One-Pion-Exchange-Current Contribution to Neutron-Proton Bremsstrahlung*

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The electromagnetic current j_2^{OPE} due to the exchange of a charged pion between a neutron and a proton is rederived by comparing the bremsstrahlung matrix element computed from the Schrödinger equation with that obtained from the lowest-order Feynman diagrams. No account is taken of renormalization effects or nucleon resonances. The operator $\int d^3x\,\vec{j}_2^{OPE}(\vec{x})e^{-i\vec{k}\cdot\vec{x}}$ which is needed for neutron-proton bremsstrahlung calculations is evaluated, and an approximation to it obtained as an expansion in powers of (photon energy)/(pion mass), keeping the first two terms. This is expected to be considerably more accurate than expanding the bremsstrahlung matrix element itself in powers of k. The second term in the expansion contributes to ${}^1S_0 \leftrightarrow {}^3S_1$ transitions and may become significant even for small values of k/μ. As a preparation for actual calculations the operator is rewritten in terms of irreducible tensors. The treatment of nonlocalities other than charged-pion exchange is discussed briefly.

I. INTRODUCTION

It has been suggested that nucleon-nucleon (N-N) bremsstrahlung experiments could be used to distinguish between the off-energy-shell behavior of different potentials which have been fitted to elastic N-N scattering data. Implicit in this notion is the assumption that one knows the N-N electromagnetic charge and current density operators. However, all of the realistic potentials contain nonlocal and exchange terms, and in their presence the usual operators, $\rho_1(\vec{x})$ and $\vec{j}_1(\vec{x})$, shown in Eq. (2) are not conserved.

It is well known that the requirement of current conservation alone does not uniquely determine the form of the operators $(\rho_2, \vec{j}_2)$ which must be added on to $(\rho_1, \vec{j}_1)$. If the nonlocal (or exchange) potential is strictly phenomenological, one will not be able to decide which is the correct[1] choice. If, on the other hand, the nonlocal potential was derived from some theory, such as the exchange of a meson between the two nucleons, then one can go back to that theory and introduce electromagnetic interaction into it (e.g., by the minimal substitution for every particle including the meson). $(\rho_2, \vec{j}_2)$ are then determined by the requirement that the complete operators ($\rho = \rho_1 + \rho_2$, $\vec{j} = \vec{j}_1 + \vec{j}_2$) used in conjunction with solutions of the Schrödinger equation containing that nonlocal potential should yield the same matrix element that the above described theory gives. $(\rho_2, \vec{j}_2)$ found in this way are, in general, not the same as those obtained by making the minimal substitution directly in the nonlocal potential.[2]

Since the realistic potentials have some nonlocal terms which are phenomenological and some which come from meson theory, it is fortunate that we are permitted to focus on one at a time. At the least this is true if ρ_2 is chosen to be zero, for then $\vec{j}_2$ is *linear* in the nonlocal potential. This is already clear from the work of Osborn and Foldy,[3] and will be shown explicitly in Sec. II. While we cannot justify setting $\rho_2 = 0$ in general, it is correct in the nonrelativistic limit for charged-scalar[2] or pseudoscalar meson exchange. The latter result will be obtained in Sec. III.

For the neutron-proton system there are terms in the realistic potentials which exchange the charge of the nucleons, and the one with the longest range arises from the exchange of a pion, V^{OPE}. We do not consider any further in this paper the contribution to the current coming from the other nonlocal terms in the potential, except for some comments in the Discussion, Sec. V.

The procedure we follow, as described above, is to write down the Feynman diagram for single-pion exchange, Fig. 1, attach a photon wherever it can go, Fig. 2, and say that this defines the total current operator to order eG^2 where G is the pion-nucleon coupling constant. (This was done in Ref. 2 for a charged-scalar meson, and regarding the potential as an exchange of the spatial coordinates. Here we are interested in the pseudoscalar case, and use isospin notation.) $(\rho_1, \vec{j}_1)$ can be identified as the portions of Figs. 2(a)–2(d) in which the radiating nucleon propagates in a positive energy state; $(\rho_2^{OPE}, \vec{j}_2^{OPE})$ is defined to be the remainder [including Fig. 2(e)].

The result of this part of the paper is not new, and was already used in a remarkable paper, considering the date, by Villars,[4] on the pion-exchange-current contribution to the magnetic moments of ^{3}H and ^{3}He. The formula for $\vec{j}_2^{OPE}$ appears for the first time, to our knowledge, in Wahlborn and Blomqvist.[5]

Although $(\rho_1 + \rho_2^{OPE},\ \vec{j}_1 + \vec{j}_2^{OPE})$ would be a con-

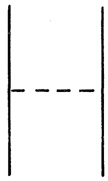

FIG. 1. The one-pion-exchange diagram.

served current if $V^{\rm OPE}$ were the only nonlocal term in the potential, this does not guarantee that this current is correct even for such a potential. For we know that diagrams other than Fig. 1 contribute to the renormalization of the coupling constant which appears in $V^{\rm OPE}$, e.g., Fig. 3, and one should insert a photon at all possible places in this diagram as well. The resulting set of diagrams is gauge invariant, as are the photon insertions on Fig. 4. It is only to the extent that diagrams such as those shown in Fig. 5 are unimportant, that we are justified in saying that Fig. 2 correctly gives the one-pion-exchange current.

If other particles or other couplings are introduced into the theory then additional diagrams are present, e.g., Fig. 6. If the heavy line in Fig. 6 represents a spin $\frac{3}{2}$ particle such as the $\Delta(1236)$, then the sum of these two diagrams is gauge invariant. The corresponding diagrams without the photon are identically zero, and therefore contribute nothing to the one-pion-exchange potential. Lacking a fundamental theory, it is unclear to us whether or not the electromagnetic current associated with Fig. 6 should be included. All we can say is that it is not forced on us by the nonlocality of $V^{\rm OPE}$. So much for the philosophy behind this paper.

In Sec. II we write down the Schrödinger equation for two particles interacting with each other via a potential which is allowed to be completely nonlocal and fairly general with respect to isospin dependence. In searching for a conserved current, we identify the customary operators $\rho_1(\vec{x})$ and $\vec{j}_1(\vec{x})$, and obtain an expression for $\vec{\nabla}\cdot\vec{j}_2(\vec{x})$.

The contribution to $\vec{j}_2$ from the exchange of a pseudoscalar meson is considered in Sec. III where the formula for $\vec{j}_2^{\rm OPE}$ is verified. In Sec. IV we find the general expression for $\int d^3x\,\vec{j}_2^{\rm OPE}(\vec{x})e^{-i\vec{k}\cdot\vec{x}}$, the operator needed for bremsstrahlung calculations. This operator is expanded in powers of k, keeping the first two terms, and is then rewritten in terms of irreducible tensors. The magnetic moment operator $\vec{m}^{\rm OPE}$ is also obtained.

A general expression for the $k=0$ limit of the operator is obtained in Appendix A. Appendix B contains a discussion of the "recoil" emission diagram. Some of the angular momentum formulas needed to obtain Eq. (17) are collected in Appendix C.

II. CONSERVED CURRENT WITH NONLOCAL AND EXCHANGE POTENTIALS

The Schrödinger equation for two nucleons of mass m is written

$$\left(-\frac{\nabla_1^2}{2m}-\frac{\nabla_2^2}{2m}\right)\psi_i(\vec{r}_1,\vec{r}_2,t)+\int d^3r_1'd^3r_2'\langle\vec{r}_1,\vec{r}_2|V|\vec{r}_1{}',\vec{r}_2{}'\rangle\psi_i(\vec{r}_1{}',\vec{r}_2{}',t)=i\frac{\partial}{\partial t}\psi_i(\vec{r}_1,\vec{r}_2,t). \tag{1}$$

The isospin dependence is made explicit by writing $V=V_D+V_E\vec{\tau}_1\cdot\vec{\tau}_2$, where the terms V_D and V_E stand for direct and exchange potentials. Other isospin dependences are possible but we shall not need them. V_D and V_E are operators in the (Pauli) spin space of the two nucleons, and the spin and isospin dependence in

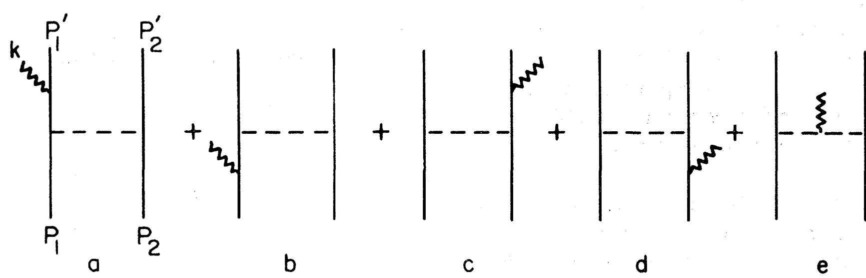

FIG. 2. (a)–(e) The five Feynman diagrams which are considered in this paper as defining the electromagnetic current operator eG^2.

the wave function has been suppressed. Translation and Galilean invariance and Hermiticity require[6] that

$$\langle \vec{r}_1, \vec{r}_2 | V | \vec{r}_1{}', \vec{r}_2{}' \rangle = \delta^{(3)}(\vec{R} - \vec{R}')\langle \vec{r} | V | \vec{r}' \rangle,$$

$$\langle \vec{r} | V | \vec{r}' \rangle = \langle \vec{r}' | V^T | \vec{r} \rangle^*,$$

where $\vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$ and $\vec{r} = \vec{r}_1 - \vec{r}_2$.

One writes down the equation corresponding to Eq. (1) for $\psi_f^\dagger$, multiplies the former on the left by $-ie\psi_f^\dagger \tau_1^p$, where $\tau^{p,n} \equiv \frac{1}{2}(1 \pm \tau^z)$, the latter on the right by $-ie\tau_1^p \psi_i$, and subtracts the two, integrates over d^3r_2, and then relabels $\vec{r}_1$ as $\vec{x}$; repeat using τ_2^p, this time integrating over d^3r_1, and then replacing $\vec{r}_2$ by $\vec{x}$. Summing the two results, and defining

$$\langle \psi_f | \rho_1(\vec{x}) | \psi_i \rangle \equiv e\left[\int d^3r_2 \psi_f^*(\vec{x}, \vec{r}_2)\tau_1^p \psi_i(\vec{x}, \vec{r}_2) + \int d^3r_1 \psi_f^*(\vec{r}_1, \vec{x})\tau_2^p \psi_i(\vec{r}_1, \vec{x})\right], \tag{2a}$$

$$\begin{aligned}\langle \psi_f | \vec{j}_1(x) | \psi_i \rangle \equiv \frac{e}{2im}&\left[\int d^3r_2 \psi_f^*(\vec{x}, \vec{r}_2)\tau_1^p \overleftrightarrow{\nabla}_x \psi_i(\vec{x}, \vec{r}_2) + \int d^3r_1 \psi_f^*(\vec{r}_1, \vec{x})\tau_2^p \overleftrightarrow{\nabla}_x \psi_i(\vec{r}_1, \vec{x})\right] \\ &+ \mathrm{curl}_x \int d^3r_2\, \psi_f^*(\vec{x}, \vec{r}_2)\vec{\mu}_1 \psi_i(\vec{x}, \vec{r}_2) + \mathrm{curl}_x \int d^3r_1 \psi_f^*(\vec{r}_1, \vec{x})\vec{\mu}_2 \psi_i(\vec{r}_1, \vec{x}),\end{aligned} \tag{2b}$$

where $\varphi \overleftrightarrow{\nabla} \chi \equiv \varphi \vec{\nabla} \chi - \chi \vec{\nabla} \varphi$, and $\vec{\mu}_i \equiv [\tau_i^p \mu^p + \tau_i^n \mu^n]\sigma_i$, the current is seen to be conserved, that is,

$$\frac{\partial}{\partial t}\langle \psi_f | \rho_1(x) | \psi_i \rangle + \vec{\nabla}_x \cdot \langle \psi_f | \vec{j}_1(x) + \vec{j}_2(x) | \psi_i \rangle = 0 \tag{3}$$

provided

$$\begin{aligned}\vec{\nabla}_x \cdot \langle \vec{r}_1, \vec{r}_2 | \vec{j}_2(\vec{x}) | \vec{r}_1{}', \vec{r}_2{}' \rangle = \tfrac{1}{2} e\delta^{(3)}(\vec{R} - \vec{R}')& \\ \times \langle \vec{r} | iV[&\delta^{(3)}(\vec{x} - \vec{r}_1) - \delta^{(3)}(\vec{x} - \vec{r}_1{}') + \delta^{(3)}(\vec{x} - \vec{r}_2) - \delta^{(3)}(\vec{x} - \vec{r}_2{}')] \\ &+ i\tau_1^z\{V_D[\delta^{(3)}(\vec{x} - \vec{r}_1) - \delta^{(3)}(\vec{x} - \vec{r}_1{}')] + V_E[\delta^{(3)}(\vec{x} - \vec{r}_2) - \delta^{(3)}(\vec{x} - \vec{r}_2{}')]\} \\ &+ i\tau_2^z\{V_D[\delta^{(3)}(\vec{x} - \vec{r}_2) - \delta^{(3)}(\vec{x} - \vec{r}_2{}')] + V_E[\delta^{(3)}(\vec{x} - \vec{r}_1) - \delta^{(3)}(\vec{x} - \vec{r}_1{}')]\} \\ &+ (\vec{\tau}_1 \times \vec{\tau}_2)^z V_E[\delta^{(3)}(\vec{x} - \vec{r}_1) - \delta^{(3)}(\vec{x} - \vec{r}_2) + \delta^{(3)}(\vec{x} - \vec{r}_1{}') - \delta^{(3)}(\vec{x} - \vec{r}_2{}')] | \vec{r}' \rangle.\end{aligned} \tag{4}$$

The second set of terms in $\vec{j}_1(\vec{x})$ is the current due to the total magnetic moments of the particles and is conserved by itself. $\mu^{n,p}$ is the total magnetic moment of the neutron (proton); $\mu^n = K^n \mu_B$, $\mu^p = (1 + K^p)\mu_B$, $\mu_B = e/2m$, and $K^n = -1.91$, $K^p = 1.79$.

The solution of Eq. (4) for $\vec{j}_2(\vec{x})$ has considerable arbitrariness, since the curl of any vector field can be freely added on. Osborn and Foldy[3] expressed some of this arbitrariness by writing

$$\vec{j}_2 = V_E(\vec{\tau}_1 \times \vec{\tau}_2)^z \vec{\xi}(\vec{x}, \vec{r}_1, \vec{r}_2, \vec{\sigma}_1, \vec{\sigma}_2) + (\text{other isospin dependence which is divergenceless}),$$

where $\vec{\nabla}_x \cdot \vec{\xi} = \delta^{(3)}(\vec{x} - \vec{r}_1) - \delta^{(3)}(\vec{x} - \vec{r}_2)$. [They were concerned with a charge-exchange potential with no spatial nonlocality, so that only the third term on the right-hand side of Eq. (4) is present. The one-pion-exchange potential is of this type; see Eq. (6).] Although the vector field $\vec{\xi}$ has considerable pedagogic value, and is simple for charged-scalar-meson exchange,[2] it is not a convenient thing to do for a pseudoscalar meson and we work directly with $\vec{j}_2$.

In the next section we obtain a unique expression for $\vec{j}_2^{\,\mathrm{OPE}}$ by the requirement that the bremsstrahlung matrix element computed with this current agrees with a certain set of Feynman diagrams in the nonrelativistic limit.

It is clear from Eq. (4) that if there is more than one nonlocal (or exchange) potential in the Hamiltonian, one may construct $\vec{j}_2$ by adding a separate contribution for each term in the potential.

Note that it is trivial to generalize all of the above formulas to a system of more than two nucleons, since $\vec{j}_2$ is just a two-particle operator.

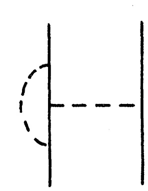

FIG. 3. A contribution to the renormalization of the pion-nucleon coupling constant.

III. ONE-PION-EXCHANGE CURRENT

We first state the result[7] for the one-pion-exchange current:

$$\langle \vec{r}_1, \vec{r}_2 | \rho_2^{\rm OPE}(\vec{x}) | \vec{r}_1{}', \vec{r}_2{}' \rangle = 0 ,$$

$$\langle \vec{r}_1, \vec{r}_2 | \vec{j}_2^{\rm OPE}(\vec{x}) | \vec{r}_1{}', \vec{r}_2{}' \rangle = \frac{ef^2}{\mu^2} (\vec{\tau}_1 \times \vec{\tau}_2)^z \delta^{(3)}(\vec{r}_1 - \vec{r}_1{}') \delta^{(3)}(\vec{r}_2 - \vec{r}_2{}') \times \Bigg[\delta^{(3)}(\vec{x} - \vec{r}_1)\vec{\sigma}_1 (\vec{\sigma}_2 \cdot \vec{\nabla}_x) \frac{e^{-\mu|\vec{x}-\vec{r}_2|}}{|\vec{x}-\vec{r}_2|} - \delta^{(3)}(\vec{x} - \vec{r}_2)\vec{\sigma}_2 (\vec{\sigma}_1 \cdot \vec{\nabla}_x) \frac{e^{-\mu|\vec{x}-\vec{r}_1|}}{|\vec{x}-\vec{r}_1|} - \frac{1}{4\pi}\left(\vec{\sigma}_1 \cdot \vec{\nabla}_x \frac{e^{-\mu|\vec{x}-\vec{r}_1|}}{|\vec{x}-\vec{r}_1|}\right) \vec{\nabla}_x \left(\vec{\sigma}_2 \cdot \vec{\nabla}_x \frac{e^{-\mu|\vec{x}-\vec{r}_2|}}{|\vec{x}-\vec{r}_2|}\right)\Bigg] , \qquad (5)$$

where $f^2 = (G^2/4\pi)(\mu/2m)^2 \simeq 0.08$, and μ and m are the pion and nucleon masses, respectively. One checks directly that $\vec{\nabla}_x \cdot \vec{j}_2(x)$ satisfies Eq. (4) using the expression

$$\langle \vec{r} | V^{\rm OPE} | \vec{r}\,' \rangle = \delta^{(3)}(\vec{r} - \vec{r}\,')\vec{\tau}_1 \cdot \vec{\tau}_2 V^{\rm OPE}(\vec{r}) , \qquad (6)$$

where

$$V^{\rm OPE}(\vec{r}) = \frac{f^2}{\mu^2} (\vec{\sigma}_1 \cdot \vec{\nabla}_r)(\vec{\sigma}_2 \cdot \vec{\nabla}_r) \frac{e^{-ur}}{r} .$$

This guarantees that $(\rho_1, \vec{j}_1 + \vec{j}_2^{\rm OPE})$ is a conserved current if $V^{\rm OPE}$ is the only nonlocal (or exchange potential in the Schrödinger equation.

We will now verify that the bremsstrahlung matrix element,

$$(2\pi)^3 \delta^{(3)}(\vec{P}_f + \vec{k} - \vec{P}_i)(\epsilon_0 M_0 - \vec{\epsilon} \cdot \vec{M}) \equiv \left\langle \psi_f^{(-)} \middle| \int d^3x e^{-i\vec{k}\cdot\vec{x}} [\epsilon_0 \rho(\vec{x}) - \vec{\epsilon} \cdot \vec{j}(x) \middle| \psi_i^{(+)} \right\rangle , \qquad (7)$$

where $\vec{k}$, $\vec{\epsilon}$ are the momentum and polarization of the photon, *computed to order* eG^2 with $\rho = \rho_1$ and $\vec{j} = \vec{j}_1 + \vec{j}_2^{\rm OPE}$, and using wave functions which are solutions of the Schrödinger equation containing $V^{\rm OPE}$ as the only potential, agrees with the field theory calculation, i.e., the five Feynman diagrams in Fig. (2). This is correct in the nonrelativistic limit for the nucleons, i.e., terms of order $(p/m)^2$ and higher are neglected.

To order G^2, the initial wave function needed in Eq. (7) can be written

$$\psi_i^{(+)}(\vec{r}_1, \vec{r}_2) = e^{i\vec{p}_1 \cdot \vec{r}_1} e^{i\vec{p}_2 \cdot \vec{r}_2} + 2m \iint \frac{d^3q_1 d^3q_2}{(2\pi)^6} e^{i\vec{q}_1 \cdot \vec{r}_1} e^{i\vec{q}_2 \cdot \vec{r}_2} \frac{\langle \vec{q}_1, \vec{q}_2 | V^{\rm OPE} | \vec{p}_1, \vec{p}_2 \rangle}{p_1{}^2 + p_2{}^2 - (q_1{}^2 + q_2{}^2) + i\epsilon} , \qquad (8a)$$

where

$$\langle \vec{q}_1, \vec{q}_2 | V^{\rm OPE} | \vec{p}_1, \vec{p}_2 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}_1 + \vec{p}_2 - \vec{q}_1 - \vec{q}_2)\vec{\tau}_1 \cdot \vec{\tau}_2 \tilde{V}(\vec{p}_1 - \vec{q}_1) ,$$

$$\tilde{V}(\vec{p}) \equiv -\frac{G^2}{(2m)^2} \frac{(\vec{\sigma}_1 \cdot \vec{p})(\vec{\sigma}_2 \cdot \vec{p})}{\vec{p}^2 + \mu^2} , \qquad (8b)$$

with a similar expression for $\psi_f^{(-)}$ involving p_1' and p_2'. Using Eqs. (2), (5), (7), and (8) one obtains

$$M^0(G^2) = em \sum_{i=1}^{2} \left[\frac{\tau_i^{\rho}(\vec{\tau}_1 \cdot \vec{\tau}_2)}{mk - \vec{k} \cdot (\vec{p}_i{}' + \frac{1}{2}\vec{k})} - \frac{(\vec{\tau}_1 \cdot \vec{\tau}_2)\tau_i^{\rho}}{mk - \vec{k} \cdot (\vec{p}_i - \frac{1}{2}\vec{k})} \right] \tilde{V}(\vec{p}_i - \vec{p}_i{}' - \vec{k}) , \qquad (9a)$$

$$\vec{M}_1(G^2) = e \sum_{i=1}^{2} \tilde{V}(\vec{p}_i - \vec{p}_i{}' - \vec{k}) \times \left[\frac{\tau_i^{\rho}(\vec{\tau}_1 \cdot \vec{\tau}_2)}{mk - \vec{k} \cdot (\vec{p}_i{}' + \frac{1}{2}\vec{k})} (\vec{p}_i{}' + \tfrac{1}{2}\vec{k}) - \frac{(\vec{\tau}_1 \cdot \vec{\tau}_2)\tau_i^{\rho}}{mk - \vec{k} \cdot (\vec{p}_i - \frac{1}{2}\vec{k})} (\vec{p}_i - \tfrac{1}{2}\vec{k}) \right] + im \sum_{i=1}^{2} \left[\frac{(\vec{k} \times \vec{\mu}_i)(\vec{\tau}_1 \cdot \vec{\tau}_2)}{mk - \vec{k} \cdot (\vec{p}_i{}' + \frac{1}{2}\vec{k})} \tilde{V}(\vec{p}_i - \vec{p}_i{}' - \vec{k}) - \tilde{V}(\vec{p}_i - \vec{p}_i{}' - \vec{k}) \frac{(\vec{\tau}_1 \cdot \vec{\tau}_2)(\vec{k} \times \vec{\mu}_i)}{mk - \vec{k} \cdot (\vec{p}_i - \frac{1}{2}\vec{k})} \right] , \qquad (9b)$$

$$\vec{M}_2^{\text{OPE}}(G^2)=ie\,\frac{G^2}{(2m)^2}\,(\vec{\tau}_1\times\vec{\tau}_2)^z\left\{\frac{\sigma_1\cdot(\vec{p}_1{}'-\vec{p}_1)}{(\vec{p}_1{}'-\vec{p}_1)^2+\mu^2}\,\vec{\sigma}_2-\frac{\vec{\sigma}_2\cdot(\vec{p}_2{}'-\vec{p}_2)}{(\vec{p}_2{}'-\vec{p}_2)^2+\mu^2}\,\vec{\sigma}_1\right.$$
$$\left.+\frac{[\vec{\sigma}_1\cdot(\vec{p}_1{}'-\vec{p}_1)][\vec{\sigma}_2\cdot(\vec{p}_2{}'-\vec{p}_2)]}{[(\vec{p}_1{}'-\vec{p}_1)^2+\mu^2][(\vec{p}_2{}'-\vec{p}_2)^2+\mu^2]}\,(\vec{p}_2-\vec{p}_1+\vec{p}_1{}'-\vec{p}_2{}')\right\}, \tag{9c}$$

where $\vec{M}_1$ is the matrix element due to $\vec{j}_1$ and $\vec{M}_2$ due to $\vec{j}_2$, with $\vec{M}=\vec{M}_1+\vec{M}_2$. The three terms in Eq. (9c) correspond, respectively, to the three terms in Eq. (5). The symbol G^2 on the left-hand side of Eq. (9) is a reminder that the calculation has been carried out only to that order.

We now wish to compare the result in Eq. (9) with the bremsstrahlung matrix elements corresponding to the Feynman diagrams of Fig. (2). In order to do this we use the following interaction Hamiltonian,

$$H_I=G\bar{\psi}(x)\gamma_5\vec{\tau}\psi(x)\cdot\vec{\varphi}(x)+e\,\bar{\psi}(x)\gamma_\mu\tau^p\psi(x)A^\mu(x)$$
$$+e\left[\vec{\varphi}(x)\times\frac{\partial\vec{\varphi}(x)}{\partial x\mu}\right]_3 A^\mu(x)+\tfrac{1}{2}\mu_B\,\bar{\psi}(x)(K^p\tau^p+K^n\tau^n)\sigma_{\mu\nu}\psi(x)F^{\mu\nu}(x), \tag{10}$$

where $\psi(x)$ is the nucleon field, $A^\mu(x)$ is the photon field, $\vec{\varphi}(x)$ is the pion field, $\sigma_{\mu\nu}=(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)/2i$, and $F^{\mu\nu}(x)=\partial A^\mu/\partial x_\nu-\partial A^\nu/\partial x_\mu$. The first three terms in Eq. (10) couple the pion and nucleon, photon and nucleon, photon and pion, respectively, while the last term is a Pauli coupling which describes the anomalous moment of the nucleon. The Feynman diagrams of Fig. 2 lead to the following results. For the time component, Fig. 2(e) contributes nothing, and the sum of Figs. 2(a)–2(d) equals the Schrödinger result, Eq. (9a).

$$M_e^0=0\,, \tag{11a}$$

$$M_a^0+M_b^0+M_c^0+M_c^0=M^0(G^2)\,. \tag{11b}$$

This is the reason for choosing $\rho_2^{\text{OPE}}=0$. For the spatial components

$$\vec{M}_a+\vec{M}_b+\vec{M}_c+\vec{M}_d=ie\,\frac{G^2}{(2m)^2}\,(\vec{\tau}_1\times\vec{\tau}_2)^z\left\{\frac{[\vec{\sigma}_1\cdot(\vec{p}_1{}'-\vec{p}_1)]}{(\vec{p}_1{}'-\vec{p}_1)^2+\mu^2}\,\vec{\sigma}_2-\frac{[\vec{\sigma}_2\cdot(\vec{p}_2{}'-\vec{p}_2)]}{(\vec{p}_2{}'-\vec{p}_2)^2+\mu^2}\,\vec{\sigma}_1\right\}+\vec{M}_1(G^2), \tag{11c}$$

$$\vec{M}_e=ie\,\frac{G^2}{(2m)^2}\,(\vec{\tau}_1\times\vec{\tau}_2)^z\,\frac{[\vec{\sigma}_2\cdot(\vec{p}_2{}'-\vec{p}_2)][\vec{\sigma}_1\cdot(\vec{p}_1{}'-\vec{p}_1)]}{[(\vec{p}_2{}'-\vec{p}_2)^2+\mu^2][(\vec{p}_1{}'-\vec{p}_1)^2+\mu^2]}\,(\vec{p}_2-\vec{p}_1+\vec{p}_1{}'-\vec{p}_2{}'), \tag{11d}$$

where $\vec{M}_1(G^2)$ is the result found in Eq. (9b). Adding Eqs. (11c) and (11d), and also Eqs. (9b) and (9c) shows that

$$\vec{M}_a+\vec{M}_b+\vec{M}_c+\vec{M}_d+\vec{M}_e=\vec{M}_1(G^2)+\vec{M}_2^{\text{OPE}}(G^2)\,. \tag{12}$$

In Eqs. (11a)–(11d) terms of order $(p/m)^2$ and higher are neglected. It is thus verified that in the nonrelativistic limit the matrix element generated by $(\rho_1, \vec{j}_1+\vec{j}_2^{\text{OPE}})$ to order eG^2 is the same as that corresponding to the Feynman diagrams of Fig. (2).

The first two terms in Eq. (11c) arise from the portions of diagrams 2(a)–2(d) in which the propagating nucleon is in a negative energy state. The positive energy portions of those diagrams yield $\vec{M}_1(G^2)$, the result obtained from the Schrödinger equation with the normal current. Comparing Eqs. (11c) and (11d) with (9c) shows that $\vec{j}_2^{\text{OPE}}$ corresponds to radiation from the exchange pion plus radiation from a nucleon in a negative energy state. The anomalous magnetic moment does not contribute in the negative-energy states, and in the positive-energy states merely adds to the Dirac term to give the total magnetic moment.

The diagrams in which a nucleon radiates while

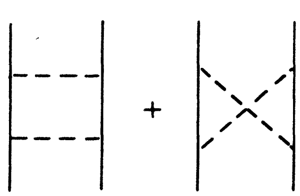

FIG. 4. The two-pion-exchange diagrams.

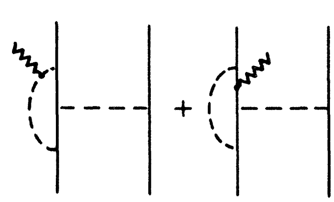

FIG. 5. Additional contributions to the one-pion-exchange current which are not considered in this paper.

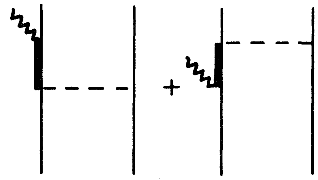

FIG. 6. Nucleon resonance contribution to the one-pion-exchange current. (Not considered in this paper.)

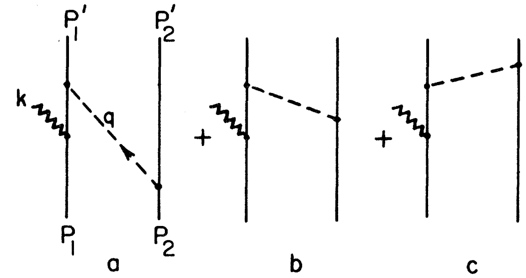

FIG. 7. (a)–(c) Time-ordered diagrams.

in a positive-energy state can be further subdivided according to the time ordering of the pion interaction on the nonradiating nucleon, as shown in Fig. 7. Figure 7(a), in which the photon is radiated while the pion is in flight, is referred to as the nucleon recoil diagram, and it is sometimes stated that this contribution is not contained in the Schrödinger matrix element $\vec{M}_1(G^2)$. We show in Appendix B by direct calculation that this statement is incorrect. The sum of the three contributions in Fig. (7) (plus the other three sets of three, where the photon comes off the other possible nucleon legs) gives a *better* approximation to $\vec{M}_1(G^2)$ than if the recoil emission diagrams are omitted. This means that one should *not* add the recoil current as as additional piece when doing calculations with the Schrödinger equation.

IV. GENERAL EXPRESSION FOR THE BREMSSTRAHLUNG MATRIX ELEMENT, $\vec{M}_2^{\text{OPE}}$

Having verified that the one-pion-exchange current is given by Eq. (5), we would now like to compute its contribution to a neutron-proton bremsstrahlung matrix element using wave functions which are solutions of the Schrödinger equation with an arbitrary potential. According to Eq. (7) this requires the calculation of

$$\int d^3x e^{-i\vec{k}\cdot\vec{x}}\, \vec{j}_2^{\text{OPE}}(x).$$

From Eq. (5) we find

$$\int d^3x e^{-i\vec{k}\cdot\vec{x}} \langle \vec{r}, \vec{R}|\vec{j}_2^{\text{OPE}}(\vec{x})|\vec{r}', \vec{R}'\rangle = \delta^3(\vec{R}-\vec{R}')\delta^3(\vec{r}-\vec{r}')e^{-i\vec{k}\cdot\vec{R}} e\frac{f^2}{\mu^2}(\vec{\tau}_1\times\vec{\tau}_2)^z$$
$$\times\left\{e^{-\frac{1}{2}i\vec{k}\cdot\vec{r}}\left(\vec{\sigma}_2\cdot\vec{\nabla}_r\frac{e^{-\mu r}}{r}\right)\vec{\sigma}_1+e^{\frac{1}{2}i\vec{k}\cdot\vec{r}}\left(\vec{\sigma}_1\cdot\vec{\nabla}_r\frac{e^{-\mu r}}{r}\right)\vec{\sigma}_2\right.$$
$$\left.-\frac{1}{\pi^2}[\vec{\sigma}_1\cdot(i\vec{\nabla}_r+\tfrac{1}{2}\vec{k})][\vec{\sigma}_2\cdot(i\vec{\nabla}_r-\tfrac{1}{2}\vec{k})]\vec{\nabla}_r I(\vec{k},\vec{r})\right\}, \tag{13a}$$

where

$$I(\vec{k},\vec{r})\equiv\int d^3q\frac{e^{i\vec{q}\cdot\vec{r}}}{[(\vec{q}+\frac{1}{2}\vec{k})^2+\mu^2][(\vec{q}-\frac{1}{2}\vec{k})^2+\mu^2]}. \tag{13b}$$

[If the spatial wave functions which multiply Eq. (13a) are taken to be plane waves, then the matrix element reduces to Eq. (9c).]

For actual bremsstrahlung calculations it is convenient to have an analytic approximation to Eq. (13b). This can be obtained by expanding the integrand in powers of $k^2/(q^2+\mu^2)$ and $[\vec{k}\cdot\vec{q}/(q^2+\mu^2)]^2$. If $k\ll\mu$ this expansion is valid over the entire range of integration. But even if $k\approx\mu$, the approximation fails only for small q where the integrand is small, so we expect to obtain good results even for moderate values of k/μ. Proceeding in this way yields

$$I(\vec{k},\vec{r})=\frac{\pi^2}{\mu}e^{-\mu r}\left[1-\frac{1}{12}\left(\frac{k}{\mu}\right)^2(1+\mu r+\tfrac{1}{2}\mu^2r^2\cos^2\theta)+O(k^4)\right], \tag{14}$$

where θ is the angle between $\vec{k}$ and $\vec{r}$.

To see how well the approximation works, we made a Legendre polynomial expansion of Eq. (13b)

$$I(\vec{k},\vec{r})=\frac{4\pi}{k}\sum_{\text{even } l=0}^{\infty}(2l+1)i^l P_l(\cos\theta)\int_0^\infty dq\frac{qj_l(qr)Q_l(z)}{(q^2+\mu^2+k^2/4)} \tag{15}$$

with $z \equiv (q^2+\mu^2+k^2/4)/kq$, and performed the integrations on q numerically. (The $l=2$ term is only two percent of the $l=0$ term even for a photon energy of 200 MeV.) The results are shown on Fig. 8. For a photon energy of 50 MeV, the first term in Eq. (14), which is independent of k and θ, is an extremely good approximation of $I(\vec{k},\vec{r})$. For $k=150$ MeV (which is larger than the photon energies measured in any of the existing neutron-proton bremsstrahlung experiments) the second term in Eq. (14) is starting to become important, and including it gives a very good approximation.

Taking only the leading term from Eq. (14), expanding the factors of $e^{\pm\frac{1}{2}i\vec{k}\cdot\vec{r}}$ in Eq. (13a) out to the linear terms, and carrying out the integration over the c.m. coordinate we obtain

$$\vec{M}_2^{\rm OPE} = -e\int d^3r\,\varphi_f^{(-)*}(\vec{r})(\vec{\tau}_1\times\vec{\tau}_2)^z\left[\vec{r}\,V^{\rm OPE}(\vec{r}) + i\,\frac{f^2}{2}\,\frac{\vec{k}}{\mu}\times\vec{N}(\vec{r}) + O(k^2)\right]\varphi_i^{(+)}(\vec{r}), \tag{16a}$$

where

$$\vec{N}(\vec{r}) \equiv \left[\left(1+\frac{1}{\mu r}\right)\frac{\vec{\sigma}_1\times\vec{\sigma}_2\cdot\vec{r}}{r^2}\,\mathrm{r} - \sigma_1\times\sigma_2\right]e^{-\mu r} \tag{16b}$$

and the φ's are the wave functions for the relative motion.

If $\vec{M}_2^{\rm OPE}$ is sufficiently large compared to the rest of the bremsstrahlung matrix element, $\vec{M}_1$, (and if the experimental accuracy is great enough), then it may become necessary to include the k^2 term in Eq. (16a). This receives contributions of the form $(\vec{k}\cdot\vec{r})^2$ and $(\vec{\sigma}_1\cdot\vec{k})(\vec{\sigma}_2\cdot\vec{k})$ from Eq. (13) as well as the k^2 term from Eq. (14), and would be a straightforward addition to the calculation. In the limit $k\to 0$, Eq. (16a) agrees with the result found in Appendix A.[8]

Note that the first two terms of the expansion of $\int d^3x\,\vec{j}_2(\vec{x})e^{-i\vec{k}\cdot\vec{x}}$ can give a very good approximation to $\vec{M}_2^{\rm OPE}$ even when the first two terms of the expansion of $\vec{M}_2^{\rm OPE}$ itself (or the full bremsstrahlung matrix element $\vec{M}$) is not a good approximation. This is due to the fact that the k dependence of the *wave functions* can be considerably more complicated than that of the operator.

In order to project out the partial waves it is convenient to write Eq. (16a) in terms of irreducible tensor operators. The details are given in Appendix C, where the square bracket notation for coupling angular momenta is defined, and the result is shown in the following expression,

$$\begin{aligned}\vec{\epsilon}\cdot\vec{M}_2^{\rm OPE} = {}& -e(4\pi)^{1/2}f^2\sum_{M=-1}^{1}(-1)^M\int d^3r\,\varphi_f^*(\vec{r})(\vec{\tau}_1\times\vec{\tau}_2)^z e^{-\mu r}\\
&\times\Big\{\vec{\epsilon}_{-M}\Big[\sqrt{\tfrac{2}{5}}\left(\tfrac{1}{3}+\frac{1}{\mu r}+\frac{1}{\mu^2r^2}\right)\big(\sqrt{3}\,\{Y_3(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^2\}^1_M\\
&\quad-\sqrt{2}\,\{Y_1(\hat{r})\times[\sigma_1\times\sigma_2]^2\}^1_M\big)-\tfrac{1}{3}\{Y_1(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^0\}^1_M\Big]\\
&\quad-\frac{1}{3\mu}(\vec{\epsilon}\times\vec{k})_{-M}\Big[\left(1+\frac{1}{r\mu}\right)\{Y_2(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^1\}^1_M\\
&\quad+\sqrt{2}\left(1-\frac{1}{2\mu r}\right)\{Y_0(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^1\}^1_M\Big]+O(k^2)\Big\}\,\varphi_i(\vec{r}).\end{aligned} \tag{17}$$

The term $\vec{\epsilon}_M$ in Eq. (17) comes from $\vec{r}\,V^{\rm OPE}(\vec{r})$ in Eq. (16), while the term $(\vec{\epsilon}\times\vec{k})_{-M}$ arises from $if^2\vec{k}\times\vec{N}(\vec{r})/2\mu$. It is seen from Eq. (17) that the latter term contributes to the transitions ${}^1S_0 \to {}^3S_1$ of the neutron-proton system (while the former does not), and because of the importance of these states this operator may become significant even for small values of k/μ.

It is of some interest to evaluate the magnetic moment operator due to the current $\vec{j}_2^{\rm OPE}$. This was first done by Villars.[4] From the definition of the magnetic moment operator

$$\begin{aligned}\vec{m} &\equiv \tfrac{1}{2}\int d^3x\,\vec{x}\times\vec{j}(\vec{x})\\
&= \lim_{k\to 0}\frac{i}{2}\,\mathrm{curl}_k\int d^3x\,e^{-i\vec{k}\cdot\vec{x}}\,\vec{j}(\vec{x})\end{aligned} \tag{18}$$

it is seen that if the quantity $\int d^3\vec{x}\, e^{-i\vec{k}\cdot\vec{x}}\, \vec{j}_2^{\,\mathrm{OPE}}(\vec{x})$ is expanded in powers of k only the term of first order in k will contribute to $\vec{m}^{\mathrm{OPE}}$. From Eqs. (13) and (14)

$$\langle \vec{r},\vec{R}|\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\,\vec{j}_2^{\,\mathrm{OPE}}(\vec{x})|\vec{r}\,',\vec{R}'\rangle = -e(\vec{\tau}_1\times\vec{\tau}_2)^z\delta^3(\vec{R}-\vec{R}')\delta^3(\vec{r}-\vec{r}\,')$$
$$\times\left[\vec{r}\,V^{\mathrm{OPE}}(\vec{r})+i\frac{f^2}{2}\frac{\vec{k}}{\mu}\times\vec{N}(\vec{r})-i(\vec{k}\cdot\vec{R})\vec{r}\,V^{\mathrm{OPE}}(\vec{r})+\mathcal{O}(k^2)\right]. \tag{19}$$

Combining Eqs. (18) and (19) gives the local operator

$$\vec{m}^{\mathrm{OPE}}=-\tfrac{1}{2}e(\vec{\tau}_1\times\vec{\tau}_2)^z\left[f^2\vec{N}(\vec{r})/\mu+(\vec{R}\times\vec{r})V^{\mathrm{OPE}}(r)\right] \tag{20}$$

which is Villars' result.

V. DISCUSSION

We have obtained the electromagnetic current operator due to the exchange of a charged pion, and have expanded its Fourier transform in powers of the photon's energy. The contribution to the bremsstrahlung matrix element from this current is shown in Eq. (16). In Eq. (17) the matrix element is rewritten in terms of irreducible tensors, and this (together with the formulas in Appendix C) is all one needs to incorporate $\vec{j}_2^{\,\mathrm{OPE}}$ into calculations of neutron-proton bremsstrahlung.

What should be done with the other nonlocal and exchange terms in the neutron-proton potential? If these potentials were obtained from Feynman diagrams, one could attempt to carry out the same program as was done here for single-pion exchange by inserting a photon at all possible places in these diagrams. For the exchange of a single heavy boson one will obtain a good approximation,[9] for values of k which are not too large, by using the $k=0$ *limit of the operator*, as given in Eq. (A2), Appendix A. [This is the generalization of the first term in Eq. (16a) to the case where the direct and/or exchange potential are nonlocal.] Equation (16a) shows that the corrections to this operator are of order (photon energy)/(meson mass), and will be small for a heavy meson.

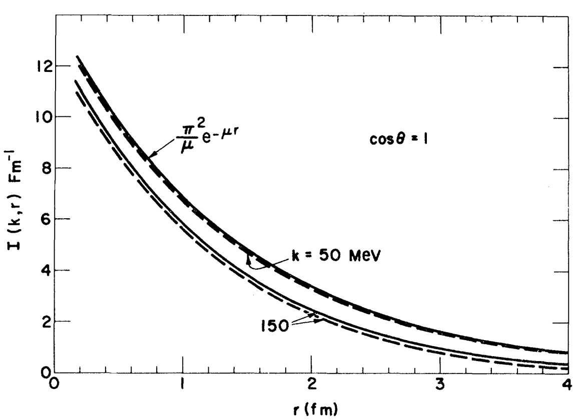

FIG. 8. The function $I(\vec{k},\vec{r})$ defined in Eq. (13b), for $\cos\theta=1$, where θ is the angle between $\vec{k}$ and $\vec{r}$. The lower solid curve is an exact evaluation [obtained from Eq. (15)] for $k=150$ MeV. The dashed curves are the approximation given by Eq. (14), in which terms of order k^4 are neglected. The upper solid curve is the simplest approximation which neglects all k (and θ) dependence. At $k=50$ MeV the exact result (not shown) falls between the two approximations. Even at $k=150$ MeV the simple approximation $\pi^2 e^{(-\mu r)}/\mu$ is not bad.

The most important diagrams for which this approximation is not adequate are those which give rise to the two-pion-exchange potential. One should compare the sum of all such diagrams involving a photon, to the Schrödinger equation calculation correct *to order* eG^4 [including $V^{\mathrm{OPE}}+V^{\mathrm{TPE}}$ and $(\rho_1,\vec{j}_1+\vec{j}_2^{\,\mathrm{OPE}})$] and thereby deduce the correct form for $(\rho_2^{\mathrm{TPE}},\vec{j}_2^{\,\mathrm{TPE}})$ to this order. This would be a moderately involved task, and would again raise the question of the correct treatment of N^* resonances.

All of the nucleon-nucleon potentials which do a fair job of fitting the elastic scattering data have some nonlocalities which are strictly phenomenological. As pointed out in the Introduction, the best one can do is to try different forms for the current $\vec{j}_2$ to be associated with each such nonlocal term in the potential and hope that the variation in the computed bremsstrahlung cross sections is small compared to the experimental uncertainty. In this connection it is worth remembering that there is one choice for $\vec{j}_2(\vec{x})$, called the "maximal" current,[2] which, although very unphysical because it falls off only as x^{-3} at infinity,[10] is especially simple to calculate with. It produces *no* addition to the bremsstrahlung matrix element.

ACKNOWLDEGMENTS

We would like to thank V. Brown and J. Franklin for a conversation about their work on exchange current. L. H. is indebted to R. Amado for a helpful discussion about gauge invariance, and would also like to thank M. Johnson and E. Lomon for their views on the recoil emission diagram. R. H. T. acknowledges a valuable discussion with M. Bolsterli about irreducible tensor operators.

APPENDIX A. ZERO PHOTON ENERGY

In the limit of zero photon energy, there is a *unique* value for the contribution of $\vec{j}_2$ to the bremsstrahlung matrix element, determined entirely by the expression for $\vec{\nabla}\cdot\vec{j}_2$, provided $\vec{j}_2$ falls off more rapidly than x^{-3} at infinity.[10] This is obtained from an integration by parts, with the surface term vanishing because of the assumed behavior at infinity:

$$\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, \vec{j}_2(\vec{x}) \underset{k\to 0}{=} \int d^3x\, \vec{j}_2(\vec{x}) = -\int d^3x\, \vec{x}\,[\vec{\nabla}\cdot\vec{j}_2(\vec{x})]\ . \tag{A1}$$

Using Eq. (4) from the text gives

$$\begin{aligned}\langle \vec{r},\vec{R}\left|\int d^3x\,\vec{j}_2(\vec{x})\right|\vec{r}',\vec{R}'\rangle = -\frac{e}{2}\,\delta^{(3)}(\vec{R}-\vec{R}')\,[\tfrac{1}{2}i(\tau_1^z-\tau_2^z)(\vec{r}-\vec{r}')\langle\vec{r}|V_D-V_E|\vec{r}'\rangle \\ +(\vec{\tau}_1\times\vec{\tau}_2)^z(\vec{r}+\vec{r}')\langle\vec{r}|V_E|r'\rangle],\end{aligned} \tag{A2}$$

and with some isospin algebra, Eqs. (7) and (A2) lead to

$$\lim_{k\to 0}\vec{M}_2 = \frac{ie}{4}\int d^3r\,\vec{r}\,[\langle\varphi_f^{(-)}|V|\vec{r}\rangle(\tau_1^z-\tau_2^z)\varphi_i^{(+)}(\vec{r}) - \varphi_f^{(-)*}(\vec{r})(\tau_1^z-\tau_2^z)\langle\vec{r}|V|\varphi_i^{(+)}\rangle]\ . \tag{A3}$$

Since the states φ_i and φ_f describing the relative motion of the two particles are solutions of the Schrödinger equation, the potential energy in Eq. (A3) can be expressed in terms of the kinetic and total energies.[11]

For the special case of a potential with no spatial nonlocality, such as V^{OPE}, the term in Eq. (A2) proportional to $(\vec{r}-\vec{r}')$ vanishes, and one obtains the first term in Eq. (16a) in the limit $k\to 0$.

APPENDIX B. RECOIL EMISSION DIAGRAM

In Sec. III of the text it is stated that the portions of the Feynman diagrams in Figs. 2(a)–2(d) in which the radiating nucleon propagates in a positive-energy state agrees with the Schrödinger equation matrix element to within a relative accuracy $\Theta(p/m)^2$. This might appear surprising when one thinks of the time ordering on the nonradiating nucleon line of Fig. 2(b) as shown on Fig. 7. [The sum of the three time-ordered contributions in Fig. 7 is just the same as the portion of the Feynman diagram Fig. 2(b) in which the radiating nucleon propagates in a positive energy state.] In Fig. 7(a) the pion is emitted *before* the photon, and absorbed *after*, and it is often stated that this "recoil" emission contribution is not contained in the Schrödinger matrix element because the potential in the Schrödinger equation acts instantaneously. We conclude that this statement is incorrect by simply evaluating the three portions of Fig. 7, called T_a, T_b, and T_c, and showing that $T_a+T_b+T_c$ is a *better* approximation to the Schrödinger matrix element than T_b+T_c.

The essential point of the demonstration has to do with the energy denominators, and this is all we write down. The additional factors coming from the spins and pseudoscalar nature of the pion do not change the conclusion.

Defining $E\equiv(\vec{p}^2+m^2)^{1/2}$, $\omega\equiv(q^2+\mu^2)^{1/2}$, where $\vec{p}$ and $\vec{q}$ are the momenta of the virtual nucleon and pion as shown on Fig. 7, the three contributions can be written as

$$\begin{aligned}T_a &= \frac{1}{2\omega}\,\frac{m}{E}\,\frac{1}{E_2-E_2'-\omega}\,\frac{1}{E_1'-E-\omega}\ ,\\ T_b &= \frac{1}{2\omega}\,\frac{m}{E}\,\frac{1}{E_1-E-k}\,\frac{1}{E_1'-E-\omega}\\ &= \frac{1}{2\omega}\,\frac{m}{E}\,\frac{1}{E_2-E_2'-\omega}\left(\frac{1}{E_1-E-k}-\frac{1}{E_1'-E-\omega}\right),\\ T_c &= \frac{1}{2\omega}\,\frac{m}{E}\,\frac{1}{E_1-E-k}\,\frac{1}{E_1-E_1'-\omega-k}\ ,\end{aligned} \tag{B1}$$

where we have made use of energy conservation. Combining terms

$$\begin{aligned}T_b+T_c = \frac{1}{2\omega}\,\frac{m}{E}\,\frac{1}{E_1-E-k}\\ \times\left[\frac{-2\omega+(E_1-E-k)}{\omega^2-\omega(E_1-E-k)+(E_1'-E)(E_1-E_1'-k)}\right]\end{aligned} \tag{B2}$$

and going to the over-all c.m. system, $\vec{p}_1+\vec{p}_2=0=\vec{p}_1'+\vec{p}_2'+\vec{k}$, where one can show that $k\lesssim 2p_1q/m$, a little algebra gives

$$\begin{aligned}T_b+T_c = -\frac{1}{\omega}\,\frac{1}{E_1-E-k}\,\frac{1}{\omega-\frac{1}{2}(E_1-E-k)}\\ \times\left[1+\Theta\left(\frac{p_1}{m}\right)^2\right].\end{aligned} \tag{B3}$$

From Eq. (B1) it is seen that when T_a and the second version of T_b are added together a cancellation occurs which leads to

$$T_a + T_b + T_c = \frac{1}{2\omega}\frac{m}{E}\frac{1}{E_1 - E - k}\times\left(\frac{1}{E_2 - E_2' - \omega} - \frac{1}{E_2 - E_2' + \omega}\right) = \frac{m}{E}\frac{1}{E_1 - E - k}\frac{1}{(E_2 - E_2')^2 - \omega^2} \tag{B4}$$

which is just the original Feynman matrix element, Fig. 2(b), with the radiating nucleon restricted to positive energy. One shows directly that $(E_2 - E_2')^2 < (p_1^2/m^2)\omega^2$ and therefore

$$T_a + T_b + T_c = -\frac{1}{E_1 - E - k}\frac{1}{\omega^2}\left[1 + \mathcal{O}\left(\frac{p_1^2}{m^2}\right)\right]. \tag{B5}$$

Except for the relativistic correction factor this is precisely the result obtained from the Schrödinger equation, so

$$T_a + T_b + T_c - T_{\rm sch} = \mathcal{O}\left(\frac{p_1^2}{m^2}\right)T_{\rm sch} = \mathcal{O}\left(\frac{p_1^2}{m^2 k\omega^2}\right). \tag{B6}$$

Returning to Eq. (B3),

$$T_b + T_c - T_{\rm sch} = -\frac{1}{2\omega^2}\frac{1}{\omega - \frac{1}{2}(E_1 - E - k)} + \mathcal{O}\left(\frac{p_1^2}{m^2 k\omega^2}\right) = \mathcal{O}\left(\frac{1}{\omega^3}\right) + \mathcal{O}\left(\frac{p_1^2}{m^2 k\omega^2}\right). \tag{B7}$$

If $(k/\omega) \lesssim (p_1/m)^2$, then the second term on the second line of Eq. (B7) is at least as important as the first, in which case the error in the Schrödinger result is essentially the same whether or not the recoil emission diagram Fig. 7(a) is included. This is due to the fact that this diagram does not have the infrared divergence. If on the other hand $k \sim p_1\omega/m$, then the error is *less* if the recoil emission diagram is *included*. This means that the Schrödinger matrix element already contains the recoil contribution, and therefore Fig. 7(a) should *not* be added on as an additional contribution.

APPENDIX C.

Two irreducible tensors, $A^{l_1}_{m_1}$ and $B^{l_2}_{m_2}$, are coupled to form a third tensor, C^L_M, as in Messiah[12]

$$C^L_M \equiv [A^{l_1} \times B^{l_2}]^L_M = (-1)^{l_1 - l_2 + M}(2L+1)^{1/2}\sum_{m_1 m_2}\begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M\end{pmatrix} A^{l_1}_{m_1} B^{l_2}_{m_2}, \tag{C1}$$

where

$$\begin{pmatrix} l_1 & l_2 & L \\ m_1 & m_2 & -M\end{pmatrix}$$

is a $3j$ symbol. The spherical components of a vector are defined by

$$a_1 = -\frac{(a_x + i a_y)}{\sqrt{2}}, \quad a_0 = a_z, \quad a_{-1} = \frac{(a_x - i a_y)}{\sqrt{2}}, \tag{C2}$$

and the spherical harmonics are related to the coordinate vector by

$$Y_1^M(\hat{r}) = \left(\frac{3}{4\pi}\right)^{1/2}\frac{r_M}{r}. \tag{C3}$$

Some useful relationships are

$$-\sqrt{3}\,[a^1 \times b^1]^0_0 = \vec{a}\cdot\vec{b}, \tag{C4}$$

$$-i\sqrt{2}\,[a^1 \times b^1]^1_M = (\vec{a}\times\vec{b})_M, \tag{C5}$$

$$\left[Y_{l_1}(\hat{r})\times Y_{l_2}(\hat{r})\right]^L = \frac{(2l_1+1)^{1/2}(2l_2+1)^{1/2}}{\sqrt{4\pi}}(-1)^{l_1 - l_2}\begin{pmatrix} l_1 & l_2 & L \\ 0 & 0 & 0\end{pmatrix} Y_L(\hat{r}), \tag{C6}$$

$$\sigma_M - 3(\vec{r}\cdot\vec{\sigma})r_M/r^2 = \sqrt{8\pi}\,[Y_2 \times \vec{\sigma}]^1_M, \tag{C7}$$

and the recoupling formula

$$[A^{j_1}\times[B^{j_2}\times C^{j_3}]^{j_{23}}]^J=\sum_{j_{12}}[(2j_{12}+1)(2j_{23}+1)]^{1/2}(-1)^{j_1+j_2+j_3+J}\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{Bmatrix}[[A^{j_1}\times B^{j_2}]^{j_{12}}\times C^{j_3}]^J\,, \tag{C8}$$

where

$$\begin{Bmatrix} j_1 & j_2 & j_{12}\\ j_3 & J & j_{23}\end{Bmatrix}$$

is a $6j$ symbol. Using the above relations one can rewrite Eq. (16) as Eq. (17). It is now a simple matter to project the partial waves from $\epsilon\cdot\vec{\mathrm{M}}_2^{\mathrm{OPE}}$, Eq. (17), using the following equations,

$$\langle lSJM|\{Y_L(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^{S''}\}^1_{M''}|l'S'J'M'\rangle=(-1)^{J-M}\begin{pmatrix} J & 1 & J'\\ -M & M'' & M'\end{pmatrix}\langle lSJ\|[Y_L(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^{S''}]^1\|l'S'J'\rangle\,, \tag{C9}$$

$$\langle lSJ\|(Y_L(\hat{r})\times[\vec{\sigma}_1\times\vec{\sigma}_2]^{S''})^1\|l'S'J'\rangle=[(2J+1)(2J'+1)3]^{1/2}\begin{Bmatrix} l' & S' & J'\\ L & S'' & 1\\ l & S & J\end{Bmatrix}\langle l\|Y_L(\hat{r})\|l'\rangle\langle S\|[\vec{\sigma}_1\times\vec{\sigma}_2]^{S''}\|S'\rangle\,, \tag{C10}$$

where

$$\langle l\|Y_L(\hat{r})\|l'\rangle=(-1)^l\left[\frac{(2l+1)(2l'+1)(2L+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l & L & l'\\ 0 & 0 & 0\end{pmatrix} \tag{C11}$$

and

$$\langle S\|[\vec{\sigma}_1\times\vec{\sigma}_2]^{S''}\|S'\rangle=6[(2S+1)(2S'+1)(2S''+1)]^{1/2}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & S'\\ 1 & 1 & S''\\ \frac{1}{2} & \frac{1}{2} & S\end{Bmatrix}. \tag{C12}$$

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[1]This is not necessarily disastrous. The whole range of possible choices may lead to only slightly different bremsstrahlung matrix elements. This will obviously be the case if the nonlocal potential is sufficiently weak, which seems to be true for the phenomenological spin-orbit potential.

[2]L. Heller, in *The Two-Body Force in Nuclei*, edited by S. M. Austin and G. M. Crawley (Plenum, New York 1972), p. 79.

[3]R. K. Osborn and L. L. Foldy, Phys. Rev. **79**, 795 (1950).

[4]F. Villars, Phys. Rev. **72**, 256 (1947).

[5]S. Wahlborn and J. Blomqvist, Nucl. Phys. **133A**, 50 (1969).

[6]† stands for complex conjugate and transposing the spin and isospin indices; T stands for transposing the spin and isospin indices.

[7]Equation (28) in Ref. 5 is equivalent to our Eq. (5) provided no derivatives of the current operator are needed. If they are, our version must be used.

[8]V. R. Brown and J. Franklin have carried out calculations of neutron-proton bremsstrahlung including the $k=0$ limit of the operator. See *The Two-Body Force in Nuclei*, (Ref. 2), p. 123; in International Conference on Few Particle Problems in Nuclear Interaction, Los Angeles, 1972 (to be published).

[9]This suggestion is due to J. Franklin.

[10]If $\vec{j}_2(\vec{x})$ does not fall off faster than x^{-3} at infinity, the soft photon theorem for bremsstrahlung is violated. See Ref. 2.

[11]A formula like this appears as Eq. (2) in the final reference of Ref. 8, but lacking the factor $\frac{1}{2}(\tau_1^z-\tau_2^z)$ from Eq. (A3). The authors say that the initial and final states must have different isospin, but it is not clear that the *sign* of the matrix element is correct. This is important for the interference with $\vec{\mathrm{M}}_1$.

[12]A. Messiah, *Quantum Mechanics* (Wiley, New York, 1966), Vol. II, Appendix C.