reaction is then probably not a nuclear structure effect, but rather a facet of the reaction mechanism. Future studies of the (^{3}He, ^{6}He) reaction on targets of ^{26}Mg and ^{14}C will be useful in answering the questions posed by the present experiment.

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Optical Potential for Scattering from a System of Finite-Mass Particles*

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We consider a projectile scattering elastically from a system of finite-mass constituents via a separable microscopic interaction. In order to reduce the situation to a solvable problem, several standard assumptions are necessary. The validity of these assumptions can be checked in a given model problem. The optical potential is identified by comparing the multiple scattering series we obtain with that obtained in the equivalent one-body problem. The optical potential explicitly exhibits the effects of the Fermi motion and finite mass of the target particles and is a generalization of an optical potential obtained earlier. Our results are in a form suitable for application to intermediate energy projectile many-body target scattering.

I. INTRODUCTION

The concept of the optical potential has been extremely useful for reducing the complexities of the many-body elastic scattering problem to the simplicity of the equivalent one-body problem. Optical potentials not only provide a convenient way of describing elastic scattering but yield valuable input, in the form of distorted waves, for currently fashionable approaches to inelastic scattering and reaction processes such as the distorted-wave Born approximation. Of course one wishes to understand the relationship between the optical potential and the more elementary interactions between the (perhaps complex) projectile and the individual constituents of the target. In this way, for example, one may limit the geometrical forms adopted for the optical potential whose parameters are to be obtained by fitting to a given experiment. In addition, by studying microscopic theories of the optical potential one gains some insight into (1) the limits of validity of the concept and (2) the dependence of the optical potential on the energy of the projectile and the detailed characteristics of the target.

The microscopic theory uses as input the elementary two-body (projectile-elementary target constituent) interaction which is usually obtained from experimental two-particle scattering. Often the concept of a two-body potential is avoided by using the experimental data directly to define the on-shell two-particle t matrix, then some multiple scattering theory is used to relate the two-body t matrix to the optical potential. In fact, of course, several important approximations are necessary in order to use the free two-particle t matrices in a tractable way in the many-body problem. The two-particle t matrix in the many-body problem can, in general, be considerably different than the one obtained from free two-body scattering (the kinematical restrictions and relations are of course much different because of the presence of more than two bodies.) It appears that for projectiles at sufficiently high energy the problem greatly simplifies because the approximations usually adopted [for example, see Kerman, McManus, and Thaler[1] (KMT)] in the impulse approximation become appropriate.

Recently Foldy and Walecka[2] (FW) have investigated, in considerable detail, the theory of the optical potential for a projectile elastically scattering from a system of A fixed scattering centers (i.e., infinite-mass constituents). Two basic assumptions utilized in their elegant discussion are first, the projectile energy is high enough so that closure may be used on the target and second, the projectile-single target particle potential is separable. The lowest-order optical potential obtained by FW not only includes in a natural way the two-particle t matrix and the bound target particle momentum distribution, but also allows for off-shell intermediate propagation of the projectile between scatterings. Using further limiting approximations FW make connection with the simple high-energy Glauber approximation[3] for the optical potential.

Our purpose in this paper is to consider the same situation as FW but not assume that the individual target particles are infinitely heavy. This requires that certain δ functions missing in the FW discussion must now be included in the scattering equations. Relatedly, the microscopic potential becomes nonlocal in both the projectile and target particle coordinates. Because of these additional complications we are forced to use a different approach than that followed by FW. However, we use exactly the same assumptions as adopted in Ref. 2 except for an important "angle average" approximation adopted in order to obtain a closed form expression at one point in the discussion. It is argued that, in nontrivial situations, this additional approximation may be valid when one does not wish to assume the individual target particles are infinitely massive. The validity of the approximation can be checked in a given model. The result we obtain for the optical potential shows explicitly the effect of the finite mass of the constituent particle and demonstrates several effects due to the target particle's Fermi momentum. While our result for the optical potential is naturally more complicated than that obtained earlier it still remains tractable and therefore should be useful in future applications (for example, in studying intermediate energy pion-nucleus scattering). It is also useful because it explicitly demonstrates how, under the assumptions adopted, an effective two-particle t matrix in the many-body problem can be expected to differ from that obtained in free two-particle scattering. Of course if one assumes that the target particles are infinitely heavy the optical potential contained herein reduces to the lowest-order optical potential obtained by FW.

In the next section we consider the problem in the simplified setting of a two-particle interaction. Drawing on the results and techniques developed in Sec. II, we tackle the many-particle situation in Sec. III. The result for the optical potential, which will correctly reproduce the elastic scattering amplitude as obtained from the multiple scattering theory, is the most important result contained in Sec. III. Finally, in the last section we review the approximations leading to the results obtained in this work, identify and discuss the main results, and suggest applications or further generalizations.

II. TWO-BODY PROBLEM

In this section we define the basic two-body separable interaction that will be assumed in the many-body problem treated in the next section. In addition we briefly review the two-body scattering problem for the case of a nonlocal separable Galilean-invariant potential. The basic procedure adopted here will appear in a more complicated setting in Sec. III.

Consider the Schrödinger equation for two particles interacting via a two-body potential

$$\left(-\frac{\hbar^2}{2m_1}\nabla^2_{\vec{r}_1}-\frac{\hbar^2}{2m_2}\nabla^2_{\vec{r}_2}\right)\psi(\vec{r}_1,\vec{r}_2)+\int v(\vec{r}_1,\vec{r}_2;\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime})\psi(\vec{r}_1^{\,\prime},\vec{r}_2^{\,\prime})\,d\vec{r}_1^{\,\prime}\,d\vec{r}_2^{\,\prime}=E\psi(\vec{r}_1,\vec{r}_2)\,. \tag{2.1}$$

A simple special case of the operator $v(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2')$ that is frequently encountered is given by

$$v(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2') = v(|\vec{r}_1 - \vec{r}_2|)\delta(\vec{r}_1 - \vec{r}_1')\delta(\vec{r}_2 - \vec{r}_2') \tag{2.2}$$

(i.e., a local, Galilean- and rotationally-invariant potential). The familiar Galilean-invariant potential which is nonlocal in the relative coordinate, $\vec{r}_1 - \vec{r}_2$, may be written in the form

$$v(\vec{r}_1, \vec{r}_2; \vec{r}_1', \vec{r}_2') = v(\vec{r}_1 - \vec{r}_2, \vec{r}_1' - \vec{r}_2')\delta\left[\frac{1}{m_1+m_2}(m_1\vec{r}_1 + m_2\vec{r}_2) - \frac{1}{m_1+m_2}(m_1\vec{r}_1' + m_2\vec{r}_2')\right]. \tag{2.3}$$

Using the standard change of variables,

$$\begin{aligned} &\vec{r} = \vec{r}_1 - \vec{r}_2, \quad \vec{R} = \frac{1}{m_1+m_2}(m_1\vec{r}_1 + m_2\vec{r}_2), \\ &\vec{r}' = \vec{r}_1' - \vec{r}_2', \quad \vec{R}' = \frac{1}{m_1+m_2}(m_1\vec{r}_1' + m_2\vec{r}_2'), \\ &M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{m_1+m_2}, \end{aligned} \tag{2.4}$$

and adopting the potential given in Eq. (2.3) we may rewrite Eq. (2.1) in the form

$$\left(-\frac{\hbar^2}{2M}\nabla_{\vec{R}}^2 - \frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2\right)\psi(\vec{r}, \vec{R}) + \int v(\vec{r}, \vec{r}')\delta(\vec{R} - \vec{R}')\psi(\vec{r}', \vec{R}')\,d\vec{r}'\,d\vec{R}' = E\psi(\vec{r}, \vec{R}). \tag{2.5}$$

Making use of the δ function appearing in Eq. (2.5), and following the standard procedure one can break up Eq. (2.5) into the two differential equations,

$$-\frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2\varphi(\vec{r}) + \int v(\vec{r}, \vec{r}')\varphi(\vec{r}')d\vec{r}' = \lambda\varphi(\vec{r}), \tag{2.6a}$$

$$-\frac{\hbar^2}{2M}\nabla_{\vec{R}}^2\chi(\vec{R}) = (E - \lambda)\chi(\vec{R}), \tag{2.6b}$$

where

$$\psi(\vec{r}, \vec{R}) = \varphi(\vec{r})\chi(\vec{R}). \tag{2.7}$$

We now investigate the equation involving only the relative coordinate [Eq. (2.6a)] and work in the center-of-momentum (c.m.) system. It is important to note that in order to reduce the two-body problem to an effective one-body problem in the c.m. system it was necessary to make use of the δ function appearing in Eq. (2.5). In the problem we consider we shall be able to obtain the potential from a knowledge of the phase shifts in the c.m. system (the inverse scattering problem). If this potential is utilized in the many-body problem we must be careful to also include the term $\delta(\vec{R} - \vec{R}')$. [Note: Since $\delta(\vec{R} - \vec{R}')$ is a two-body c.m. coordinate δ function it will not, in general, trivially disappear in the many-body problem.] If the energy originally available in the lab system (where particle 2 is originally at rest) is denoted by E_{lab} then the energy available in the c.m. system is

$$E_{\text{c.m.}} = \frac{m_2}{m_1+m_2}E_{\text{lab}}. \tag{2.8}$$

We may rewrite Eq. (2.6a) as an integral equation (with the outgoing wave boundary condition)

$$\varphi^{(+)}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \int \frac{e^{i\vec{t}\cdot(\vec{r}-\vec{r}')}}{t^2 - k^2 - i\epsilon}\frac{d\vec{t}}{(2\pi)^3}\mathcal{V}(\vec{r}', \vec{r}'')\varphi(\vec{r}'')d\vec{r}'\,d\vec{r}'', \tag{2.9}$$

where

$$\mathcal{V}(\vec{r}', \vec{r}'') \equiv \frac{2\mu}{\hbar^2}v(\vec{r}', \vec{r}''), \tag{2.10}$$

and where

$$k^2 = \frac{2\mu}{\hbar^2}E_{\text{c.m.}} = \frac{2\mu}{\hbar^2}\frac{m_2}{M}E_{\text{lab}} = \left(\frac{m_2}{m_1+m_2}\right)^2 k_{1(\text{lab})}{}^2. \tag{2.11}$$

The elastic scattering amplitude may be determined from

$$f(\vec{k}',\vec{k}) = -\frac{1}{4\pi}\int e^{-i\vec{k}'\cdot\vec{r}}\,\mathcal{V}(\vec{r},\vec{r}\,')\varphi^{(+)}(\vec{r}\,')d\vec{r}\,d\vec{r}\,' . \tag{2.12}$$

The particular nonlocal potential $v(\vec{r},\vec{r}\,')$ assumed in this paper is both separable and rotationally invariant and may be written as

$$v(\vec{r},\vec{r}\,') = \sum_{lm} 4\pi\lambda_l v_l(r)v_l(r')Y_{lm}(\Omega_{\vec{r}})Y^*_{lm}(\Omega_{\vec{r}\,'}) . \tag{2.13}$$

Now substituting the expression (2.13), for $v(\vec{r},\vec{r}\,')$, into Eqs. (2.9) and 2.12), iterating Eq. (2.9) and substituting the result into Eq. (2.12) allows the scattering amplitude to be written as

$$\begin{aligned}
-4\pi f(k',k) = &\int e^{-i\vec{k}'\cdot\vec{r}_0}\frac{2\mu}{\hbar^2}\sum_{l_0m_0}4\pi\lambda_{l_0}v_{l_0}(r_0)v_{l_0}(r_1)Y_{l_0m_0}(\Omega_{\vec{r}_0})Y^*_{l_0m_0}(\Omega_{\vec{r}_1})e^{i\vec{k}\cdot\vec{r}_1}d\vec{r}_0\,d\vec{r}_1\\
&-\int e^{-i\vec{k}'\cdot\vec{r}_0}\frac{2\mu}{\hbar^2}\sum_{l_0m_0}4\pi\lambda_{l_0}v_{l_0}(r_0)v_{l_0}(r_1)Y_{l_0m_0}(\Omega_{\vec{r}_0})Y^*_{l_0m_0}(\Omega_{\vec{r}_1})\\
&\times\int\frac{d\vec{t}_1}{(2\pi)^3}\frac{e^{i\vec{t}_1\cdot(\vec{r}_1-\vec{r}_2)}}{t_1^2-k^2-i\epsilon}\frac{2\mu}{\hbar^2}\sum_{l_1m_1}4\pi\lambda_{l_1}v_{l_1}(r_2)v_{l_1}(r_3)Y_{l_1m_1}(\Omega_{\vec{r}_2})Y^*_{l_1m_1}(\Omega_{\vec{r}_3})e^{i\vec{k}\cdot\vec{r}_3}\prod_{j=0}^{3}d\vec{r}_j\\
&+\cdots+\int\cdots\int\left(\prod_{j=0}^{2n-1}d\vec{r}_j\right)\Big\{(-1)^{n-1}\int e^{-i\vec{k}'\cdot\vec{r}_0}\frac{2\mu}{\hbar^2}\sum_{l_0m_0}4\pi\lambda_{l_0}v_{l_0}(r_0)v_{l_0}(r_1)Y_{l_0m_0}(\Omega_{\vec{r}_0})Y^*_{l_0m_0}(\Omega_{\vec{r}_1})\\
&\qquad\times\prod_{i=1}^{n-1}\Bigg[\int\frac{d\vec{t}_i}{(2\pi)^3}\frac{e^{i\vec{t}_i\cdot(\vec{r}_{2i-1}-\vec{r}_{2i})}}{t_i^2-k^2-i\epsilon}\frac{2\mu}{\hbar^2}\\
&\qquad\qquad\times\sum_{l_im_i}4\pi\lambda_{l_i}v_{l_i}(r_{2i})v_{l_i}(r_{2i+1})Y_{l_im_i}(\Omega_{\vec{r}_{2i}})Y^*_{l_im_i}(\Omega_{\vec{r}_{2i+1}})\Bigg]_i\\
&\qquad\qquad\times e^{i\vec{k}\cdot\vec{r}_{2n-1}}\Big\}+\cdots .
\end{aligned}\tag{2.14}$$

We now concentrate on the nth order term. First substitute the partial wave expansion for a plane wave

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi\sum_{lm}(i)^l j_l(kr)Y_{lm}(\Omega_{\vec{r}})Y^*_{lm}(\Omega_{\vec{k}}) \tag{2.15}$$

into Eq. (2.14) everywhere a plane-wave term appears. The resulting expression can be easily simplified by carrying out the integrations over the angles of $\vec{r}_0$, $\vec{r}_n$, and all the $\vec{t}_i$ and $\vec{r}_i$. Because of the orthogonality relation for spherical harmonics the angular integrations collapse the many sums over various l and m's to a single sum, say over l and m. The resulting expression for the nth order term ($n>1$) in Eq. (2.14) is given by

$$\begin{aligned}
-4\pi f_n(\vec{k}',\vec{k}) = &\int\cdots\int\left(\prod_{j=0}^{2n-1}r_j^2\,dr_j\right)\Big\{(-1)^{n-1}\sum_{lm}4\pi\int(-i)^l j_l(k'r_0)\frac{2\mu}{\hbar^2}4\pi\lambda_l v_l(r_0)Y_{lm}(\Omega_{\vec{k}'})v_l(r_1)\\
&\times\prod_{i=1}^{n-1}\Bigg[\frac{(4\pi)^2}{(2\pi)^3}\int\frac{t_i^2dt_i}{t_i^2-k^2-i\epsilon}j_l(t_ir_{2i-1})j_l(t_ir_{2i})\frac{2\mu}{\hbar^2}\\
&\times4\pi\lambda_l v_l(r_{2i})v_l(r_{2i+1})\Bigg]_i\Big\}4\pi(i)^l j_l(kr_{2n-1})Y^*_{lm}(\Omega_{\vec{k}}) .
\end{aligned}\tag{2.16}$$

Now defining

$$v_{lm}(\vec{k}) \equiv \sqrt{4\pi}\,i^l\,Y^*_{lm}(\Omega_{\vec{k}})v_l(k) \tag{2.17a}$$

and

$$v_l(k) \equiv 4\pi\int v_l(x)j_l(kx)x^2\,dx \tag{2.17b}$$

and utilizing the property

$$\sum_m v^*_{lm}(\vec{k}')v_{lm}(\vec{k}) = (2l+1)[v_l(k)]^2P_l(\cos\theta_{k'k}) , \tag{2.17c}$$

we can rewrite Eq. (2.16) as (note $|\vec{k}|=|\vec{k}'|$)

$$-4\pi f_n(\vec{k}',\vec{k})=(-1)^{n-1}\frac{2\mu}{\hbar^2}\sum_l \lambda_l|v_l(k)|^2(2l+1)P_l(\cos\theta_{k'k})\left(\frac{2\mu}{\hbar^2}\frac{\lambda_l}{(2\pi)^3}\int\frac{t^2dt4\pi|v_l(t)|^2}{t^2-k^2-i\epsilon}\right)^{n-1}. \tag{2.18}$$

Now if we define

$$X\equiv\left(\frac{2\mu}{\hbar^2}\frac{\lambda_l}{(2\pi)^3}\int\frac{d\vec{t}}{t^2-k^2-i\epsilon}|v_l(t)|^2\right), \tag{2.19}$$

then using Eq. (2.18) (the result for the nth order term) we see that the scattering amplitude, Eq. (2.14), may be written

$$f(\vec{k}',\vec{k})=-\frac{1}{4\pi}\sum_l\lambda_l\frac{2\mu}{\hbar^2}|v_l(k)|^2(2l+1)P_l(\cos\theta_{k',k})\times[1-X+X^2\cdots]$$

$$=-\frac{1}{4\pi}\sum_l\frac{\lambda_l(2\mu/\hbar^2)|v_l(k)|^2(2l+1)P_l(\cos\theta_{k',k})}{1+[\lambda_l/(2\pi)^3](2\mu/\hbar^2)\int[d\vec{t}|v_l(t)|^2/(t^2-k^2-i\epsilon)]}. \tag{2.20}$$

The familiar expansion for the scattering amplitude in terms of the c.m. scattering phase shifts is written

$$f(k',k)=\sum_l\frac{e^{i\delta_l(k)}\sin\delta_l(k)}{k}(2l+1)P_l(\cos\theta_{k'k}) \tag{2.21}$$

so that we make the identification [using Eq. (2.20)]

$$\frac{e^{i\delta_l(k)}\sin\delta_l(k)}{k}=\frac{-(\lambda_l/4\pi)(2\mu/\hbar^2)|v_l(k)|^2}{1+[\lambda_l/(2\pi)^3](2\mu/\hbar^2)\int[d\vec{t}\,|v_l(t)|^2/(t^2-k^2-i\epsilon)]}. \tag{2.22}$$

We remind the reader that these results are standard and our main purposes here are to define some basic quantities and to apply the basic techniques, to be used in the next section, in a situation where the notation is not overly burdensome. Equation (2.22), which relates the phase shifts to the separable potential may be "inverted" so that given the phase shift in the lth partial wave at all energies allows one to obtain $v_l(k)$. The inverse scattering problem for separable potentials has been the subject of considerable study[4-6] and often allows particularly simple solutions. For example if one assumes $\delta_l(k=0)-\delta_l(k=\infty)=0$ then several authors have shown that the potential may be obtained from the phase shifts via[4-6]

$$-\lambda_l\frac{2\mu}{\hbar^2}[v_l(k)]^2=4\pi\frac{\sin\delta_l(k)}{k}\exp\left(-\frac{2P}{\pi}\int_0^\infty\frac{\delta_l(k')k'dk'}{k'^2-k^2}\right). \tag{2.23}$$

Of course the expression for the potential is more complicated if there is a bound state or if $\delta_l(0)-\delta_l(\infty)=N\pi(N\neq 0)$.[5] However, one can still easily work backwards from the phase shifts and obtain a parametrization of the on-shell two-body data with a definite prescription for going off shell (i.e., obtain the separable potential).

Although we shall not make use, in this paper, of the *two-body* scattering amplitude in the lab, it may be instructive to calculate it using similar approximations that will be adopted in the next section. We begin by writing the Lippman-Schwinger equation describing the scattering in the lab system

$$\psi^{(+)}_{(\vec{r}_1^0,\vec{r}_2^0)}=e^{i\vec{k}\cdot\vec{r}_1^0}\Phi_0(\vec{r}_2^0)-\frac{2m_1}{\hbar^2}\sum_n\Phi_n(\vec{r}_2^0)\int\int d\vec{r}_1^1\,d\vec{r}_2^1\int\frac{d\vec{t}_1}{(2\pi)^3}\frac{e^{i\vec{t}_1\cdot(\vec{r}_1^0-\vec{r}_1^1)}}{t_1^2-k_n^2-i\epsilon}\Phi_n^*(\vec{r}_2^1)$$

$$\times\sum_{lm}4\pi\lambda_l v_l(|\vec{r}_1^1-\vec{r}_2^1|)v_l(|\vec{r}_1^2-\vec{r}_2^2|)Y_{lm}(\Omega_{\vec{r}_1^1-\vec{r}_2^1})Y^*_{lm}(\Omega_{\vec{r}_1^2-\vec{r}_2^2})$$

$$\times\delta\left[\frac{1}{m_1+m_2}(m_1\vec{r}_1^1+m_2\vec{r}_2^1)-\frac{1}{m_1+m_2}(m_1\vec{r}_1^2+m_2\vec{r}_2^2)\right]\psi^{(+)}_{(\vec{r}_1^2,\vec{r}_2^2)}\,d\vec{r}_1^2\,d\vec{r}_2^2, \tag{2.24}$$

where $\Phi_n(\vec{r}_2)$ represents the total wave function, in the lab system, for the target (which would be a plane wave for an elementary system without internal degrees of freedom). The sum over the complete set of states of the target (including c.m. motion as well as internal excitations) is only symbolically denoted by

$\sum_n$. (Naturally integration over continuum states is required.) The symbol r refers to the usual position variable associated with a given "particle." The subscript on the variable r refers to particle 1 or 2 while the superscript keeps track of the different dummy integration variables for a given particle and a given coordinate. (Thus $\vec{r}_1^2$ and $\vec{r}_1^3$ are both the elementary particle 1 coordinate variable but refer to different dummy integration variables.) The quantity k_n^2 depends on the state Φ_n of particle 2 and the initial kinetic energy (in the lab) of particle 1 $[E_1(\text{lab}) = \hbar^2 k_1^2/2m_1]$,

$$k_n^2 = k_1^2 - \frac{2m_1}{\hbar^2}[E_n(\text{particle 2}) - E_0(\text{particle 2})], \tag{2.25}$$

where E_n is the energy (including recoil) associated with the state n of particle 2. The quantity E_0 is the initial (here no c.m. motion and the internal ground state) energy of particle 2. Now let us assume that

$$k_1^2 \gg \frac{2m_1}{\hbar^2}(E_n - E_0) \tag{2.26}$$

so that

$$k_n^2 \approx k_1^2 . \tag{2.27}$$

The condition assumed in Eq. (2.26) is one of the basic approximations made by FW and we discuss its implications more fully in the next section. However, we note for an *elementary* target the condition is most likely to be satisfied at forward angles. In fact, FW obtain the following condition on the scattering angle

$$4\sin^2\tfrac{1}{2}\theta \ll \frac{m_2}{m_1} . \tag{2.28}$$

Thus if m_2 is very large compared to m_1 the condition can be satisfied at all angles. (We shall use the condition in the next section only for elementary particle-*nucleus* interactions and thus in that case the m_2 would refer to the A body *nucleus* mass and not the mass of a single elementary particle.) Replacing k_n^2 by k^2 (or k_{av}^2) in Eq. (2.24) and using closure on the target wave functions

$$\sum_n \Phi_n(\vec{r}_2^{\,0})\Phi_n^*(\vec{r}_2^{\,1}) = \delta(\vec{r}_2^{\,0} - \vec{r}_2^{\,1}) \tag{2.29}$$

permits the integral equation [(2.24)] to be simplified

$$\psi^{(+)}(\vec{r}_1^{\,0}, \vec{r}_2^{\,0}) = e^{i\vec{k}\cdot\vec{r}_1^{\,0}}\Phi_0(\vec{r}_2^{\,0}) - \frac{2m_1}{\hbar^2}\int\int\int \frac{d\vec{t}_1}{(2\pi)^3}\frac{e^{i\vec{t}_1\cdot(\vec{r}_1^{\,0}-\vec{r}_1^{\,1})}}{t_1^2 - k^2 - i\epsilon}$$

$$\times \sum_{lm} 4\pi\lambda_l v_l(|\vec{r}_1^{\,1} - \vec{r}_2^{\,0}|)v_l(|\vec{r}_1^{\,2} - \vec{r}_2^{\,1}|)Y_{lm}(\Omega_{\vec{r}_1^{\,1}-\vec{r}_2^{\,0}})Y_{lm}^*(\Omega_{\vec{r}_1^{\,2}-\vec{r}_2^{\,1}})$$

$$\times \delta\left[\frac{1}{m_1+m_2}(m_1\vec{r}_1^{\,1} + m_2\vec{r}_2^{\,0}) - \frac{1}{m_1+m_2}(m_1\vec{r}_1^{\,2} + m_2\vec{r}_2^{\,1})\right]\psi^{(+)}(\vec{r}_1^{\,2}, \vec{r}_2^{\,1})\,d\vec{r}_1^{\,2}\,d\vec{r}_1^{\,1}\,d\vec{r}_2^{\,1} . \tag{2.30}$$

Thus by essentially neglecting the final kinetic energy of the target particle we obtain the laboratory scattering amplitude

$$f_{\text{lab}}(\vec{k}', \vec{k}) = -\frac{1}{4\pi}\int\int \Phi_0^*(\vec{r}_2^{\,0})e^{-i\vec{k}'\cdot\vec{r}_1^{\,0}}\frac{2m_1}{\hbar^2}\sum_{lm} 4\pi\lambda_l v_l(|\vec{r}_1^{\,0} - \vec{r}_2^{\,0}|)v_l(|\vec{r}_1^{\,1} - \vec{r}_2^{\,1}|)Y_{lm}(\Omega_{\vec{r}_1^{\,0}-\vec{r}_2^{\,0}})Y_{lm}^*(\Omega_{\vec{r}_1^{\,1}-\vec{r}_2^{\,1}})$$

$$\times \delta\left[\frac{1}{m_1+m_2}(m_1\vec{r}_1^{\,0} + m_2\vec{r}_2^{\,0}) - \frac{1}{m_1+m_2}(m_1\vec{r}_1^{\,1} + m_2\vec{r}_2^{\,1})\right]\psi^{(+)}(\vec{r}_1^{\,1}, \vec{r}_2^{\,1})\,d\vec{r}_1^{\,0}\,d\vec{r}_1^{\,1}\,d\vec{r}_2^{\,0}\,d\vec{r}_2^{\,1}, \tag{2.31}$$

where $\psi^{(+)}$ is given by Eq. (2.30). Iterating Eq. (2.30) and substituting the result into Eq. (2.31) yields

$$
\begin{aligned}
-4\pi f_{\text{lab}}(\vec{k}',\vec{k}) = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{2m_1}{\hbar^2}\int \frac{d\vec{k}_2^0}{(2\pi)^{3/2}} e^{-i\vec{k}_2^0\cdot\vec{r}_2^0}\varphi^*(\vec{k}_2^0)\, e^{-i\vec{k}'\cdot\vec{r}_1^0} \\
\times \sum_{lm} 4\pi\lambda_l v_l(|\vec{r}_1^0-\vec{r}_2^0|)v_l(|\vec{r}_1^1-\vec{r}_2^1|)Y_{lm}(\Omega_{\vec{r}_1^0-\vec{r}_2^0})Y_{lm}^*(\Omega_{\vec{r}_1^1-\vec{r}_2^1}) \\
\times \delta\left[\frac{1}{m_1+m_2}(m_1\vec{r}_1^0+m_2\vec{r}_2^0)-\frac{1}{m_1+m_2}(m_1\vec{r}_1^1+m_2\vec{r}_2^1)\right] \\
\times \Bigg(\delta_{1,n}\int\frac{d\vec{k}_2^1}{(2\pi)^{3/2}} e^{i\vec{k}_2^1\cdot\vec{r}_2^1}\varphi(\vec{k}_2^1)\, e^{i\vec{k}\cdot\vec{r}_1^1}\, d\vec{r}_1^0\, d\vec{r}_1^1\, d\vec{r}_2^0\, d\vec{r}_2^1 \\
+\theta(n-1)\prod_{i=1}^{n-1}\Bigg\{\int\frac{d\vec{t}_i}{(2\pi)^3}\frac{e^{i\vec{t}_i\cdot(\vec{r}_1^{2i-1}-\vec{r}_1^{2i})}}{t_i^2-k^2-i\epsilon}\frac{2m_1}{\hbar^2}\sum_{l_i m_i}4\pi\lambda_{l_i}v_{l_i}(|\vec{r}_1^{2i}-\vec{r}_2^{i}|) \\
\times v_{l_i}(|\vec{r}_1^{2i+1}-\vec{r}_2^{i+1}|)Y_{l_i m_i}(\Omega_{\vec{r}_1^{2i}-\vec{r}_2^{i}})Y_{l_i m_i}^*(\Omega_{\vec{r}_1^{2i+1}-\vec{r}_2^{i+1}}) \\
\times\delta\left(\left[\frac{1}{m_1+m_2}(m_1\vec{r}_1^{2i}+m_2\vec{r}_2^{i})-\frac{1}{m_1+m_2}(m_1\vec{r}_1^{2i+1}+m_2\vec{r}_2^{i+1})\right]\right)\Bigg\}_i \\
\times\int\frac{d\vec{k}_2^n}{(2\pi)^{3/2}}e^{i\vec{k}_2^n\cdot\vec{r}_2^n}\varphi(\vec{k}_2^n)\,e^{i\vec{k}\cdot\vec{r}_1^{2n-1}}\prod_{j=0}^{2n-1}d\vec{r}_1^j\prod_{q=0}^{n}d\vec{r}_2^q\Bigg),
\end{aligned}
\tag{2.32}
$$

where $\varphi(k_2)$ is the Fourier transform of the target particle coordinate space wave function

$$\varphi(\vec{k}_2)=\frac{1}{(2\pi)^{3/2}}\int d\vec{r}_2\, e^{-i\vec{k}_2\cdot\vec{r}_2}\varphi(\vec{r}_2).$$

The θ function is zero unless its argument is greater than zero in which case it is unity.

Now we make the following variable changes

$$
\begin{aligned}
&\vec{\mathcal{R}}_0=\vec{r}_1^0-\vec{r}_2^0, && \vec{R}_0=\frac{1}{m_1+m_2}(m_1\vec{r}_1^0+m_2\vec{r}_2^0),\\
&\vec{\mathcal{R}}_1=\vec{r}_1^1-\vec{r}_2^1, && \vec{R}_1=\frac{1}{m_1+m_2}(m_1\vec{r}_1^{2n-1}+m_2\vec{r}_2^n),\\
&\vec{\mathcal{R}}_2=\vec{r}_1^2-\vec{r}_2^1, && \vec{r}_2^1=\vec{r}_2^1,\\
&\vdots && \\
&\vec{\mathcal{R}}_{2i-1}=\vec{r}_1^{2i-1}-\vec{r}_2^i, && \vec{r}_2^2=\vec{r}_2^2,\\
&\vec{\mathcal{R}}_{2i}=\vec{r}_1^{2i}-\vec{r}_2^i, && \vec{r}_2^3=\vec{r}_2^3,\\
&\vdots && \vdots\\
&\vec{\mathcal{R}}_{2n-2}=\vec{r}_1^{2n-2}-\vec{r}_2^{n-1}, && \vec{r}_2^{n-1}=\vec{r}_2^{n-1}.\\
&\vec{\mathcal{R}}_{2n-1}=\vec{r}_1^{2n-1}-\vec{r}_2^n, &&
\end{aligned}
\tag{2.33}
$$

The Jacobian, denoted by J, of this transformation is unity. (The Jacobian appears in the integrations via the transformation

$$\prod_{j=0}^{2n-1}d^3r_i^j\prod_{q=0}^{n}d^3r_2^q\to J\,d^3R_1\,d^3R_0\prod_{j=0}^{2n-1}d^3r_1^j\prod_{q=1}^{n-1}d^3r_2^q\quad .) \tag{2.34}$$

The exponential involving $\vec{t}_i$ is easily changed from the r variables to the $\mathcal{R}$ variables by making use of

$$e^{i\vec{t}_i\cdot(\vec{r}_1^{2i-1}-\vec{r}_1^{2i})}=e^{i\vec{t}_i\cdot(\vec{\mathcal{R}}_{2i-1}-\vec{\mathcal{R}}_{2i})}. \tag{2.35}$$

The potential functions are, of course, already only a function of the $\mathcal{R}$ variables. The δ functions may be used to eliminate the $\vec{R}_1$ and $\vec{r}_2^{\,1}$, $\vec{r}_2^{\,2}\cdots\vec{r}_2^{\,n-1}$ integrations, since the δ functions require

$$\vec{R}_1=\frac{1}{m_1+m_2}(m_1\vec{r}_1^{\,2n-2}+m_2\vec{r}_2^{\,n-1}) \tag{2.36a}$$

$$=\frac{m_1}{m_1+m_2}\vec{\mathcal{R}}_{2n-2}+\vec{r}_2^{\,n-1} \tag{2.36b}$$

$$=\frac{m_1}{m_1+m_2}(\vec{\mathcal{R}}_{2n-2}-\vec{\mathcal{R}}_{2n-3}+\vec{\mathcal{R}}_{2n-4})+\vec{r}_2^{\,n-2} \tag{2.36c}$$

$$=\frac{m_1}{m_1+m_2}(\vec{\mathcal{R}}_{2n-2}-\vec{\mathcal{R}}_{2n-3}+\vec{\mathcal{R}}_{2n-4}-\vec{\mathcal{R}}_{2n-5}+\cdots-\vec{\mathcal{R}}_3+\vec{\mathcal{R}}_2-\vec{\mathcal{R}}_1)+\vec{R}_0\,. \tag{2.36d}$$

The exponential expressions originally depending on r_2^n and r_1^{2n-1} may be reexpressed using (2.33) and (2.36d) so that

$$\begin{aligned} e^{i\vec{k}_2^n\cdot\vec{r}_2^{\,n}}e^{i\vec{k}\cdot\vec{r}_1^{\,2n-1}} &= e^{i(\vec{k}_2^n+\vec{k})\cdot\vec{R}_1}\exp\left[i\left(\frac{m_2}{m_1+m_2}\vec{k}-\frac{m_1}{m_1+m_2}\vec{k}_2^n\right)\cdot\vec{\mathcal{R}}_{2n-1}\right] \\ &= \exp\left[i\left(\frac{m_2}{m_1+m_2}\vec{k}-\frac{m_1}{m_1+m_2}\vec{k}_2^n\right)\cdot\vec{\mathcal{R}}_{2n-1}\right] \\ &\quad\times\exp\left[i(\vec{k}_2^n+\vec{k})\cdot\left(\frac{m_1}{m_1+m_2}(\vec{\mathcal{R}}_{2n-2}-\vec{\mathcal{R}}_{2n-3}+\vec{\mathcal{R}}_{2n-4}\cdots+\vec{\mathcal{R}}_2-\vec{\mathcal{R}}_1)+\vec{R}_0\right)\right]. \end{aligned} \tag{2.37}$$

Similarly we find

$$e^{-i\vec{k}_2^0\cdot\vec{r}_2^{\,0}}e^{-i\vec{k}'\cdot\vec{r}_1^{\,0}}=e^{-i(\vec{k}_2^0+\vec{k}')\cdot\vec{R}_0}\exp\left[-i\left(\frac{m_2}{m_1+m_2}\vec{k}'-\frac{m_1}{m_1+m_2}\vec{k}_2^0\right)\cdot\vec{\mathcal{R}}_0\right]. \tag{2.38}$$

Substituting Eqs. (2.37) and (2.38) into Eq. (2.32) and carrying out the integration over $\vec{R}_0$ (which yields an over-all momentum conserving δ function) gives $[\mathcal{R}\equiv|\vec{\mathcal{R}}|]$

$$\begin{aligned} -4\pi f_{\text{lab}}(\vec{k}',\vec{k}) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2m_1}{\hbar^2}\int\int\cdots\int\int d\vec{k}_2^n\,d\vec{k}_2^0\,\delta^3(\vec{k}_2^0+\vec{k}'-\vec{k}_2^n-\vec{k})\,\varphi^*(\vec{k}_2^0)\varphi(\vec{k}_2^n) \\ &\quad\times\exp\left[-i\left(\frac{m_2}{m_1+m_2}\vec{k}'-\frac{m_1}{m_1+m_2}\vec{k}_2^0\right)\cdot\vec{\mathcal{R}}_0\right] \\ &\quad\times\exp\left[i\left(\frac{m_2}{m_1+m_2}\vec{k}-\frac{m_1}{m_1+m_2}\vec{k}_2^n\right)\cdot\vec{\mathcal{R}}_{2n-1}\right]\sum_{lm}4\pi\lambda_l v_l(\mathcal{R}_0)\,v_l(\mathcal{R}_1)\,Y_{lm}(\Omega_{\vec{\mathcal{R}}_0})Y^*_{lm}(\Omega_{\vec{\mathcal{R}}_1}) \\ &\quad\times\left[\delta_{1,n}+\theta(n-1)\prod_{i=1}^{n-1}\left(\int\frac{d\vec{t}_i}{(2\pi)^3}\frac{\exp\{i[\vec{t}_i-m_1/(m_1+m_2)(\vec{k}_2^n+\vec{k})]\cdot(\vec{\mathcal{R}}_{2i-1}-\vec{\mathcal{R}}_{2i})\}}{t_i^2-k^2-i\epsilon}\frac{2m_1}{\hbar^2}\right.\right. \\ &\qquad\left.\left.\times\sum_{l_i m_i}4\pi\lambda_{l_i}v_{l_i}(\mathcal{R}_{2i})v_{l_i}(\mathcal{R}_{2i+1})Y_{l_i m_i}(\Omega_{\vec{\mathcal{R}}_{2i}})Y^*_{l_i m_i}(\Omega_{\vec{\mathcal{R}}_{2i+1}})\right)_i\right]\prod_{j=0}^{2n-1}d\vec{\mathcal{R}}_j\,. \end{aligned} \tag{2.39}$$

The procedure followed in reducing Eq. (2.14) to the simple form given by Eq. (2.20) is not immediately applicable in the present situation because of the more complicated expressions appearing in the exponentials in Eq. (2.39). In the present case the procedure of expanding all the exponentials in a partial wave expansion and then carrying out the angular integrations over all the $\vec{t}_i$ and $\vec{\mathcal{R}}_j$ does not contract all the sums over various l_i down to a single sum because $Y_{LM}Y^*_{LM}$ terms that in the previous case [i.e., in going from (2.14) to (2.20)] were functions of the same variable are now functions of somewhat different variables. We proceed by making the following variable changes

$$\begin{aligned} \vec{p}_i &= \vec{t}_i-\frac{m_1}{m_1+m_2}(\vec{k}_2^n+\vec{k})\,,\quad \text{for all } i \\ \vec{k}_2^n &= \vec{k}_2^n\,, \\ \vec{k} &= \vec{k}\,. \end{aligned} \tag{2.40}$$

The Jacobian of this transformation is one. The net effect of this variable change is that in Eq. (2.39) we replace the expression

$$\int \frac{d\vec{t}_i}{(2\pi)^3} \frac{\exp\{i[\vec{t}_i - m_1/(m_1+m_2)(\vec{k}_2^n+\vec{k})]\cdot(\vec{\mathcal{R}}_{2i-1}-\vec{\mathcal{R}}_{2i})\}}{t_i^2 - k^2 - i\epsilon}$$

by

$$\int \frac{d\vec{p}_i}{(2\pi)^3} \frac{e^{i\vec{p}_i\cdot(\vec{\mathcal{R}}_{2i-1}-\vec{\mathcal{R}}_{2i})}}{p_i^2 + 2\alpha\vec{p}_i\cdot(\vec{k}_2^n+\vec{k}) + \alpha^2(\vec{k}_2^n+\vec{k})^2 - k^2 - i\epsilon}, \tag{2.41}$$

where

$$\alpha = \frac{m_1}{m_1+m_2}.$$

Now if we replace the $\vec{p}_i$ angle dependence in the denominator of Eq. (2.41) by some "average" value denoted by $\langle \vec{p}_i\cdot(\vec{k}_2^n+\vec{k})\rangle$ then significant simplification is possible. We delay until the next section the discussion of an alternate approximation whose validity is more easily tested in a given model. We note here, however, that an alternative approach is to assume that diagonal contributions in the l, m sums dominate and thereby motivate contracting all the sums into a single summation. (We will actually do this in the next section.) Replacing

$$[\vec{p}_i\cdot(\vec{k}_2^n+\vec{k})]$$

by

$$\langle[\vec{p}_i\cdot(\vec{k}_2^n+\vec{k})]\rangle_{\text{av}} \tag{2.42}$$

and subsequently carrying out the angular integrations for the $\vec{\mathcal{R}}_j$ and $\vec{p}_i$ leads to the following expression for the nth order contribution ($n>1$) in Eq. (2.39)

$$\begin{aligned} -4\pi f_n(\vec{k}',\vec{k}) = (-1)^{n-1}\frac{2m_1}{\hbar^2}\int d\vec{k}_2^n\,\varphi^*(\vec{k}_2^n+\vec{k}-\vec{k}')\varphi(\vec{k}_2^n)\sum_{lm}\lambda_l v_l\left(\left|\vec{k}' - \frac{m_1}{m_1+m_2}\vec{k}_2^n - \frac{m_1}{m_1+m_2}\vec{k}\right|\right)4\pi \\ \times v_l\left(\left|\frac{m_2}{m_1+m_2}\vec{k} - \frac{m_1}{m_1+m_2}\vec{k}_2^n\right|\right) Y_{lm}(\Omega_{\vec{k}'-m_1/(m_1+m_2)\vec{k}_2^n - m_1/(m_1+m_2)\vec{k}})\,Y^*_{lm}(\Omega_{m_2/(m_1+m_2)\vec{k}-m_1/(m_1+m_2)\vec{k}_2^n}) \\ \times\left[\frac{2m_1}{\hbar^2}\frac{\lambda_l}{(2\pi)^3}\int\frac{p^2dp4\pi|v_l(p)|^2}{p^2+2\alpha\langle\vec{p}\cdot(\vec{k}_2^n+\vec{k})\rangle + \alpha^2(\vec{k}_2^n+\vec{k})^2 - k^2 - i\epsilon}\right]^{n-1}, \end{aligned} \tag{2.43}$$

where we have also made use of the definitions Eqs. (2.17a) and (2.17b). Now, in analogy with the discussion that led from Eq. (2.18) to Eq. (2.20) we may immediately write $[\beta \equiv m_2/(m_1+m_2)]$

$$\begin{aligned} f_{\text{lab}}(\vec{k}',\vec{k}) = -\sum_{lm}\lambda_l\frac{2m_1}{\hbar^2}\int d\vec{k}_2^n\,\varphi^*(\vec{k}_2^n+\vec{k}-\vec{k}')\varphi(\vec{k}_2^n) \\ \times\frac{v_l(|\vec{k}'-\alpha(\vec{k}_2^n+\vec{k}|)v_l(|\beta\vec{k}-\alpha\vec{k}_2^n|)Y_{lm}(\Omega_{\vec{k}'-\alpha(\vec{k}_2^n+\vec{k})})Y^*_{lm}(\Omega_{\beta\vec{k}-\alpha\vec{k}_2^n})}{1+[\lambda_l/(2\pi)^3](2m_1/\hbar^2)\int d\vec{p}|v_l(p)|^2/[p^2+2\alpha\langle\vec{p}\cdot(\vec{k}_2^n+\vec{k})\rangle+\alpha^2(\vec{k}_2^n+\vec{k})^2-k^2-i\epsilon]}. \end{aligned} \tag{2.44}$$

Although somewhat more complicated in appearance, Eq. (2.44) is similar in form to Eq. (2.20) [the expression for $f_{\text{c.m.}}(\vec{k}',\vec{k})$]. The complications appearing in the present case include the appearance of the Fourier transforms of the initial and final target wave functions in the numerator, the sum over m (which can be trivially carried out even though the two spherical harmonics depend on somewhat different variables), and finally the integral in the denominator has a slightly more involved integrand. An expression of this general form will be obtained (and indeed is one of the principal results) in the many-body problem treated in the next section. Note that in the particular case that the target particle is initially at rest in the lab system that $\varphi(k_2^n) = \delta^3(\vec{k}_2^n)$ and the integration over $\vec{k}_2^n$ is trivially accomplished in Eq. (2.44) resulting in significant simplification. Of course if m_2 is infinitely heavy the c.m. and lab systems coincide and in that case ($\alpha = 0, \beta = 1$) Eqs. (2.44) and (2.20) coincide. For $m_2 \to \infty$ Eq. (2.44) also coincides, naturally, with the expression (in the lab system) given by FW for scattering via a separable potential from a fixed scattering center.

Several variations of useful approximations may be possible in obtaining a tractable reduction of Eq. (2.39) and the reader is encouraged to search for alternatives.

III. MANY-BODY PROBLEM

At the beginning of this section we follow closely the discussion of FW in order to demonstrate why inclusion of the two-particle c.m. δ function requires a departure from their approach. The starting point is the many-body scattering integral equation in the laboratory system [compare Eq. (2.24) and the subsequent discussion and see Eq. (2.7) in FW]

$$\psi^{(+)}(\vec{x}_1^0, \ldots, \vec{x}_A^0, \vec{x}_0^0) = \Phi_0(\vec{x}_1^0 \cdots \vec{x}_A^0)e^{i\vec{k}\cdot\vec{x}_0^0} - \frac{2m_0}{\hbar^2}\sum_n \Phi_n(\vec{x}_1^0, \ldots, \vec{x}_A^0)\int\cdots\int d\vec{x}_1^1\cdots d\vec{x}_A^1 \int \frac{d\vec{t}}{(2\pi)^3}\frac{e^{i\vec{t}\cdot(\vec{x}_0^0-\vec{x}_0^1)}}{t^2-k_n^2-i\epsilon}$$
$$\times \Phi_n^*(\vec{x}_1^1, \ldots, \vec{x}_A^1)\sum_{i=1}^{A}\sum_{lm} 4\pi\lambda_l v_l(|\vec{x}_0^1-\vec{x}_i^1|)v_l(|\vec{x}_0^2-\vec{x}_i^2|)Y_{lm}[\Omega_{(\vec{x}_0^1-\vec{x}_i^1)}]Y_{lm}^*[\Omega_{(\vec{x}_0^2-\vec{x}_i^2)}]$$
$$\times \delta\left[\frac{1}{m_0+m_i}(m_0\vec{x}_0^1+m_i\vec{x}_i^1) - \frac{1}{m_0+m_i}(m_0\vec{x}_0^2+m_i\vec{x}_i^2)\right]\psi^{(+)}(\vec{x}_1^1, \ldots, \vec{x}_i^2, \ldots, \vec{x}_A^1, \vec{x}_0^2)\,d\vec{x}_i^2\,d\vec{x}_0^1\,d\vec{x}_0^2, \tag{3.1}$$

where the states Φ_n represent a set of energy eigenstates of the A particle target satisfying

$$H_n|\Phi_n\rangle = E_n|\Phi_n\rangle . \tag{3.2}$$

(H_n is the *total* free nuclear Hamiltonian including both c.m. and intrinsic nuclear coordinates.) The quantity k_n^2 is defined by

$$k_n^2 \equiv k^2 + \frac{2m_0}{\hbar^2}[E_n(\text{target}) - E_0(\text{target})], \tag{3.3}$$

where E_0 is the initial nuclear target state (i.e., at rest in the lab with the nucleus in its intrinsic ground state). In Eq. (3.1) we have used the integral form for the free particle Green's function. All variables referring to the projectile have a subscript zero (i.e., x_0) while variables which refer to the ith target particle have a subscript i (i.e., x_i). Superscripts are used to distinguish between different dummy integration variables. Now along with FW, we make the basic approximation that

$$k^2 \gg \frac{2m_0}{\hbar^2}(E_n - E_0), \tag{3.4}$$

so that we approximate k_n^2 by k^2 in Eq. (3.1). In the context of the many-body problem this means we assume for the "important" intermediate scatterings that the energy transferred to the target as a whole is small compared to the incident projectile energy. There are two contributions to E_n, the recoil of the *entire A*-particle nucleus and the intrinsic excitation energy of the nucleus. We ignore recoil of the whole nucleus because we assume it to be considerably more massive than the projectile (or a single nucleon). As far as the intrinsic excitations are concerned, if the "important" intermediate nuclear excited states are, for example, low-lying collective nuclear states then one might imagine this approximation would be valid. [Of course all of this depends on the sensitivity of the integral to very small changes in the denominator. One check of the approximation, as FW point out, is to evaluate the appropriate off-diagonal matrix elements in Eq. (3.1) and see if the important states n of the target [(i.e., those yielding appreciable off-diagonal matrix elements) satisfy Eq. (3.4)]. The important point is that there may be, for example, because of the difference between the mass of the whole target and one of its constituents and because of the average excitation energy of important intermediate intrinsic nuclear excited states, a problem where one does not wish to treat the mass of an individual target particle (nucleon) as infinitely heavy but still may adopt the approximation $k_n^2 = k^2$ or $k_n^2 = k_{av}^2$. Of course our main purpose here is only to see what modifications are required if the δ function in Eq. (3.1) is included in arguments leading to a derivation of the optical potential. Replacing k_n^2 by k^2 in Eq. (3.1) allows one to carry out the sum over n making use of the completeness of the nuclear states. Since

$$\sum_n \Phi_n(\vec{x}_1^0 \cdots \vec{x}_A^0)\Phi_n^*(\vec{x}_1^1 \cdots \vec{x}_A^1) = \delta(\vec{x}_1^0 - \vec{x}_1^1)\cdots\delta(\vec{x}_A^0 - \vec{x}_A^1), \tag{3.5}$$

we may rewrite Eq. (3.1) as

$$\psi^{(+)}(\vec{x}_1^0,\ldots,\vec{x}_A^0,\vec{x}_0^0)=\Phi_0(\vec{x}_1^0,\ldots,\vec{x}_A^0)e^{i\vec{k}\cdot\vec{x}_0^0}-\frac{2m_0}{\hbar^2}\int\frac{d\vec{t}_1}{(2\pi)^3}\frac{e^{i\vec{t}\cdot(\vec{x}_0^0-\vec{x}_0^1)}}{t^2-k_n^2-i\epsilon}\sum_{i=1}^{A}\sum_{lm}4\pi\lambda_l v_l(|\vec{x}_0^1-\vec{x}_i^0|)v_l(|\vec{x}_0^2-\vec{x}_i^1|)$$

$$\times Y_{lm}[\Omega_{(\vec{x}_0^1-\vec{x}_i^0)}]Y_{lm}^*[\Omega_{(\vec{x}_0^2-\vec{x}_i^1)}]\delta\left[\frac{1}{m_0+m_i}(m_0\vec{x}_0^1+m_i\vec{x}_i^0)-\frac{1}{m_0+m_i}(m_0\vec{x}_0^2+m_i\vec{x}_i^1)\right]$$

$$\times\psi^{(+)}(\vec{x}_1^0,\ldots,\vec{x}_i^1,\ldots,\vec{x}_A^0,\vec{x}_0^2)\,d\vec{x}_i^1\,d\vec{x}_0^1\,d\vec{x}_0^2. \tag{3.6}$$

Now if, for the moment, we were to let the mass m_i of the elementary constituents go to infinity the δ functions in Eq. (3.6) simply require

$$\vec{x}_i^0=\vec{x}_i^1. \tag{3.7}$$

This means the potential is essentially only nonlocal in the projectile coordinate x_0. Thus as $m_i\to\infty$ ($i=1$ to A) Eq. (3.6) may be rewritten as

$$\psi^{(+)}(\vec{x}_1^0\cdots\vec{x}_A^0,\vec{x}_0^0)=\Phi_0(\vec{x}_1^0\cdots\vec{x}_A^0)e^{i\vec{k}\cdot\vec{x}_0^0}-\frac{2m_0}{\hbar^2}\int\int\frac{d\vec{t}}{(2\pi)^3}\frac{e^{it\cdot(\vec{x}_0^0-\vec{x}_0^1)}}{t^2-k^2-i\epsilon}$$

$$\times\sum_{i=1}^{A}\sum_{lm}v_{lm}(\vec{x}_0^1-\vec{x}_i^0)v_{lm}(\vec{x}_0^2-\vec{x}_i^0)\,\psi^{(+)}(\vec{x}_1^0\cdots\vec{x}_i^0\cdots\vec{x}_A^0,\vec{x}_0^2)\,d\vec{x}_0^1. \tag{3.8}$$

FW note that the solutions of Eq. (3.8) are of the form

$$\psi^{(+)}(\vec{x}_1^0\cdots\vec{x}_A^0;\vec{x}_0^0)=\Phi_0(\vec{x}_1^0\cdots\vec{x}_A^0)\psi_k^{(+)}(\vec{x}_0^0), \tag{3.9}$$

where

$$\psi_k^{(+)}(\vec{x}_0^0)=e^{i\vec{k}\cdot\vec{x}_0^0}-\frac{2m_0}{\hbar^2}\int\int d\vec{x}_0^1\,d\vec{x}_0^2\frac{d\vec{t}}{(2\pi)^3}\Big|\int\frac{d\vec{t}}{(2\pi)^3}\frac{e^{i\vec{t}\cdot(\vec{x}_0^0-\vec{x}_0^1)}}{t^2-k^2-i\epsilon}\sum_{i=1}^{A}\sum_{lm}v_{lm}(\vec{x}_0^1-\vec{x}_i^0)v_{lm}(\vec{x}_0^2-\vec{x}_i^0)\,\psi_k^{(+)}(\vec{x}_0^2). \tag{3.10}$$

Note that although the integral equation (3.10) contains the x_i variables, the integration variables all refer to the projectile. Thus Eq. (3.10) is an integral equation only involving the projectile coordinates in an essential way. The elastic scattering amplitude may be written

$$f(\vec{k}',\vec{k})=-\frac{1}{4\pi}\frac{2m_0}{\hbar^2}\int\cdots\int d\vec{x}_1^0\cdots d\vec{x}_A^0\,d\vec{x}_0^0\,\Phi_0^*(\vec{x}_1^0,\ldots,\vec{x}_A^0)e^{-i\vec{k}'\cdot\vec{x}_0^0}$$

$$\times\sum_{i=1}\sum_{lm}v_{lm}(\vec{x}_0^0-\vec{x}_i^0)v_{lm}(\vec{x}_0^1-\vec{x}_i^0)\,\psi^{(+)}(\vec{x}_1^0\cdots\vec{x}_A^0,\vec{x}_0^1)\,d\vec{x}_0^1. \tag{3.11}$$

Combining Eqs. (3.9) and (3.11) allows

$$f(\vec{k}',\vec{k})=-\frac{1}{4\pi}\frac{2m_0}{\hbar^2}\int\cdots\int d\vec{x}_1^0\cdots d\vec{x}_A^0\,\Phi_0^*(\vec{x}_1^0,\ldots,\vec{x}_A^0)$$

$$\times\left[\int e^{-i\vec{k}'\cdot\vec{x}_0^0}\int\sum_{i=1}^{A}\sum_{lm}v_{lm}(\vec{x}_0^0-\vec{x}_i^0)v_{lm}(\vec{x}_0^1-\vec{x}_i^0)\,\psi_k(\vec{x}_0^1)\,d\vec{x}_0^0\,d\vec{x}_0^1\right]\Phi_0(\vec{x}_1^0,\ldots,\vec{x}_A^0). \tag{3.12}$$

One can think of the term in brackets in Eq. (3.12) as a many-body scattering operator $f_{k'k}(\vec{x}_1^0\cdots\vec{x}_A^0)$. Thus what has been accomplished in Eq. (3.12) is a separation of the problem into two parts; first, one may treat the part in brackets involving only the coordinate of the projectile as an integration variable and second, average the bracketed expression, $f_{k'k}(\vec{x}_1^0\cdots\vec{x}_A^0)$, (which after integration over projectile coordinates is a function only of the target constituent variables) over the target initial-state-probability distribution. This technique is basic to the elegant discussion in FW.

Returning to Eq. (3.6) and not assuming $m_i=\infty$ we find that the technique discussed above needs to be modified. The basic problem is that the potential is now nonlocal in both coordinates (with the δ function providing an additional constraint). Thus, for example, a solution of the form of Eq. (3.9) [where $\psi_k^{(+)}(\vec{x}_0^0)$ satisfies an integral equation involving only the projectile coordinates as integration variables] does not seem possible. Therefore, we shall adopt a somewhat different approach.

Our procedure will be to write down the series expansion for the many-body elastic scattering amplitude incorporating Eq. (3.6) for $\psi^{(+)}$. We shall make the assumption that although a particle may be multiply struck (in fact we shall sum a certain class of projectile-target particle interactions to infinite order) that once the projectile leaves the jth particle in the nucleus and scatters from the kth particle ($j \neq k$), the projectile does not return to interact again with the jth particle.

First consider the expression for multiple scattering *from the jth target particle*. Here we are motivated by the usual desire to replace the projectile-single target particle v matrix by something akin to the two-particle t matrix. We obtain

$$
\begin{aligned}
-4\pi f^j_{\text{lab}}(\vec{k}',\vec{k}) = &\sum_{n=1}^{\infty}(-1)^{n-1}\int \Phi_0^*(\vec{x}_1^0\cdots\vec{x}_A^0)e^{-i\vec{k}'\cdot\vec{x}_0^0}\frac{2m_0}{\hbar^2}\sum_{lm}4\pi\lambda_l v_l(|\vec{x}_0^0-\vec{x}_j^0|)v_l(|\vec{x}_0^1-\vec{x}_j^1|)\\
&\times Y_{lm}[\Omega_{(\vec{x}_0^0-\vec{x}_j^0)}]\,Y^*_{lm}[\Omega_{(\vec{x}_0^1-\vec{x}_j^1)}]\,\delta\left[\frac{1}{m_0+m_j}(m_0\vec{x}_0^0+m_j\vec{x}_j^0)-\frac{1}{m_0+m_j}(m_0\vec{x}_0^1+m_j\vec{x}_j^1)\right]\\
&\times\Bigg\{\delta_{1,n}\;\Phi_0(\vec{x}_1^0,\ldots,\vec{x}_j^1,\ldots,\vec{x}_A^0)e^{i\vec{k}\cdot\vec{x}_0^1}\left(\prod_{r=0}^{A}d\vec{x}_r^0\right)d\vec{x}_j^1 d\vec{x}_0^1\\
&+\theta(n-1)\prod_{i=1}^{n-1}\int\frac{d\vec{t}_i}{(2\pi)^3}\frac{e^{i\vec{t}_i\cdot(\vec{x}_0^{2i-1}-\vec{x}_0^{2i})}}{t_i^2-k^2-i\epsilon}\frac{2m_0}{\hbar^2}\sum_{l_i m_i}4\pi\lambda_{l_i}v_{l_i}(|\vec{x}_0^{2i}-\vec{x}_j^i|)\\
&\times v_{l_i}(|\vec{x}_0^{2i+1}-\vec{x}_j^{i+1}|)Y_{l_i m_i}(\Omega_{\vec{x}_0^{2i}-\vec{x}_j^i})Y^*_{l_i m_i}(\Omega_{\vec{x}_0^{2i+1}-\vec{x}_j^{i+1}})\delta\left[\frac{1}{m_0+m_j}(m_0\vec{x}_0^{2i}+m_j\vec{x}_j^i-m_0\vec{x}_0^{2i+1}-m_j\vec{x}_j^{i+1})\right]\\
&\times\Phi_0(\vec{x}_1^0\cdots\vec{x}_j^n\cdots\vec{x}_A^0)e^{i\vec{k}\cdot\vec{x}_0^{2n-1}}\prod_{g=1}^{2n-1}d\vec{x}_0^g\prod_{q=1}^{n}d\vec{x}_j^q\prod_{r=0}^{A}d\vec{x}_r^0\Bigg\}.
\end{aligned}
\tag{3.13}
$$

The integrations over the variables $\vec{x}_r^0$ ($r=1$ to A, not j) may be easily carried out since these variables appear only in the function Φ_0. The resulting expression is quite similar to that obtained for two-particle scattering in the lab system [see Eq. (2.32)]. The similarity with Eq. (2.32) may be made completely transparent by simply making the identification $\vec{x}_0 \to \vec{r}_1$ and $\vec{x}_j \to \vec{r}_2$ and defining

$$\varphi(\vec{k}_j)=\frac{1}{(2\pi)^{3/2}}\int d\vec{x}_j\,e^{-i\vec{k}_j\cdot\vec{x}_j}\,\varphi(\vec{x}_j),$$

where $\varphi(x_j)$ is a "single-particle" wave function obtained after integrating out the dependence of Φ_0 on $\vec{x}_1^0$ to $\vec{x}_A^0$ (excluding of course $\vec{x}_j^0$). The function $\varphi(k_j)$ then is a measure of the momentum distribution of the jth target particle [i.e., $\varphi^*(\vec{k}_j)\varphi(\vec{k}_j)$ yields the target particle momentum probability distribution]. Then making the variable changes given in Eq. (2.33), and repeating the discussion leading to Eq. (2.44) allows the following closed form expression for scattering from the jth target particle:

$$
\begin{aligned}
t^j_{\text{lab}}(\vec{k}',\vec{k})\equiv f^j_{\text{lab}}(\vec{k}',\vec{k}) = &-\sum_{lm}\lambda_l\frac{2m_0}{\hbar^2}\int d\vec{k}_j\,\varphi^*(\vec{k}_j+\vec{k}-\vec{k}')\varphi(\vec{k}_j)\,v_l(|\vec{k}'-\alpha(\vec{k}_j+\vec{k})|)\\
&\times\frac{v_l(|\beta\vec{k}-\alpha\vec{k}_j|)Y_{lm}(\Omega_{\vec{k}'-\alpha(\vec{k}_j+\vec{k})})Y^*_{lm}(\Omega_{\beta\vec{k}-\alpha\vec{k}_j})}{1+[\lambda_l/(2\pi)^3](2m_0/\hbar^2)\int d\vec{p}\,|v_l(p)|^2/[p^2+2\alpha\langle p\cdot(\vec{k}_j+\vec{k})\rangle+\alpha^2(\vec{k}_j+\vec{k})^2-k^2-i\epsilon]}.
\end{aligned}
\tag{3.14}
$$

We wish to consider the usual multiple scattering expansion, where after each scattering with the ith particle the projectile may interact with other target particles. If we denote a single-projectile ith-target-particle interaction by v_i and adopt the common restriction (as in FW) that the projectile never returns to a target particle once it has interacted with it and subsequently interacted with another target particle (for each term in the multiple scattering series, we call this the never-come-back approximation) then the multiple scattering series may be written symbolically

$$S(k',k)=\sum_{i=1}^{A}v_i+\sum_{i\neq j=1}^{A}v_iv_j+\sum_{i=1}^{A}v_iv_i+\sum_{i\neq j\neq k=1}^{A}v_iv_jv_k+\sum_{i\neq j=1}^{A}v_iv_iv_j+\sum_{i\neq j=1}^{A}v_iv_jv_j+\sum_{i=1}^{A}v_iv_iv_i+\cdots. \tag{3.15}$$

If we define the usual infinite sum of interactions with particle i [i.e., the "t" matrix we have obtained in closed form in Eq. (3.14)] as

$$t_i=v_i+v_iv_i+v_iv_iv_i+\cdots \tag{3.16}$$

then the multiple scattering series [Eq. (3.15)] may be regrouped and written as

$$S(k',k)=\sum_{i=1}^{A} t_i+\sum_{i\neq j=1}^{A} t_i t_j+\sum_{i\neq j\neq k=1}^{A} t_i t_j t_k+\cdots . \tag{3.17}$$

Of course this is just a greatly oversimplified summary of the usual expansion of the many-body scattering amplitude in terms of sums of products of two-particle t matrices. In the usual discussion one understands that these are two-particle t matrices imbedded in the many-body problem and so are different ("off the two-body energy shell") than the free two-particle t matrix. In fact, one often wishes to relate the two-body t matrix in the many-body problem to the free two-body t matrix. {In this way one can, for example, eliminate the difficulties due to the "hard core" in the nucleon-nucleon problem by using experimental data for nucleon-nucleon scattering in the nucleon-nucleus many-body scattering problem. In the standard approach, in order to relate the free two-particle t matrix to the many-body two-particle t matrix one must make certain approximations that lead to uncertainties in the validity of the approach. For example, the free two-particle t matrix is often written $t(k, k', \omega)$, where k and k' are the initial and final relative momenta and ω is the energy available in the c.m. system; (note the relation between ω, k, and k' is here determined by two-body kinematics) for the two-particle t matrix in the many-body problem [which one would also like to write as $t'(k, k', \omega')$] one is confronted with the many-body complications which makes the ω' appearing in t' uncertain and also allows of course $|\vec{k}|\neq|\vec{k}'|$.} Our philosophy is somewhat different. We have a definite potential that is presumed to fit the two-body data. We wish to see in some detail how the final expression we shall obtain for the optical potential is related to the expression we found for the free two-particle t matrix (as well as comparing our result with that of FW).

We wish to generalize Eq. (3.13) to include scattering from more than one target particle (in the never-come-back approximation) using a multiple scattering expansion of the form (3.17). Of course we start with Eq. (3.15) and then just regroup terms. The question is will we get terms involving simple products of $t^{j}_{\text{lab}}(k', k)$ as given by Eq. (3.14) or will the resulting expression be so complicated that it becomes intractable. The answer is that by making one further standard assumption (which FW also adopt) the expression is quite simple and we believe instructive. Although the general expressions are quite lengthy, the main details can be understood from studying the (relatively) simple expression for single scattering from particle i and a subsequent single scattering from particle j. This contribution to the elastic scattering amplitude is given by

$$\begin{aligned}-4\pi f^{i,j(2)}_{\text{lab}}=\frac{2m_0}{\hbar^2}\int &\varphi^*(\vec{x}^0_j\vec{x}^0_i)e^{-i\vec{k}'\cdot\vec{x}^0_0}\sum_{lm}4\pi\lambda_l v_l(\vec{x}^0_0-\vec{x}^0_j)v_l(\vec{x}^1_0-\vec{x}^1_j)\\ &\times\delta\left[\frac{1}{m_0+m_j}(m_0\vec{x}^0_0+m_j\vec{x}^0_j)-\frac{1}{m_0+m_j}(m_0\vec{x}^1_0+m_j\vec{x}^1_j)\right]\left[-\frac{2m_0}{\hbar^2}\int\frac{d\vec{t}}{(2\pi)^3}\frac{e^{i\vec{t}\cdot(\vec{x}^1_0-\vec{x}^2_0)}}{t^2-k^2-i\epsilon}\right]\\ &\times\sum_{l'm'}4\pi\lambda_l v_l(\vec{x}^2_0-\vec{x}^0_i)v_l(\vec{x}^3_0-\vec{x}^1_i)\delta\left[\frac{1}{m_0+m_i}(m_0\vec{x}^2_0+m_i\vec{x}^0_i)-\frac{1}{m_0+m_i}(m_0\vec{x}^3_0+m_i\vec{x}^1_i)\right]\\ &\times e^{i\vec{k}\cdot\vec{x}^3_0}\varphi(\vec{x}^1_j,\vec{x}^1_i)\,d\vec{x}^0_j\,d\vec{x}^0_i\,d\vec{x}^1_j\,d\vec{x}^1_i\prod_{n=0}^{3}d\vec{x}^n_0 .\end{aligned} \tag{3.18}$$

There are several interesting features present in Eq. (3.18). First the expression easily breaks up into two terms each involving a single scattering from a given target particle. The term $e^{-i\vec{t}\cdot\vec{x}^2_0}$ acts as the *final* projectile wave function with respect to scattering from particle i while the term $e^{i\vec{t}\cdot\vec{x}^1_0}$ acts as the *initial* projectile wave function for scattering from particle j. Interestingly enough the δ *functions* do not impose constraints between the variables associated with the projectile scattering from *different* target particles. These features are not limited to single scatterings from just two target particles. One can easily show that the same simple partial factorization holds for any number of scatterings involving any number of target particles (in the never-come-back approximation).

There is one difficulty which makes the simple factorization incomplete – correlations among the target particles which complicates the many-particle ground-state wave function. A procedure previously adopted[2] is to expand $\Phi^*\Phi(\vec{x}_1\cdots\vec{x}_A)$ in terms of products of $1, 2\cdots n$ particle density functions and then keep only the term involving products of single-particle density functions. We shall adopt this approximation for obtaining the optical potential. Thus we *assume* that the many-particle density may be written

$$\varphi_0^*(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_j, \ldots, \vec{x}_A)\varphi_0(\vec{x}_1', \vec{x}_2', \ldots, \vec{x}_j', \ldots, \vec{x}_A') = \rho(\vec{x}_1, \vec{x}_1')\rho(\vec{x}_2, \vec{x}_2')\cdots\rho(\vec{x}_j, \vec{x}_j')\cdots\rho(\vec{x}_A, \vec{x}_A') + (\text{other terms which can be neglected})$$
$$\cong \varphi^*(\vec{x}_1)\varphi(\vec{x}_1')\varphi^*(\vec{x}_2)\varphi(\vec{x}_2')\cdots\varphi^*(\vec{x}_A)\varphi(\vec{x}_A') . \quad (3.19)$$

Under these circumstances a complete factorization is possible and Eq. (3.18) can be written

$$-4\pi f_{\text{lab}}^{i,j(2)} = \int \frac{d\vec{t}}{(2\pi)^3} f_{\text{lab}}^j (\vec{k}', \vec{t} \text{ single scattering}) \frac{1}{t^2 - k^2 - i\epsilon} f_{\text{lab}}^i (\vec{t}, \vec{k}, \text{ single scattering}) . \quad (3.20)$$

As mentioned earlier the factorization is not affected by the number of scatterings involving a single target particle or by the number of target particles with which the projectile interacts (again these statements are in the context of the never-come-back approximation). The expression for the multiple scattering amplitude Eq. (3.15) is easily obtained and can then be regrouped to yield the sum Eq. (3.17) in a straightforward manner. The final result for projectile-complex target elastic scattering may be written in the form

$$-4\pi f_{\text{lab}}(\vec{k}', \vec{k}) = \sum_{i=1}^{A} f^i(\vec{k}', \vec{k}) - \sum_{i=1}^{A} \sum_{\substack{j=1 \\ j\neq i}}^{A} \int \frac{d\vec{t}}{(2\pi)^3} f^j(\vec{k}', \vec{t}) \frac{1}{t^2 - k^2 - i\epsilon} f^i(\vec{t}, \vec{k})$$
$$+ \sum_{i=1}^{A} \sum_{\substack{j=1 \\ j\neq i}}^{A} \sum_{\substack{k=1 \\ k\neq i, k\neq j}}^{A} \int\int \frac{d\vec{t}_1}{(2\pi)^3} \frac{d\vec{t}_2}{(2\pi)^3} f^k(\vec{k}', \vec{t}_2) \frac{1}{t_2^2 - k^2 - i\epsilon} f^j(\vec{t}_2, \vec{t}_1) \frac{1}{t_1^2 - k^2 - i\epsilon} f^i(\vec{t}_1, \vec{k}) \cdots , \quad (3.21)$$

where [compare Eq. (3.14)]

$$f^j(\vec{q}', \vec{q}) = \sum_{lm} \lambda_l \frac{2m_0}{\hbar^2} \int d\vec{k}_j \varphi^*(\vec{k}_j + \vec{q} - \vec{q}')\varphi(\vec{k}_j)$$
$$\times \frac{4\pi v_l(|\vec{q}' - \alpha(\vec{k}_j + \vec{q})|) v_l(|\beta\vec{q} - \alpha\vec{k}_j|) Y_{lm}(\Omega_{\vec{q}' - \alpha(\vec{k}_j + \vec{q})}) Y_{lm}^*(\Omega_{\beta\vec{q} - \alpha\vec{k}_j})}{1 + [\lambda_l/(2\pi)^3](2m_0/\hbar^2) \int d\vec{p} |v_l(p)|^2 / [p^2 + 2\alpha\langle\vec{p}\cdot(\vec{k}_j + \vec{q})\rangle + \alpha^2(\vec{k}_j + \vec{q})^2 - k^2 - i\epsilon]} . \quad (3.22)$$

[Note: $\sum_m Y_{lm}(\Omega_{\vec{q}' - \alpha(\vec{k}_j + \vec{q})}) Y_{lm}^*(\Omega_{\beta\vec{q} - \alpha\vec{k}_j}) \neq (1/4\pi)(2l+1)P_l(\cos\theta_{q',q})$.]

In order to define the optical potential we consider the equivalent one-body problem assuming a nonlocal optical potential. We wish an optical potential, U, which when inserted into the one-body projectile Schrödinger equation

$$\psi_k^{(+)}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \int\int\int \frac{d\vec{t}}{(2\pi)^3} \frac{e^{i\vec{t}\cdot(\vec{r} - \vec{r}')}}{t^2 - k^2 - i\eta} U(\vec{r}', \vec{r}'')\psi(\vec{r}'') d\vec{r}' d\vec{r}'' \quad (3.23)$$

will yield a scattering amplitude

$$-4\pi f(\vec{k}', \vec{k}) = \int e^{-i\vec{k}'\cdot\vec{r}} U(\vec{r}, \vec{r}')\psi_k^{(+)}(\vec{r}') d\vec{r} d\vec{r}' \quad (3.24)$$

which is identical with the elastic scattering amplitude calculated from Eq. (3.21). Substituting Eq. (3.23) into Eq. (3.24) and iterating yields

$$-4\pi f(k', k) = \int e^{-i\vec{k}'\cdot\vec{r}} U(\vec{r}, \vec{r}') e^{i\vec{k}\cdot\vec{r}'} d\vec{r} d\vec{r}'$$
$$- \int \cdots \int e^{-i\vec{k}'\cdot\vec{r}} U(\vec{r}, \vec{r}') \frac{e^{i\vec{t}\cdot(\vec{r}' - \vec{r}'')}}{t^2 - k^2 - i\eta} U(\vec{r}'', \vec{r}''') e^{i\vec{k}\cdot\vec{r}'''} \frac{d\vec{t}}{(2\pi)^3} d\vec{r} d\vec{r}' d\vec{r}'' d\vec{r}''' + \cdots . \quad (3.25)$$

Now defining

$$U(\vec{p}, \vec{q}) = \int\int e^{-i\vec{p}\cdot\vec{x}} U(\vec{x}, \vec{y}) e^{i\vec{q}\cdot\vec{y}} d\vec{x} d\vec{y} \quad (3.26)$$

Eq. (3.25) may be rewritten

$$-4\pi f(\vec{k}',\vec{k}) = U(\vec{k}',\vec{k}) - \int \frac{d\vec{t}}{(2\pi)^3} U(\vec{k}',\vec{t}) \frac{1}{t^2 - k^2 - i\eta} U(\vec{t},\vec{k})$$

$$+ \int\int \frac{d\vec{t}_1}{(2\pi)^3}\frac{d\vec{t}_2}{(2\pi)^3} U(\vec{k}',\vec{t}_2)\frac{1}{t_2^2 - k^2 - i\eta} U(\vec{t}_2,\vec{t}_1)\frac{1}{t_1^2 - k^2 - i\eta} U(\vec{t}_1,\vec{k}) \cdots . \tag{3.27}$$

Note that Eq. (3.27) is very similar in form to Eq. (3.21) except for the summations appearing in the latter equation. If we treat all the target particles as equivalent, the summations in Eq. (3.21) are easily carried out and we obtain (for an A particle target)

$$-4\pi f(\vec{k}',\vec{k}) = Af(\vec{k}',\vec{k}) - A(A-1)\int \frac{d\vec{t}}{(2\pi)^3} f(\vec{k}',\vec{t})\frac{1}{t^2-k^2-i\eta} f(\vec{t},\vec{k})$$

$$+ A(A-1)(A-2)\int\int \frac{d\vec{t}_1}{(2\pi)^3}\frac{d\vec{t}_2}{(2\pi)^3} f(\vec{k}',\vec{t}_2)\frac{1}{t_2^2-k^2-i\eta} f(\vec{t}_2,\vec{t}_1)\frac{1}{t_1^2-k^2-i\eta} f(\vec{t}_1,\vec{k}) - \cdots . \tag{3.28}$$

If we assume the number of iterations of Eq. (3.28) needed for an accurate result is small compared to A, then by comparing Eqs. (3.27) and (3.28) we immediately identify

$$U(\vec{q}',\vec{q}) = Af(\vec{q}',\vec{q}), \tag{3.29}$$

where $f(q',q)$ is given by Eq. (3.22).

If we assume that the individual target particles have infinite mass, then in Eq. (3.22) $\alpha \to 0$ and $\beta \to 1$ so that the optical potential becomes

$$U(q',q) = A\rho(\vec{q}'-\vec{q})\,\frac{2m_0}{\hbar^2}\sum_l \frac{\lambda_l v_l(q')v_l(q)(2l+1)P_l(\cos\theta_{q'q})}{1+[\lambda_l/(2\pi)^3](2m_0/\hbar^2)\int d\vec{p}\,|v_l(p)|^2/(p^2-k^2-i\epsilon)}, \tag{3.30}$$

where

$$\rho(\vec{q}'-\vec{q}) = \int d\vec{k}_j\,\varphi^*(\vec{k}_j+\vec{q}-\vec{q}')\varphi(k_j)$$

$$= \frac{1}{(2\pi)^3}\int d\vec{k}_j\,d\vec{x}_1\,d\vec{x}_1'\,e^{i(\vec{k}_j+\vec{q}-\vec{q}')\cdot\vec{x}_1}\,\varphi^*(\vec{x}_1)\,e^{-i(\vec{k}_j)\cdot x_1'}\,\varphi(\vec{x}_1')$$

$$= \int d\vec{x}_1\,e^{i(\vec{q}-\vec{q}')\cdot\vec{x}_1}\rho(\vec{x}_1). \tag{3.31}$$

If we define

$$f_l \equiv -\frac{1}{4\pi}\left[\frac{2m_0}{\hbar^2}\,\frac{\lambda_l v_l^2(k)}{1+[\lambda_l/(2\pi)^3](2m_0/\hbar^2)\int d\vec{p}\,|v_l(p)|^2/(p^2-k^2-i\epsilon)}\right] \tag{3.32}$$

then Eq. (3.30) may be rewritten

$$U(q',q) = -4\pi A\rho(\vec{q}'-\vec{q})\sum_l \lambda_l(2l+1)\frac{v_l(q')v_l(q)}{|v_l(k)|^2} f_l(k)P_l(\cos\theta q'q) \tag{3.33}$$

which is the result obtained for the optical potential in Ref. 2. If one assumes that $\rho(\vec{q}'-\vec{q})$ falls off much more quickly as a function of momentum transfer than the other terms in Eq. (3.33), then setting $k\approx q' \simeq q \simeq k'$ in all the other terms yields the usual Glauber[3] or eikonal result

$$U_G(q'q) \cong -4\pi A\rho(\vec{q}'-\vec{q})\,f_k(0) \tag{3.34a}$$

or

$$U_G(x,y) \cong -4\pi A\rho(x)\,f_k(0)\delta(\vec{x}-\vec{y}). \tag{3.34b}$$

We now consider several approximations which allow the optical potential [Eqs. (3.29) and (3.22)] to be reduced to a more tractable form. Studying Eq. (3.22) we see that the initial target particle momentum $\vec{k}_j$ enters in a complicated way in both the numerator and denominator. Thus if one wishes to study the effect of the Fermi momentum in a more realistic manner, Eq. (3.22) provides a good starting point. However, we note that $\vec{k}_j$ is everywhere multiplied by $\alpha(\sim\frac{1}{8}$ for pion-nucleus scattering) except in the target particle density. If we assume that the most important effect of the Fermi momentum is contained simply in the nuclear momentum probability distribution $\varphi^*(\vec{k}_j+\vec{q}-\vec{q}\,')\varphi(k_j)$, (the standard assumption – usually this is the only place it appears in previous treatments[1,2]) then eliminating k_j everywhere in Eq. (3.22) except in the wave function and carrying out the integration over k_j yields

$$U(\vec{q}\,',\vec{q})=A\rho(\vec{q}\,'-\vec{q})\frac{2m_0}{\hbar^2}\sum_{lm}\lambda_l v_l(|\vec{q}\,'-\alpha\vec{q}|)v_l(|\beta\vec{q}|)\times\frac{4\pi Y_{lm}(\Omega_{\vec{q}\,'-\alpha\vec{q}})Y^*_{lm}(\Omega_{\beta\vec{q}})}{1+[\lambda_l/(2\pi)^3](2m_0/\hbar^2)\int d\vec{p}[v_l(p)]^2/(p^2+2\alpha\langle\vec{p}\cdot\vec{q}\rangle+\alpha^2q^2-k^2)}. \tag{3.35}$$

The c.m. of the nucleus is assumed at rest in the lab. We ignore in this discussion the $1/A$ effect resulting from $\sum_{j=1}^{A}\vec{k}_j=0$.

Now of course one can introduce an adjustable parameter (to be fitted to experiment) to take care of the term $\langle\vec{p}\cdot\vec{q}\rangle$. We do not attempt to motivate a realistic choice for the parameter in this paper.

Since one is averaging over the angles of $\vec{p}$ a rough first approximation would be $\langle\vec{p}\cdot\vec{q}\rangle\approx 0$. It is probably consistent in this situation to ignore the term depending on the square of α. Under these limiting assumptions the only effect of the finite mass is in the numerator.

We recall that the reason for the angle average approximation was that one wanted a closed form expression for the repeated scattering of the projectile from a given target particle. The usual technique did not suffice because certain angular integrations, which before had collapsed many sums over partial waves into a single sum, were now more complicated and hence did not lead to a simple solution [see Eq. (2.39) and the discussion immediately following]. The basic difficulty resulted from terms of the form [we have taken terms from the i and $i-1$ integrations in Eq. (2.39)]

$$\begin{aligned}\sum_{l_i'm_i'}f(l_i'm_i')\sum_{l_im_i}\int\!\!\int\!\!\int\frac{d\vec{t}_i}{(2\pi)^3}\,d\vec{\mathcal{R}}_{2i-1}d\vec{\mathcal{R}}_{2i}\,&\frac{\exp\{i[\vec{t}_i-\alpha(\vec{k}_2^n+\vec{k})]\cdot(\vec{\mathcal{R}}_{2i-1}-\vec{\mathcal{R}}_{2i})\}}{t_i^2-k^2-i\epsilon}\\
&\times\lambda_{l_i}\lambda_{l_i'}(4\pi)v_{l_i'}(\mathcal{R}_{2i-1})Y^*_{l_i'm_i'}(\Omega_{\vec{\mathcal{R}}_{2i-1}})v_{l_i}(\mathcal{R}_{2i})\,Y_{l_im_i}(\Omega_{\vec{\mathcal{R}}_{2i}})\\
&=\sum_{l_i'm_i'}f(l_i'm_i')\sum_{l_im_i}\int\!\!\int\!\!\int\frac{d\vec{t}_i}{(2\pi)^3}\,d\vec{\mathcal{R}}_{2i-1}d\vec{\mathcal{R}}_{2i}\frac{1}{t_i^2-k^2-i\epsilon}\sum_{lm}4\pi(i)^l j_l[|\vec{t}-\alpha(\vec{k}_2^n+\vec{k})|\mathcal{R}_{2i-1}]\\
&\times Y_{lm}(\Omega_{\vec{\mathcal{R}}_{2i-1}})Y^*_{lm}(\Omega_{\vec{t}-\alpha(\vec{k}_2^n+\vec{k})})\sum_{l'm'}4\pi(-i)^{l'}j_{l'}[|\vec{t}-\alpha(\vec{k}_2^n+\vec{k})|\mathcal{R}_{2i}]\\
&\times Y^*_{l'm'}(\Omega_{\vec{\mathcal{R}}_{2i}})Y_{l'm'}(\Omega_{\vec{t}-\alpha(\vec{k}_2^n+\vec{k})})\lambda_{l_i}\lambda_{l_i}'(4\pi)v_{l_i}'(\mathcal{R}_{2i-1})\,Y^*_{l_i'm_i'}(\Omega_{\vec{\mathcal{R}}_{2i-1}})v_{l_i}(\mathcal{R}_{2i})Y_{l_im_i}(\Omega_{\vec{\mathcal{R}}_{2i}})\,.\end{aligned} \tag{3.36}$$

Carrying out the angular integrations over $\vec{\mathcal{R}}_{2i}$ and $\vec{\mathcal{R}}_{2i-1}$ in Eq. (3.36) results in the requirements (because of the orthonormality of the spherical harmonics)

$$\begin{aligned}&l=l_i',\qquad l'=l_i,\\&m=m_i',\quad m'=m_i.\end{aligned} \tag{3.37}$$

In order to completely collapse the sums we must also have $l=l'$, $m=m'$ which in the limit $\alpha\to 0$ can be obtained from the angular $\vec{t}$ integration. We can use the assumption that although there could be off-diagonal contributions $(l'\neq l)$, $(m'\neq m)$, on the average their effect will be small (note the off-diagonal terms are down by an order α from the diagonal terms and have less sign correlation), and therefore keep only diagonal contributions. This results finally in the following form for the optical potential [under the stated assumptions insert Eq. (3.36) in Eq. (2.39), a closed form expression for the two-particle amplitude is then easily obtained, and the optical potential discussion proceeds as before]

$$U(q',q)=A\int d\vec{k}_2^n\varphi^*(\vec{k}_2^n+\vec{q}-\vec{q}\,')\varphi(\vec{k}_2^n)\times\sum_{lm}\frac{(2m_0/\hbar^2)\lambda_l v^*_{lm}[\vec{q}\,'-\alpha(\vec{k}_2^n+\vec{q})]v_{lm}(\beta\vec{q}-\alpha\vec{k}_2^n)}{1+[\lambda_l/(2\pi)^3](2m_0/\hbar^2)\int d\vec{t}|v_{lm}[\vec{t}-\alpha(\vec{q}+\vec{k}_n)]|^2/(t^2-k^2-i\epsilon)}, \tag{3.38}$$

where $v_{lm}(\vec{q})$ is defined in Eq. (2.17a). If the microscopic potential is a purely s-wave potential ($v_l = 0$, $l \neq 0$) the diagonal approximation leading to Eq. (3.38) is exact. We regard Eqs. (3.22), (3.29), (3.35), and (3.38) the most important results for determining the extension of the optical potential possibly required to include target particles of finite mass.

In concluding this section, we note there are at least two distinct and important results. First we see that it is possible to include the effect of finite target particle mass and obtain an expression for the optical potential which reduces to that given by FW in the limit of infinitely massive target particles (so no pathologies result from the inclusion of the finite mass).

Secondly, we see that it is possible to obtain a closed form expression for the optical potential, when m_i is finite, making additional assumptions about the angular integrations involved in the intermediate scatterings. An important consideration in this context is whether or not one expects the assumption of closure and the replacement of $\vec{p} \cdot \vec{k}$ by $\langle \vec{p} \cdot \vec{k} \rangle$ or keeping only the diagonal terms in the intermediate partial wave sums to be valid in a situation where taking into account the finite mass of the target particle is important. Some further discussion is included in the next section. Two important effects that can (and should) be checked given a nuclear model and a form for the separable potential are the off-diagonal matrix elements in Eq. (3.1) between the nuclear ground state and the excited states n, and the off-diagonal matrix elements ($l \neq l'$, $m \neq m'$) in Eq. (3.36). One should be able to evaluate the effect on the scattering wave function (after one or two iterations) resulting from relaxing the closure and ($l = l'$, $m = m'$) restrictions. Such investigations are planned as part of the application of this formal discussion to pion-nucleus scattering.

IV. CONCLUSIONS AND DISCUSSION

In this paper we have considered the alterations produced in a particular theory of the optical potential[2] by extending the formalism to include a many-particle target composed of finite-mass particles (constituents). Most of the approximations adopted are the same as those used in Ref. 2. They include:

(a) use of a nonrelativistic potential model;
(b) a nonlocal separable microscopic interaction acting between the projectile and the target constituents [see Eqs. (2.3) and (2.13)];
(c) the assumption that the projectile energy is high enough so that closure can be used on the target [see Eq. (3.4) and the discussion immediately thereafter];
(d) use of a product of single-particle densities for the nuclear ground-state matter distribution;
(e) the assumption that the intermediate scatterings are mainly forward so that once the projectile scatters from a given target particle i and then leaves it to scatter from another particle j, it does not return to particle i;
(f) the number of iterations of the optical potential required for a given degree of accuracy is small compared to A, the number of constituents in the target.

The main features introduced by the finite-mass constituents are:
(a) The microscopic potential becomes nonlocal in both the projectile and constituent coordinates;
(b) some δ functions which correlate certain of the coordinates must be included in the formalism.

These complications require a different approach than that used in Ref. 2, where it was possible to cast the problem in terms of finding the ground-state expectation value of a particular many-body scattering operator. The approach used here was simply to iterate the many-particle integral equation for the wave function and then substitute this infinite series into the expression for the elastic scattering amplitude. When repeated scattering of the projectile from a given target particle was considered (i.e., the two-particle t matrix in the many-body environment) it was found that one additional assumption was apparently required in order to find a closed form expression for the two-particle repeated scattering series. This additional approximation was required because certain partial wave sums did not collapse into a single sum. [See Eqs. (2.39) or (3.36) and the discussion immediately following each of these equations.] One assumption involves assuming the ($l \neq l'$), ($m \neq m'$) off-diagonal contribution of various integrals of the form

$$\int Y^*_{l'm'}(\theta, \varphi) Y_{lm}(\theta, \varphi) f(\theta, \varphi) d\Omega$$

are negligible compared to the diagonal terms. This assumption can be tested in a given model. Another possible assumption, that results in a simple closed form expression for the optical potential, involves replacing a certain angle dependent function by its angle averaged expectation value – in practice probably introducing a "small" adjustable parameter into the theory.

We find that as the mass of the individual target particles approaches infinity the optical potential obtained by Foldy and Walecka is recovered. With further limiting assumptions, this potential reduces to the familiar Glauber form.

In general, it is certainly possible that the assumptions made in obtaining the closed form for the optical potential are no more valid than assuming the individual target constituents have infinite mass. Hopefully, this can be tested in a given application.

There is a widespread belief that the field of intermediate energy nuclear physics shows promise of yielding considerable additional understanding concerning scattering mechanisms and the structure of many-particle strongly interacting systems. Part of this promise depends on one's understanding of the role of the optical potential and its relation to the basic two-body data. The advantage of the particular detailed potential model considered in this paper (and in part Ref. 2) is that it provides an optical potential which is explicitly related to a choice of microscopic potential, the Fermi momentum, the finite mass of the constituents, and if one wishes, the two-particle scattering amplitude.

Thus by studying the validity of the several simplifying assumptions and by comparing the elastic scattering predicted by various limiting forms of the optical potential, one should be able to more critically evaluate the sensitivity of the experimental results to the features of strongly interacting systems we hope to learn about.

From a more practical point of view, we have already begun to study pion-nucleus elastic scattering using this formalism. In addition, some progress has been made in working in the c.m. projectile many-body target system as opposed to the lab system. This latter approach should be particularly useful for studying the scattering of two many-particle systems of comparable mass. Of course further investigation to obtain tractable forms for the optical potential under less restrictive conditions (for example, dropping the closure approximation) continues.

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