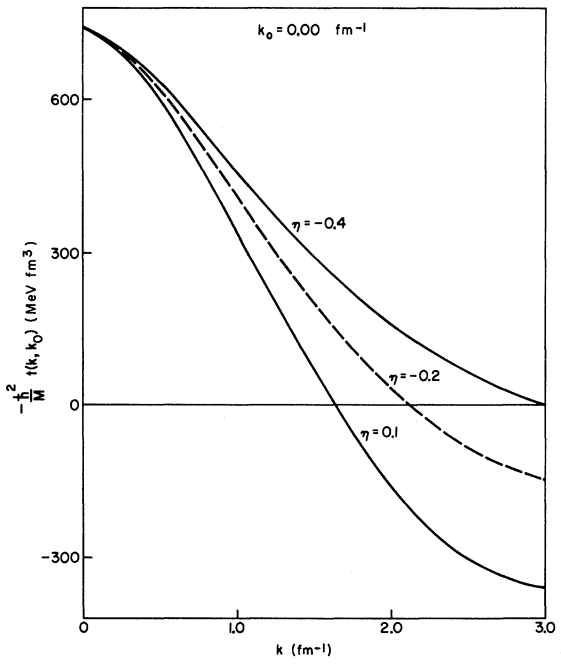

FIG. 8. The zero-energy $^1S_0$ half-shell T matrix in the model of Picker, Redish, and Stephenson. The parameter η is defined in Ref. 30. The values of $\eta = 0.1$ and $\eta = -0.4$ represent the maximum and minimum values of η permitted by the smoothness constraints of the model.

of Picker, Redish, and Stephenson. Aside from normalization, the zero-energy half-shell T matrix elements are the $^1S_0$ analog to the functions $g_d(k)$ of Fig. 1. Notice that in comparison to Fig. 1, the half-shell T matrix curves $\eta = -0.4$ and $\eta = 0.1$ of Fig. 8 show even a wider variation in their k dependence than curves $U0$ and $U7$ of Fig. 1. This wide variation in $t(k, 0)$ in the $^1S_0$ state occurs even though the wave functions at short distances are constrained to be relatively smooth. If the scale-distortion techniques of Sec. V are general and can be extrapolated to realistic interactions, the T matrix variations of Fig. 8 could account for about 4-MeV or more uncertainty in E_T. Further experimental constraints on the $^1S_0$ interaction are necessary.[32] The estimates of this section indicate that the triton binding energy itself could provide a very useful constraint on the singlet-deuteron form factor.

The situation with regards to the discrepancy between theory and experiment for the form factor of ^{3}He is more complicated. Results from this paper suggest that it may be impossible to find a potential that simultaneously predicts the correct N-N phase shifts, triton binding energy, and the ^{3}He form factor near and past the diffraction minimum. Of course further off-shell studies with more general off-shell variations and with more realistic potentials are necessary to verify the observed trend between E_T and $F(q^2)$. If the trend is a general one, it may indicate an important role for three-body forces[33] or charge-dependent nuclear forces[34] in the N-N interaction. One must remember, however, that the variational calculation of the form factor of ^{3}He of Yang and Jackson[26] with the Reid potential yields better agreement with experiment than do Faddeev calculations.[15] Also, Hadjimichael and Jackson,[13] in a variational calculation with phase-shift-equivalent potentials, find one potential (their potential 3) that yields less binding in the triton than the Reid potential yet has the diffraction minimum at slightly larger q^2. Obviously, the disagreement between Faddeev calculations and variational calculations of the form factor of ^{3}He should be resolved.

VII. SUMMARY

The principal factor dominating the off-shell variations in the triton-binding-energy results are changes in the pole behavior of the N-N T matrix. A maximum variation of 6 MeV occurs in triton-binding-energy predictions for the phase-shift-equivalent potentials studied here with all but one MeV being attributable to changes in the deuteron wave function. The discussion of Sec. V suggests that off-shell changes in the triton binding energy

is likely related to the relative distortion in the deuteron wave function. For potentials whose deuteron wave functions differ mainly for separations of less than 1.5 fm, the changes in triton-binding-energy results are probably attributable to changes in the Fourier components for 1 $\text{fm}^{-1} < k < 2$ fm^{-1}. Potentials that yield more "compressed" deuteron wave functions for $k < 2$ fm^{-1}, i.e., fall off more rapidly in k, give less binding in the triton. The scale-distortion techniques discussed in Sec. V provide a possible means of estimating in advance how potentials with different deuteron wave functions may differ in triton predictions.

One important result of this paper is the apparent incompatibility of the experimentally measured triton binding energy and the experimental position of the diffraction minimum of the charge form factor of ^{3}He. The phase-shift-equivalent potentials that predict less binding in the triton predict the diffraction minimum at lower values of the momentum transfer. This trend is opposite to what we would hope for to simultaneously improve the binding energy and form-factor predictions of the current potential models (e.g., the Reid potential) *vis a vis* experiment.

Two main conclusions follow from the discussion of Sec. VI of this work. The first is that the current experimental measurements of the deuteron electric form factor may be sufficiently accurate as to eliminate most of the off-shell uncertainty in the triplet-even interaction relevant to the triton problem. The lack of experimental or theoretical constraints on the $^1S_0$ interaction, however, could lead to considerable uncertainty in the predictions of triton (or ^{3}He) observables. The second conclusion is that the incompatibility between the binding energy and form-factor results mentioned in the preceding paragraph point the way for a role for three-body forces in the three-nucleon problem. Of course both of the conclusions mentioned here should be subjected to further tests with more realistic forces, i.e., with spin-dependent interactions with tensor forces. Also, the role of higher partial waves requires examination, possibly by perturbation theory.

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The author appreciates the helpful discussions with J. Lavine, G. J. Stephenson, A. D. Jackson, and E. F. Redish. He would like to thank E. F. Redish for providing some unpublished T matrix elements used in Sec. VI. The author is greatly indebted to M. Rosen for critically reading the manuscript and offering helpful suggestions.

APPENDIX. SCALE DISTORTION IN TWO-BODY PROBLEM

We consider the effect on the two-body binding energy of a "scaled" potential $V_\lambda = \Lambda S(\lambda) V S^{-1}(\lambda)$, where $\lambda = \lambda(k)$, $S(\lambda)\psi(k) = \lambda'^{5/2}(k)\psi(\lambda(k))$, and $\Lambda\psi(k) = 1/(k^2)\lambda^2(k)\psi(k)/\lambda'^4(k)$. The choices of S and Λ are such as to make V_λ Hermitian and reduce to V_λ of Sec. V for $\lambda(k) = \lambda k$. Specifically, the matrix elements of V_λ are

$$V_\lambda(k,k') = \frac{1}{k^2 k'^2}\lambda^2(k)\lambda^2(k') V(\lambda(k),\lambda(k')/[\lambda'^{3/2}(k)\lambda'^{3/2}(k')].$$

We shall derive an expression for the eigenvalue E_λ that (1) holds pertubatively, i.e., for $\lambda(k) = k + \delta\lambda(k)$, where $\delta\lambda(k)$ is infinitesmal, and (2) reduces to Eqs. (21) and (22) for $\lambda(k) = \lambda k$ [here $\lambda'(k) = \lambda$].

We first write down the Schröedinger equation

$$E_\lambda \psi_\lambda = (H_0 + \Lambda S V S^{-1})\psi_\lambda \tag{A.1}$$

and consider the effect of letting $\lambda(k)$ go to $\lambda(k) + \delta\lambda(k)$ [we take the limit of $\lambda(k) = k$ later]. Using Eq. (A.1) and differentiating both sides of Eq. (A.1) we obtain

$$\delta E = \langle \psi_\lambda | \, \delta\Lambda\Lambda^{-1}(E - H_0) + \Lambda\delta S S^{-1}\Lambda^{-1}(E - H_0) - (E - H_0)\delta S S^{-1} | \, \psi_\lambda \rangle , \tag{A.2}$$

where $\delta E = E_{\lambda+\delta\lambda} - E_\lambda$, with similar expressions for $\delta\Lambda$ and δS. If we assume λ has an inverse function λ_- (i.e., $k = \lambda_-[\lambda(k)]$), we can derive,

$$\delta S S^{-1}\psi_\lambda(k) = [\lambda'(k) + \delta\lambda'(k)]^{5/2}\{\lambda_-[\lambda(k) + \delta\lambda(k)]/\lambda'^{5/2}(k)\}\psi_\lambda(\lambda_-[\lambda(k) + \delta\lambda(k)]) - \psi_\lambda(k) . \tag{A.3}$$

With the fact that $d\lambda_-(\lambda)/d\lambda = dk/d\lambda = 1/\lambda'(k)$, Eq. (A.3) becomes to first order in $\delta\lambda, \delta\lambda'$

$$\delta S S^{-1}\psi_\lambda(k) = \tfrac{5}{2}[\delta\lambda'(k)/\lambda'(k) - \lambda''(k)\delta\lambda(k)/\lambda'^2(k)]\psi_\lambda(k) + \delta\lambda(k)\psi_\lambda'(k)/\lambda'(k) . \tag{A.4}$$

With Eq. (A.4) and the definitions of $\delta\Lambda, \Lambda$, we can evaluate the three terms of Eq. (A.2). After some lengthy algebra we can combine the three terms and obtain, to first order in $\delta\lambda, \delta\lambda'$ and for $\lambda(k) = k$

$$\delta E = \langle \psi(k) | \, -4\delta\lambda'(k)(E - k^2) + \frac{2}{k}\delta\lambda(k)(E - 2k^2) | \, \psi(k) \rangle . \tag{A.5}$$

Now note that the expression

$$E_{\lambda(k)} = E/[\langle\psi(k)|\, 2\lambda'(k) - \lambda(k)/k\,|\,\psi(k)\rangle]^2 - \langle\psi(k)|\, 4k[\,\lambda(k) - k\lambda'(k)\,]\psi(k)\rangle\,, \tag{A.6}$$

where ψ, E are the wave function and eigenvalue for $\lambda(k) = k$, holds [to first order in $\delta\lambda(k)$, $\delta\lambda'(k)$] for a perturbative scale distortion $[\lambda(k) = k + \delta\lambda(k)]$ and for a constant scale transformation $[\lambda(k) = \lambda k]$. With a slowly varying scale distortion, i.e., $k\lambda'(k) \approx \lambda(k)$, Eq. (A.6) becomes

$$\begin{aligned} E_{\lambda(k)} &\approx E/[\langle\psi(k)|\,\lambda'(k)\,|\,\psi(k)\rangle]^2\,, \\ &\approx E/[\langle\psi(k)|\,\lambda(k)/k\,|\,\psi(k)\rangle]^2\,, \end{aligned} \tag{A.7}$$

i.e., the energy scales as the expectation value of the distortion function. [The distortion function $\lambda(k)$ here corresponds to $k\lambda(k)$ of Sec. V.]

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$$P_l\left[\frac{q' - qx/\sqrt{3}}{q_0(x)}\right] = \sum_{\lambda=0}^{l} (-1)^\lambda \left(\frac{q}{\sqrt{3}}\right)^\lambda \frac{q'^{l-\lambda} l!\, P_\lambda(x)}{(l-\lambda)!\lambda!\, q_0^{\,l}(x)}\,.$$

One can easily derive this relation by induction.

[22]The form factor $f(q^2)$ is the body form factor of the deuteron. To obtain the electric form factor, $f_{\mathrm{EL}}(q^2)$, one must multiply $f(q^2)$ by the sum of the neutron and proton form factors $[G_{E_n}(q^2) + G_{E_p}(q^2)]$. We shall denote deuteron form factors by lower case f and trinucleon form factors by capital F.

[23]Instead of interpolating on $\Psi(p, q)$ with respect to p, q, we interpolate on $G(p)H(q)\Psi(p, q)$ with respect to $y(p^2)$, $y'(q^2)$, where $y(p^2) = 2/\pi \tan^{-1}(p^2/\alpha^2)$; $y'(q^2) = 2/\pi \times \tan^{-1}(q^2/\beta^2)$; $G(p) = 1 + p^2/a^2$; $H(q) = (1 + q^2/b^2)^n$. The parameters α, β, a, b, n are adjusted (for each potential) as to give a good interpolation "fit." Linear interpolation with this technique is about four times more accurate than standard linear interpolation and more than twice as accurate as LaGrangian quadratic interpolation.

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[28]The definition of the half-shell T matrix element $t(k,k_0)$ is $t(k,k_0)\equiv e^{-i\delta_0(k_0)}t(k,k_0;k_0^2+i\epsilon)$, where δ_0 is the S-wave phase shift.

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New Integrodifferential Equation and Integral Constraint for the Correlation Function Between Two Nucleons in Nuclear Matter

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A nonlinear integrodifferential equation for the correlation function between two nucleons in nuclear matter is obtained. This equation results by writing the (trial) expression of the energy per particle in nuclear matter in a suitable form and applying subsequently the variational principle. The investigation of the behavior of the equation for large internucleon distances r_{12} leads to a new integral constraint on the correlation function.

1. INTRODUCTION

The determination of the nucleon-nucleon correlation function, which is used in the variational or "Jastrow" approach to nuclear matter[1-3] has been a thorny problem for a long time.

In this approach the trial many-body wave function

$$\Psi_N = S_N \prod_{i<j}^{N} f(r_{ij}) \tag{1}$$

is employed for the calculation of the (trial) expression of the energy per particle E/N.

In expression (1), S_N is a Slater determinant in which the orbital parts of the single-particle wave functions are plane waves and f the nucleon-nucleon correlation function. This should be chosen to be zero inside and at the hard-core radius c of the nucleon-nucleon potential and to approach unity sufficiently rapidly for large internucleon distances.

The E/N is written in the form of a cluster expansion

$$\frac{E}{N} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \cdots \tag{2}$$

in which only the first few terms are retained. The expansion is often truncated at $\mathcal{E}_2$ and the energy per particle depends upon the correlation function only through the second term, which is proportional to the constant density ρ of the (infinite) nuclear matter. The first term is just the Fermi energy:

$$\mathcal{E}_1 = \frac{3}{5}\frac{\hbar^2 K_F^2}{2m}. \tag{3}$$

The problem of determining f has been faced in two ways:

In the first, a suitable analytic form is assumed for the f in which there are certain parameters and the E/N is minimized with respect to them. It was realized, however, that the correlation function has to be restricted in order to avoid the so called "Emery difficulty,"[3,4] which is due to the absence of normalization in the above-mentioned truncated cluster expansion of E/N. Various restrictions have been used. The early types of them have been suggested in Ref. 3. The conditions which were employed recently are of integral form, and one of the parameters in the expression of the f is fixed by the requirement that the considered condition is satisfied. It should be noted, however, that it is not clear which of the various conditions is the appropriate one to be used, and consequently there is a degree of arbi-