PHYSICAL REVIEW C VOLUME 7, NUMBER 2 FEBRUARY 1973

Charge Symmetry and Nonlocality of the Two-Nucleon Interaction

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(Received 18 August 1972)

An analytic model-independent description of the nonlocal-Coulomb-correction effect in the two-nucleon system is given. A first-order estimate of the effect is found to be consistent with recent measurements of the two-neutron scattering length which suggest that a_{nn} is slightly less negative than a_{pp}. A zero-energy nonlocal correlation length of $0.6\hbar/m_\pi c$ is deduced from charge-symmetric models and the mean (−16.4 fm) of the measured values of a_{nn}. A larger value of the correlation length is obtained when symmetry-violating terms necessary to reconcile the ^{3}He-^{3}H binding-energy-difference discrepancy are included in the interaction.

INTRODUCTION

It is generally accepted on the basis of theoretical arguments[1,2] that the interaction between two nucleons is nonlocal within a distance of $\hbar/2m_\pi c$, but experimental effects that are a direct result of nonlocality have proved difficult to isolate. The best evidence[2] for an effect due to nonlocality comes from the observation[3,4] that it is not possible to fit the $^1S_0$, $^1D_2$, and $^1G_4$ p-p phase shifts with an angular-momentum-independent local potential, but this observation does not require a nonlocal potential. While a nonlocal potential will always lead to an angular momentum dependence in the form of different interactions in each partial wave, there are no results which suggest that a test can be devised to distinguish it from an intrinsically angular-momentum-dependent local interaction. The complicated nature of the two-nucleon system, the limited number of angular momentum states accessible to study, and the abundance of state-dependent local potentials that have been constructed to give a good fit to the observables render the prospects of such a test doubtful.

A test to distinguish between nonlocal and local two-nucleon interactions should be based on experimental results for a single angular momentum state. At the present time, off-shell information has not been decomposed into partial waves. On-shell information is subject to restrictions imposed by the theory of the inverse scattering problem; namely, given the phase shift at all energies in a partial wave that has no bound states, it is possible to construct an energy-independent local potential that reproduces the phase shift. The result of any test with a single phase shift is negative. External probes of the deuteron wave function are not subject to the inverse-problem restriction and cannot be easily dismissed as a source of information on nonlocality, but if the present uncertainty in the D-state percentage is any indication, orbital momentum coupling in the $^3S_1$ state will be a formidable obstacle to progress in this direction for some time. The only other way to get around the inverse-problem restriction is to consider phase shifts for different isobars of the two-nucleon system in the same angular momentum state. In this case, it is necessary to invoke either charge symmetry or independence. Charge independence is known to be violated by electromagnetic mass splitting of the mesons responsible for the nuclear interaction. Estimates of the violation are subject to the uncertainties of the meson theory of nuclear forces and cannot be simply disentangled from the question of nonlocality. Charge symmetry, on the other hand, is not affected by meson mass splitting because of TCP invariance.

A test to distinguish between nonlocal and local two-nucleon interactions that is based on charge symmetry for a single angular momentum state will be reasonably free from other uncertainties in the interaction. Charge symmetry is presently tested, if not defined, in terms of local-potential models.[5] In order for a test of nonlocality to be successful, it must be shown that a charge-symmetric nonlocal interaction provides an effect that is equivalent to a violation of charge symmetry as defined by local-potential models. Such an effect occurs in the nuclear optical model where the proton optical potential is slightly more attractive than the neutron potential on account of the nonlocality (or velocity dependence) of the nucleon-nucleus interaction.[6,7] It is called the Coulomb correction, and its existence, which depends only on the nonlocality of the interaction, is confirmed by a number of optical-model studies.[8] There is no reason to believe the Coulomb correction should be absent from the two-nucleon system, and there, its small size would be magnified in the $^1S_0$ state

by the large scattering length. The most recent measurements[9,10] of $a_{nn} \approx -15 \pm 1$ fm are systematically smaller than the nominal value of -17 fm determined from charge-symmetric local-potential descriptions of the $^1S_0$ state.[11-13] The magnitude (≈ 1 fm) and sign of the difference are consistent with a simple estimate[14] of the Coulomb-correction effect in the two-nucleon system.

[*Note added:* More recent results for a_{nn}, reported after this paper was submitted for publication, are considered in an Addendum.]

The Coulomb-correction effect gives an apparent violation of charge symmetry relative to charge-symmetric local-potential models. It is a model-dependent effect and does not represent a true violation of charge symmetry for nuclear interactions. However, it cannot be distinguished from an effect due to a symmetry-breaking local interaction of the same magnitude and sign as the Coulomb-correction potential. The questions of charge symmetry and nonlocality of the two-nucleon interaction are not independent.

The purpose of this paper is to give an analytic description of the Coulomb-correction effect in the two-nucleon system. Exact results are derived in the next section. The following section is devoted to a first-order estimate of the effect; sources of error in the estimate are discussed at length. The last section is concerned with several other questions related to the charge symmetry and nonlocality of the two-nucleon interaction.

EXACT RESULTS

We restrict our attention to a nonrelativistic description of the $^1S_0$ state. We assume that the relative motion of two protons is described by the equation

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2}+E_p - V_p^C(r) - V_p^f(r)\right)\Psi_p^C(k_p, r) = \int_0^\infty V_p(r, s)\Psi_p^C(k_p, s)ds\,, \tag{1}$$

and two neutrons by

$$\left(\frac{\hbar^2}{m_n}\frac{d^2}{dr^2}+E_n - V_n^f(r)\right)\Psi_n^f(k_n, r) = \int_0^\infty V_n(r, s)\Psi_n^f(k_n, s)ds\,. \tag{2}$$

The c.m. energies E are related to the wave numbers k by $E = \hbar^2 k^2/m$. The potentials $V_p^C(r) + V_p^f(r)$ and $V_n^f(r)$ are the p-p and n-n direct electromagnetic interactions. They are present if there is no nuclear interaction. The p-p interaction is separated into $V_p^C(r)$, a long-range electric interaction between two-point protons,[15] and $V_p^f(r)$, the difference between the electromagnetic and point-charge electric interactions. The purpose of introducing $V_p^C(r)$ is to provide a model-independent frame of reference for extracting the nuclear-plus-nucleon electromagnetic-structure phase shifts from the p-p scattering data. The kernels $V_p(r, s)$ and $V_n(r, s)$ are the p-p and n-n nonlocal nuclear interactions for the $^1S_0$ state. We assume they are charge symmetric, i.e., $V_p(r, s) = V_n(r, s)$. The symbol U is used for all potentials to denote the product $mV/\hbar^2$.

An essential part of a test for charge symmetry is the hypothetical system of two uncharged protons. We define the $^1S_0$ state of this system according to the equation

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2}+E_p - V_p^f(r)\right)\Psi_p^f(k_p, r) = \int_0^\infty V_p(r, s)\Psi_p^f(k_p, s)ds\,. \tag{3}$$

The differences between Eqs. (2) and (3), namely the n-p mass difference and the short-range electromagnetic structure potentials, violate charge symmetry. They need to be considered explicitly in a test to distinguish between charge-symmetric nonlocal and local nuclear interactions. It is convenient to treat these effects separately by introducing the equations

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2}+E_p\right)\Psi_p(k_p, r) = \int_0^\infty V_p(r, s)\Psi_p(k_p, s)ds \tag{4}$$

and

$$\left(\frac{\hbar^2}{m_n}\frac{d^2}{dr^2}+E_n\right)\Psi_n(k_n, r) = \int_0^\infty V_n(r, s)\Psi_n(r, s)ds\,. \tag{5}$$

The phase shifts for Eqs. (1)–(5) can be related to each other through effective-range functions. Approximate relations between effective-range functions were derived some years ago,[16] and form the basis for many studies of charge symmetry.[5] However, with the availability of a number of realistic models of the $^1S_0$ state, it is more desirable to use exact relations and to define model-dependent parameters which give a measure of model differences. The effective-range functions used in this paper are described in the Appendix. The notation introduced for the solutions to Eqs. (1)–(5) is also used for the effective-range functions.

The relation between the effective-range functions for charged and uncharged protons depends on the potential used for the long-range electric interaction. We use the Coulomb potential as in-

dicated by the superscript C in Eq. (1). Vacuum polarization is considered separately and the results given later. The relation between the effective-range functions for Eqs. (1) and (3) is

$$X_p^C(k_p) - X_p^f(k_p) = -R^{-1}[\ln(\bar{r}/R) + 2\gamma - 1 + C(k_p)], \tag{6}$$

where $R = \hbar^2/m_p e^2$,[17] $\bar{r}$ is a characteristic nuclear length, γ is Euler's constant, and $C(k)$ is a model-dependent function defined by Eqs. (A25) and (A36). It is customary to take $\bar{r} = \rho_{pp}^C \approx 2.8$ fm. We take $\bar{r} = 2\lambda\!\!\!^{-}_0$, where $\lambda\!\!\!^{-}_0 = \hbar/m_{\pi 0}c$, in order to eliminate the experimental uncertainty of ρ_{pp}^C from the relation. The effective range functions for Eqs. (2)–(5) are related through Eq. (A16). We define the following quantities[18]:

$$X_p^f(k_p) - X_p(k_p) = \int_0^\infty \Psi_p^f(k_p, r) U_p^f(r) \Psi_p(k_p, r) dr \equiv R^{-1} F_p(k_p), \tag{7}$$

$$X_n^f(k_n) - X_n(k_n) = \int_0^\infty \Psi_n^f(k_n, r) U_n^f(r) \Psi_n(k_n, r) dr \equiv R^{-1} F_n(k_n), \tag{8}$$

$$X_p(k_p) - X_n(k_p) = \int_0^\infty \int_0^\infty \Psi_p(k_p, r)[U_p(r, s) - U_n(r, s)] \times \Psi_n(k_p, s) ds dr \equiv \lambda\!\!\!^{-1}_0 \delta m M(k_p), \tag{9}$$

where $\delta m \equiv (m_n - m_p)/m_p$. The reason for using k_p as the argument of $X_n(k)$ in Eq. (9) is given in the Appendix. With these definitions, the effective range functions for the $^1S_0$ states of the physical p-p and n-n systems are related by

$$X_p^C(k_p) - X_n^f(k_p) = -R^{-1}[\ln(2\tilde{\alpha}) + 2\gamma - 1 + C(k_p) - F_p(k_p) + F_n(k_p) - (\delta m/\tilde{\alpha}) M(k_p)], \tag{10}$$

where $\tilde{\alpha} = \alpha m_p/m_{\pi 0}$ and $\alpha = e^2/\hbar c$. Vacuum polarization can be taken into account by using Eq. (A42). The result is

$$X^e(k_p) - X^C(k_p) = -\lambda R^{-1} V(k_p). \tag{11}$$

The functions C, F_p, F_n, M, and V can be evaluated during the course of numerical calculations with a given potential by direct comparison of the effective range functions obtained from solving Eqs. (1)–(5), subject to the constraint that the phase shifts from Eq. (1) are fitted to the empirical $^1S_0$ phase shifts. The range of variation of these functions for a given class of potentials is indicative of the potential dependence of the class. It has already been established[11–13] that $C(0)$, $F_p(0)$, $F_n(0)$, $M(0)$, and $V(0)$ are relatively constant for a variety of local potentials that have been constructed by fitting the $^1S_0$ p-p phase shifts, but there have been few studies with nonlocal potentials. We have defined these functions for nonlocal potentials, and our next step is to relate them to their counterparts for local potentials.

Each of Eqs. (1)–(5) and its solutions can be transformed into an equivalent local equation and its solutions.[19,20] The equivalent local equations corresponding to Eqs. (1)–(5) are

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2} + E_p - V_p^C(r) - V_p^f(r)\right)\psi_p^C(k_p, r) = V_p^C(k_p, r)\psi_p^C(k_p, r), \tag{12}$$

$$\left(\frac{\hbar^2}{m_n}\frac{d^2}{dr^2} + E_n - V_n^f(r)\right)\psi_n^f(k_n, r) = V_n^f(k_n, r)\psi_n^f(k_n, r), \tag{13}$$

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2} + E_p - V_p^f(r)\right)\psi_p^f(k_p, r) = V_p^f(k_p, r)\psi_p^f(k_p, r), \tag{14}$$

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2} + E_p\right)\psi_p(k_p, r) = V_p(k_p, r)\psi_p(k_p, r), \tag{15}$$

$$\left(\frac{\hbar^2}{m_n}\frac{d^2}{dr^2} + E_n\right)\psi_n(k_n, r) = V_n(k_n, r)\psi_n(k_n, r). \tag{16}$$

The potentials in Eqs. (12)–(16) are called equivalent local potentials. A solution to one of the nonlocal equations with a specified boundary condition is related to the solution of the equivalent local equation satisfying the same boundary condition by[21]

$$\Psi(k, r) = A(k, r)\psi(k, r), \tag{17}$$

where $A(k, r)$ is called the damping function.

A sufficient condition for the construction of the equivalent local potential and damping function is the existence for all r of a pair of linearly independent solutions to the nonlocal equation.[20] We assume that the nonlocal $^1S_0$ two-nucleon potential is such that the solutions of the nonlocal equation satisfy this condition in the energy range of interest.[22] We choose the Jost solutions defined in the Appendix as the independent pair. Their Wronskian, normalized to one at $r = \infty$, is

$$J(k, r) = (2ik)^{-1}\left(f^-(k, r)\frac{d}{dr}f^+(k, r) - f^+(k, r)\frac{d}{dr}f^-(k, r)\right). \tag{18}$$

It can be shown that $J(k, 0) = 1$. The equivalent

local potential is given by[20]

$$V(k,r)=\frac{\hbar^2}{m}\left[-\frac{1}{2}\frac{d^2}{dr^2}\ln J(k,r)+\frac{1}{4}\left(\frac{d}{dr}\ln J(k,r)\right)^2\right]+\int_0^\infty V(r,s)\frac{f^-(k,s)\frac{d}{dr}f^+(k,r)-f^+(k,s)\frac{d}{dr}f^-(k,r)}{f^-(k,r)\frac{d}{dr}f^+(k,r)-f^+(k,r)\frac{d}{dr}f^-(k,r)}ds \tag{19}$$

and the damping function by

$$A(k,r)=J(k,r)^{1/2}. \tag{20}$$

Both depend parametrically on k^2 and are real for real k^2. The bound-state energies and scattering phase shifts obtained from the nonlocal equation and its local equivalent are identical.

The equivalent local potentials in Eqs. (12)–(16) cannot be equal because the solutions of the different nonlocal equations used to construct them are not equal. For example, Eqs. (12) and (14) show an apparent charge asymmetry. The equation obtained by "turning off" $V_p^C(r)$ in Eq. (12) without altering $V_p^C(k_p,r)$,

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2}+E_p-V_p^f(r)\right)\chi_p^f(k_p,r)=V_p^C(k_p,r)\chi_p^f(k_p,r), \tag{21}$$

corresponds to the equation for the $^1S_0$ state of the hypothetical system of two uncharged protons with the local interaction $V_p^C(k_p,r)$. The difference between $V_p^C(k_p,r)$ and $V_p^f(k_p,r)$ from Eq. (14),

$$V_p^{Cf}(k_p,r)=V_p^C(k_p,r)-V_p^f(k_p,r), \tag{22}$$

is called the equivalent local Coulomb-correction potential.[23] It is responsible for part of the apparent asymmetry between the physical p-p and n-n systems that is characteristic of nonlocal potentials. There are also equivalent local correction potentials for the short-range electromagnetic-structure potentials, the n-p mass difference, and the vacuum-polarization potential. By comparing the relations between various effective-range functions, the effect of the equivalent local correction potentials can be isolated, and the model-dependent terms in Eqs. (10) and (11) can be separated into terms characteristic of local-potential models and nonlocal corrections. The Coulomb correction is treated first.

The effective-range functions for Eqs. (1) and (12) are equal, since the phase shifts obtained from these equations are identical. The same is true of the effective-range functions for Eqs. (3) and (14), so the effective-range functions for Eqs. (12) and (14) are related by Eq. (6) with a model-dependent function $C^N(k_p)$ representative of nonlocal-potential models. The effective-range functions for Eqs. (12) and (21) are related by

$$X_p^C(k_p)-\bar{X}_p^f(k_p)=-R^{-1}[\ln(\bar{r}/R)+2\gamma-1+C^L(k_p)] \tag{23}$$

with a model-dependent function $C^L(k_p)$ representative of local-potential models. The effective-range functions for Eqs. (14) and (21) are related by

$$\bar{X}_p^f(k_p)-X_p^f(k_p)=\int_0^\infty \chi_p^f(k_p)U_p^{Cf}(k_p,r)\psi_p^f(k_p,r)dr. \tag{24}$$

The result for the Coulomb correction,

$$C^N(k_p)=C^L(k_p)-R\int_0^\infty \chi_p^f(k_p)U_p^{Cf}(k_p,r)\psi_p^f(k_p,r)dr, \tag{25}$$

is obtained by comparing Eq. (6) with Eqs. (23) and (24). The other corrections can be treated individually or in various combinations by following the same procedure.

One of the equivalent local potentials in Eqs. (12)–(16) needs to be chosen as a charge-symmetric local potential relative to which the net effect of the different equivalent local correction potentials is determined. The potential $V_p^C(k_p,r)$ in Eq. (12), or $V_p^e(k_p,r)$ if vacuum polarization is taken into account, is most likely to resemble the local potentials that have been constructed by fitting the $^1S_0$ p-p phase shifts.[24] These empirical potentials contain the Coulomb-correction potential if the two-nucleon interaction is nonlocal. We take $V_p^C(k_p,r)$ as the charge-symmetric local potential, and define equations corresponding to Eqs. (13)–(16) with $V_p^C(k,r)$ as the potential. We denote the solutions to these equations by χ_n^f, χ_p^f, χ_p, and χ_n. Equation (21) is one of these equations. We denote the sum of the model-dependent corrections for the nonlocal potential $V(r,s)$ given in Eq. (10) by $N(k_p)$, and the sum of the model-dependent correction for the local potential $V_p^C(k,r)$, defined according to Eqs. (7)–(9), by $L(k_p)$. The quantities $N(k_p)$ and $L(k_p)$ are given by

$$N(k_p)=-R^{-1}[C^N(k_p)-F_p^N(k_p)+F_n^N(k_p)-(\delta m/\tilde{\alpha})M^N(k_p)] \tag{26}$$

and

$$L(k_p) = -R^{-1}[C^L(k_p) - F_p^L(k_p) + F_n^L(k_p) - (\delta m/\tilde{\alpha})M^L(k_p)], \tag{27}$$

where, for example

$$M^L(k_p) = -\chi_0 \int_0^\infty \chi_p(k_p, r) U_p^C(k_p, r) \chi_n(k_p, r) dr \tag{28}$$

and

$$F_n^L(k_p) = R \int_0^\infty \chi_n^f(k_p, r) U_n^f(r) \chi_n(k_p, r) dr . \tag{29}$$

They are related by

$$\begin{aligned} N(k_p) - L(k_p) &= \int_0^\infty \chi_n^f(k_p, r)[U_n^C(k_p, r) - U_n^f(k_p, r)] \\ &\quad \times \psi_n^f(k_p, r) dr \\ &\equiv -R^{-1}\Gamma(k_p) , \end{aligned} \tag{30}$$

where $U_n^C(k, r) \equiv m_n V_p^C(k, r)/\hbar^2$. The quantity $\Gamma(k_p)$ represents the model-dependent correction due to the nonlocality of the two-nucleon interaction.

It is convenient to break up $\Gamma(k_p)$ into a sum of three terms. We define

$$\Gamma_p(k_p) \equiv C^N(k_p) - F_p^N(k_p) - C^L(k_p) + F_p^L(k_p) , \tag{31}$$

$$\Gamma_n(k_p) \equiv F_n^N(k_p) - F_n^L(k_p) , \tag{32}$$

$$\Gamma_m(k_p) \equiv -(\delta m/\tilde{\alpha})[M^N(k_p) - M^L(k_p)] , \tag{33}$$

and write

$$\Gamma(k_p) = \Gamma_p(k_p) + \Gamma_n(k_p) + \Gamma_m(k_p) . \tag{34}$$

The first term in Eq. (34) includes both the equivalent local Coulomb correction given by Eq. (22) and the p-p electromagnetic structure correction, $V_p^f(k_p, r) - V_p(k_p, r)$. The vacuum-polarization correction, $V_p^e(k_p, r) - V_p^C(k_p, r)$, can be included by starting from $V_p^e(k_p, r)$ and $E(k_p)$ instead of $V_p^C(k_p, r)$ and $C(k_p)$ in the development from Eqs. (26)–(34). The first term in Eq. (34) is the equivalent local correction for the entire p-p direct electromagnetic interaction; it can be written

$$\begin{aligned} \Gamma_p(k_p) &= -R \int_0^\infty \chi_p(k_p, r)[U_p^C(k_p, r) - U_p(k_p, r)] \\ &\quad \times \psi_p(k_p, r) dr . \end{aligned} \tag{35}$$

The second term in Eq. (34) represents the equivalent local correction for the n-n direct electromagnetic interaction. An expression for $\Gamma_n(k_p)$ is obtained by introducing the equation

$$\left(\frac{\hbar^2}{m_n}\frac{d^2}{dr^2} + E_n - V_n^f(r)\right)\phi_n^f(k_n, r) = V_n^f(k_n, r)\phi_n^f(k_n, r) \tag{36}$$

that is the result of turning on $V_n^f(r)$ in Eq. (16) without altering $V_n(k_n, r)$. The effective-range functions for Eqs. (16) and (36) are related by

$$\begin{aligned} \bar{X}_n^f(k_n) - X_n(k_n) &= \int_0^\infty \phi_n^f(k_n, r) U_n^f(r) \psi_n(k_n, r) dr \\ &\equiv R^{-1}\bar{F}_n^L(k_n) , \end{aligned} \tag{37}$$

and those for Eqs. (13) and (36) by

$$\begin{aligned} X_n(k_n) - \bar{X}_n^f(k_n) &= \int_0^\infty \psi_n^f(k_n, r)[U_n^f(k_n, r) - U_n(k_n, r)] \\ &\quad \times \phi_n^f(k_n, r) dr , \end{aligned} \tag{38}$$

in which case $\Gamma_n(k_p)$ is given by

$$\begin{aligned} \Gamma_n(k_p) &= \bar{F}_n^L(k_p) - F_n^L(k_p) \\ &\quad + R \int_0^\infty \psi_n^f(k_p, r)[U_n^f(k_p, r) - U_n(k_p, r)] \\ &\quad \times \phi_n^f(k_p, r) dr . \end{aligned} \tag{39}$$

The first two terms in Eq. (39) represent the difference between the symmetry-violating n-n direct-electromagnetic interaction corrections for the two local potentials $V_n(k_p, r)$ and $V_p^C(k_p, r)$. They occur because $V_p^C(k_p, r)$ was chosen as the charge-symmetric local potential relative to which $\Gamma(k_p)$ is defined. The last term in Eq. (39) is the equivalent local correction for the n-n direct electromagnetic interaction relative to the local potential $V_n(k_p, r)$.

The third term in Eq. (34) represents the equivalent local correction for the n-p mass difference. It can be written as

$$\begin{aligned} \Gamma_m(k_p) &= -(\delta m/\tilde{\alpha})[\bar{M}^L(k_p) - M^L(k_p)] \\ &\quad - R \int_0^\infty \psi_p(k_p, r)\left(U_p(k_p, r) - \frac{m_p}{m_n} U_n(k_p, r)\right) \\ &\quad \times \phi_p(k_p, r) dr , \end{aligned} \tag{40}$$

where

$$\bar{M}^L(k_p) = -\chi_0 \frac{m_p}{m_n} \int_0^\infty \phi_p(k_p, r) U_n(k_p, r) \psi_n(k_p, r) dr \tag{41}$$

and $\phi_p(k_p, r)$ satisfies

$$\left(\frac{\hbar^2}{m_p}\frac{d^2}{dr^2} + E_p\right)\phi_p(k_p, r) = V_n(k_p, r)\phi_p(k_p, r) . \tag{42}$$

The origin of the three terms in Eq. (40) is similar to the origin of the terms in Eq. (39). The quantities $\Gamma_p(k_p)$ and $\Gamma_n(k_p)$ have the characteristics of a symmetry-violating indirect electromagnetic effect in that they vanish when the nuclear interaction is turned off; $\Gamma_m(k_p)$ behaves like a symmetry-violating nuclear interaction.

The result of the preceding analysis is a relation between the physical p-p and n-n effective

range functions:

$$X_p^C(k_p) - X_n^f(k_p) = -R^{-1}[\ln(2\tilde{\alpha}) + 2\gamma - 1 + C^L(k_p) - F_p^L(k_p) + F_n^L(k_p) - (\delta m/\tilde{\alpha})M(k_p) + \Gamma_p(k_p) + \Gamma_n(k_p) + \Gamma_m(k_p)]. \quad (43)$$

The first three terms on the right are independent of two-nucleon-interaction models, while the remaining ones are model-dependent corrections with values characteristic of charge-symmetric nonlocal-potential models. Of the remaining terms, the first four have values characteristic of charge-symmetric local-potential models and the last three are corrections due to the nonlocality of the model interaction. The nonlocal corrections are identically zero for charge-symmetric local-potential models; they are also zero if the electromagnetic interactions and n-p mass difference are turned off. Equation (43) is exact. Within the framework of our assumptions, it is independent of a theory of nuclear forces.

FIRST-ORDER ESTIMATES

In this section we make first-order estimates of the nonlocal corrections in the zero-wave-number limit and compare these estimates with the measured p-p and n-n scattering lengths. Two intermediate scattering lengths are introduced to simplify the discussion. The first is a model-independent n-n scattering length a_{nn}^I defined by

$$\frac{1}{a_{nn}^I} - \frac{1}{a_{pp}^C} = -R^{-1}[\ln(2\tilde{\alpha}) + 2\gamma - 1] = 0.074\,034\,6\ \text{fm}^{-1}, \quad (44)$$

where a_{pp}^C is the p-p Coulomb scattering length.[25] We use the value $a_{pp}^C = -7.82 \pm 0.01$ fm which is approximately the mean of the values (-7.814 ± 0.004 fm, -7.823 ± 0.01 fm) determined in recent analyses.[11,12] We obtain $a_{nn}^I = -18.57 \pm 0.06$ fm from Eq. (44). The purpose of introducing a_{nn}^I, whose magnitude is dependent on the choice of $\bar{r}$ made in the last section, is to magnify the uncertainty in a_{pp}^C and to keep it separate from the model-dependent estimates. The second intermediate n-n scattering length a_{nn}^L is defined by

$$\frac{1}{a_{nn}^L} - \frac{1}{a_{nn}^I} = -R^{-1}[C^L - F_p^L + F_n^L - (\delta m/\tilde{\alpha})M^L] \quad (45)$$

and is related to the measured n-n scattering length by

$$\frac{1}{a_{nn}} - \frac{1}{a_{nn}^L} = -R^{-1}(\Gamma_p + \Gamma_n + \Gamma_m). \quad (46)$$

The arguments of the model-dependent terms are omitted with the understanding that they are evaluated at $k_p = 0$. The magnitude and sign of the nonlocal corrections in Eq. (46) can be determined from the experimental n-n scattering length when the value of a_{nn}^L is established from Eq. (45).

The nominal value of $a_{nn}^L \approx -17$ fm determined from local-potential models[11-13] is larger in magnitude than the recent measurements of a_{nn}. Kühn *et al.*[9] have deduced $a_{nn} = -15.0 \pm 1$ fm from a Watson-Migdal analysis of the $^3\text{H}(t, n\,^4\text{He})n$ reaction. Zeitnitz *et al.*[10] have deduced $a_{nn} = -15.2 \pm 0.9$ fm from a similar analysis of the $^2\text{H}(n, 2n)\text{H}$ reaction; they also subjected their data to an analysis using three-body calculations[26] and found $a_{nn} = -14.5 \pm 0.8$ fm. Earlier measurements of a_{nn} tend to give values slightly more negative than -16 fm, but the uncertainties are large enough to be consistent with the latest results.[9,10] We find the value $a_{nn} = -15.0 \pm 1.0$ fm to be representative of a trend in the various measured values of this quantity. There is a discrepancy between the local-potential models and experiment of about 1 fm. *More recent results for a_{nn}, which indicate a smaller discrepancy, are considered in the Addendum to this paper.*

In order to estimate the importance of the nonlocal corrections in Eq. (46), it is necessary to have a better understanding of the model-dependent terms used to calculate a_{nn}^L. The Michigan State group[11] has constructed several local potentials, including both hard- and soft-core potentials, that give quantitative fits to a_{pp}^C and the $^1S_0$ p-p phase shifts for energies up to 350 MeV. They find a surprising degree of model independence, as is indicated by the range of the value of $C^L = 0.120 \pm 0.006$ that we have extracted from their results. If the other model-dependent terms in Eq. (45) are neglected, the value $C^L = 0.126$ would give $a_{nn}^L = -17.17 \pm 0.05$ fm. Kermode and Sprung[13] have examined a number of $^1S_0$ state p-p potentials. The largest value of C^L obtained from their work is less than 0.120, but values as small as 0.083 were encountered for potentials with supersoft cores. We take $C^L = 0.126$ as the empirical upper limit of this parameter for local-potential models. The remaining model-dependent corrections in Eq. (45) are small and tend to cancel each other. The short-range electromagnetic potentials $V_p^f(r)$ and $V_n^f(r)$ in Ref. 11 are weakly repulsive with similar magnitudes for $r \gtrsim 0.8$ fm, but $V_p^r(r)$ changes sign and becomes attractive for smaller distances. The change in sign of $V_p^f(r)$ renders F_p^L small compared to F_n^L. The n-n electromagnetic potential $V_n^f(r)$ contributes about +0.3 fm to a_{nn}^L, while the n-p mass difference gives roughly -0.3 fm. The two terms F_n^L and $(\delta m/\tilde{\alpha})M^L$ cancel each other remarkably well. We note for

future reference that $1.0 \leq M^L \leq 1.4$ for the local potential models of Ref. 11.

The available results for the scattering lengths of charge-symmetric local-potential models of the $^1S_0$ state satisfy the inequality

$$\frac{1}{a^L_{nn}} - \frac{1}{a^C_{pp}} > 0.069 \text{ fm}^{-1}, \tag{47}$$

from which we obtain $a^L_{nn} < -16.98 \pm 0.05$ fm. This limiting value of a^L_{nn} is an empirical upper bound. It is necessary for a local-potential model to violate Eq. (47) by about 0.003 fm^{-1} to attain a value of -16 fm for a^L_{nn}.

The equivalent local potentials constructed from a nonlocal-potential model of the $^1S_0$ state depend on the wave number, and represent a larger class of local potentials than the empirical ones which satisfy the above inequality. It is not obvious that we can extrapolate Eq. (47) to include the class of equivalent local potentials. The importance of distinguishing between these classes of local potentials is illustrated quite simply by noting that the theory of the inverse-scattering problem mentioned in the Introduction applies only to the class of energy-independent local potentials. With reference to Eq. (47), it is legitimate to expect a result of this type from energy-independent local-potential models that have been constructed by fitting the $^1S_0$ p-p phase shifts from 0 to 350 MeV because the inversion process over this finite energy interval defines a reasonably unique potential at large distances. No corresponding result holds for energy-dependent local potentials, and the extrapolation of Eq. (47) to include this class is objectionable. However, the empirical potentials constructed thus far define a fairly unique local potential only for distances greater than about 1.2 fm. These potentials show marked differences at smaller distances which do not affect the inequality in Eq. (47). If we impose the reasonable restriction that a nonlocal-potential model of the $^1S_0$ state must have a local equivalent which, for distances greater than 1.2 fm, is practically independent of energy and has the same shape as the local-potential models, then it is unlikely that Eq. (47) will be violated. This restriction is consistent with the view[1,2] that nonlocal effects are small corrections at large distances, but become increasingly important as the two-nucleon separation gets smaller. We extrapolate the inequality established for the empirical local potentials with the above reservations in mind.

The various terms that make up the nonlocal correction in Eq. (46) can be divided into two groups. The first group consists of the terms $\bar{F}^L_n - F^L_n$ from Γ_n in Eq. (39) and $-(\delta m/\tilde{\alpha})(\bar{M}^L - M^L)$ from Γ_m in Eq. (40) which can be combined in the form

$$\Gamma_1 = [\bar{F}^L_n - (\delta m/\tilde{\alpha})\bar{M}^L] - [F^L_n - (\delta m/\alpha)M^L], \tag{48}$$

and estimated from the results for local-potential models. Each of the bracketed terms in Γ_1 is roughly zero due to the cancellation of the n-n electromagnetic and n-p mass-difference corrections that occurs for local-potential models,[27] and their difference probably gives further cancellation. The contribution to the nonlocal correction from Γ_1 is negligible.

The second group consists of the remaining terms in the nonlocal correction, each of which has the form of an integral over the difference between two equivalent local potentials. The local-energy approximation[28] is used to obtain a simple formula for the difference in the potentials that allows the integrals to be estimated from the known results for local-potential models. The local-energy approximation is summarized as follows.

Let $V(k,r)$ be the equivalent local potential constructed according to Eq. (19) from the solutions of the equation for the nonlocal potential $V(r,s)$. Let $V'(k,r)$ be the equivalent local potential constructed from the solutions of the equation for the nonlocal potential plus a local potential $V(r)$. In the local-energy approximation, $V'_{\text{LE}}(k,r)$ and $V_{\text{LE}}(k,r)$ are related by

$$V'_{\text{LE}}(k,r) = V_{\text{LE}}(k,r) - U(r)\frac{d}{dk^2}V_{\text{LE}}(k,r), \tag{49}$$

where LE denotes the local-energy approximation to the equivalent local potential. There are many nonlocal potentials for which $V_{\text{LE}}(k,r)$ is not a good approximation to $V(k,r)$. We have observed in several numerical calculations that the exact equivalent local potentials satisfy

$$V'(k,r) \approx V(k,r) - U(r)\frac{d}{dk^2}V(k,r) \tag{50}$$

rather well, even when the local-energy approximation to the potentials is not accurate. The error in Eq. (50) is typically less than 10% of $V'(k,r) - V(k,r)$, while in Eq. (49) $U(r)d/dk^2 V_{\text{LE}}(k,r)$ may differ from $V'(k,r) - V(k,r)$ by as much as a factor of 2. We make another approximation, also motivated by results established in the local-energy approximation, by writing

$$\frac{d}{dk^2}V(k,r) \approx \left(\frac{d}{dk^2}\ln A(\vec{k},\vec{k})\right)V(k,r), \tag{51}$$

where $A(\vec{k},\vec{k})$ is the Born-approximation scattering amplitude in the forward direction for a nonlocal potential $V(\vec{r},\vec{s})$. The potential $V(r,s)$ that we have been considering is the $l=0$ partial-wave component of $V(\vec{r},\vec{s})$. The importance of the loga-

rithmic derivative of $A(\vec{k},\vec{k})$ in connection with nonlocal interactions has been demonstrated by Monahan and Thaler.[29] They introduce the quantity $R_N(k)$, defined by

$$R_N(k)^2 = -6\frac{d}{dk^2}\ln A(\vec{k},\vec{k}), \tag{52}$$

and call it the nonlocal correlation length. For local potentials, $R_N(k)$ is identically zero. We have found the nonlocal correlation length to be a convenient measure of the nonlocality of a potential. The difference between the two equivalent local potentials is written

$$V'(k,r) - V(k,r) \approx \tfrac{1}{6}R_N(k)^2 U(r) V(k,r). \tag{53}$$

Equation (53) is slightly less accurate than Eq. (50), but more useful.

The nonlocal correction for the p-p direct electromagnetic interaction is given by

$$\Gamma_p \approx -\tfrac{1}{6}RR_N(0)^2 \int_0^\infty \chi_p(0,r)[U_p^C(r)+U_p^f(r)]U_p(0,r) \times \psi_p(0,r)dr \tag{54}$$

when Eq. (53) is used in Eq. (35). To evaluate this integral, we note that $U_p^C(r)+U_p^f(r)$ varies slowly with r for $r \gtrsim 0.3$ fm and is not singular at the origin in current models (see Fig. 5 of Ref. 11). We estimate the mean value of the integral with a uniform-charged-sphere average potential,

$$U_p^C(r)+U_p^f(r) \approx \frac{6}{5RR_c}, \tag{55}$$

where $R_c \approx 1.3$ fm is the radius of the uniform charged sphere. We take $R_c = \lambda\!\!\!^{-}_0$ for convenience. The remaining integral in Eq. (54) is

$$\int_0^\infty \chi_p(0,r)\, U_p(0,r)\psi_p(0,r)dr \simeq -\lambda\!\!\!^{-}_0{}^{-1}M^L + \Gamma_p/R, \tag{56}$$

where the first-order estimate of Eq. (28),

$$M^L \approx -\lambda\!\!\!^{-}_0 \int_0^\infty \chi_p(0,r)^2 U_p^C(0,r)dr, \tag{57}$$

has been used. The second term on the right in Eq. (55) is due to the difference between $U_p(0,r)$ in Eq. (54) and $U_p^C(0,r)$ in Eq. (57). By combining Eqs. (54)–(57) we obtain

$$\Gamma_p \approx \tfrac{1}{5}X_N{}^2 M^L/(1+\lambda\!\!\!^{-}_0 X_N{}^2/5R) \approx \tfrac{1}{5}X_N{}^2 M^L, \tag{58}$$

where $X_N = R_N(0)/\lambda\!\!\!^{-}_0$.

The same approximations can be applied to the third term of the nonlocal corrections for the n-n direct electromagnetic interaction in Eq. (39) and the n-p mass difference in Eq. (40). The third term in Γ_n is much smaller than Γ_p, since $V_n^f(r) \ll V_p^C(r)+V_p^f(r)$ for $r \gtrsim 0.3$ fm. It has the opposite sign, however, and would tend to cancel Γ_p if it were not so small. The third term in Eq. (39) can be written[30]

$$\Gamma_n \approx -\tfrac{1}{6}R\lambda\!\!\!^{-}_0 \langle U_n^f(r)\rangle X_N{}^2 M^L, \tag{59}$$

where $\langle U_n^f(r)\rangle$ is a suitable average potential energy. We estimate, with the help of Eq. (37), $\langle U_n^f(r)\rangle \approx \overline{F}_n{}^L/R\lambda\!\!\!^{-}_0$, in which case

$$\Gamma_n \approx -\tfrac{1}{6}X_N{}^2 \overline{F}_n{}^L M^L. \tag{60}$$

The difference between the potentials in the integrand of the third term in Eq. (40) is given by

$$\frac{m_p}{\hbar^2}[V_p(k_p,r) - V_n(k_p,r)] \approx -\frac{\delta m R_N(k_p)^2 m_p}{6m_n} U_p(k_p,r)U_n(k_p,r) \tag{61}$$

and results in[30]

$$\Gamma_m \approx -\frac{R\lambda\!\!\!^{-}_0 m_p}{6m_n}\delta m \langle U_n(0,r)\rangle X_N{}^2 M^L. \tag{62}$$

We estimate $\langle U_p(0,r)\rangle \approx -M^L/\lambda\!\!\!^{-}_0{}^2$ by the method used for $\langle U_n^f(r)\rangle$, and obtain

$$\Gamma_m \approx \tfrac{1}{6}X_N{}^2(\delta m/\bar{\alpha})\overline{M}^L M^L. \tag{63}$$

With these estimates, it follows that Γ_n and Γ_m cancel because $\overline{F}_n^L$ and $(\delta m/\bar{\alpha})\overline{M}^L$ cancel.

The preceding estimates based on the local-energy approximation are valid only if the nonlocality is in some sense small, but no precise criterion of smallness has been given. An indication of the validity of the approximation can be obtained by comparing the individual nonlocal corrections with their corresponding local-potential-model terms. The ratios

$$\frac{\Gamma_m}{-(\delta m/\bar{\alpha})\overline{M}^L} = \frac{\Gamma_n}{\overline{F}_n^L} \approx -\frac{X_n{}^2}{6}, \tag{64}$$

where $M^L \approx 1$, indicate that the estimates should be reliable when $X_n = 1$, but that they are likely to deteriorate rapidly for $X_N > 1$. In addition, it is unlikely that they are appropriate when the nonlocal correlation length is much larger than the range of the nuclear interaction[31] if for no other reason than that $X_N > 1$ is inconsistent with the view[1,2] that nonlocal corrections are small at large distances. For $X_N = \frac{1}{2}$, which corresponds to the often quoted distance $\hbar/2m_\pi c$ where nonlocal effects become important, $\frac{1}{6}X_N{}^2 \approx -0.04$, and the estimates are likely to be quite accurate.

The small size and cancellation of Γ_n and Γ_m render these terms unimportant for $X_N \lesssim 1$. The importance of the remaining term Γ_p can be seen by comparing it with the local-potential model

n-p mass-difference term. The ratio

$$\frac{\Gamma_p}{-(\delta m/\tilde{\alpha})M^L} \approx -\frac{\tilde{\alpha}}{5\delta m} X_N^2 = -7.36\, X_N^2 \qquad (65)$$

shows that the nonlocal correction for the p-p direct electromagnetic interaction is important. This nonlocal correction is almost twice the size of the local n-p mass-difference term for $X_N = \frac{1}{2}$, and would still be important in the event that our estimate of Γ_p is too large by a factor of 2. It is opposite in sign to the local n-p mass-difference term and will increase the n-n scattering length. We obtain

$$\frac{1}{a_{nn}} - \frac{1}{a_{nn}^L} \approx -\frac{X_N^2 M^L}{5R} = -0.007\, X_N^2 \text{ fm}^{-1} \qquad (66)$$

when Eq. (58) is substituted into Eq. (46) and M^L is taken to be one. The final result of our estimates,

$$\frac{1}{a_{nn}} - \frac{1}{a_{pp}^C} \approx 0.070\,(1 - X_N^2/10) \text{ fm}^{-1}, \qquad (67)$$

yields $a_{nn} = -16.5$ fm for $X_N = \frac{1}{2}$ and $a_{nn} = -15.2$ fm for $X_N = 1$. Equation (67) is valid only if the nuclear interaction is charge symmetric; the effect of violations of charge symmetry is considered later.

In arriving at the estimate in Eq. (66), we have attempted to minimize the size of the nonlocal correction by choosing a somewhat large value for R_c in Eq. (55) and taking the smallest value we could find for M^L. The magnitude of the correction for a given value of X_N is sensitive to these choices, as well as to the errors in the approximations we have made. We believe the uncertainty in the numerical factor on the right side of Eq. (66) to be less than 30%.

RELATED QUESTIONS

A. Discrepancy Between Local-Potential Models and Experiment

We have weighted heavily the most recent measurements[9,10] in determining an average value for a_{nn}. Had we adopted a more uniform weighting, we would have obtained a value of a_{nn} which would have reduced the discrepancy between local-potential models and experiment. Clearly, additional measurements of a_{nn} from kinematically complete experiments are desirable. The empirical bound we have established in Eq. (47) from available local-potential models may not be absolute. This point could be tested by attempting to construct a local-potential model which violates the bound. If one were so inclined, one might attribute most of the 1-fm difference, which we consider to be a nonlocal effect, to the uncertainties in the measured values of a_{nn} and the results for local-potential models. We think such a view is overly pessimistic if one considers the absence of other means of determining the nonlocality of the interaction. The following argument, based on the proven methods of isolating nonlocal effects in the nucleon-nucleus interaction, is the source of our belief that the effect of a nonlocal interaction is most likely to be observed in a comparison between the measured values of a_{nn} and the results from local-potential models.

The first evidence for the nonlocality of the nucleon-nucleus interaction came from the observation that the real part of the local optical potential determined from nucleon-nucleus scattering is energy-dependent. The difference in the strengths of the neutron and proton optical potentials was also noted in the early studies, but the explanation of a substantial part of this difference as a nonlocal Coulomb correction was hindered by the presence of an effect of similar magnitude due to the Pauli exclusion principle and the spin dependence of the two-nucleon interaction. By analogy, one might expect that the nonlocality of the two-nucleon interaction could be established by looking for an energy dependence in local model potentials. The theory of the inverse-scattering problem, which guarantees the existence of an energy-independent local potential, renders this analogy impotent. For the same reason, it is unlikely that the energy dependence of the nucleon-nucleus optical potential could be established by attempting to construct a potential from data for a single nucleus. The intended purpose of constructing nuclear optical potentials is to describe the average behavior of nucleon-nucleus scattering throughout the Periodic Table. The effect of this global construction procedure, as far as establishing the existence of nonlocal effects is concerned, is to circumvent any restrictions that might be imposed by the theory of the inverse-scattering problem and to permit a determination of the energy dependence.

Since the global construction procedure cannot be meaningfully extended to the $A = 2$ system, we do not have a method for determining the energy dependence of the two-nucleon interaction. This leaves the nonlocal Coulomb correction effect as the only possibility for observing a nonlocal effect in the two-nucleon system. The spin dependence of the two-nucleon interaction, which tends to obscure the nonlocal Coulomb correction in the nucleon-nucleus system, presents no problem in the analysis of a single angular momentum state in the two-nucleon system, and since the scattering lengths in the $^1S_0$ state are extremely sensitive to

small changes in the interaction, this state is the obvious place to look for the nonlocal-Coulomb-correction effect. The effect would be obscured in a comparison of the $^1S_0$ states of the n-p and p-p systems by violations of charge independence. A comparison of the $^1S_0$ states of the n-n and p-p systems is the remaining possibility. If one accepts the picture[1,2] that the two-nucleon interaction is nonlocal within a distance of about $\hbar/2m_\pi c$, then on the basis of the results of the previous section, one would expect a difference between the measured values of a_{nn} and those obtained from local potential models.

B. Binding-Energy Differences of Certain Mirror Nuclei

The experimental results[9,10] for a_{nn} and the nonlocal Coulomb correction can be thought of as being due to an n-n interaction that is slightly weaker than the p-p interaction. Theoretical calculations[32,33] of the binding-energy differences of the mirror nuclei ^{3}He-^{3}H and ^{41}Sc-^{41}Ca with charge-symmetric local nuclear interactions do not agree with the observed binding-energy differences, and can be brought into agreement only by making the n-n interaction slightly stronger than the p-p interaction. Thus, both the experimental results for a_{nn} and the first-order nonlocal Coulomb correction effect appear to be inconsistent with the indirect evidence from these mirror nuclei. The following argument resolves the inconsistency.

A comparison of a_{nn} and a_{pp}^C is equivalent to a comparison of the average strengths of the equivalent local n-n and p-p interactions at the same energy, while the binding-energy differences of mirror nuclei reflect a comparison of the n-n interaction at an energy E_1 with the p-p interaction at an energy $E_1+\Delta_C$, where Δ_C is the average Coulomb energy of the two protons. As is well known from the independent-particle model,[34] the first-order nonlocal Coulomb correction is canceled by the increased energy Δ_C at which the p-p interaction is evaluated. More precisely, the average potential energies of the p-p and n-n equivalent local potentials satisfy

$$\langle V_{pp}(E+\Delta_C)\rangle \approx \langle V_{nn}(E)\rangle \tag{68}$$

to first order. Thus, one would expect to see the apparent charge asymmetry due to nonlocality in a comparison of a_{nn} and a_{pp}^C, but not in the binding-energy differences of mirror nuclei which should exhibit charge symmetry to first order in the nonlocal Coulomb correction.

If the two-nucleon interaction is nonlocal, there are three possible sources for the disagreement between the theoretical and experimental binding-energy differences of the mirror nuclei ^{3}He-^{3}H and ^{41}Sc-^{41}Ca, instead of one source for local potentials. First there is the customary explanation that the two-nucleon interaction is not charge symmetric. Second, the discrepancy may be due to an incomplete cancellation of the first-order nonlocal Coulomb correction due to many-body effects not taken into account in the independent-particle model; and third, it may be due to a second-order nonlocal Coulomb correction. We are unable to eliminate the latter two possibilities, but we suspect that they are not the primary source of the discrepancy. Friar[35] has obtained an estimate of the ^{3}He-^{3}H binding-energy difference from an approach that does not involve the direct use of the two-nucleon interaction. This estimate is in very good agreement with the results obtained from local-potential models, and we do not see any way in which Friar's approach would be subject to modification by allowing the two-nucleon interaction to be nonlocal. The situation is less clear[36] for ^{41}Sc-^{41}Ca, but the ^{3}He-^{3}H case is sufficient for our purposes. While it may be premature to conclude that a violation of charge symmetry is responsible for the binding-energy-difference discrepancy, it appears to be the only viable candidate in connection with the question of the nonlocality of the two-nucleon interaction.

C. Symmetry-Violating Interactions

We have not considered a real violation of charge symmetry in the analysis of the last two sections, but we pointed out in the Introduction that a real violation of charge symmetry of the same magnitude and sign as the nonlocal Coulomb-correction potential cannot be distinguished from the apparent violation characteristic of nonlocal-potential models. If one were to accept the explanation of the discrepancy in binding-energy differences as being due to an n-n interaction that is slightly stronger than the p-p interaction, then it would be possible to distinguish between this real violation of charge symmetry and the apparent violation due to the nonlocality of the interaction. This explanation is most appealing from the point of view of determining the magnitude of the nonlocality of the two-nucleon interaction.

Symmetry-violating terms in the two-nucleon interaction are expected because of the mixing of isoscalar and isovector mesons,[5,37,38] but there is considerable uncertainty with regard to the effect of these terms on a_{nn}. Okamoto and Pask[37] have argued that the discrepancy in the ^{3}He-^{3}H binding-energy difference could be resolved without substantially altering the value of a_{nn} obtained from charge-symmetric models. On the other hand, Henley and Keliher[38] have estimated that the mixing of isoscalar and isovector mesons

necessary to resolve the discrepancy would result in a decrease of a_{nn}^L by roughly 0.8 fm.

The present analysis can be extended to include a symmetry-violating interaction. The result of such an extension is two additional terms in Eq. (43) which would modify Eqs. (45)–(46) as follows:

$$\frac{1}{a_{nn}^L} - \frac{1}{a_{nn}^I} = -R^{-1}\left[C^L - F_p^L + F_n^L - (\delta m/\bar{\alpha})M^L - S^L \right], \tag{69}$$

$$\frac{1}{a_{nn}} - \frac{1}{a_{nn}^L} = -R^{-1}(\Gamma_p + \Gamma_n + \Gamma_m + \Gamma_S). \tag{70}$$

In Eq. (69), S^L is the local-potential model-dependent coefficient due to a symmetry-violating term in the interaction. The value of S^L that corresponds to a decrease of a_{nn}^L by 0.8 fm is about 0.07. In Eq. (70), Γ_S is the correction to S^L due to the nonlocality of the interaction. The term Γ_S, which exists in principle, is taken to be zero, since it must be smaller than S^L and is probably much smaller than the present uncertainty in S^L. It then follows that a symmetry-violating term in the interaction does not alter our estimate of the nonlocal correction given by Eq. (66). The only change is the value of a_{nn}^L obtained from local-potential models. Thus, any decrease in a_{nn}^L due to a symmetry-violating interaction necessary to reconcile the ^{3}He-^{3}H binding-energy-difference discrepancy will tend to cancel the nonlocal correction described in this paper. If we take $a_{nn}^L = -17.9$ fm, as suggested by Henley and Keliher,[38] then Eq. (66) yields $a_{nn} = -17.4$ fm for $X_N = \frac{1}{2}$ and $a_{nn} = -15.9$ fm for $X_n = 1$. The charge-symmetric local-potential model value of -17 fm is obtained when $X_N \approx 0.6$. For reasonable values, the nonlocal and symmetry-violating corrections almost completely cancel each other.

D. One-Pion-Exchange Potential

We have not specifically included the one-pion-exchange potential (OPEP) in the analysis of this paper. This potential is the long-range part of most local-potential models, and its role in the discussion of the nonlocal-potential models was not made clear in the first section. The OPEP potential can be added to Eqs. (1)–(5) and Eqs. (12)–(16) without altering the results in an essential way.[39] Alternatively, it can be retained in the charge-symmetric nonlocal interaction as a component with zero nonlocal correlation length.

CONCLUSIONS

If the two-nucleon interaction is charge symmetric and nonlocal, then one should expect the measured value of a_{nn} to be smaller in magnitude than the value of a_{nn} obtained from charge-symmetric local-potential models. Recent measurements[9, 10] are systematically smaller than the available results[11–13] from the local-potential models, and are consistent with a zero-energy nonlocal correlation length of about $0.6\hbar/m_\pi c$. This value of the correlation length is in agreement with theoretical expectations.[1, 2]

The apparent violation of charge symmetry due to the nonlocality of the two-nucleon interaction is canceled to first order by the Coulomb-energy difference in the binding energies of mirror nuclei. Therefore, there is no reason to expect agreement between the measured value of a_{nn} and violations of charge symmetry deduced from existing theoretical calculations[32, 33] of these binding-energy differences.

ADDENDUM

Since this paper was submitted for publication, several new results were reported in contributions to the conference on Few Particle Problems in the Nuclear Interaction. These results suggest that the discrepancy between the measured values of a_{nn} and the value of -17 fm deduced from charge-symmetric local-potential models fitted to the p-p phase shifts is not as large as the measurements[9, 10] weighted heavily in this paper indicate. More important, however, there is now better agreement between the values of a_{nn} obtained from different experiments and a consensus that a_{nn} is slightly less negative than -17 fm. The purpose of this Addendum is to compare our estimate with the results reported at the conference.

Wilkinson[40] has suggested that at present one adopt $a_{nn} = -16.4 \pm 0.9$ fm, which is the mean of the measured values reported at the conference, and which is to be compared with -17.1 ± 0.2 fm deduced from charge-symmetric local-potential models. The estimate of the present work, Eq. (66), is compared with these suggested values in Fig. 1. The solid curve in Fig. 1 is the estimated value of a_{nn} as a function of the zero-energy nonlocal correlation length $R_N(0)$ with $a_{nn}^L = -17.1$ fm; the dashed curve is a similar estimate with $a_{nn}^L = -17.9$ fm. The 0.8-fm difference between the two values of a_{nn}^L represents the estimate by Henley and Keliher[38] of the change in a_{nn}^L due to a violation of charge symmetry necessary to reconcile the ^{3}He-^{3}H binding-energy-difference discrepancy. The dotted lines in Fig. 1 denote the suggested value of a_{nn} and its 0.9-fm uncertainty. While the uncertainty in the suggested value of a_{nn} is too large to make a definitive statement, it appears that there is only marginal consistency with local-potential models for which $R_N(0)$ is identically

zero. The mean of the measured values of a_{nn} points toward $R_N(0) \approx 0.6\hbar/m_\pi c$, and it is of some interest to note that a definite statement about the nonlocality of the two-nucleon interaction could be made if the uncertainty in the measured values can be reduced by a factor of 2. The conclusions of the present paper are consistent with the suggested value of a_{nn}.

APPENDIX: EFFECTIVE-RANGE FUNCTIONS

Consider the radial equation for the s-wave relative motion of two identical particles of mass m interacting through the potential $V(r) = \hbar^2/m\, U(r)$:

$$\left(\frac{d^2}{dr^2} + k^2\right)\psi(k,r) = U(r)\psi(k,r). \tag{A1}$$

The solutions to this equation which satisfy the boundary conditions

$$\lim_{r\to\infty} e^{\mp ikr} f^{\pm}(k,r) = 1 \tag{A2}$$

are called Jost solutions. The solution which satisfies the boundary condition

$$\lim_{r\to 0} r^{-1}\varphi(k,r) = 1 \tag{A3}$$

is called the regular solution. It is related to the Jost solutions by

$$\varphi(k,r) = (2ik)^{-1}[\,\mathfrak{F}^-(k) f^+(k,r) - \mathfrak{F}^+(k) f^-(k,r)\,], \tag{A4}$$

where

$$\mathfrak{F}^{\pm}(k) = \lim_{r\to 0} f^{\pm}(k,r) \tag{A5}$$

is called the Jost function. The Jost function is related to the phase shift $\delta(k)$ by

$$\mathfrak{F}^{\pm}(k) = |\mathfrak{F}^{\pm}(k)|\, e^{\mp i\delta(k)}. \tag{A6}$$

The quantity $|\mathfrak{F}^{\pm}(k)|^{-2}$ is the probability of finding the two particles at $r = 0$ in the presence of the interaction $V(r)$ relative to what it would be if there were no interaction. An integral representation for the Jost function is

$$\mathfrak{F}^{\pm}(k) = 1 + \int_0^\infty f_0^{\pm}(k,r) U(r) \varphi(k,r)\, dr, \tag{A7}$$

where $f_0^{\pm}(k,r) = e^{\pm ikr}$ are the Jost solutions for a free particle. The regular solution for a free particle is $\varphi_0(k,r) = k^{-1}\sin kr$, and its Jost function is unity. The solution defined by

$$\theta(k,r) = 2^{-1}[\,\mathfrak{F}^+(k)^{-1} f^+(k,r) + \mathfrak{F}^-(k)^{-1} f^-(k,r)\,] \tag{A8}$$

is called the irregular solution. The irregular solution for a free particle is $\theta_0(k,r) = \cos kr$. Both $\theta(k,r)$ and $\varphi(k,r)$ are even functions of k. The inverse of Eqs. (A4) and (A8) is

$$\mathfrak{F}^{\pm}(k)^{-1} f^{\pm}(k,r) = \theta(k,r) \pm ik\,|\mathfrak{F}^{\pm}(k)|^{-2}\varphi(k,r). \tag{A9}$$

The above defined notation and nomenclature will be used throughout this Appendix.

The above results are easily generalized to the radial equation for a nonlocal potential:

$$\left(\frac{d^2}{dr^2} + k^2\right)\psi(k,r) = \int_0^\infty U(r,s)\psi(k,s)\, ds. \tag{A10}$$

The integral representation for the Jost function that corresponds to Eq. (A7) is

$$\mathfrak{F}^{\pm}(k) = 1 + \int_0^\infty \int_0^\infty f_0^{\pm}(k,r) U(r,s) \varphi(k,s)\, ds\, dr. \tag{A11}$$

It reduces to Eq. (A7) if the local potential $U(r)$ is written in the nonlocal form $U(r,s) = U((r+s)/2) \times \delta(r-s)$, where $\delta(r-s)$ is a radial δ function.

The effective-range function for a short-range

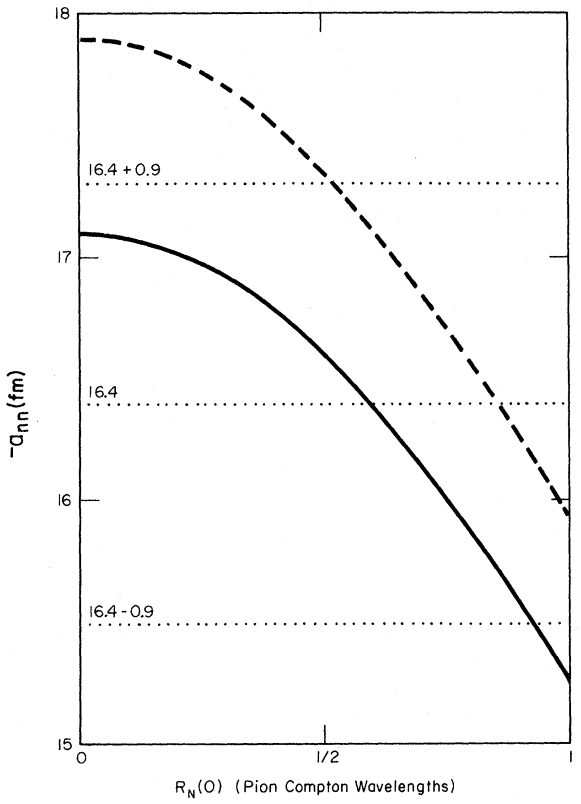

FIG. 1. Comparison of a_{nn} given by Eq. (66) for $a_{nn}^L = -17.1$ fm (——) and $a_{nn}^L = -17.9$ fm (----) with the suggested experimental value of −16.4 fm and its 0.9 fm uncertainty (.....).

potential is defined by

$$X(k)\equiv k\cot\delta(k)=-[1+I_{\theta_0}(k)]/I_{\varphi_0}(k), \tag{A12}$$

where

$$I_\chi(k)\equiv\int_0^\infty\int_0^\infty\chi(k,r)U(r,s)\varphi(k,s)dsdr \tag{A13}$$

and χ denotes a function of k and r. The effective-range function is an even function of k. The scattering length is defined by

$$\lim_{k\to0}X(k)\equiv\frac{-1}{a}, \tag{A14}$$

and the effective range by

$$\lim_{k\to0}\frac{d^2}{dk^2}X(k)\equiv\rho. \tag{A15}$$

The effective-range functions for equations with different potentials $V(r,s)$ and $\bar{V}(r,s)$ or different masses m and $\bar{m}$ are related by

$$\bar{X}(k)-X(k)=\int_0^\infty\int_0^\infty\bar{\psi}(k,r)[\bar{U}(r,s)-U(r,s)]\times\psi(k,s)dsdr, \tag{A16}$$

where

$$\psi(k,r)\equiv\varphi(k,r)/I_{\varphi_0}(k) \tag{A17}$$

is the solution that behaves like $\sin[kr+\delta(k)]/\sin\delta(k)$ as r becomes large.[18] The standard first-order or variational formula for $\bar{X}(k)-X(k)$ is obtained by replacing $\bar{\psi}(k,r)$ by $\psi(k,r)$, or vice versa. It should also be mentioned that Eq. (A16) is a fixed wave-number relation. When it is applied to equations with different masses, the energies $\bar{E}$ and E are related by $\bar{m}\bar{E}=mE$.

In charged-particle scattering the phase shift is determined relative to the phase shift for the long-range electric potential, and the effective-range function defined by Eq. (A12) must be modified. The way it is modified can be seen by considering the simpler case of a short-range reference potential. Let $\mathfrak{F}_1^\pm(k)$ be the Jost function for the reference potential $V_1(r)$. Let $\mathfrak{F}_2^\pm(k)$ be the Jost function for the potential $V_2(r)=V_1(r)+V(r)$. If we write $\mathfrak{F}_2^\pm(k)=\mathfrak{F}_1^\pm(k)\mathfrak{F}_r^\pm(k)$, the phase $\delta_r(k)$ defined by $\mathfrak{F}_r^\pm(k)=|\mathfrak{F}_r^\pm(k)|e^{\mp i\delta_r(k)}$ is the phase shift for the potential $V_2(r)$ relative to the phase shift for the potential $V_1(r)$; i.e., $\delta_r(k)=\delta_2(k)-\delta_1(k)$. Also, $|\mathfrak{F}_r^\pm(k)|^{-2}$ is the probability of finding two particles at $r=0$ in the presence of the interaction $V_2(r)$ relative to the probability for $V_1(r)$.

The Jost function $\mathfrak{F}_2^\pm(k)$ can be written

$$\mathfrak{F}_2^\pm(k)=\mathfrak{F}_1^\pm(k)+\int_0^\infty f_1^\pm(k,r)U(r)\varphi_2(k,r)dr, \tag{A18}$$

in which case the relative Jost function is

$$\mathfrak{F}_r^\pm(k)=1+\mathfrak{F}_1^\pm(k)^{-1}\int_0^\infty f_1^\pm(k,r)U(r)\varphi_2(k,r). \tag{A19}$$

A relative effective-range function, defined by analogy with Eq. (A12), is

$$X_r(k)\equiv-[1+I_{\theta_1}^2(k)]/I_{\varphi_1}^2(k)=k\,|\,\mathfrak{F}_1^\pm(k)\,|^{-2}\cot\delta_r(k). \tag{A20}$$

The superscript in $I_\chi^2(k)$ denotes the solution $\varphi_2(k,r)$ in the integrand of Eq. (A13). Equation (A20) is not an optimum choice for a relative-effective-range function in that $\theta_1(k,r)$ contains a component of the regular solution $\varphi_1(k,r)$ which results in terms of $X_r(k)$ that are independent of $V(r)$ through cancellation in the ratio $I_{\theta_1}^2(k)/I_{\varphi_1}^2(k)$. Specifically, $\theta_1(k,r)$ can be written as[41]

$$\theta_1(k,r)=\theta_0(k,r)+[\alpha(r)+\beta(k)]\varphi_1(k,r)+\tilde{\theta}_1(k,r), \tag{A21}$$

where the remainder $\tilde{\theta}_1(k,r)$ and its derivative are zero at the origin, and $\alpha(r)+\beta(k)$ is determined from the behavior of $d\theta_1(k,r)/dr$ for small r. The quantity $\alpha(r)$ is equal to a constant plus singular terms, less singular at $r=0$ than r^{-1}, if the latter are needed. The quantity $\beta(k)$ is defined to be 0 at $k=0$. A representation of $\alpha(r)+\beta(k)$ can be obtained by examining the behavior of the integral equation for $\theta_1(k,r)$ in the limit as $r\to0$. The customary definition of the relative-effective-range function is

$$X^r(k)\equiv k\,|\,\mathfrak{F}^\pm(k)\,|^{-2}\cot\delta_r(k)+\beta(k)=-\langle\alpha(r)\rangle-[1+I_{\theta_0}^2(k)+I_{\tilde{\theta}_1}^2(k)]/I_{\varphi_1}^2(k), \tag{A22}$$

where

$$\langle\alpha(r)\rangle=I_{\varphi_1}^2(k)^{-1}\int_0^\infty\alpha(r)\varphi_1(k,r)U(r)\varphi_2(k,r)dr. \tag{A23}$$

The relative scattering length a^r and effective range ρ^r are defined from Eq. (A22) according to Eqs. (A14) and (A15).

The effective-range functions $X^r(k)$ and $X(k)$ are related by

$$X^r(k)-X(k)=-\alpha(\bar{r})-R(k). \tag{A24}$$

Equation (A24) defines the function $R(k)$, which is given by

$$R(k)=\langle\alpha(r)\rangle-\alpha(\bar{r})+[1+I_{\theta_0}^2(k)+I_{\tilde{\theta}_1}^2(k)]/I_{\varphi_1}^2(k)-[1+I_{\theta_0}(k)]/I_{\varphi_0}(k). \tag{A25}$$

The value of $\bar{r}$ for the argument of $\alpha(r)$ in Eq. (A24) is arbitrary; it is usually chosen to be a length that is characteristic of the system under study, for instance, $\bar{r} \equiv \rho^r$. The above results for the relative-effective-range function can be used with the long-range reference potentials if the reference Jost function $\mathfrak{F}_1^{\pm}(k)$ is known. The remainder of this Appendix is devoted to the Coulomb and vacuum-polarization reference potentials.

The radial equation for the s-wave relative motion of two identical particles interacting through the Coulomb potential is

$$\left(\frac{d^2}{dr^2} + k^2 - \frac{1}{rR}\right)\psi_{0C}(k, r) = 0, \tag{A26}$$

where $R = \hbar^2/me^2$. Since the Coulomb potential has a long range, the boundary conditions at $r = \infty$ must be modified. The solutions to Eq. (A26) which satisfy the boundary conditions

$$\lim_{r\to\infty} e^{\mp i(kr - \eta \ln 2kr)} f_{0C}^{\pm}(k, r) = 1, \tag{A27}$$

where $\eta = (2kR)^{-1}$, are called the Coulomb Jost solutions. The solution to Eq. (A26) which satisfies the boundary condition (A3) is called the regular Coulomb solution. It is related to the Coulomb Jost solutions by

$$\varphi_{0C}(k,r) = (2ik)^{-1}[\,\mathfrak{F}_{0C}^{-}(k) f_{0C}^{+}(k,r) - \mathfrak{F}_{0C}^{+}(k) f_{0C}^{-}(k,r)\,], \tag{A28}$$

where

$$\mathfrak{F}_{0C}^{\pm}(k) = \frac{e^{\pi\eta/2}}{\Gamma(1 \pm i\eta)} \tag{A29}$$

is the Coulomb Jost function. The Coulomb phase shift is $\sigma_0 = \arg\Gamma(1 + i\eta)$ and the quantity

$$C_0(\eta)^2 = |\,\mathfrak{F}_{0C}^{\pm}|^{-2} = \frac{2\pi\eta}{e^{2\pi\eta} - 1} \tag{A30}$$

is the probability of finding two charged particles at $r = 0$ relative to what it would be if the particles were uncharged. The regular Coulomb solution is related to the regular Coulomb wave function[42] $F_0(\eta, \rho)$ by $F_0(\eta, \rho) = kC_0(\eta)\varphi_{0C}(k, r)$, where $\rho = kr$. The irregular Coulomb solution is defined by

$$\theta_{0C}(k, r) = 2^{-1}[\,\mathfrak{F}_{0C}^{+}(k)^{-1} f_{0C}^{+}(k,r) + \mathfrak{F}_{0C}^{-}(k)^{-1} f_{0C}^{-}(k, r)\,]. \tag{A31}$$

It is related to the irregular Coulomb wave function[42] $G_0(\eta, \rho)$ by $G_0(\eta, \rho) = C_0(\eta)^{-1}\theta_{0C}(k, r)$. The irregular Coulomb solution can be written[43]

$$\theta_{0C}(k,r) = \theta_0(k,r) + R^{-1}[\,\ln(r/R) + 2\gamma - 1 + h(\eta)\,]\,\varphi_{0C}(k,r) + R^{-1}\sum_{l=1}^{\infty} \frac{2l+1}{l(l+1)}\,[2^{-1} - (-1)^l]\,\mathrm{Re}[q_l^{\pm}(k)]\varphi_{lC}(k,r), \tag{A32}$$

where γ is Euler's constant,

$$h(\eta) = \mathrm{Re}[\,\psi(1 + i\eta)] - \ln\eta, \tag{A33}$$

$$q_l^{\pm}(k) = \prod_{p=1}^{l} \frac{-(2pR)^{-1} \pm ik}{2p+1}, \tag{A34}$$

and $\varphi_{lC}(k, r)$ is the regular Coulomb solution for orbital momentum l.

The relative-effective-range function for the Coulomb potential is obtained by comparing Eqs. (A21) and (A32). This gives $\alpha(r) = R^{-1}[\,\ln(r/R) + 2\gamma - 1\,]$ and $\beta(k) = h(\eta)/R$. The standard expression for the relative-effective-range function,

$$kC_0(\eta)^2 \cot\delta^C(k) + h(\eta)/R = X^C(k) \tag{A35}$$

is obtained from Eqs. (A22) and (A30). The effective-range functions $X^C(k)$ and $X(k)$ are related by

$$X^C(k) - X(k) = -R^{-1}[\,\ln(\bar{r}/R) + 2\gamma - 1 + C(k)\,], \tag{A36}$$

where $C(k) \equiv R\,R^C(k)$ with $R^C(k)$ given by Eq. (A25).

The generalization of the relative-effective-range function for the Coulomb potential to include vacuum polarization is straightforward. Equations (A21) and (A22) yield Heller's formula[44]

$$\frac{kC_0(\eta)^2}{1 - \varphi_0}\,[\,(1 + \chi_0)\cot\delta^e(k) - \tan\tau_0\,] + [h(\eta) + l_0(\eta)]/R = X^e(k). \tag{A37}$$

The quantity $(1 - \varphi_0)/(1 + \chi_0)$ is the magnitude squared of the relative Jost function for the vacuum-polarization potential with respect to the Coulomb potential. The phase of this Jost function is $-\tau_0$. The quantity $\alpha(r)$ from Eq. (A21) is

$$\alpha(r) = R^{-1}[\,\ln(r/R) + 2\gamma - 1\,] - \lambda R^{-1}[\,(\ln(\kappa r) - 1)(\gamma - \tfrac{1}{6}) + \tfrac{1}{2}\ln^2(\kappa r) + v\,], \tag{A38}$$

where $\lambda = 2e^2/3\pi\hbar c$, $\kappa^{-1} = \hbar/m_e c$, and

$$v = \lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\infty} \theta_{0C}(0,r)\,\theta_{0e}(0,r) I(r)\frac{dr}{r} - \ln(\kappa\epsilon)[\gamma + \tfrac{5}{6} + \tfrac{1}{2}\ln(\kappa\epsilon)]\right). \tag{A39}$$

In this last expression, $\theta_{0e}(k, r)$ is the irregular solution for the Coulomb-plus-vacuum-polarization potential, $[1 + \lambda I(r)]/Rr$, with $I(r)$ given by

$$I(r) = \int_1^{\infty} e^{-2\kappa r x}\left(\frac{1}{x^2} + \frac{1}{2x^4}\right)(x^2 - 1)^{1/2} dx. \tag{A40}$$

The relative-effective-range functions $X^e(k)$ and

$X(k)$ are related by

$$X^e(k) - X(k) = -R^{-1}[\ln(\bar{r}_C/R) + 2\gamma - 1] + \lambda R^{-1}\{[\ln(\kappa\bar{r}_v) - 1](\gamma - \tfrac{1}{6}) + \tfrac{1}{2}\ln^2(\kappa\bar{r}_v) + v\} - R^{-1}E(k), \quad \text{(A41)}$$

where $E(k) \equiv RR^e(k)$ with $R^e(k)$ given by Eq. (A25). The first term on the right in Eq. (A41) is the Coulomb term and $\bar{r}_C$ may be chosen to have the same value it would have in the absence of vacuum polarization. There is no present need to keep the second and third terms separate as vacuum polarization is a small correction to the Coulomb potential; it is sufficient to redefine $E(k)$ to include the second term. With this latter definition, the relative-effective-range functions $X^e(k)$ and $X^C(k)$ are related by

$$X^e(k) - X^C(k) = -\lambda R^{-1}V(k), \quad \text{(A42)}$$

where $\lambda V(k) = E(k) - C(k)$.

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[18]We use the symbol that defines a generic solution to a given wave equation interchangably with the particular solution to the equation defined according to Eq. (A17) For example, $\psi_p(k_p, r)$ refers to any solution without consideration of boundary conditions in Eq. (4); in Eq. (7), or in any other relation between effective-range functions, $\psi_p(k_p, r)$ refers to the particular solution defined by Eq. (A17). Different symbols are used to denote other particular solutions.

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[21]We omit the superscripts and subscripts from Eqs. (1)–(5) and (12)–(16) in the discussion of the equivalent-local-potential method.

[22]This assumption is restrictive in that it is easy to devise nonlocal potentials which do not satisfy it for a range of energies. Whether or not the assumption is satisfied in a given energy range depends as much on the values for parameters of the potential as it does on its analytic form.

[23]Equation (22) reduces to the approximate result for the Coulomb correction potential given in Ref. 7 when the nonlocality of $V(r, s)$ is small.

[24]The empirical p-p potentials differ from the equivalent local potentials in that the latter depend on the wave number. We will consider this difference later.

[25]We are omitting vacuum polarization from the estimates since the difference between the electric scattering length and the Coulomb scattering length is smaller than the uncertainty we have assigned to the latter.

[26]The three-body calculations described in Ref. 10 utilize separable nonlocal two-nucleon interactions. The analysis of the $^2\mathrm{H}(n, 2n)\mathrm{H}$ reaction with these calculations is not affected by the results of this paper, since the reaction does not involve charged-particle interactions.

[27]The modest difference between $V_p^C(k, r)$ and $V_n(k, r)$ should not make the cancellation less complete in $\bar{F}_n^L - (\delta m/\tilde{\alpha})\bar{M}^L$ than in $F_n^L - (\delta m/\tilde{\alpha})M^L$. The factor m_p/m_n in Eq. (41) for $\bar{M}^L$ is also of no consequence.

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[31]More generally, X_N can be defined as the ratio of the zero-energy nonlocal correlation length to the rms radius of the potential; for example

$$X_N^2 \equiv \frac{\int d\vec{r}_1 d\vec{r}_2 r^2 V(\vec{r}_1, \vec{r}_2)}{\int d\vec{r}_1 d\vec{r}_2 R^2 V(\vec{r}_1, \vec{r}_2)},$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$ and $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$. With this definition, $X_N = 2$ for an s-state separable potential $V(\vec{r}_1, \vec{r}_2) = \sum_i \lambda_i f_i(r_1) f_i(r_2)$. The large value of X_N for this class of nonlocal potentials indicates that our estimates are not applicable to potentials in this class.

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