

Embedding of the Brueckner Approximation in the Extended Jastrow Scheme*

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A nonperturbative derivation of the Brueckner approximation to the ground-state energy of nuclear matter (with standard choice of hole and particle energies) is given within the framework of the method of correlated basis functions. The structure of the factor-cluster (FIY) expansion of the expectation value of the Hamiltonian with respect to a trial function incorporating short-range correlations in a general fashion, is analyzed for a uniform extended medium of fermions. With a special choice of correlated wave function, the Brueckner approximation is extracted from the FIY expansion by selective summation of two-body combination terms to all orders in the smallness parameter κ associated, in this case, with both Brueckner-Bethe-Goldstone and FIY energy expansions. A reexamination of the numerical comparison of conventional Brueckner and Jastrow methods carried out earlier for two simple models of nuclear matter leads to the conclusion that it is inadvisable, at least in a Jastrow calculation satisfying the average Pauli condition, to incorporate the $O(\kappa)$ contribution of the analog of the dispersion correction of reaction-matrix theory without at the same time including the $O(\kappa)$ contribution of the analog of the Bethe-Faddeev term. For the uniform extended medium for the Jastrow choice of wave function, there is a formal cancellation of these two three-body contributions, the scale of this cancellation being governed by the size of κ . In the numerical example considered, κ is about 0.22 and the cancellation goes something like $+14 \text{ MeV} - 15 \text{ MeV} = -1 \text{ MeV}$. These findings, when juxtaposed with the standard Brueckner results for the simple models in question, suggest that a departure from the standard prescription of hole and particle energies in reaction-matrix theory may be necessary when κ is sizeable – as in liquid ^3He and the high-density neutron gas, as well as equilibrium nuclear matter described with some potentials.

I. INTRODUCTION

The most thoroughly explored device for evaluating the ground-state energy of a normal system of A strongly interacting fermions is the Brueckner-Bethe-Goldstone (BBG) expansion. The first two terms of this expansion, the one-hole-line term plus the two-hole-line term, comprise the famous Brueckner reaction-matrix approximation,^{1,2} which has been energetically applied to nuclear matter and finite nuclei. Formation of the third term involves Bethe-Faddeev summation^{3,4} of an infinite series of three-hole-line diagrams. Extensive literature dealing with the formal and numerical aspects of the BBG scheme is available.¹⁻¹⁰

The last few years have seen a marked revival of interest in the search, initiated by de-Shalit and Weisskopf¹¹ and Dabrowski¹² and carried on by Moszkowski,¹³ for connections between the BBG theory, which is built upon perturbation theory with respect to dynamically uncorrelated basis functions, and other many-body theories which

start with dynamically *correlated* basis functions (CBF).

These latter methods are called, generically, CBF methods. As a first step one considers the expectation value of the Hamiltonian with respect to the basis function which has the best chance of describing the ground state. The simplest example is the familiar Jastrow variational approach; another well-known example is the unitary-model-operator scheme. Obviously we have in mind “true” rather than “model” CBF theories, to use the terminology of Prange and Klein¹⁴ and Brandom.⁹ Extensive discussions of formal and practical aspects of CBF methods belonging to this class are to be found in Refs. 15–24.

It was established by Day that the BBG expansion can be related directly to the simple Jastrow example of CBF if a certain “classical” approximation is made.²⁵ At about the same time, Wong²⁶ extended the numerical comparison of reaction-matrix and Jastrow methods begun by Bäckman, Chakkalakal, and Clark (BCC)²⁷ and pointed out the absence of an explicit dispersion contribution

to the Jastrow pair approximation to the energy. By introducing (adding in one place and subtracting in another) an auxiliary single-particle potential, he was able to generate the Brueckner reaction-matrix approximation from the Iwamoto-Yamada^{28, 17} (IY) cluster expansion of the energy expectation value with respect to a trial wave function containing state-dependent correlations.²⁹ Other workers have arrived at the same result by essentially the same technique, and have gone on to consider the connection between the cluster-expanded CBF expectation value (the "extended Jastrow scheme") and BBG scheme in higher orders.^{30, 31}

Wong's result was given more consistent interpretation in an earlier contribution by the present authors.³² Here we shall look further into the formal and numerical aspects of the CBF-BBG connection at the two-body level.

In Sec. II we present the fundamentals of the extended Jastrow scheme. The factor-cluster formalism^{17-19, 32} is used to develop the expectation value of the nuclear Hamiltonian with respect to a general correlated trial wave function $\Psi = F\Phi$. Here Φ is an independent-particle-model ground-state wave function which would be adequate for an initial description of the system in the absence of strong short-range interactions (in our case the ground-state wave function of the Fermi gas) and F is a symmetric correlation operator satisfying the cluster property. This cluster procedure effects an automatic decomposition of the expectation value into linked one-body, two-body, ..., n -body, ..., A -body parts. We then establish, for the uniform infinite system, a further decomposition of the $n(\geq 2)$ -body part of this expansion [the so-called factorized Iwamoto-Yamada (FIY) expansion¹⁷⁻¹⁹] into two-body, three-body, ..., n -body combination terms.^{33, 19} The m -body combination term, $2 \leq m \leq n$, contains only matrix elements referring to m or fewer-body subsystems. (The n -body combination term is called the "proper" n -body part.) An explicit classification of the contributions to one-, two-, three-, and four-body parts, or clusters, is provided.

As a first step in the general program of expressing BBG theory in CBF (more specifically, FIY) language, a certain class of two-body combination terms contributing to two-body, three-body, ... clusters will be singled out for closer study. Upon taking the two-body correlation operator of the CBF scheme to coincide with the wave operator of the Brueckner reaction-matrix approximation (at least in the Fermi-sea subspace of two-body states), we shall, in Sec. III, demonstrate two theorems which accomplish an embedding of the Brueckner approximation (with standard choice

of propagator renormalization) into the FIY cluster expansions of Sec. II and amount to a non-perturbative derivation of this approximation which gives it a variational interpretation. Especially, we shall see how Wong's connection can be obtained without recourse to the introduction of an auxiliary single-particle potential into the cluster formalism. The two-body combination terms which are singled out from the three-body, four-body, ... clusters comprise the dispersion correction of the Brueckner approximation.^{34, 35, 26, 29, 30} They may be considered as arising from a "dispersion effect" even when the two-body correlation operator is not specialized to the Brueckner choice, thus allowing one to make a "dispersion correction" to a simple two-body Jastrow calculation.

Some observations on the structural or diagrammatic relations between higher-order FIY and BBG (or perturbation-theoretic) terms are collected in Sec. IV.

In Sec. V we reconsider, in the light of the formal relations uncovered in Secs. II and III, the numerical comparison of conventional Brueckner and Jastrow methods carried out for a simplified model of nuclear matter by BCC²⁷ and subsequently examined by Wong.²⁶ It is found quite unwise to supplement the two-body Jastrow result by a dispersion correction, without incorporating the remaining three-body contributions, because of a near cancellation of the two-body combination term of the three-body cluster by its proper three-body part. There remains a substantial discrepancy between Brueckner and Jastrow results for the simple central potentials assumed. This discrepancy must be due to the large values of the wound parameter κ ("smallness parameter" of BBG theory) associated with these potentials. It is suggested that in circumstances where κ becomes sizeable (in high-density nuclear and neutron matter, hypernuclear matter, and liquid ³He; in studies of off-shell effects in nuclear matter), a more careful examination of the three-body Bethe-Faddeev term of the BBG expansion is in order.

II. STRUCTURAL ANALYSIS OF CONTRIBUTIONS TO THE FIY EXPANSION— A SUMMARY

We begin with the "extended Jastrow" ansatz^{19, 32} for the trial ground-state ket of the A -nucleon system,

$$|\Psi\rangle = F|\Phi\rangle, \quad (\text{II.1})$$

where $|\Phi\rangle$ is the totally antisymmetric ground-state ket of a suitable independent-nucleon model and F is a symmetrical operator designed to

introduce short-range correlations. The correlation operator F , which we presume to be defined for an arbitrary number n of particles, $1 \leq n \leq A$, is required to possess the cluster property. By the cluster property is meant a sufficiently rapid factorization according to

$$F(1 \cdots l m \cdots n) \rightarrow F(1 \cdots l)F(m \cdots n) \quad (\text{II.2})$$

as the particle set $\{1 \cdots l\}$ is moved far from the particle set $\{m \cdots n\}$. We take $F(1)=1$ for simplicity, it being understood that we have already made the best choice of single-nucleon orbitals. Special examples fulfilling the stated requirements are the ordinary Jastrow ansatz^{36, 15} and the unitary-model-operator ansatz.^{37, 21-24}

The expectation value

$$E = \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \quad (\text{II.3})$$

of the ground-state energy with respect to the trial ket $|\Psi\rangle$ can be evaluated by means of a linked-cluster expansion, in particular the FIY expansion^{17-19, 32} in which the terms are arranged according to the number of occupied orbitals – “hole orbitals” – involved:

$$E = E_0 + (\Delta E)_2 + (\Delta E)_3 + \cdots + (\Delta E)_A. \quad (\text{II.4})$$

The term E_0 is the ground-state energy of the input independent-nucleon model and the index n on $(\Delta E)_n$ counts hole orbitals. The nuclear Hamiltonian H is herein assumed to have the usual form

$$H = T + V = \sum_{i=1}^A t(i) + \sum_{1 \leq i < j \leq A} v(ij), \quad (\text{II.5})$$

with $t(i)$ the kinetic-energy operator for nucleon i and $v(ij)$ the interaction operator for nucleons i, j .

Specializing to uniform, extended nuclear matter, the model ket $|\Phi\rangle$ is taken to be the unit-normalized ground-state ket of a Fermi gas of A nucleons. With this choice, the term E_0 in (II.4) is the kinetic energy of the Fermi-gas ground state, $T|\Phi\rangle = E_0|\Phi\rangle$, and the term $(\Delta E)_n$, $2 \leq n \leq A$, is the n -body cluster contribution to the energy shift as described in Refs. 19 and 32. In passing to the many-body limit, we may drop all contributions to E which are of order unity or less compared to A . All surviving contributions to a given $(\Delta E)_n$ can be expressed in terms of matrix elements of induced or effective two-, three-, four-, ..., n -body potentials $w_2(12)$, $w_3(123)$, $w_4(1234)$, ..., $w_n(1 \cdots n)$; this produces a very natural separation of each cluster addend into structurally distinct parts. Such a classification was carried out in detail for the two- and three-body terms in Ref. 19. Here we shall merely collect

the earlier results along with the results of a corresponding analysis of $(\Delta E)_4$. A more elaborate discussion of the structure of the FIY expansion, and therewith of the more familiar IY expansion^{28, 15, 17, 18, 32} in which the terms are arranged according to the number of independent hole orbitals involved, will be given in another paper.

The two-body term $(\Delta E)_2$ represents a simple pair approximation; it may be written

$$(\Delta E)_2 = \sum_{i < j} \langle ij | w_2 | ij \rangle_a, \quad (\text{II.6})$$

with

$$w_2(12) = \frac{1}{2} F^\dagger(12) [t(1) + t(2), F(12)] + \frac{1}{2} F^\dagger(12) v(12) F(12) + \text{adj.} \quad (\text{II.7})$$

Some notational conventions must be explained at this point. We will always use the letters i, j, k, l in labeling the unit-normalized one-body kets of which $|\Phi\rangle$ is built. Thus $|i\rangle, |j\rangle, |k\rangle, |l\rangle$ represent “occupied” or “hole” orbitals, in the present instance Fermi-sea orbitals. The letters p, q will be used to tag arbitrary one-body kets of the assumed independent-nucleon model. The subscript a , when attached to an independent-nucleon ket $|pq \cdots\rangle$, turns that ket into a unit-normalized, antisymmetrized multibody ket constructed from the one-body kets $|p\rangle, |q\rangle, \dots$. When attached to a matrix element $\langle p'q' \cdots | \text{operator} | pq \cdots \rangle$, the subscript a turns that matrix element into the matrix element computed with respect to the bra $(|p'q' \cdots\rangle_a)^\dagger$ and the ket $|pq \cdots\rangle_a$. A sum over i, j, k , or l will mean a sum over *all* occupied, i.e., Fermi-sea, orbitals.

The three-body term $(\Delta E)_3$ is composed of a proper three-body part $(\Delta E)_3^{(3)}$ and a two-body combination part or term^{33, 19} $(\Delta E)_3^{(2)}$:

$$(\Delta E)_3 = (\Delta E)_3^{(3)} + (\Delta E)_3^{(2)}, \quad (\text{II.8})$$

where

$$(\Delta E)_3^{(3)} = \sum_{i < j < k} \langle ijk | w_3 | ijk \rangle_a \quad (\text{II.9})$$

and

$$(\Delta E)_3^{(2)} = - \sum_{ijk} \eta_{ij} \langle ik | w_2 | ik \rangle_a. \quad (\text{II.10})$$

Here we have defined

$$\eta_{ij} \equiv \langle ij | F^\dagger(12)F(12) - 1 | ij \rangle_a, \quad (\text{II.11})$$

which may be recognized as the departure of the norm of the correlated two-body ket $F(12)|ij\rangle_a$ from unity. The induced or effective three-body potential is a straightforward extension of the two-body expression (II.7) to the case of three

particles,

$$w_3(123) = \frac{1}{2} F^\dagger(123)[t(3), F(123)] + \frac{1}{2} F^\dagger(123)v(12)F(123) - \frac{1}{2} w_2(12) + \text{cycl.} + \text{adj.} \quad (\text{II.12})$$

By means of the same procedure as used in Ref. 19, $(\Delta E)_4$ can be classified into two- and three-body combination parts $(\Delta E)_4^{(2)}$ and $(\Delta E)_4^{(3)}$ and a proper four-body part $(\Delta E)_4^{(4)}$:

$$(\Delta E)_4 = (\Delta E)_4^{(4)} + (\Delta E)_4^{(3)} + (\Delta E)_4^{(2)}, \quad (\text{II.13})$$

where

$$(\Delta E)_4^{(4)} = \sum_{i < j < k < l} \langle ijkl | w_4 | ijkl \rangle_a \quad (\text{II.14})$$

and

$$(\Delta E)_4^{(3)} = (\Delta E)_4^{(3)(1)} + (\Delta E)_4^{(3)(2)} \quad (\text{II.15})$$

with

$$(\Delta E)_4^{(3)(1)} = -\frac{1}{2} \sum_{ijkl} \eta_{ijk} \langle il | w_2 | il \rangle_a, \quad (\text{II.16})$$

$$(\Delta E)_4^{(3)(2)} = -\frac{1}{2} \sum_{ijkl} \eta_{ijl} \langle ik | w_3 | ik \rangle_a. \quad (\text{II.17})$$

The new ingredients are

$$\eta_{ijk} = \langle ij | k | [F^\dagger(123)F(123) - [F^\dagger(12)F(12) + \text{cycl.}] + 2] | ij \rangle_a \quad (\text{II.18})$$

and the diagonal Fermi-sea matrix elements of the effective four-body potential $w_4(1234)$ defined by

$$\begin{aligned} w_4(1234) + w_4^R(1234) &= \frac{1}{2} F^\dagger(1234)[t(1) + t(2) + t(3) + t(4), F(1234)] + \frac{1}{2} [F^\dagger(1234), t(1) + t(2) + t(3) + t(4)] F(1234) \\ &\quad + F^\dagger(1234)[v(12) + v(13) + v(14) + v(23) + v(24) + v(34)] F(1234) \\ &\quad - [w_3(123) + w_3(124) + w_3(134) + w_3(234)] - [w_2(12) + w_2(13) + w_2(14) + w_2(23) + w_2(24) + w_2(34)], \\ w_4^R(1234) &= [F^\dagger(12)F(12) - 1]w_2(34) + [F^\dagger(13)F(13) - 1]w_2(24) + [F^\dagger(14)F(14) - 1]w_2(23) \\ &\quad + [F^\dagger(23)F(23) - 1]w_2(14) + [F^\dagger(24)F(24) - 1]w_2(13) + [F^\dagger(34)F(34) - 1]w_2(12). \end{aligned} \quad (\text{II.19})$$

The two-body combination part begins in this order to assume a richer structure,

$$(\Delta E)_4^{(2)} = (\Delta E)_4^{(2)(1)} + (\Delta E)_4^{(2)(2)} + (\Delta E)_4^{(2)R}, \quad (\text{II.20})$$

with

$$(\Delta E)_4^{(2)(1)} = \sum_{ijkl} \eta_{ij} \eta_{ik} [\langle il | w_2 | il \rangle_a + \langle j | l | w_2 | j | l \rangle_a], \quad (\text{II.21})$$

$$(\Delta E)_4^{(2)(2)} = \frac{1}{2} \sum_{ijkl} \eta_{ij} \eta_{kl} \langle ik | w_2 | ik \rangle_a, \quad (\text{II.22})$$

$$(\Delta E)_4^{(2)R} = \frac{1}{4} \sum_{ijkl} \langle ij | F^\dagger(12)F(12) - 1 | kl \rangle_a \langle kl | w_2 | ij \rangle_a. \quad (\text{II.23})$$

It is easy to detect that (II.23) is a reducible four-body part in the sense that it involves only three *independent* hole orbitals or "hole lines"; it therefore contributes to the three-body term of the *IY expansion* in independent hole lines. The rest of the *IY* three-body term is, of course, given by $(\Delta E)_3$.

A "smallness parameter" ξ for the *IY* or independent-hole-line expansion^{28, 15, 38, 18} may be defined in terms of a suitable average, with respect to Fermi sea orbitals, of the quantity η_{ij} . It is convenient to average over the Fermi sea, and

define

$$\xi \equiv \frac{1}{A} \sum_{ij} \eta_{ij}. \quad (\text{II.24})$$

It is useful to define, in addition to η_{ij} of (II.15), a quantity

$$\nu_{ij} \equiv \langle ij | [1 - F^\dagger(12)][1 - F(12)] | ij \rangle_a, \quad (\text{II.25})$$

the norm of a sort of two-body defect ket representing the distortion of the two-body ket $F(12)|ij\rangle_a$ due to correlations, along with its

Fermi-sea average $A\kappa$,^{27, 20}

$$\kappa \equiv \frac{1}{A} \sum_{ij} \nu_{ij}. \quad (\text{II.26})$$

The point is that κ is the analog of the (superficial) "smallness parameter" of the BBG expansion.^{6, 7}

It must be emphasized that the IY expansion is not a strict Taylor-series expansion in ξ , since the various contributions to the cluster development are in general of too complicated structure to allow integrals like η_{ij} , not to mention averages over them like ξ/A , to be factored out. More properly, ξ assumes the role of an ordering parameter, the two-hole-line term being $O(\xi^0)$, the three-independent-hole-line term, $O(\xi^1)$, ..., the n -independent-hole-line term, $O(\xi^{n-2})$, One has here merely a device for classifying contributions and defining successive cluster approximants to the energy. Whether or not the ξ -classification scheme provides a good rule for grouping cluster terms rests on the convergence obtained with it in numerical application to the given physical problem.

Further caution is in order in regarding ξ as a smallness parameter: unlike κ , it is not in general a positive-definite quantity. However, we shall be primarily interested in special cases in which the correlation factor $F(12)$ satisfies the "average Pauli condition," meaning

$$\frac{1}{A} \sum_{ij} \langle ij | 1 - F(12) | ij \rangle_a = 0. \quad (\text{II.27})$$

Then the smallness parameter ξ collapses to the wound parameter κ .

III. EMBEDDING OF THE BRUECKNER APPROXIMATION INTO THE FIY FORMALISM—TWO THEOREMS

In the Brueckner reaction-matrix approximation, the two-body approximation of BBG theory,

the ground-state energy is given by¹

$$E_B = E_0 + \sum_{i < j} \langle ij | G(\epsilon_i + \epsilon_j) | ij \rangle_a = E_0 + \frac{1}{2} \sum_i u_i, \quad (\text{III.1})$$

with the reaction operator G taken "on the energy shell" and $\epsilon_i = t_i + u_i$. The one-body potential $u(i)$ appearing in the propagator of G is supposed to be specified in the conventional manner, $\langle p | u | q \rangle = u_p \delta_{pq}$,

$$u_p = \begin{cases} \sum_j \langle pj | G(\epsilon_p + \epsilon_j) | pj \rangle_a, & p \text{ inside Fermi sea,} \\ 0, & p \text{ outside.} \end{cases} \quad (\text{III.2})$$

Accordingly, the hole potential is to be determined self-consistently from on-energy-shell diagonal reaction-matrix elements with respect to Fermi-sea orbitals, a prescription urged and motivated by Bethe, Brandow, and Petschek⁸ and others^{39, 40} which is almost universally accepted.⁴¹ The best choice of particle potential is more a subject of debate.^{6, 4, 40, 42-44}

To relate this scheme to the CBF method in its extended-Jastrow guise, we make the identification already asserted in Refs. 26, 29, 30, and 32,

$$F(12) | ij \rangle_a = v^{-1} G(\epsilon_i + \epsilon_j) | ij \rangle_a. \quad (\text{III.3})$$

Thus we specialize the general state-dependent two-body correlation operator $F(12)$ of the extended Jastrow formalism in such a way that it gives precisely the Bethe-Goldstone ket when operating on the pair ket $|ij\rangle_a$.

It is now elementary to express the hole potential energies u_i and the Brueckner ground-state energy E_B in the language of the FIY cluster formalism.

Application of the Pauli principle for virtual excitations in the medium, viz.

$$\langle pq | 1 - F(12) | ij \rangle_a = 0, \quad \text{one or both of } p, q \text{ inside Fermi sea,} \quad (\text{III.4})$$

and of the familiar Bethe-Goldstone equation, satisfied by $F(12) | ij \rangle_a$, produces²⁹

$$u_i = \sum_j \langle ij | F^\dagger(12) v(12) F(12) | ij \rangle_a + \sum_j \langle ij | F^\dagger(12) [t(1) + t(2) + u(1) + u(2) - \epsilon_i - \epsilon_j] F(12) | ij \rangle_a, \quad (\text{III.5})$$

or, adopting the notation of (II.7),

$$u_i = \sum_j \langle ij | w_2 | ij \rangle_a + \sum_j \langle ij | F^\dagger(12) [u(1) + u(2) - u_i - u_j] F(12) | ij \rangle_a. \quad (\text{III.6})$$

Thus we see that the Brueckner energy may be expressed as

$$E_B = E_0 + (\Delta E)_2 + (\Delta U)_2, \quad (\text{III.7})$$

in which $(\Delta E)_2$ and $(\Delta U)_2$ have just the form of ordinary two-body cluster terms (simple pair form),

$$(\Delta E)_2 = \sum_{i < j} \langle ij | w_2 | ij \rangle_a, \quad (\text{III.8})$$

$$(\Delta U)_2 = \sum_{i < j} \langle ij | F^\dagger(12)[u(1) + u(2) - u_i - u_j] F(12) | ij \rangle_a. \quad (\text{III.9})$$

These last two formulas fit precisely into the schemes devised in Ref. 32 for inclusion of dispersion effects, i.e., for the inclusion of effects of an external potential, within the extended Jastrow framework. Appealing to that analysis [especially to "Example (c)"], we arrive at the theorem:

(i) *The Brueckner approximation E_B to the ground-state energy can be viewed as the two-body cluster approximation to the expectation value $\langle \Psi | H_B | \Psi \rangle / \langle \Psi | \Psi \rangle$ of the Hamiltonian $H_B = T + V + U - U_0$, where $U - U_0$ is an auxiliary external potential defined by*

$$U = \sum_i u(i), \quad U|\Phi\rangle = U_0|\Phi\rangle. \quad (\text{III.10})$$

Obviously a complete knowledge of the correlated ket $|\Psi\rangle = F|\Phi\rangle$ is not called for; all that is needed is a specification of the mode of action of $F(12)$ on the two-body Fermi-sea kets $|ij\rangle_a$ in accordance with (III.3).

We now establish a second theorem which comprises a revealing amplification of the first, and allows us to dispense with the artifice of an external potential. Exploiting (III.2), Eq. (III.6) may be cast into the form

$$u_i = \sum_j \langle ij | w_2 | ij \rangle_a - \sum_j \eta_{ij}(u_i + u_j). \quad (\text{III.11})$$

With w_2 and η_{ij} fixed, this provides an integral equation for u_i . We presume there is no non-trivial solution of the homogeneous equation. Then, iterating (III.11) and inserting the resultant expansion into (III.1), we come to

$$\begin{aligned} E_B = E_0 + \sum_{i < j} \langle ij | w_2 | ij \rangle_a \\ - \sum_{ijk} \eta_{ij} \langle ik | w_2 | ik \rangle_a + \sum_{ijk} \eta_{ij} \eta_{ik} [\langle il | w_2 | il \rangle_a \\ + \langle jl | w_2 | jl \rangle_a] + \dots \end{aligned} \quad (\text{III.12})$$

Comparison of this expression with (II.6), (II.10), and (II.21) yields

$$E_B = E_0 + (\Delta E)_2 + (\Delta E)_3^{(2)} + (\Delta E)_4^{(2)(1)} + \dots \quad (\text{III.13})$$

Although an explicit demonstration has not been given, there is every reason to believe that the higher-order terms represented by dots in (III.13) are also present, as two-body combination terms of fifth and higher order, in the cluster expansion

(II.4). Therefore we assert:

(ii) *The Brueckner approximation E_B to the ground-state energy can be viewed as a selective summation of two-body combination terms of the cluster expansion of the expectation value $\langle \Psi | H | \Psi \rangle / \langle \Psi | \Psi \rangle$ of the given Hamiltonian.*

The selected cluster contributions of theorem (ii), when of three-body order or higher, will be referred to as dispersion contributions, since their sum reproduces precisely the dispersion correction contained in the Brueckner approximation based on the conventional choice (III.2) of one-body potential.

The variational interpretation of the Brueckner approximation can be carried to a deeper level within the FIY scheme by following Schäfer and Schütte³¹: the Euler-Lagrange equation obtained by varying the expression $E_0 + (\Delta E)_2 + (\Delta U)_2$ with respect to $F(12)$, subject to the Pauli condition (III.4) as Lagrangian constraint, is just the Bethe-Goldstone equation. Thus one may achieve a variational underpinning of the choice (III.3) of the two-body correlation factor.

In the concluding section the numerical study of BCC²⁷ will be carefully reexamined in the light of the formal connections between Brueckner and extended Jastrow methods which have been explicated above. To set the scene, we shall at this point extend the terminology "dispersion contributions of conventional type" to the higher-order (three- or more-hole-orbital) two-body combination terms

$$\begin{aligned} [(\Delta E)_3^{(2)}]_d &= (\Delta E)_3^{(2)}, \\ [(\Delta E)_4^{(2)}]_d &= (\Delta E)_4^{(2)(1)}, \dots \\ [(\Delta E)_n^{(2)}]_d &= \dots, \dots \end{aligned}$$

isolated in theorem (ii), irrespective of the choice of $F(12)$. These contributions have a simple structure and can readily be incorporated into an FIY evaluation of the energy for arbitrary $F(12)$. One can capture the complete "dispersion correction" for given $F(12)$ by solving the integral equation (III.11) and substituting into $E_0 + \frac{1}{2} \sum_i u_i$. But this may not actually be required in practice, if the parameter $|\xi|$ is sufficiently small. Truncation of the expansion $\sum_{n=2}^{\infty} [(\Delta E)_n^{(2)}]_d$ {with $[(\Delta E)_2^{(2)}]_d \equiv (\Delta E)_2$ } at the term $O(\xi)$, or at most the term $O(\xi^2)$, may be adequate. Furthermore, if the ma-

trix elements η_{ij} remain near enough to their Fermi-sea average, ξ/A , then the approximation

$$[(\Delta E)_n^{(2)}]_a \approx (-2\xi)^{n-2}(\Delta E)_2 \quad (\text{III.14})$$

should be sensible. If so, then the "dispersion correction" may be simply but effectively absorbed by summing a geometric series,

$$\sum_{n=2}^{\infty} [(\Delta E)_n^{(2)}]_a \approx (\Delta E)_2 - 2\xi(\Delta E)_2 + \cdots = (1 + 2\xi)^{-1}(\Delta E)_2, \quad (\text{III.15})$$

even when $|\xi|$ is not particularly small (but still less than $\frac{1}{2}$). In Sec. V we shall apply these formulas for the Jastrow ansatz, $F(12) = f(r_{12})$, in the special context of the test models of nuclear matter considered in Ref. 27.

IV. STRUCTURAL CORRESPONDENCES BETWEEN FIY AND BBG CONTRIBUTIONS

With the choice (III.3) of two-body correlation operator (which we adopt throughout this section) the matrix element η_{ij} , first appearing in $(\Delta E)_3^{(2)}$, can be rewritten as

$$\eta_{ij} = \langle ij | [1 - F^\dagger(12)][1 - F(12)] | ij \rangle_a. \quad (\text{IV.1})$$

Evidently, it describes the excitation of a pair from Fermi-sea orbitals i, j to orbitals outside the Fermi sea, with subsequent deexcitation to the original pair state. The factor $1 - F(12)$ creates a two-particle-two-hole state; the factor $1 - F^\dagger(12)$ destroys it. The matrix element $\langle ik | w_2 | ik \rangle_a$ in $(\Delta E)_3^{(2)}$ describes the (effective) interaction, at an intermediate stage, of the hole in i with a vacuum particle in the Fermi-sea orbital k . In such terms, adequately generalized, a structural correspondence of certain of the contributions to the FIY expansion of Sec. II with contributions to the BBG theory or to the general reaction-matrix expansion or to the ordinary linked perturbation expansion (these being conventionally represented by Goldstone or Brandow diagrams^{5, 1, 9, 6, 7, 33}) may be set up.

Obviously $(\Delta E)_2$, being in pure pair form and describing the interaction of two vacuum particles in Fermi-sea orbitals i, j through the irreducible effective two-body interaction $w_2(12)$, has the same structure as a first-order perturbation correction, or as the lowest-order reaction-matrix contribution to the BBG energy shift. From the above it is now clear that $(\Delta E)_3^{(2)}$ has the same structure as the hole-bubble contribution (including all exchanges) of third order in the Goldstone perturbation expansion or in the general G -matrix expansion.¹ The proper three-body part, $(\Delta E)_3^{(3)}$, corresponds structurally to the three-hole-line Bethe-Faddeev sum of BBG theory,¹ describing, as it

does, the interaction of three vacuum particles in Fermi-sea orbitals ijk , through the (in general irreducible) effective three-body interaction $w_3(123)$.³³ To go further, the reducible portion $(\Delta E)_4^{(2)R}$ of the four-body FIY cluster has a structure identical to that of the hole-hole contribution to the three-independent-hole-line portion of the BBG energy.¹ In the same spirit, $(\Delta E)_4^{(2)(2)}$ may be likened to the four-hole-line "saturation-potential" contribution to the BBG energy, discussed by Brandow.^{6, 7, 33} The quantity $(\Delta E)_4^{(2)(1)}$ has the structure of the BBG diagram (type A3) of Fig. 8 of Ref. 33, modified by a hole-bubble insertion (corresponding to $\langle il | w_2 | il \rangle_a$ or $\langle jl | w_2 | jl \rangle_a$) into one or another of the propagating hole lines. The quantity $(\Delta E)_4^{(3)(2)}$ is like the aforementioned third-order hole-bubble contribution, except that the hole in orbital i interacts simultaneously with *two* vacuum particles in orbitals k and l via the irreducible effective *three-body* interaction w_3 . To interpret $(\Delta E)_4^{(3)(1)}$ in BBG language, we insist that $F(123)$ be defined so that the operator in curly brackets in (II.18) creates and subsequently destroys a three-particle-three-hole state (Pauli condition generalized to three particles). Then $(\Delta E)_4^{(3)(1)}$ has the structure of the Bethe-Faddeev sum, but with a hole-bubble attached to one of the hole lines at a level where three particles are still excited. Finally, $(\Delta E)_4^{(4)}$ corresponds structurally to the four-body analog of the Bethe-Faddeev sum.³³ It is clear that a diagrammatic representation of the contributions to the FIY expansion may be set up which closely resembles those devised by Goldstone⁵ for ordinary linked perturbation theory and for the general G -matrix expansion, but it is also clear that additional graphical ingredients, similar in nature to Brandow's or Day's compact-cluster boxes,^{6, 7, 33} would have to be introduced. An important distinction between the FIY and the G -matrix expansion should be noticed: in the former there is (owing to the absence of energy denominators) nowhere any direct reference to off-energy-shell quantities, all such complications being buried in the $F(1 \cdots n)$ and the $w_n(1 \cdots n)$. Some may regard this feature as an advantage, albeit only a formal one. [In practice, one has to construct the $F(1 \cdots n)$ and the $w_n(1 \cdots n)$ starting from the interaction $v(12)$, and this process will in general take one off the energy shell.]

V. NUMERICAL COMPARISON OF BRUECKNER AND ORDINARY JASTROW METHODS—A REAPPRAISAL

The findings of Sec. III show the way to a fair comparison of the Brueckner reaction-matrix

approximation and the ordinary Jastrow method, which obtains upon specialization of the FIY formalism via the ansatz

$$F(1 \cdots n) = \prod_{1 \leq i < j \leq n} f(r_{ij}), \quad n=2, \dots, A. \quad (\text{V.1})$$

First of all one may compare

$$E_0 + (\Delta E)_2 + \sum_{n=3}^{\infty} [(\Delta E)_n^{(2)}]_d \quad (\text{V.2})$$

with the Brueckner energy E_B . (Throughout this section all energies are to be regarded as energies *per particle*.) Alternatively, one may compare

$$E_0 + (\Delta E)_2 \quad (\text{V.3})$$

with E_B sans dispersion correction. To make the Jastrow two-body correlation function $f(r_{12})$ resemble the Brueckner $F(12)$ of (III.3) as closely as possible, one is led to impose the average Pauli condition (II.27) on this function. In practice, $f(r)$ is determined by minimization of a truncated cluster expansion for the Jastrow energy expectation value, subject to this and other physically or mathematically motivated restrictions.^{27, 20, 45}

The proposed comparisons would, however, only tell how well a state-independent Jastrow correlation factor, so determined, can simulate the Brueckner correlation factor of (III.3), as measured by how well (V.2) reproduces the Brueckner energy and (V.3) the Brueckner energy without dispersion correction. We would have to know the *correct* energy for the given potential with sufficient accuracy, to say whether a calculation with or without dispersion correction is preferable. It is therefore obvious that a consideration of higher-order effects (*at least* the three-body effects in both BBG and Jastrow approaches) is necessary before one or the other conclusion can be drawn for a given physical problem (symmetrical nuclear matter, neutron gas, liquid ^3He , ...). This point was brought out forcefully in Ref. 32.

With this fact in mind, we shall reexamine the numerical comparison of Brueckner and Jastrow methods which was carried out some time ago by BCC²⁷ for two simplified models of symmetrical nuclear matter.

BCC performed standard Brueckner and Jastrow calculations for two central test potentials designed more or less to bracket the physical situation in nuclear matter. Both potentials have state-independent hard cores⁴⁶ of radius $c = 0.6$ fm. The potential designated OMY has, in even states, a spin-dependent exponential attraction outside the hard core, with range and depth parameters adjusted so as to fit the low-energy two-nucleon

data.⁴⁷ In the potential designated IY, the singlet-even-state OMY potential is assumed to act in both singlet- and triplet-even states.⁴⁸ The extra-core interaction in odd states is taken zero in both examples.

In the Brueckner calculation, the recipe (III.2) was adopted for the one-body potential $u(i)$. In the Jastrow calculation, the three parameters of the analytic form

$$f(r) = \begin{cases} 0, & r \leq c, \\ \{1 - \exp[-\mu_1(r - c)]\} \{1 + \nu \exp[-\mu_2(r - c)]\}, & r > c, \end{cases} \quad (\text{V.4})$$

were determined by minimization of

$$E_0 + (\Delta E)_2 + (\Delta E)_3 + (\Delta E)_4^{(2)R} \quad (\text{V.5})$$

subject to the average Pauli condition and to certain desirable inequalities involving the cluster-expanded Jastrow radial distribution function.^{45, 27} We shall refer to the correlation function determined in this fashion as the Chakkalakal f . In terms of the IY arrangement of cluster contributions to the energy expectation value, viz.,

$$E = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \cdots + \mathcal{E}_n \cdots, \quad (\text{V.6})$$

in which \mathcal{E}_n is the part of E implicating exactly n independent hole orbitals, the approximation (V.5) retains all contributions involving *three* or fewer independent hole orbitals, with $E_0 = \mathcal{E}_1 = \frac{3}{5} \hbar^2 k_F^2 / 2m$, $(\Delta E)_2 = \mathcal{E}_2$, and $(\Delta E)_3 + (\Delta E)_4^{(2)R} = \mathcal{E}_3$. Chakkalakal's $f(r_{12})$ bears a close resemblance to the Brueckner $F(12)$, if one judges by the degree of agreement of Jastrow and Brueckner smallness parameters. The two κ values agree to surprising accuracy, over the whole density range studied ($k_F = 1.2$ – 1.7 fm⁻¹ for the OMY and 1.1 – 1.6 fm⁻¹ for the IY potential). The same may be said of the $f(r_{12})$ of form (V.4) determined by Wong,²⁶ who minimized

$$E_0 + (1 + 2\kappa)^{-1} (\Delta E)_2 \quad (\text{V.7})$$

subject to the average Pauli condition. In both f determinations, higher-order cluster contributions (past two-body) were allowed to influence the correlation parameters – the full three-independent-hole-orbital contribution in the case of the Chakkalakal procedure, the dispersive two-body combination series in the case of the Wong procedure.

It was Wong who first called attention to the absence of an explicit dispersion correction from the Jastrow two-body energy.^{26, 29} Of course this fact is obvious from the outset, since the explicit dispersion correction to the energy is $O(\kappa^1)$, while the two-body Jastrow energy (with $\xi = \kappa$) is by definition $O(\kappa^0)$. On the other hand, if disper-

sion contributions, of first and/or higher orders in κ , are allowed to influence the determination of f , one *can* say that some dispersion effects are *implicitly* present in the two-body Jastrow energy; they would presumably encourage the healing of f .^{49, 26} Be that as it may, Wong has argued that a better approximation to the energy would be obtained if the two-body Jastrow result were supplemented by an *explicit* dispersion correction; basically, Wong infers that expression (V.2) is preferred over expression (V.3).^{29, 26} This argument is predicated on the assumption that the Brueckner energy E_B , with choice (III.2) of propagator renormalization, is fairly accurate, or at least a lower bound on the true energy.

Let us see how this suggestion works for the IY potential at $k_F = 1.36 \text{ fm}^{-1}$, the equilibrium Fermi wave number given by the Jastrow calculation of BCC based on Chakkalakal's f determination. (This k_F corresponds closely to the empirical equilibrium density of "realistic" symmetrical nuclear matter.) According to Wong,²⁶ the Brueckner calculation yields $E_B = -0.4 \text{ MeV}$, with $\kappa = 0.224$. For Chakkalakal's f (specified by $\mu_1 = 1.76 \text{ fm}^{-1}$, $\mu_2 = 1.72 \text{ fm}^{-1}$, $\nu = 1.310$) one has $E_0 + (\Delta E)_2 = \mathcal{E}_1 + \mathcal{E}_2 = -6.4 \text{ MeV}$, $E_0 + (\Delta E)_2 + (\Delta E)_3^{(2)} = +6.8 \text{ MeV}$, and $E_0 + (1 + 2\kappa)^{-1}(\Delta E)_2 = 2.7 \text{ MeV}$, with $\kappa = 0.224$ and of course $E_0 = 23.0 \text{ MeV}$. For Wong's f (specified by $\mu_1 = 1.34 \text{ fm}^{-1}$, $\mu_2 = 1.70 \text{ fm}^{-1}$, $\nu = 2.555$) one has $E_0 + (\Delta E)_2 = -8.1 \text{ MeV}$, $E_0 + (\Delta E)_2 + (\Delta E)_3^{(2)} = +6.4 \text{ MeV}$, and $E_0 + (1 + 2\kappa)^{-1}(\Delta E)_2 = 1.4 \text{ MeV}$, with $\kappa = 0.219$. A numerical study by Ter Louw⁵⁰ has shown that approximation (III.14), for $n = 3$, is good to within 1% over the full density ranges considered, for both IY and OMY potentials and the respective density-dependent Chakkalakal correlation functions. (This approximation is not as good for Wong's f .) We may expect (III.15) to be quite adequate here, as a means for incorporating the complete Jastrow dispersion correction.

These results are consistent with, but certainly do not imply, the following situation: the revised Jastrow energy estimate (V.2) possesses the upper-bound property, lying some few (here 2–3) MeV above the Brueckner energy, which constitutes an accurate approximation to the true ground-state energy. A state-independent correlation factor of Jastrow type is not realistic enough to provide an accurate energy.

A numerical examination of the *three-hole-orbital* contributions to the Jastrow energy calls this viewpoint into dispute, and shows rather clearly that Wong's suggestion for revising the simple two-body Jastrow energy is a poor one for the model problems of BCC.

For his f , Chakkalakal⁵¹ finds $(\Delta E)_3 = -1.1 \text{ MeV}$. [BCC gives $\mathcal{E}_3 = -0.07 \text{ MeV}$, but in their calcula-

tion $(\Delta E)_4^{(2)R}$ was overestimated, by perhaps as much as an MeV. By the same arguments as applied to the hole-hole diagram in BBG theory,³³ $(\Delta E)_4^{(2)R}$ should be quite small, positive for the Chakkalakal f , and perhaps 0.3 MeV. We neglect it here, approximating \mathcal{E}_3 by $(\Delta E)_3$.] Similarly, for Wong's f , Ter Louw⁵⁰ finds $(\Delta E)_3 = 3.5 \text{ MeV} \approx \mathcal{E}_3$.

The relatively small magnitude of these $(\Delta E)_3$ values has drastic implications, in view of the large size of the dispersion contributions $(\Delta E)_3^{(2)}$ which they contain: $(\Delta E)_3^{(2)} = 13.2 \text{ MeV}$ (Chakkalakal f) and $(\Delta E)_3^{(2)} = 14.5 \text{ MeV}$ (Wong f). The remainder of $(\Delta E)_3$, viz. the proper three-body part $(\Delta E)_3^{(3)}$, which Wong implies may be neglected, must be large and negative, some -14 MeV in the case of the Chakkalakal f and some -11 MeV in the case of the Wong f .

The higher-order dispersion contributions, $O(\kappa^n)$, $n \geq 2$, reduce the energy by a few (4–5) MeV, but do nothing to alter the conclusion that 11–14 MeV of attraction from $(\Delta E)_3 \approx \mathcal{E}_3$ would be ignored. As the density increases, matters grow worse: at $k_F = 1.6 \text{ fm}^{-1}$, $(\Delta E)_3^{(2)} \approx 25 \text{ MeV}$, and a contribution of some -20 MeV would be discarded. The same behavior is found for the OMY potential.

In the face of the large-scale cancellations involved in $(\Delta E)_3$, it is, contrary to Wong's suggestion, preferable to keep *all* three-body contributions or drop them all rather than split up the small quantity $(\Delta E)_3 \approx \mathcal{E}_3$ into two large ones of opposite sign and discard the negative one. Observing this injunction, we have

$$E_0 + (\Delta E)_2 \approx E_0 + (\Delta E)_2 + (\Delta E)_3 < E_B. \quad (\text{V.8})$$

These results are consistent with the following situation: the Jastrow energy (two body or three body) possesses the upper-bound property and, lying some several MeV below the Brueckner energy, constitutes a more accurate approximation to the true ground-state energy.

The striking neutralization of the $O(\kappa)$ contribution to the dispersion correction by the other $O(\kappa)$ terms, in particular by $(\Delta E)_3^{(3)}$, leads us to believe that this situation is more likely to prevail than the one proposed earlier. An ordering of the terms in the cluster expansion of the Jastrow energy-expectation value in accord with the Brueckner approximation is surely a bad one, presuming this expansion to have convergence properties such as to make it at all useful. Since the smallness parameter κ is the same, both numerically and in formal role, in BBG and Jastrow cases is it not conceivable that this kind of ordering is bad in the BBG context as well? At least an explicit evaluation of the $O(\kappa)$ terms in the BBG expansion, in particular, the Bethe-Faddeev sum,

is required before the issue can be decided. In order for the three-body approximations of the two approaches to agree, the Bethe-Faddeev sum must yield some -7 MeV for the IY potential at $k_F = 1.36 \text{ fm}^{-1}$, presuming the Chakkalakal f to be a superior (nearly optimal) choice for the Jastrow case. [Remember that in a conventional BBG calculation with choice (III.2) of $u(i)$, there are dispersion contributions to the Brueckner energy $O(\kappa^2)$, $O(\kappa^3)$, ..., as well as $O(\kappa)$, and dispersive contributions to the Bethe-Faddeev term $O(\kappa^2)$, $O(\kappa^3)$, ..., the κ dependence of the dispersion correction in the Brueckner energy being such that, for the projected agreement, the Bethe-Faddeev sum need not be so large in magnitude as the proper three-body part of the Jastrow energy.] Actually one would expect that if both three-body approximations provide accurate evaluations of the energy series concerned (which are presumed to converge), the Bethe-Faddeev sum should give a bit more attraction than this because the exact BBG energy should lie below the exact Jastrow energy.

If, on the contrary, it were to turn out that the Bethe-Faddeev term is small and repulsive⁵² for the hard-core OMY and IY potentials near equilibrium, one would have to think more deeply into the connection between Jastrow and BBG methods in order to understand the Brueckner-Jastrow energy discrepancy we have discussed. One would be forced to regard with skepticism the claim that κ , when as large as ~ 0.3 , plays, in any useful sense, the role of a smallness parameter. Certainly one would have to conclude that, even if the two-body correlations of BBG and Jastrow approaches are quite similar, notably in the respect of producing a common wound integral, the specifically three-body correlations of the two approaches may still be profoundly different.

It is important to realize that sensible cancellation of contributions to the Jastrow three-body term occurs *formally* as well as numerically. The interesting point is that with a Jastrow wave function the cluster expansion of the energy shift for the uniform infinite medium simplifies neatly into a sum of two-body, three-body, ..., integrals of very tight structure. The integrand of any such survivor past the two-body term must have the property that each particle label present is joined to every other (by a product of dynamical or statistical bond functions), via *at least two* chains having no particle in common. Contributions not possessing this tight structure *do* appear superficially if we write out the FIY formulas of Sec. II for the Jastrow ansatz (there are reducible violators buried in the combination terms and in the proper terms) but they all cancel out order

by order in ξ . In the $O(\kappa)$ cancellation whose numerical consequences have been mentioned above the part of the three-body contribution $(\Delta E)_3^{(2)}$ which corresponds structurally to the third-order v or G Goldstone diagram containing a direct hole-bubble insertion is compensated precisely by the part of $(\Delta E)_3^{(3)}$ which corresponds structurally to the third-order v or G Goldstone diagram containing a direct particle-bubble insertion. Indeed, one may state further that no contribution to $(\Delta E)_3$ survives which corresponds structurally to a third-order v (or G) diagram which can be split apart by cutting only one or two v (or G) lines. The proof leans on the state independence of the Jastrow $F(12)$ and $F(123)$. We emphasize that this cancellation phenomenon is a general feature of the ordinary Jastrow method, not restricted to the special model under consideration and not restricted to the three-body cluster. Just as in the classical Ursell-Mayer cluster expansion for the logarithm of the partition function,⁵³ each term $\mathcal{E}_{n \geq 2}$ in the IY expansion with the Jastrow ansatz (V.1) may, for the uniform infinite medium, be collapsed to a sum of *irreducible* n -body integrals, irreducible in the sense that they cannot be evaluated in finite terms from fewer-body integrals. Such a collapsed version of (V.6) was used in the calculations of Refs. 45 and 27, masking the (here largely artificial) presence of a dispersion contribution $(\Delta E)_3^{(2)}$ in the three-body cluster.

It seems not unreasonable that some remnant of this cancellation phenomenon, in particular, substantial neutralization of $(\Delta E)_3^{(2)}$ by a portion of $(\Delta E)_3^{(3)}$ of particle-bubble character, should persist when one goes over from Jastrow two- and three-body correlation factors $f(r_{12})$, $f(r_{12}) \times f(r_{23})f(r_{31})$ to more elaborate "more realistic" state-dependent $F(12)$, $F(123)$ (for example the BBG choices). After all, the formalism of Sec. II is variational in spirit and therefore supposed to be in some way optimal in structure; the presence of a negative component of $(\Delta E)_3^{(3)}$ which acts to diminish the repulsive dispersion contribution $(\Delta E)_3^{(2)}$ may be a manifestation of this optimal structure.

Even so, the dramatic scale of the internal cancellation we have observed in $(\Delta E)_3$ is a special feature of the model problems considered and may be traced immediately to the inordinately large values of the "smallness parameter" κ associated with the IY and OMY potentials. For the IY (OMY) potential, κ grows from 0.224 (0.300) at $\kappa_F = 1.36 \text{ fm}^{-1}$ (1.54 fm^{-1}) to 0.324 (0.368) at $\kappa_F = 1.6 \text{ fm}^{-1}$ (1.7 fm^{-1}). (These values are based on the Chakkalakal f 's.) The value of κ for the realistic Reid soft-core potential⁵⁴ is only about half as large, 0.135 at $\kappa_F = 1.36 \text{ fm}^{-1}$.⁵⁵ Therefore we

do not wish to call into question the accuracy of the conventional Brueckner calculations on nuclear matter near equilibrium density with realistic static two-nucleon potentials. The qualitative situation is further altered, from that we have explored, by the existence of strong noncentral components in the realistic potentials.⁴³

On the other hand, it would surely be disturbing if the success of the Brueckner approximation in the nuclear-matter problem (as evidenced by small higher-order terms in the BBG expansion) were to hinge delicately on the details of the two-nucleon interaction assumed. Our results underscore this possibility and suggest that in studies of the sensitivity of the nuclear-matter ground-state energy to off-shell and deuteron properties of classes of phase-equivalent potentials,⁵⁶ it would be advisable to supplement the Brueckner approximation by an adequate evaluation of the three-body part of the BBG expansion, for those potentials having large wound parameters κ .

Our findings may have more direct bearing on Brueckner-Jastrow comparisons for certain other

problems involving large wound parameters, viz. the calculation of the ground-state energy and density of liquid ^3He ⁵⁷ and of the binding energy of a Λ particle in nuclear matter.^{58, 38, 59}

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Pion-Oxygen Elastic Scattering in the 3-3 Resonance Region*

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The total and differential cross sections for π^- - ^{16}O elastic scattering in the 3-3 resonance region are calculated using the first-order optical potential derived from the multiple-scattering theory of Kerman, McManus, and Thaler. In order to include the 3-3 resonance and generate a reasonable off-shell behavior of the π -nucleon transition matrix, a separable model is used to construct the π -nucleon transition matrix. The optical potential also includes a proper transformation of the π -nucleon transition matrix from the π -nucleon to the π -nucleus c.m. frames. The results are compared with recent π^- - ^{16}O elastic-scattering data and with earlier π^- - ^{12}C calculations.

I. INTRODUCTION

With the advent of meson factories, there will be an increase in the study of nuclear structure using the π meson as a probe. The basic π -nucleon interaction has relatively well-determined phase shifts that show the interaction to be resonant and dominated by the $J = \frac{3}{2}$, $T = \frac{3}{2}$ channel over a fairly wide energy range. In order to extract detailed information about nuclear structure, a reliable method of calculation should be estab-

lished, in which the resonant features and the general dependence of the π -nucleon interaction are correctly incorporated.

In an earlier paper,¹ we studied first-order π -nucleus optical potentials defined in the multiple-scattering theory of Kerman, McManus, and Thaler²; several models for the off-shell π -nucleon transition matrices were used. The π -nucleon collision matrices employed in the Kisslinger and Laplacian (or local) optical models were seen to diverge with increasing off-shell momenta, where-