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Relativistic Corrections in the Nuclear-Plus-Electromagnetic Hamiltonian

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We investigate the problem of relativistic corrections to the Hamiltonian of two nucleons interacting with each other and with an external electromagnetic field. We find, in addition to well-known corrections to the kinetic energy and to the Foldy-Wouthuysen reduction, a correction to the potential energy. This effect depends on the way in which the nonrelativistic limit of the dynamical variables is taken. The correction vanishes in the c.m. system, but leads to a nonzero correction to the electromagnetic Hamiltonian. We define the current and charge densities, verify the conservation of current, and verify the low-energy theorem for bremsstrahlung.

1. INTRODUCTION

The problem of combining the electromagnetic interaction and the strong interaction is an old and difficult one in theoretical physics. It is particularly important in nuclear physics, where electromagnetic transitions have been a primary source of information about nuclei since the very beginning of the subject. In the simplest nuclear problem, the two-nucleon system, electromagnetic processes have been studied in connection with Coulomb effects in *NN* elastic scattering, the photodisintegration of the deuteron, and *NN* bremsstrahlung.

In general these problems have been treated nonrelativistically in the sense that the nucleons are nonrelativistic, and the strong interaction is represented by a potential used with the Schrödinger equation. In this paper we discuss the way in which the electromagnetic-plus-nuclear-interaction problem may be treated relativistically. One approach is to use field theory, but difficulties arise because of the incomplete nature of a field theory of strong interactions. A second approach, which is the one we investigate here, considers *direct* interactions, not mediated by a field. One tries to determine a Hamiltonian which, along with the other generators of the transformations of the Lorentz group, obeys the commutation relations of that group. These generators are all functions of the particle dynamical variables, and the Hamiltonian is then used in a many-particle relativistic Schrödinger equation.[1] This group approach has been developed by Bakamjian and Thomas,[2] who first described the treatment of a system of interacting particles, and also obtained the center-of-mass and internal dynamical variables including first-order relativistic corrections.[3]

The problem of relativistic effects for a system of particles interacting with an external electromagnetic field has been discussed by numerous authors.[4] We refer particularly to the work of Osborn,[5] who first obtained the exact form for relativistic center-of-mass and internal dynamical variables for a two-body system, and to the work of Close and Copley[6] who generalized these expressions to a system with an arbitrary number of particles. The latter authors also investigate the first-order corrections to the electromagnetic interaction, and, among other things, derive the correction term to the Foldy-Wouthuysen (FW) reduction.[4-8]

In this paper we combine the strong interaction and the electromagnetic interaction and obtain the form of the complete Hamiltonian correct to first order in c^{-2}, for a two-nucleon system. This includes relativistic corrections to the kinetic energy and the potential energy, electromagnetic corrections derived from these, and the correction to the FW reduction. Thus the Hamiltonian satisfies the fundamental condition of Lorentz covariance (to order c^{-2}). A second fundamental condition which ought to be satisfied is current conservation. We define the current and charge densities, and verify that the current is conserved.

This first-order relativistic treatment is appropriate for applications to nuclear physics. Roughly speaking, an energy up to 200 MeV is possible for excited nucleons in nuclei. Furthermore, elastic *NN* scattering data up to 300 MeV are used in determining two-nucleon potential models, and *NN* bremsstrahlung experiments have been done at energies up to 200 MeV. For a 200-MeV nucleon, $p^2/m^2 \sim 0.4$, so that first-order corrections may well be important, although an exact relativistic treatment should not be necessary.

In one sense it may be said that any existing *NN* Hamiltonian is relativistic, in that it is fitted to the experimental (therefore, relativistic) data, through the nonrelativistic Schrödinger equation. Thus one may regard this procedure (Hamiltonian-plus-Schrödinger equation) as simply a model for extrapolating off the energy shell, and the model may then be used consistently to calculate some off-shell phenomenon – three-nucleon scattering, for example. On the other hand, such a model has a definite form, $H=\vec{p}^{\,2}/2m$ + a function local in r space at large distances, and this form is physically unrealistic at high energies. At least the nonlocal part should contain the kinetic energy correction term, $-(\vec{p}^{\,2})^2/8m^3$. To obtain a more physically meaningful model, one should write a Hamiltonian with a relativistically correct form, and fit free parameters in it to the data, obtaining thereby a new potential.

In Sec. 2 we discuss the relativistic correction to the nuclear interaction, including spin dependence, and taking account of parity, time reversal, and rotational invariance. In order to derive the corrections to the potential properly one must be careful to use a consistent definition for the nonrelativistic limit of the variables. Our assumption here is that the individual particle dynamical variables (such as momenta $\vec{p}_1$ and $\vec{p}_2$) are independent of c, while the center-of-mass and internal variables are functions of c. An alternative terminology is used in Refs. 4–6, although these papers do not discuss the correction terms we are interested in. This matter is discussed further in Sec. 2 and in detail in Appendix A.

In Sec. 3 we discuss the electromagnetic part of the interaction. In Sec. 4 we verify the conservation of the current derived from our Hamiltonian. In Sec. 5 we discuss the low-energy theorem. It is known that when the current is conserved the amplitude for bremsstrahlung of soft photons depends only on the on-shell two-body t matrix, and in this section we indicate how this theorem is allowed to hold in the presence of a potential model with relativistic terms. In Appendix A we define our dynamical variables, and give an alternative proof of the correction to the FW reduction. Appendix B derives some identities useful in the proof of current conservation.

2. NUCLEAR INTERACTION

We consider the interaction of two nucleons through a potential V such that the total Hamiltonian of the system can be written in the form of Bakamjian and Thomas[2]

$$H=E\equiv(M'^2+\vec{P}^2)^{1/2}. \tag{2.1}$$

Here $\vec{P}=\vec{p}_1+\vec{p}_2$ is the total momentum operator of the system, $\vec{p}_i$ is the momentum operator for the ith nucleon, and the operator M' is given in terms of the potential V, and the energy operator of the ith nucleon, ω_i, by

$$M'=\omega_1+\omega_2+V. \tag{2.2}$$

Letting m_i be the mass of the ith nucleon and $\vec{q}$ be the relative momentum in the c.m. frame, we can write ω_i as

$$\omega_i=(m_i^{\,2}+\vec{q}^{\,2})^{1/2}. \tag{2.3}$$

If we assume the interaction to be parity and time-reversal invariant, then the potential has the following general form:

$$\begin{aligned}V= {} & V_1(r,q,l)+V_2(r,q,l)\vec{s}_1'\cdot\vec{s}_2' \\ & +V_3(r,q,l)\vec{l}\cdot\vec{s}_1'+V_4(r,q,l)\vec{l}\cdot\vec{s}_2' \\ & +V_5(r,q,l)\vec{l}\cdot\vec{s}_1'\,\vec{l}\cdot\vec{s}_2' \\ & +V_6(r,q,l)\vec{r}\cdot\vec{s}_1'\,\vec{r}\cdot\vec{s}_2' \\ & +V_7(r,q,l)\vec{q}\cdot\vec{s}_1'\,\vec{q}\cdot\vec{s}_2',\end{aligned} \tag{2.4}$$

where $\vec{l}=\vec{r}\times\vec{q}$. The explicit relativistic expressions for the internal variables $\vec{q}$, $\vec{r}$, $\vec{s}_1'$, and $\vec{s}_2'$ were obtained by Osborn.[5] The expressions to order c^{-2} are given in Appendix A. Fong and Sucher[9] have proved that if the Hamiltonian of the system has the form given by Eq. (2.1) and if V_i ($i=1,\ldots,7$) of Eq. (2.4) vanish sufficiently rapidly for large r, then the associated S matrix is covariant. An alternative proof can also be given by using the method introduced by Weinberg[10] and generalized by Kazes.[11,12]

Having given the form of the Hamiltonian for the relativistic two-nucleon interaction, the next problem is to obtain the relativistic corrections to the nonrelativistic Hamiltonian. We first expand Eq. (2.1) in powers of m_i^{-1}, to obtain

$$H=m+H_0+V_N, \tag{2.5}$$

where

$$m=m_1+m_2, \tag{2.6a}$$

$$H_0=\frac{\vec{P}^2}{2m}-\frac{(\vec{P}^2)^2}{8m^3}+\frac{\vec{q}^{\,2}}{2m_1}+\frac{\vec{q}^{\,2}}{2m_2}-\frac{\vec{P}^2}{2m^2}\left(\frac{\vec{q}^{\,2}}{2m_1}+\frac{\vec{q}^{\,2}}{2m_2}\right)-\left(\frac{(\vec{q}^{\,2})^2}{8m_1^{\,3}}+\frac{(\vec{q}^{\,2})^2}{8m_2^{\,3}}\right)+\cdots, \tag{2.6b}$$

$$V_N=V-\frac{\vec{P}^2V}{2m^2}+\cdots. \tag{2.6c}$$

Now H_0 and V [through Eq. (2.4)] are functions of the relativistic variables $\vec{r}$, $\vec{q}$, $\vec{s}_1'$, $\vec{s}_2'$, and $\vec{l}$ and these must be expanded about the nonrelativistic variables. Dealing with H_0 first, we write $\vec{q}=\vec{q}^{\,\rm NR}+\delta\vec{q}$

from Eqs. (A2) and (A3) of Appendix A and find

$$H_0=\frac{\vec{P}^2}{2m}-\frac{(\vec{P}^2)^2}{8m^3}+\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)\vec{q}^{\,NR\,2}+2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)\vec{q}^{\,NR}\cdot\delta\vec{q}-\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)\frac{\vec{P}^2\vec{q}^{\,NR\,2}}{2m^2}-\left(\frac{1}{8m_1^{\,3}}+\frac{1}{8m_2^{\,3}}\right)(\vec{q}^{\,NR\,2})^2, \tag{2.7}$$

or in the c.m. frame, with $\vec{P}=0$,

$$H_0=\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)\vec{q}^{\,NR\,2}+\left[2\left(\frac{1}{2m_1}+\frac{1}{2m_2}\right)\vec{q}^{\,NR}\cdot\delta\vec{q}-\left(\frac{1}{8m_1^{\,3}}+\frac{1}{8m_2^{\,3}}\right)(\vec{q}^{\,NR\,2})^2\right]. \tag{2.8}$$

The first term gives the kinetic energy operator used in the Schrödinger equation, and the second term is the relativistic correction to the kinetic energy operator. This correction ordinarily has not been used in determination of phenomenological potentials.

For future reference we note that, if we express $\vec{q}^{\,NR}$, $\vec{P}$, and $\delta\vec{q}$ in terms of the individual particle momenta again [Eqs. (A2) amd (A3)], Eq. (2.7) becomes

$$H_0=T_1+T_2, \tag{2.9}$$

where

$$T_i=\frac{\vec{p}_i^{\,2}}{2m_i}-\frac{(\vec{p}_i^{\,2})^2}{8m_i^{\,3}}+\cdots. \tag{2.9a}$$

This is of course the expected result.

Now returning to V_N of Eq. (2.6c), it is expressed in terms of the relativistic variables $\vec{r}$, $\vec{q}$, $\vec{s}_1'$, $\vec{s}_2'$, and $\vec{l}$. The nonrelativistic limits of these variables are given in Eq. (A2) of Appendix A. If we further define

$$\vec{l}^{\,NR}=\vec{r}^{\,NR}\times\vec{q}^{\,NR} \tag{2.10}$$

then we may define the corrections $\delta\vec{l}$, r_c, q_c, and l_c by

$$\begin{aligned}\vec{l}&=\vec{l}^{\,NR}+\delta\vec{l},\\ r&=r^{NR}+r_c,\\ q&=q^{NR}+q_c,\\ l&=l^{NR}+l_c.\end{aligned} \tag{2.11}$$

Only the terms in m^{-2} in these corrections are kept. Now, for each of the potentials in Eq. (2.4) we expand

$$V_i(r,q,l)=V_i(r^{NR},q^{NR},l^{NR})+\frac{1}{2}\left(\frac{\partial V_i}{\partial r}r_c+r_c\frac{\partial V_i}{\partial r}\right)+\frac{1}{2}\left(\frac{\partial V_i}{\partial q}q_c+q_c\frac{\partial V_i}{\partial q}\right)+\frac{1}{2}\left(\frac{\partial V_i}{\partial l}l_c+l_c\frac{\partial V_i}{\partial l}\right). \tag{2.12}$$

The derivatives are evaluated at $r=r^{NR}$, $q=q^{NR}$, and $l=l^{NR}$. Similar expressions are given for the spin factors, $\vec{s}_1'\cdot\vec{s}_2'$, $\vec{l}\cdot\vec{s}_1'$, etc., and we obtain the form

$$V(\vec{r},\vec{q},\vec{s}_1',\vec{s}_2',\vec{l})=V_N^{NR}+V_c. \tag{2.13}$$

Substituting Eq. (2.13) into Eq. (2.6c), we get

$$V_N=V_N^{NR}+V_N^{\Delta}, \tag{2.14}$$

where

$$\begin{aligned}V_N^{\Delta}&=V_c-\frac{\vec{P}^2}{2m^2}(V_N^{NR}+V_c)\\ &=V_c-\frac{\vec{P}^2}{2m^2}V_N^{NR}+O(m^{-4}).\end{aligned} \tag{2.15}$$

The first term of Eq. (2.14), V_N^{NR}, is identified as a realistic phenomenological potential fixed by low-energy scattering data and the Schrödinger equation. The second term, V_N^{Δ}, is proportional to the total momentum, $\vec{P}$, in each term. Referring to Eq. (A3), we see that $\delta\vec{r}$, $\delta\vec{q}$, and $\delta\vec{s}_j$ vanish when $\vec{P}=0$. It follows that r_c, q_c, and l_c also vanish with $\vec{P}$, and hence, according to Eq. (2.12) so does V_c, and the entire V_N^{Δ}. Thus the correction to the potential vanishes in the c.m. system. In other words, if an analysis of scattering is performed in the c.m. system, and a potential is to be determined, only the relativistic correction to the kinetic energy need be included in order for the determination of the potential to be relativistically correct.

On the other hand, if one considers interaction with an electromagnetic field, the term V_N^{Δ} must be taken into account even though it vanishes in the c.m. frame. The minimal coupling $\vec{p}_i\to\vec{p}_i-e_i\vec{A}(\vec{r}_i)$ must be applied to the momenta in V_N^{Δ}, and one obtains, as discussed in the next section, an additional electromagnetic term.

A term similar to V_N^{Δ}, but not precisely the same, has been found by Shirokov,[13] on the basis of invariance properties of the S matrix.

We stress that the term V_c appears only because the internal variables on which the potential depends [in Eq. (2.4)] are relativistic; we obtain corrections proportional to $\delta\vec{r}$, $\delta\vec{q}$, and $\delta\vec{s}_j$. If $\vec{q}$ and $\vec{r}$ were considered to be independent of c

(see Appendix A for further discussion), there would not be any term V_c. The term $-(\vec{P}^2/2m^2)V_N^{\rm NR}$, in Eq. (2.15) would remain however, as a correction to the potential.

For completeness we mention that there is a relativistic correction to the Coulomb interaction, occurring in the proton-proton case. This has been derived by Close and Osborn,[4] and the reader is referred to this paper for details.

For an A-particle system, the analysis is very similar but the corrections to the potentials do not vanish. Therefore, they must be included in the A-particle problem.

3. ELECTROMAGNETIC INTERACTION

If the two nucleons interact with an external electromagnetic field $(\Phi, \vec{A})$, the total Hamiltonian may be written

$$H_T = H_0 + V_N + H_I \ . \qquad (3.1)$$

Here H_0 is the kinetic energy operator for the nucleons, given by Eq. (2.9). V_N is the two-nucleon interaction potential, including the relativistic correction, given by Eq. (2.14). H_I is the total electromagnetic interaction Hamiltonian which we decompose into six parts:

$$H_I(\Phi, \vec{A}, \vec{E}) = H_I^{T1}(\vec{A}) + H_I^{T2}(\vec{A}) + H_I^{\mu 1}(\Phi, \vec{A}, \vec{E}) + H_I^{\mu 2}(\Phi, \vec{A}, \vec{E}) + H_I^{\Delta}(\vec{A}, \vec{E}) + H_I^{V}(\vec{A}) \ . \qquad (3.2)$$

Here Φ, $\vec{A}$, and $\vec{E}$ are the scalar potential, vector potential, and electric field intensity, respectively. $H_I^{Tj}(j=1,2)$ are obtained from T_j, the kinetic energy operators by the standard gauge-invariant substitution,

$$\vec{p}_j \to \vec{\Pi}_j \equiv \vec{p}_j - e_j \vec{A}(\vec{r}_j) \ . \qquad (3.3)$$

Here e_j is the charge of the j th nucleon. If the kinetic energy operator is written as a function of momentum, $T_j = T(\vec{p}_j)$, then $H_I^{Tj}(\vec{A})$ is obtained from the difference,

$$\begin{aligned} H_I^{Tj}(\vec{A}) &= T(\vec{\Pi}_j) - T(\vec{p}_j) \\ &= -\frac{e_j}{2m_j}[\vec{p}_j\cdot\vec{A}(\vec{r}_j)+\vec{A}(\vec{r}_j)\cdot\vec{p}_j] + \frac{e_j^2}{2m_j}\vec{A}^2(\vec{r}_j) \\ &+ \frac{e_j}{8m_j^3}\{\vec{p}_j^{\,2}[\vec{p}_j\cdot\vec{A}(\vec{r}_j)+\vec{A}(\vec{r}_j)\cdot\vec{p}_j] + [\vec{p}_j\cdot\vec{A}(\vec{r}_j)+\vec{A}(\vec{r}_j)\cdot\vec{p}_j]\vec{p}_j^{\,2}\} \\ &- \frac{e_j^2}{8m_j^3}\{\vec{p}_j\cdot\vec{A}(\vec{r}_j)\vec{p}_j\cdot\vec{A}(\vec{r}_j) + \vec{p}_j\cdot\vec{A}(\vec{r}_j)\vec{A}(\vec{r}_j)\cdot\vec{p}_j + \vec{A}(\vec{r}_j)\cdot\vec{p}_j\vec{p}_j\cdot\vec{A}(\vec{r}_j) \\ &\qquad + \vec{A}(\vec{r}_j)\cdot\vec{p}_j\vec{A}(\vec{r}_j)\cdot\vec{p}_j + \vec{p}_j^{\,2}\vec{A}^2(\vec{r}_j) + \vec{A}^2(\vec{r}_j)\vec{p}_j^{\,2}\} \\ &+ \frac{e_j^3}{8m_j^3}\{\vec{A}^2(\vec{r}_j)[\vec{p}_j\cdot\vec{A}(\vec{r}_j)+\vec{A}(\vec{r}_j)\cdot\vec{p}_j] + [\vec{p}_j\cdot\vec{A}(\vec{r}_j)+\vec{A}(\vec{r}_j)\cdot\vec{p}_j]\vec{A}^2(\vec{r}_j)\} \\ &- \frac{e_j^4}{8m_j^3}\vec{A}^4(\vec{r}_j) + \cdots \ . \end{aligned} \qquad (3.4)$$

$H_I^{\mu j}$, which are functions of Φ, $\vec{A}$, and $\vec{E}$, have the form

$$\begin{aligned} H_I^{\mu j}(\Phi, \vec{A}, \vec{E}) &= e_j\Phi(\vec{r}_j) - \frac{\kappa_j e + e_j}{2m_j}\vec{\sigma}_j\cdot\vec{\nabla}_j\times\vec{A}(\vec{r}_j) \\ &- \frac{2\kappa_j e + e_j}{8m_j^2}[\vec{\nabla}_j\circ\vec{E}(\vec{r}_j) + \vec{\sigma}_j\circ\vec{E}(\vec{r}_j)\times\vec{p}_j \\ &- \vec{\sigma}_j\circ\vec{p}_j\times\vec{E}(\vec{r}_j) + 2e_j\vec{\sigma}_j\cdot\vec{A}(\vec{r}_j)\times\vec{E}(\vec{r}_j)] \ . \end{aligned} \qquad (3.5)$$

Here κ_j is the anomalous magnetic moment for the jth nucleon ($\kappa = 1.79$ for the proton, -1.91 for the neutron). It should be emphasized that H_I^{Tj} together with $H_I^{\mu j}$ are just the results obtained from the FW transformation[14] of the Dirac equation for the jth nucleon in the electromagnetic field. The separation of H_I^{Tj} from $H_I^{\mu j}$ is for convenience in the discussion of current conservation in the next section. Since the sum of the two FW reductions does not give the complete electromagnetic Hamiltonian, an additional term must be added. This term is called H_I^{Δ}, the correction to the FW reduction.[4-8] It is derived in Appendix A, and it has the form

$$H_I^{\Delta}(\vec{A}, \vec{E}) = -\tfrac{1}{2}\sum_{j=1}^{2} e_j[\vec{\eta}_j'\cdot\vec{E}(\vec{r}_j) + \vec{E}(\vec{r}_j)\cdot\vec{\eta}_j'] \ . \qquad (3.6)$$

$\vec{\eta}_j'$ are given by Eqs. (A8a) and (A8b) of Appendix A, with the gauge-invariant substitution.

Finally, since the potential V_N of Eq. (2.14) has momentum dependence, the electromagnetic Hamiltonian due to V_N can be obtained from it by the minimal substitution. Writing V_N as $V_N(\vec{p}_1,\vec{p}_2)$, we obtain

$$H_I^V(\vec{A}) = V_N(\vec{\Pi}_1, \vec{\Pi}_2) - V_N(\vec{p}_1, \vec{p}_2)\,. \tag{3.7}$$

The detailed expression for H_I^V is not particularly interesting, and will be omitted.

4. CURRENT CONSERVATION

The charge and current densities corresponding to the electromagnetic Hamiltonian given in the last section will be derived here. The relativistic correction to order c^{-2} will be included in these densities. Current conservation to this order can be proved by using these densities and some useful relations derived in Appendix B.

For the electromagnetic interaction Hamiltonian H_I defined by Eq. (3.2), the charge density is defined as

$$\rho(\vec{x}) = \frac{\delta H_I(\Phi, \vec{A}, \vec{E})}{\delta\Phi(\vec{x})}\,. \tag{4.1}$$

Since the Hamiltonians $H_I^{T_j}(\vec{A})$ and $H_I^V(\vec{A})$ do not depend on the scalar potential $\Phi(\vec{r})$, they make no contribution to the charge density. Therefore, $\rho(\vec{x})$ contains only three terms,

$$\rho(\vec{x}) = \rho^1(\vec{x}) + \rho^2(\vec{x}) + \rho^\Delta(\vec{x})\,, \tag{4.2}$$

where

$$\rho^j(\vec{x}) = \frac{\delta H_I^{\mu_j}}{\delta\Phi(\vec{x})}$$

$$= e_j\delta^3(\vec{r}_j - \vec{x}) + \frac{2\kappa_j e + e_j}{8m_j^2}\{\vec{\nabla}_j^2\delta^3(\vec{r}_j - \vec{x})$$

$$-\vec{\sigma}_j\cdot[\mathrm{p}_j\times\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})] + \vec{\sigma}_j\cdot[\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})\times\vec{p}_j]$$

$$+2e_j\vec{\sigma}_j\cdot[\vec{A}(\vec{r}_j)\times\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})]\}\,, \tag{4.3}$$

$$\rho^\Delta(\vec{x}) = \frac{\delta H_I^\Delta}{\delta\Phi(\vec{x})}$$

$$= \tfrac{1}{2}\sum_{j=1}^{2} e_j\{\vec{\eta}_j'\cdot[\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})]$$

$$+[\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})]\cdot\vec{\eta}_j'\}\,. \tag{4.4}$$

The current density corresponding to H_I is defined as

$$\vec{J}(\vec{x}) = -\frac{\delta H_I}{\delta\vec{A}(\vec{x})} - \frac{d}{dt}\left(\frac{\delta H_I}{\delta\vec{E}(\vec{x})}\right) \tag{4.5a}$$

$$= \vec{J}^{T_1} + \vec{J}^{T_2} + \vec{J}^{\mu_1} + \vec{J}^{\mu_2} + \vec{J}^\Delta + \vec{J}^V\,, \tag{4.5b}$$

where

$$\vec{J}^{T_j} = -\frac{\delta H_I^{T_j}}{\delta\vec{A}(\vec{x})} \tag{4.6a}$$

$$= \frac{e_j}{2m_j}[\vec{p}_j\delta^3(\vec{r}_j-\vec{x}) + \delta^3(\vec{r}_j-\vec{x})\vec{p}_j]$$

$$-\frac{e_j^2}{m_j}\vec{A}(\vec{r}_j)\delta^3(\vec{r}_j-\vec{x})$$

$$-\frac{e_j}{8m_j^3}\{\vec{p}_j^2[\vec{p}_j\delta^3(\vec{r}_j-\vec{x}) + \delta^3(\vec{r}_j-\vec{x})\vec{p}_j]$$

$$+[\vec{p}_j\delta^3(\vec{r}_j-\vec{x}) + \delta^3(\vec{r}_j-\vec{x})\vec{p}_j]\vec{p}_j^2\} + \cdots\,, \tag{4.6b}$$

$$\vec{J}^{\mu_j} = -\frac{\delta H_I^{\mu_j}}{\delta\vec{A}(\vec{x})} - \frac{d}{dt}\left(\frac{\delta H_I^{\mu_j}}{\delta\vec{E}(\vec{x})}\right) \tag{4.7a}$$

$$= \frac{\kappa_j e + e_j}{2m_j}\vec{\sigma}_j\times\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})$$

$$-\frac{2\kappa_j e + e_j}{4m_j^2}\vec{\sigma}_j\times\vec{E}(\vec{r}_j)\delta^3(\vec{r}_j-\vec{x})$$

$$+\frac{2\kappa_j e + e_j}{8m_j^2}\frac{d}{dt}\{\vec{\nabla}_j\delta^3(\vec{r}_j-\vec{x})$$

$$-\vec{\sigma}_j\times\vec{p}_j\delta^3(\vec{r}_j-\vec{x}) - \delta^3(\vec{r}_j-\vec{x})\vec{\sigma}_j\times\vec{p}_j$$

$$+2e_j\vec{\sigma}_j\times\vec{A}(\vec{r}_j)\delta^3(\vec{r}_j-\vec{x})\}\,, \tag{4.7b}$$

$$\vec{J}^\Delta = -\frac{\delta H_I^\Delta}{\delta\vec{A}(\vec{x})} - \frac{d}{dt}\left(\frac{\delta H_I^\Delta}{\delta\vec{E}(\vec{x})}\right) \tag{4.8a}$$

$$= \frac{\delta}{\delta\vec{A}(\vec{x})}\left\{+\tfrac{1}{2}\sum_{j=1}^{2}e_j[\vec{\eta}_j'\cdot\vec{E}(\vec{r}_j) + \vec{E}(\vec{r}_j)\cdot\vec{\eta}_j']\right\}$$

$$+\frac{d}{dt}\left\{\frac{\delta}{\delta\vec{E}(\vec{x})}\left[+\tfrac{1}{2}\sum_{j=1}^{2}e_j(\vec{\eta}_j'\cdot\vec{E} + \vec{E}\cdot\vec{\eta}_j')\right]\right\}\,, \tag{4.8b}$$

$$\vec{J}^V = -\frac{\delta H_I^V}{\delta\vec{A}(\vec{x})} \tag{4.9a}$$

$$= -\frac{\delta V_N(\vec{\Pi}_1, \vec{\Pi}_2)}{\delta\vec{A}(\vec{x})}\,. \tag{4.9b}$$

We omit the detailed form of the currents $\vec{J}^\Delta$ and $\vec{J}^V$.

We next show that the charge density and current density defined above satisfy the current conservation equation. The proof is very general since we use the general relations derived in Appendix B. First, using Eqs. (B19) and (B20), we rewrite Eq. (4.2) as

$$\frac{d\rho(\vec{x})}{dt} = \frac{d\rho^1(\vec{x})}{dt} + \frac{d\rho^2(\vec{x})}{dt} + \frac{d\rho^\Delta(\vec{x})}{dt}$$

$$= \vec{\nabla}_x\cdot\frac{d}{dt}\left(\frac{\delta H_I^\Delta}{\delta\vec{E}(\vec{x})}\right) + \vec{\nabla}_x\cdot\frac{d}{dt}\left(\frac{\delta H_I^{\mu_1}}{\delta\vec{E}(\vec{x})}\right) \tag{4.10}$$

$$+\vec{\nabla}_x\cdot\frac{d}{dt}\left(\frac{\delta H_I^{\mu_2}}{\delta\vec{E}(\vec{x})}\right) + \frac{d}{dt}\left[\sum_{j=1}^{2}e_j\delta^3(\vec{r}_j-\vec{x})\right].$$

Applying Eqs. (4.7a) and (4.8a), and Eqs. (B16) and (B17b) of Appendix B, this becomes

$$\frac{d\rho(\vec{x})}{dt} = -\vec{\nabla}_x\cdot(\vec{J}^{\mu_1}+\vec{J}^{\mu_2}+\vec{J}^{\Delta}) - i\left[H_I^{\mu_1}+H_I^{\mu_2}+H_I^{\Delta}, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right] + \frac{d}{dt}\left[\sum_{j=1}^{2} e_j\,\delta^3(\vec{r}_j-\vec{x})\right]. \quad (4.11)$$

If we combine Eq. (4.6a) with Eq. (B12) of Appendix B, and also combine Eq. (4.9a) with Eq. (B13) of Appendix B, we get

$$-\vec{\nabla}_x\cdot\vec{J}^{T_j} = i\left[T_j + H_I^{T_j}, e_j\delta^3(\vec{r}_j-\vec{x})\right], \quad (4.12)$$

$$-\vec{\nabla}_x\cdot\vec{J}^{V} = i\left[H_I^{V} + V_N, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad (4.13)$$

Since T_1 and $H_I^{T_1}$ commute with $\delta^3(\vec{r}_2-\vec{x})$, and T_2 and $H_I^{T_2}$ commute with $\delta^3(\vec{r}_1-\vec{x})$, we have

$$-\vec{\nabla}_x\cdot(\vec{J}^{T_1}+\vec{J}^{T_2}) = i\left[T_1+T_2+H_I^{T_1}+H_I^{T_2}, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right] \quad (4.14)$$

and

$$-\vec{\nabla}_x\cdot(\vec{J}^{T_1}+\vec{J}^{T_2}+\vec{J}^{V}) = i\left[T_1+T_2+V_N+H_I^{T_1}+H_I^{T_2}+H_I^{V}, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right] = i\left[H_T - H_I^{\mu_1} - H_I^{\mu_2} - H_I^{\Delta}, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad (4.15)$$

We have used Eq. (3.1) in obtaining the last step. If we further combine Eq. (4.11) with Eq. (4.15), and use $\vec{J}$ defined by Eq. (4.5b), we get

$$\frac{d\rho(\vec{x})}{dt} = -\vec{\nabla}\cdot\vec{J} - i\left[H_T, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right] + \frac{d}{dt}\left[\sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad (4.16)$$

Since

$$\frac{d}{dt}\left[\sum_{j=1}^{2} e_j\,\delta^3(\vec{r}_j-\vec{x})\right] = i\left[H_T, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right] \quad (4.17)$$

we obtain, finally,

$$\frac{d\rho(\vec{x})}{dt} + \vec{\nabla}\cdot\vec{J} = 0 \quad (4.18)$$

which is the current conservation equation. Thus the charge and current densities derived from the electromagnetic Hamiltonian in Eq. (3.2), which includes relativistic effects to order c^{-2}, satisfy current conservation to the same order.

Comment on Exchange Currents

It has been tacitly assumed in Secs. 2 and 3 that the potential used does not contain exchange forces. If such forces exist, some modifications in defining the current and charge densities arising from the potential are needed. If we impose current conservation, we find that additional terms in the current density are implied. Specifically, using the Hamiltonian of Eqs. (3.1) and (3.2), current conservation leads to

$$\vec{\nabla}_x\cdot\vec{J}^{V} \equiv -\vec{\nabla}_x\cdot\frac{\delta H_I^V}{\delta\vec{A}} - \vec{\nabla}_x\cdot\frac{d}{dt}\left(\frac{\delta H_I^V}{\delta\vec{E}}\right) = -i\left[V_N + H_I^V, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad (4.19)$$

If we further define

$$\vec{\nabla}_x\cdot\vec{J}_2 = \vec{\nabla}_x\cdot\vec{J}^{V} + i\left[H_I^V, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right] \quad (4.20)$$

we obtain

$$\vec{\nabla}_x\cdot\vec{J}_2 = -i\left[V_N, \sum_{j=1}^{2} e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad (4.21)$$

This result is very similar to that obtained by Heller.[15] Here relativistic effects to order c^{-2} are included. The expression for $\vec{\nabla}_x\cdot\vec{J}_2$ is unique, but of course $\vec{J}_2$ itself is not. The fact that the exchange current is not uniquely defined by current conservation is well known.

5. LOW-ENERGY THEOREM

The low-energy theorem for bremsstrahlung in a potential model has been derived in previous papers,[16-19] for the case of nonrelativistic particles. Since the theorem follows from general principles it must be valid for relativistic particles.[20] The verification of this point from our Hamiltonian is somewhat complicated in the case of the terms like $H_I^{\mu_j}$ and H_I^{Δ}. Here we illustrate the validity of the theorem for the relativistic corrections to the kinetic energy. The method is similar to Ref. 18.

For simplicity we use the Coulomb gauge ($\Phi=0$, $\vec{\nabla}\cdot\vec{A}=0$) and assume only particle 1 is charged. Then we are interested in the term

$$(\vec{K}|H_I^{T_1}|0) = Ne^{-i\vec{K}\cdot\vec{r}_1}\vec{p}_1\cdot\hat{\epsilon}\left[1-\frac{1}{4m_1^2}(2\vec{p}_1^{\,2}-2\vec{K}\cdot\vec{p}_1+\vec{K}^2)\right], \quad (5.1a)$$

$$= N\left[1-\frac{1}{4m_1^2}(2\vec{p}_1^{\,2}+2\vec{K}\cdot\vec{p}_1+\vec{K}^2)\right]\vec{p}_1\cdot\hat{\epsilon}e^{-i\vec{K}\cdot\vec{r}_1} \quad (5.1b)$$

where $N=-e_1/2\pi m_1\sqrt{K}$, $\vec{K}$ is the photon momentum and $\hat{\epsilon}$ is the photon polarization vector. Expression (5.1a) is used for the amplitude representing photon emission after the strong interaction, and expression (5.1b) for the amplitude representing photon emission before the strong interaction. The K^2 terms will be dropped because they do not enter the low-energy theorem. The second kind of electromagnetic interaction is the Hamiltonian arising out of the potential given by Eq. (2.14). It is given by

$$\langle \vec{K} | H_I^V | 0 \rangle = -m_1 N \hat{\epsilon} \cdot \vec{\nabla}_K [V_N, e^{-i\vec{K}\cdot\vec{r}_1}] . \tag{5.2}$$

The Green's function, without relativistic effects, is

$$G_0^{NR}(E^{NR}) = (E^{NR} - T_1^{NR} - T_2^{NR} + i\epsilon)^{-1} . \tag{5.3}$$

Here, E^{NR} is the nonrelativistic total energy of the system, and T_j^{NR} are the nonrelativistic kinetic energy operators defined as

$$T_j^{NR} = \frac{\vec{p}_j^{\,2}}{2m_j} . \tag{5.4}$$

With relativistic effects, the T_j^{NR} are modified to

$$T_j = \frac{\vec{p}_j^{\,2}}{2m_j} - \frac{(\vec{p}_j^{\,2})^2}{8m_j^3} , \tag{5.5}$$

and the Green's function becomes

$$G_0(E) = (E - T_1 - T_2 + i\epsilon)^{-1} . \tag{5.6}$$

E is the relativistic total energy of the system. Now, using the potential V_N given by Eq. (2.14), and the Green's function given by Eq. (5.6), the t matrix is determined by the Lippmann-Schwinger equation,

$$t(E) = V_N + V_N G_0(E) t(E) . \tag{5.7}$$

The external scattering term can be written in terms of $H_I^{T_1}$, G_0 and t as

$$\begin{aligned} \hat{\epsilon}\cdot\vec{M}_E = {} & \frac{\hat{\epsilon}\cdot\vec{p}_1'}{\Delta}\left(1 - \frac{2\vec{p}_1'^{\,2} + 2\vec{p}_1'\cdot\vec{K} + \vec{K}^2}{4m_1^2}\right) \\ & \times \langle \vec{p}_1' + \vec{K}, \vec{p}_2' | t(E) | \vec{p}_1, \vec{p}_2 \rangle \\ & + \langle \vec{p}_1', \vec{p}_2' | t(E') | \vec{p}_1 - \vec{K}, \vec{p}_2 \rangle \\ & \times \left(1 - \frac{2\vec{p}_1^{\,2} - 2\vec{p}_1\cdot\vec{K} + \vec{K}^2}{4m_1^2}\right)\frac{\hat{\epsilon}\cdot\vec{p}_1}{\Delta'} , \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} \Delta = \vec{K}\cdot\Bigg[& \frac{\vec{K}}{K} - \frac{\vec{p}_1'}{m_1}\left(1 - \frac{2\vec{p}_1'^{\,2} + 2\vec{p}_1'\cdot\vec{K} + 2\vec{K}^2}{4m_1^2}\right) \\ & - \frac{\vec{K}}{2m_1}\left(1 - \frac{2\vec{p}_1'^{\,2} + \vec{K}^2}{4m_1^2}\right)\Bigg] , \end{aligned} \tag{5.9a}$$

$$\begin{aligned} \Delta' = \vec{K}\cdot\Bigg[& -\frac{\vec{K}}{K} + \frac{\vec{p}_1}{m_1}\left(1 - \frac{2\vec{p}_1^{\,2} - 2\vec{p}_1\cdot\vec{K} + 2\vec{K}^2}{4m_1^2}\right) \\ & - \frac{\vec{K}}{2m_1}\left(1 - \frac{2\vec{p}_1^{\,2} + \vec{K}^2}{4m_1^2}\right)\Bigg] . \end{aligned} \tag{5.9b}$$

If we parametrize the t matrix elements in terms of the following scalar variables: the average of the initial and final kinetic energies in the c.m. system ν, the square of the momentum transfer u, and the amount that the initial (final) state is off the energy shell $\Delta_i(\Delta_f)$; i.e.,

$$t = t(\nu, u, \Delta_i, \Delta_f) , \tag{5.10}$$

and then expand them in powers of K, we obtain

$$\begin{aligned} \hat{\epsilon}\cdot\vec{M}_E = {} & (\text{on-shell terms}) \\ & + \hat{\epsilon}\cdot\vec{p}_1'\left[1 - \frac{1}{2m_1^2}(\vec{p}_1'^{\,2} + \vec{p}_1'\cdot\vec{K})\right]\frac{\partial t}{\partial\Delta_f} \\ & + \frac{\partial t}{\partial\Delta_i}\hat{\epsilon}\cdot\vec{p}_1\left[1 - \frac{1}{2m_1^2}(\vec{p}_1^{\,2} - \vec{p}_1\cdot\vec{K})\right] + O(K) . \end{aligned} \tag{5.11}$$

The on-shell terms which are not of interest to us here are very similar to the results obtained in Ref. 18. The internal scattering amplitude consists of $\vec{M}_R$ and $\vec{M}_G$

$$\hat{\epsilon}\cdot\vec{M}_I = \hat{\epsilon}\cdot\vec{M}_R + \hat{\epsilon}\cdot\vec{M}_G , \tag{5.12}$$

where

$$\begin{aligned} N\hat{\epsilon}\cdot\vec{M}_R &= \langle \vec{p}_1', \vec{p}_2' | t(E') G_0(E') \langle \vec{K} | H_I^{T_1} | 0 \rangle G_0(E) t(E) | \vec{p}_1, \vec{p}_2 \rangle , \\ N\hat{\epsilon}\cdot\vec{M}_G &= \langle \vec{p}_1', \vec{p}_2' | [1 + t(E') G_0(E')] \langle \vec{K} | H_I^V | 0 \rangle [1 + G_0(E) t(E)] | \vec{p}_1, \vec{p}_2 \rangle . \end{aligned} \tag{5.13}$$

It is easy to show that

$$\begin{aligned} \hat{\epsilon}\cdot\vec{M}_R &= -m_1 \hat{\epsilon}\cdot\vec{\nabla}_K \langle p_1', p_2' | t(E') G_0(E') [Q_0 G_0^{-1}(E) - G_0^{-1}(E') Q_0] G_0(E) t(E) | \vec{p}_1, \vec{p}_2 \rangle , \\ Q_0 &= e^{-i\vec{K}\cdot\vec{r}_1} . \end{aligned} \tag{5.14}$$

We now follow Ref. 18: Apply the operator identity

$$\begin{aligned} \hat{\epsilon}\cdot\vec{\nabla}_K \langle \vec{p}_1', \vec{p}_2' | t(E') G_0(E') [Q_0 G_0^{-1}(E) - G_0^{-1}(E') Q_0] G_0(E) t(E) | \vec{p}_1, \vec{p}_2 \rangle \\ = \hat{\epsilon}\cdot\vec{\nabla}_K \langle \vec{p}_1', \vec{p}_2' | t(E') Q_0 - Q_0 t(E) | \vec{p}_1, \vec{p}_2 \rangle \\ - \hat{\epsilon}\cdot\vec{\nabla}_K \langle \vec{p}_1', \vec{p}_2' | [1 + t(E') G_0(E')] [V_N, Q_0] [1 + G_0(E) t(E)] | \vec{p}_1, \vec{p}_2 \rangle \end{aligned} \tag{5.15}$$

to Eq. (5.14); use the relation

$$\hat{\epsilon}\cdot\vec{M}_G+m_1\hat{\epsilon}\cdot\vec{M}_0=0\,, \tag{5.16}$$

where

$$\hat{\epsilon}\cdot\vec{M}_0=\hat{\epsilon}\cdot\vec{\nabla}_K\langle\vec{p}_1'\vec{p}_2'\,|\,[1+t(E')G_0(E')][\,V_N,\,Q_0][1+G_0(E)t(E)]\,|\,\vec{p}_1,\vec{p}_2\rangle\,. \tag{5.17}$$

We obtain finally

$$\begin{aligned}\hat{\epsilon}\cdot\vec{M}_I&=-m_1\hat{\epsilon}\cdot\vec{\nabla}_K\langle\vec{p}_1',\vec{p}_2'\,|\,t(E')Q_0-Q_0t(E)\,|\,\vec{p}_1,\vec{p}_2\rangle+O(K)\\&=-m_1\hat{\epsilon}\cdot\vec{\nabla}_K\{\langle\vec{p}_1',\vec{p}_2'|\,t(E')|\,\vec{p}_1-\vec{K},\vec{p}_2\rangle-\langle\vec{p}_1'+\vec{K},\vec{p}_2|\,t(E)|\,\vec{p}_1,\vec{p}_2\rangle\}+O(K)\,.\end{aligned} \tag{5.18}$$

Parametrizing $\vec{M}_I$ and expanding then in powers of K, as we did for $\vec{M}_E$, leads to

$$\begin{aligned}\hat{\epsilon}\cdot\vec{M}_I=&\;(\text{on-shell terms})\\&-\frac{\partial t}{\partial\Delta_i}\,\hat{\epsilon}\cdot\vec{p}_1\left(1-\frac{1}{2m_1^2}(\vec{p}_1^{\,2}-\vec{p}_1\cdot\vec{K})\right)\\&-\hat{\epsilon}\cdot\vec{p}_1'\left(1-\frac{1}{2m_1^2}(\vec{p}_1'^{\,2}+\vec{p}_1'\cdot\vec{K})\right)\frac{\partial t}{\partial\Delta_f}+O(K)\,.\end{aligned} \tag{5.19}$$

Combining Eqs. (5.11) and (5.19), we obtain the total bremsstrahlung amplitude. The off-shell derivatives are cancelled precisely. Thus the first two terms of the total amplitude are independent of off-energy shell effects, completing the proof of the low-energy theorem.

Note added in proof: A recent report prior to publication by Richard H. Thompson and Leon Heller discusses the exchange current in the case of a one-pion exchange potential which is isospin dependent but not momentum dependent. If V_N has the form $V_D+\vec{\tau}_1\cdot\vec{\tau}_2V_E$, then the exchange current obeys

$$\vec{\nabla}_x\cdot\vec{J}_2=-ie(\vec{\tau}_1\times\vec{\tau}_2)_z[\,\delta^3(\vec{r}_1-\vec{x})-\delta^3(\vec{r}_2-\vec{x})]$$

as originally derived by Osborn and Foldy.[20a] We observe that this expression follows from our Eq. (4.21) if we replace e_j by $\frac{1}{2}(1+\tau_{jz})e$.

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APPENDIX A

We discuss relativistic and nonrelativistic definitions of the center-of-mass and internal dynamical variables, and also derive the correction terms to the FW reduction. For greater clarity we will not set $c=1$ here, but rather keep c in our equations. The nonrelativistic limit is then $c\to\infty$, and the relativistic corrections are terms in c^{-2}.

Osborn[5] has defined the relativistic center-of-mass and internal variables for a two-particle system with spin. He has shown that if one starts from the individual particle dynamical variables $(\vec{p}_1,\ \vec{p}_2,\ \vec{r}_1,\ \vec{r}_2,\ \vec{s}_1,\ \vec{s}_2)$, one can define the c.m. variables $(\vec{P},\vec{R},\vec{S})$ and, via the Gartenhaus-Schwartz transformation, the internal variables $(\vec{q},\vec{r},\vec{s}_1',\vec{s}_2')$. In defining the nonrelativistic limit of these variables some care must be taken. Both sets of variables, the individual particle variables on one hand and the c.m. plus internal variables on the other, are relativistic. Thus we can take a nonrelativistic limit for either set. But, because of the relation that exists between the two sets, we cannot take the nonrelativistic limit for both. If we do we will find a contradictory term of order c^{-2}. In this paper we let the set of individual particle dynamical variables be independent of c. Then the expressions for the internal and c.m. variables are functions of c, and taking $c\to\infty$, we obtain unambiguous definitions of $\vec{R}^{NR}$, $\vec{q}^{NR}$, and $\vec{r}^{NR}$, the nonrelativistic variables. There are no such things as $\vec{p}_j^{NR}$ and $\vec{r}_j^{NR}$ in our definitions. An alternative definition is used in Refs. 4–6, where $\vec{R}$, $\vec{q}$, and $\vec{r}$ are treated as independent of c, while $\vec{p}_j$ and $\vec{r}_j$ are functions of c. In that case nonrelativistic variables $\vec{p}_j^{NR}$ and $\vec{r}_j^{NR}$ are defined. Either definition can be considered correct. But one must use the definitions consistently, since they differ by terms of order c^{-2}.

For the convenience of the reader and for use in Sec. 2, we summarize these various kinematic relations. In terms of single-particle momenta $\vec{p}_j$, coordinates $\vec{r}_j$, masses m_j, and spins $\vec{s}_j$, the internal and c.m. variables are

$$\begin{aligned}\vec{q}&=\vec{q}^{NR}+\delta\vec{q}\,,\\\vec{r}&=\vec{r}^{NR}+\delta\vec{r}\,,\\\vec{P}&=\vec{P}^{NR},\\\vec{R}&=\vec{R}^{NR}+\delta\vec{R}\,,\\\vec{s}_j'&=\vec{s}_j^{NR}+\delta\vec{s}_j\,,\end{aligned} \tag{A1}$$

where the nonrelativistic quantities are

$$\vec{q}^{NR}=\frac{m_2\vec{p}_1-m_1\vec{p}_2}{m},$$

$$\vec{r}^{NR}=\vec{r}_1-\vec{r}_2,$$

$$\vec{P}^{NR}=\vec{p}_1+\vec{p}_2, \tag{A2}$$

$$\vec{R}^{NR}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m},$$

$$\vec{s}_j^{NR}=\vec{s}_j,$$

with $m=m_1+m_2$. The relativistic corrections are, to order c^{-2},

$$\delta\vec{q}=\left(\frac{m_1-m_2}{2m_1m_2mc^2}\vec{q}^{NR\,2}-\frac{1}{2m^2c^2}\vec{q}^{NR}\cdot\vec{P}\right)\vec{P},$$

$$\delta\vec{r}=\frac{1}{2mc^2}\vec{r}^{NR}\cdot\vec{P}\left[\frac{1}{2m}\vec{P}+\left(\frac{1}{m_1}-\frac{1}{m_2}\right)\vec{q}^{NR}\right]+\text{H.c.}-\frac{1}{2mc^2}\vec{P}\times\left(\frac{\vec{s}_1}{m_1}-\frac{\vec{s}_2}{m_2}\right), \tag{A3}$$

$$\delta\vec{R}=\frac{1}{4mc^2}\left[\frac{m_2-m_1}{m_1m_2}\vec{r}^{NR}\vec{q}^{NR\,2}+\vec{r}^{NR}\frac{\vec{P}\cdot\vec{q}^{NR}}{m}+\frac{\vec{r}^{NR}\cdot\vec{P}}{m}\vec{q}^{NR}\right]+\text{H.c.}+\frac{1}{2mc^2}\vec{q}^{NR}\times\left(\frac{\vec{s}_1}{m_1}-\frac{\vec{s}_2}{m_2}\right),$$

$$\delta\vec{s}_j=-\frac{1}{2m_jmc^2}(\vec{q}^{NR}\times\vec{P})\times\vec{s}_j.$$

Inverting Eqs. (A2), we obtain the single-particle variables

$$\vec{p}_1=\frac{m_1}{m}\vec{P}+\vec{q}^{NR},$$

$$\vec{p}_2=\frac{m_2}{m}\vec{P}-\vec{q}^{NR},$$

$$\vec{r}_1=\vec{R}^{NR}+\frac{m_2}{m}\vec{r}^{NR},$$

$$\vec{r}_2=\vec{R}^{NR}-\frac{m_1}{m}\vec{r}^{NR}. \tag{A4}$$

We now derive the correction to the FW reduction. These terms have been derived previously by Krajcik and Foldy[7] by a different method, and by Close and Copley.[6] Our derivation is similar in form to Ref. 6, but there is an important difference, resulting from our different definition of the nonrelativistic limit.

For an electromagnetic transition from state i to f, the amplitude is

$$M_{fi}=\int d^3Rd^3r\psi_f^*(\vec{R},\vec{r})H_{\text{e.m.}}(\vec{r}_1,\vec{r}_2)\psi_i(\vec{R},\vec{r}), \tag{A5}$$

where $H_{\text{e.m.}}$ is the electromagnetic Hamiltonian. For simplicity the spin factors in the wave function are suppressed, since the treatment of spin is the same as in Ref. 6. We take a scalar potential only,

$$H_{\text{e.m.}}(\vec{r}_1,\vec{r}_2)=\sum_{j=1}^{2}e_j\,\Phi(\vec{r}_j), \tag{A6}$$

since the generalization to a vector potential can be done as in Ref. 6. Combining Eqs. (A1) and (A4) we have

$$\vec{r}_1=\left(\vec{R}+\frac{m_2}{m}\vec{r}\right)+\vec{\eta}_1,$$

$$\vec{r}_2=\left(\vec{R}-\frac{m_1}{m}\vec{r}\right)+\vec{\eta}_2 \tag{A7}$$

with

$$\vec{\eta}_1=-\left(\delta\vec{R}+\frac{m_2}{m}\delta\vec{r}\right)$$

$$=\frac{1}{4mc^2}(\vec{r}_1-\vec{r}_2)\left(\frac{m_1-m_2}{m_1m_2}\vec{q}^{NR\,2}-\frac{\vec{P}\cdot\vec{q}^{NR}}{m}\right)+\frac{1}{2m^2c^2}(\vec{r}_1-\vec{r}_2)\cdot\vec{P}\left(\frac{\vec{q}^{NR}}{2}-\frac{m_2}{m_1}\vec{q}^{NR}-\frac{m_2}{2m}\vec{P}\right)+\text{H.c.}+\frac{m_2}{2m^2c^2}\left(\vec{P}-\frac{m}{m_2}\vec{q}^{NR}\right)\times\left(\frac{\vec{s}_1}{m_2}-\frac{\vec{s}_2}{m_2}\right), \tag{A8a}$$

$$\vec{\eta}_2=-\left(\delta\vec{R}-\frac{m_1}{m}\delta\vec{r}\right)$$

$$=\frac{1}{4mc^2}(\vec{r}_1-\vec{r}_2)\left(\frac{m_1-m_2}{m_1m_2}\vec{q}^{NR\,2}-\frac{\vec{P}\cdot\vec{q}^{NR}}{m}\right)+\frac{1}{2m^2c^2}(\vec{r}_1-\vec{r}_2)\cdot\vec{P}\left(\frac{\vec{q}^{NR}}{2}-\frac{m_1}{m_2}\vec{q}^{NR}+\frac{m_1}{2m}\vec{P}\right)+\text{H.c.}-\frac{m_1}{2m^2c^2}\left(\vec{P}+\frac{m}{m_1}\vec{q}^{NR}\right)\times\left(\frac{\vec{s}_1}{m_1}-\frac{\vec{s}_2}{m_2}\right). \tag{A8b}$$

Eq. (A5) now becomes

$$M_{fi} = \int d^3R d^3r\, \psi_f^*(\vec{R},\vec{r})\left[e_1\Phi\left(\vec{R}+\frac{m_2}{m}\vec{r}\right)+e_2\Phi\left(\vec{R}-\frac{m_1}{m}\vec{r}\right)\right]\psi_i(\vec{R},\vec{r})$$

$$-\int d^3Rd^3r\,\psi_f^*(\vec{R},\vec{r})\left\{\tfrac{1}{2}e_1\left[\vec{\eta}_1\cdot\vec{E}\left(\vec{R}+\frac{m_2}{m}\vec{r}\right)+\vec{E}\left(\vec{R}+\frac{m_2}{m}\vec{r}\right)\cdot\vec{\eta}_1\right]\right.$$

$$\left.+\tfrac{1}{2}e_2\left[\vec{\eta}_2\cdot\vec{E}\left(\vec{R}-\frac{m_1}{m}\vec{r}\right)+\vec{E}\left(\vec{R}-\frac{m_1}{m}\vec{r}\right)\cdot\vec{\eta}_2\right]\right\}\psi_i(\vec{R},\vec{r}) \qquad \text{(A9)}$$

where $\vec{E}=-\vec{\nabla}\Phi$ is the electric field. This amplitude may be compared with the nonrelativistic limit of M_{fi},

$$M_{fi}^{\rm NR}=\lim_{c\to\infty}M_{fi}$$

$$=\int d^3R^{\rm NR}d^3r^{\rm NR}\psi_f^*(\vec{R}^{\rm NR},\vec{r}^{\rm NR})\left[e_1\Phi\left(\vec{R}^{\rm NR}+\frac{m_2}{m}\vec{r}^{\rm NR}\right)\right.$$

$$\left.+e_2\Phi\left(\vec{R}^{\rm NR}-\frac{m_1}{m}\vec{r}^{\rm NR}\right)\right]\psi_i(\vec{R}^{\rm NR},\vec{r}^{\rm NR})\,. \qquad \text{(A10)}$$

Thus, because of the fact that $\vec{R}$, $\vec{r}$, $\vec{R}^{\rm NR}$, and $\vec{r}^{\rm NR}$ are integrated over all space, we see that the Φ terms in Eq. (A9) are precisely equal to $M_{fi}^{\rm NR}$. We find therefore the relativistic correction given by

$$M_{fi}=M_{fi}^{\rm NR}+\delta M_{fi}+O(c^{-4})\,, \qquad \text{(A11)}$$

$$\delta M_{fi}=\frac{1}{c^2}\lim_{c\to\infty}c^2(M_{fi}-M_{fi}^{\rm NR})$$

$$=-\frac{1}{2}\int d^3r_1d^3r_2\psi_f^*(\vec{r}_1,\vec{r}_2)\left\{\sum_{j=1}^{2}e_j[\vec{\eta}_j\cdot\vec{E}(\vec{r}_j)+\vec{E}(\vec{r}_j)\cdot\vec{\eta}_j]\right\}\psi_i(\vec{r}_1,\vec{r}_2)\,, \qquad \text{(A12)}$$

where we have used Eq. (A4). This result implies that the correction to the Hamiltonian $H_{\rm e.m.}$ given by Eq. (A6) can be written as

$$H_I^{\Delta}=-\tfrac{1}{2}\sum_{j=1}^{2}e_j[\vec{\eta}_j'\cdot\vec{E}(\vec{r}_j)+\vec{E}(\vec{r}_j)\cdot\vec{\eta}_j']\,. \qquad \text{(A13)}$$

Here $\vec{\eta}_j'$ is the function obtained from $\vec{\eta}_j$ by the substitution $\vec{p}_j\to\vec{p}_j-\frac{e_j}{c}\vec{A}(\vec{r}_j)$, necessary to preserve gauge invariance.

For completeness, we write $M_{fi}^{\rm NR}$ in terms of the individual particle variables, using Eq. (A4),

$$M_{fi}^{\rm NR}=\int d^3r_1d^3r_2\,\psi_f^*(\vec{r}_1,\vec{r}_2)[e_1\Phi(\vec{r}_1)+e_2\Phi(\vec{r}_2)]\psi_i(\vec{r}_1,\vec{r}_2)\,. \qquad \text{(A14)}$$

APPENDIX B

We here discuss some useful identities related to the electromagnetic Hamiltonian defined in Sec. 3. These identities are used in Sec. 4.

(I) Let $F(\vec{p})$ be a function of momentum operator $\vec{p}$, and $F(\vec{\Pi})$ be a function obtained from $F(\vec{p})$ by the gauge invariant substitution:

$$\vec{p}\to\vec{\Pi}\equiv\vec{p}-e\vec{A}(\vec{r})\,.$$

If we define

$$H_I^F(\vec{A})=F(\vec{\Pi})-F(\vec{p}) \qquad \text{(B1)}$$

then we want to show that

$$\vec{\nabla}_x\cdot\frac{\delta H_I^F}{\delta\vec{A}(\vec{x})}=i[F(\vec{\Pi}),e\delta^3(\vec{r}-\vec{x})] \qquad \text{(B2a)}$$

$$=i[F(\vec{p})+H_I^F(\vec{A}),e\delta^3(\vec{r}-\vec{x})]\,. \qquad \text{(B2b)}$$

Proof. This result is true for a function with an arbitrary number of factors of the momentum operator.

Let us first consider a function of the form[21]

$$F(\vec{p})=[\vec{G}_{n1}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\cdot\vec{p}Q_{n1}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)][\vec{G}_{n2}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\cdot\vec{p}Q_{n2}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)]\times\cdots\times[\vec{G}_{nn}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\cdot\vec{p}Q_{nn}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)]$$
$$=\prod_{i=1}^{n}[\vec{G}_{ni}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\cdot\vec{p}Q_{ni}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)] \quad \text{(B3)}$$

and define $H_I^F(\vec{A})$ by Eq. (B1) with the function $F(\vec{\Pi})$ given by

$$F(\vec{\Pi})=\prod_{i=1}^{n}[\vec{G}_{ni}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\cdot\vec{\Pi}\,Q_{ni}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)]\,. \quad \text{(B4)}$$

The functional derivative of H_I^F with respect to $A(\vec{x})$ can be written as

$$\frac{\delta H_I^F[\vec{A}(\vec{r})]}{\delta\vec{A}(\vec{x})}=(-e)\Big\{[\vec{G}_{n1}\delta^3(\vec{r}-\vec{x})Q_{n1}]\prod_{i=2}^{n}[\vec{G}_{ni}\cdot\vec{\Pi}\,Q_{ni}]$$
$$+\sum_{s=2}^{n-1}\left[\prod_{i=1}^{s-1}[\vec{G}_{ni}\cdot\vec{\Pi}\,Q_{ni})\right][\vec{G}_{ns}\delta^3(\vec{r}-\vec{x})Q_{ns}]\left[\prod_{j=s+1}^{n}(\vec{G}_{nj}\cdot\vec{\Pi}Q_{nj})\right]$$
$$+\prod_{i=1}^{n-1}[\vec{G}_{ni}\cdot\vec{\Pi}Q_{ni}][\vec{G}_{nn}\delta^3(\vec{r}-\vec{x})Q_{nn}]\Big\}\,. \quad \text{(B5)}$$

Using the relation

$$\vec{\nabla}_x\,\delta^3(\vec{r}-\vec{x})=-\vec{\nabla}_r\,\delta^3(\vec{r}-\vec{x})\,,$$

we obtain

$$\vec{\nabla}_x\cdot\frac{\delta H_I^F}{\delta\vec{A}}=e\Big\{\vec{G}_{n1}\cdot[\vec{\nabla}_r\delta^3(\vec{r}-\vec{x})]\,Q_{n1}\prod_{i=2}^{n}[\vec{G}_{ni}\cdot\vec{\Pi}\,Q_{ni}]$$
$$+\sum_{s=2}^{n-1}\prod_{i=1}^{s-1}(\vec{G}_{ni}\cdot\vec{\Pi}Q_{ni})\vec{G}_{ns}\cdot[\vec{\nabla}_r\delta^3(\vec{r}-\vec{x})]\,Q_{ns}\prod_{j=s+1}^{n}(\vec{G}_{nj}\cdot\vec{\Pi}Q_{nj})$$
$$+\prod_{i=1}^{n-1}(\vec{G}_{ni}\cdot\vec{\Pi}Q_{ni})\vec{G}_{nn}\cdot[\vec{\nabla}_r\delta^3(\vec{r}-\vec{x})]\,Q_{nn}\Big\}\,. \quad \text{(B6)}$$

Now, we can use the identity

$$\left[\prod_{i=1}^{n}A_i,\,C\right]=[A_1,C]\prod_{i=2}^{n}A_i$$
$$+\sum_{s=2}^{n-1}\Big\{\prod_{i=1}^{s-1}A_i\,[A_s,C]\prod_{j=s+1}^{n}A_j\Big\}$$
$$+\prod_{i=1}^{n-1}A_i\,[A_n,C]\,. \quad \text{(B7)}$$

Applying this identity, we obtain

$$i[F(\vec{\Pi}),e\delta^3(\vec{r}-\vec{x})]=ie\Big\{[\vec{G}_{n1}\cdot\vec{\Pi}\,Q_{n1},\delta^3(\vec{r}-\vec{x})]\prod_{i=2}^{n}(\vec{G}_{ni}\cdot\vec{\Pi}\,Q_{ni})$$
$$+\sum_{s=2}^{n-1}\prod_{i=1}^{s-1}(\vec{G}_{ni}\cdot\vec{\Pi}\,Q_{ni})[\vec{G}_{ns}\cdot\vec{\Pi}\,Q_{ns},\delta^3(\vec{r}-\vec{x})]\prod_{j=s+1}^{n}(\vec{G}_{nj}\cdot\vec{\Pi}\,Q_{nj})$$
$$+\prod_{i=1}^{n-1}(\vec{G}_{ni}\cdot\vec{\Pi}\,Q_{ni})[\vec{G}_{nn}\circ\vec{\Pi}Q_{nn},\delta^3(\vec{r}-\vec{x})]\Big\}\,. \quad \text{(B8)}$$

Since

$$[\vec{G}_{nm}\cdot\vec{\Pi}\,Q_{nm},\delta^3(\vec{r}-\vec{x})]=[\vec{G}_{nm}\cdot\vec{p}Q_{nm},\delta^3(\vec{r}-\vec{x})]$$
$$=-i\vec{G}_{nm}\cdot[\vec{\nabla}_r\delta^3(\vec{r}-\vec{x})]\,Q_{nm}\,, \quad \text{(B9)}$$

the right-hand side of Eq. (B8) is exactly equal to the right-hand side of Eq. (B6), and the relation given by Eq. (B2a) follows at once. Eq. (B2b) is obtained from Eq. (B2a) by making use of Eq. (B1). Various forms of F can be constructed, and all of these can be shown to satisfy Eq. (B2). For example, we can replace $\vec{p}$ by $\vec{l}=\vec{r}\times\vec{p}$ and write

$$F'(\vec{l})=\prod_{i=1}^{n}\left[\vec{G}'_{ni}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\cdot\vec{l}\,Q'_{ni}(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)\right]. \quad \text{(B10)}$$

We conclude, therefore, the relation given by Eq. (B2) is true for a general function of the form

$$F(\vec{p})=F_0(\vec{r},\vec{\sigma}_1,\vec{\sigma}_2)+\sum_{n=1}^{\infty}\prod_{i=1}^{n}(\mathbf{G}_{ni}\cdot\vec{p}\,Q_{ni})+\sum_{n=1}^{\infty}\prod_{i=1}^{n}(\vec{G}'_{ni}\cdot\vec{l}\,Q'_{ni})+(\text{similar combinations}). \quad \text{(B11)}$$

Now, we apply this result to the electromagnetic Hamiltonian defined in Sec. 3. $H_I^{Tj}(\vec{A})$ and $H_I^V(\vec{A})$ are defined by Eqs. (3.4) and (3.7), respectively. Applying the result given by Eq. (B2), we get

$$\vec{\nabla}_x\cdot\frac{\delta H_I^{Tj}}{\delta\vec{A}(\vec{x})}=i\left[T_j(\vec{p}_j)+H_I^{Tj}(\vec{A}),\,e_j\delta^3(\vec{r}_j-\vec{x})\right], \quad \text{(B12)}$$

$$\vec{\nabla}_x\cdot\frac{\delta H_I^{V}}{\delta\vec{A}(\vec{x})}=i\left[H_I^V(\vec{A})+V_N(\vec{p}_1,\vec{p}_2),\,\sum_{j=1}^{2}e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad \text{(B13)}$$

(II) In some cases, the electromagnetic interaction Hamiltonian is not defined by Eq. (B1). The relations given by Eqs. (B2a) and (B2b) are no longer valid. However, if the electromagnetic Hamiltonian is known as a function of $\vec{\Pi}$,

$$H_I^f(\vec{A})=f(\vec{\Pi}) \quad \text{(B14)}$$

and the function f has a form similar to Eq. (B11), then we have

$$\vec{\nabla}_x\cdot\frac{\delta H_I^f}{\delta\vec{A}(\vec{x})}=i\left[f(\vec{\Pi}),\,e\delta^3(\vec{r}-\vec{x})\right]. \quad \text{(B15)}$$

The proof is very similar to the one given in (I). An example of this case is the Hamiltonian H_I^{Δ} defined by Eq. (3.6) of Sec. 3. Applying (B15), we get

$$\vec{\nabla}_x\cdot\frac{\delta H_I^{\Delta}}{\delta\vec{A}(\vec{x})}=i\left[H_I^{\Delta},\sum_{j=1}^{2}e_j\,\delta^3(\vec{r}_j-\vec{x})\right]. \quad \text{(B16)}$$

(III) If an electromagnetic Hamiltonian is a function of $\vec{A}$ and $\vec{E}$ $[\vec{E}=-\partial\vec{A}/\partial t-\vec{\nabla}_r\Phi(\vec{r})]$, then additional terms due to the $\vec{E}$ dependence can be obtained. An example of this case is $H_I^{\mu j}$ $(j=1,2)$ defined by Eq. (3.5) of Sec. 3. The additional current arising from the $\vec{E}$ dependence of $H_I^{\mu j}$ will be discussed in next subsection of this Appendix. A straightforward calculation gives

$$\vec{\nabla}_x\cdot\frac{\delta H_I^{\mu j}}{\delta\vec{A}(\vec{x})}=i\left[H_I^{\mu j},\,e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad \text{(B17a)}$$

Since $H_I^{\mu 1}$ commutes with $\delta^3(\vec{r}_2-\vec{x})$ and $H_I^{\mu 2}$ commutes with $\delta^3(\vec{r}_1-\vec{x})$, we can rewrite Eq. (B17a) as

$$\vec{\nabla}_x\cdot\frac{\delta H_I^{\mu j}}{\delta\vec{A}(\vec{x})}=i\left[H_I^{\mu j},\,\sum_{j=1}^{2}e_j\delta^3(\vec{r}_j-\vec{x})\right]. \quad \text{(B17b)}$$

(IV) If $F(\vec{E})$ is a function of electric field intensity $\vec{E}$, and if F has a form similar to Eq. (B11), then we have

$$\vec{\nabla}_x\cdot\frac{\delta F}{\delta\vec{E}(\vec{x})}=\rho(\vec{x}), \quad \text{(B18a)}$$

where

$$\rho(\vec{x})=\frac{\delta F(\mathbf{E}(\vec{r}))}{\delta\Phi(\vec{x})} \quad \text{(B18b)}$$

or

$$\vec{\nabla}_x\cdot\frac{d}{dt}\left(\frac{\delta F}{\delta\vec{E}(\vec{x})}\right)=\frac{d\rho(\vec{x})}{dt}. \quad \text{(B18c)}$$

The proof is very straightforward and is omitted here. We now consider the application of these results to the electromagnetic Hamiltonian in Sec. 3. As shown in Sec. 4, $\rho(\vec{x})$ is identified as the charge density. The Hamiltonian H_I^{Δ} defined by Eq. (3.6) is a function of $\vec{E}$. Applying the relation (B18c), we get

$$\frac{d\rho^{\Delta}}{dt}=\vec{\nabla}_x\cdot\frac{d}{dt}\left(\frac{\delta H_I^{\Delta}}{\delta\vec{E}(\vec{x})}\right), \quad \text{(B19)}$$

where

$$\rho^{\Delta}=\frac{\delta H_I^{\Delta}}{\delta\Phi(\vec{x})}. \quad \text{(B19a)}$$

A second example is to apply the relation (B18c) to the Hamiltonian $H_I^{\mu j}$ $(j=1,2)$ defined by Eq. (3.5). We have

$$\frac{d\rho^j(\vec{x})}{dt}=\vec{\nabla}_x\circ\frac{d}{dt}\left(\frac{\delta H_I^{\mu j}}{\delta\vec{E}(\vec{x})}\right)+e_j\frac{d}{dt}\delta^3(\vec{r}_j-\vec{x}), \quad \text{(B20)}$$

where

$$\rho^j(\vec{x})=\frac{\delta H_I^{\mu j}}{\delta\Phi(\vec{x})}. \quad \text{(B20a)}$$

The additional term, $e_j\,d/dt\,\delta^3(\vec{r}_j-\vec{x})$ in Eq. (B20) is obtained from the term $e_j\Phi(\vec{r}_j)$ of Eq. (3.5).

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the first-order correction. In this paper, except the Appendix A, we set $c=1$. The expansion is then made, equivalently, in powers of $(1/m^2)$, where m is the mass of nucleon. We set $\hbar=1$ everywhere.

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Mössbauer Effect Following Coulomb Excitation of the 43.8-keV State of ^{161}Dy†

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The Mössbauer effect following Coulomb excitation of the 43.8-keV state of ^{161}Dy was studied for several Dy compounds and Dy metal. At 85°K, using 3.3-MeV α particles, effects as large as 3.5% were observed; a scintillation counter gave typical counting rates of 2000 counts/sec; 43.8-keV γ rays made up about one third of the total count rate. The half-life of this state predicts $2\Gamma_0=8$ mm/sec; the narrowest single line observed was 16 mm/sec. Dy-metal spectra taken at various temperatures were interpreted assuming a normal hyperfine interaction plus a central peak due to relaxation effects. The extracted value of the nuclear magnetic-dipole moment for this level of $(-0.134\pm 0.005)\mu_N$ agrees with the value calculated using the Nilsson model. The extracted value of the ratio of the intrinsic quadrupole moments of the ground and excited states of 1.12 ± 0.27 agrees with the value of 1.0 predicted by the Nilsson model. The extracted difference in the nuclear radius for the ground and excited states is $\delta R/R=(-1.2\pm 0.6)\times 10^{-5}$. A radiation-damage induced isomer shift corresponding to Dy^{4+} was observed in the DyF_3 target at liquid-nitrogen temperature.

I. INTRODUCTION

In recent years the Mössbauer effect (ME) following Coulomb excitation (CE) has proved to be a useful method of investigating nuclear transitions which cannot be otherwise observed due to a lack of an appropriate radioactive parent. One candidate for the application of the CE technique is ^{161}Dy. Although the Mössbauer effect has been observed in ^{161}Dy for the 25.7-keV[1-4] and 74.6-keV[4-6] transitions using radioactive sources, this is not the case for the 43.8-keV transition, since this level is only sparsely populated in the decay of ^{161}Tb and ^{161}Ho, and since the half-lives of these parent nuclei are short (see Fig. 1 and Ref. 7). In an earlier note[8] we reported the first observation of the Mössbauer effect following Coulomb excitation (CEME) of that level; in the present article we present more detailed ME studies of this transition.

From CEME spectra of Dy-metal, $DyNi_2$, and DyF_3 absorbers versus DyF_3 targets, we have been able to determine the magnetic-dipole and electric-quadrupole moments of the 43.8-keV state, together with the change in nuclear radius between the excited and ground states. In addition we have observed a radiation-damage (RD) isomer shift in the target.

In reporting these studies we will first present in Sec. II the general experimental considerations;