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Off-Shell Correction to the High-Energy Optical Potential*

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We obtain the leading off-shell correction to the high-energy optical potential: $v = v_0 \times [1 - (\gamma/2k^2)v_0]$, where $\gamma = k[(\partial/\partial q)\ln f(q;k)]_{q=k}$ for $f(q;k) = f(\vec{q},\vec{q};k)$ the off-shell projectile-nucleon scattering amplitude. A slow energy variation of the amplitude at very high energy gives $\gamma = k(\partial/\partial k)\ln f_k(0)$, where $f_k(0)$ is the on-shell amplitude, and the correction is small. In the region of projectile-nucleon resonances, we describe the two-body interaction by a separable potential and relate γ to the scattering phase shifts. The leading off-shell correction is substantial for an absorptive resonance but "anomalously" small at an elastic resonance.

I. INTRODUCTION

The problem of scattering on composite systems is an extremely difficult one, and a standard approach in the analysis of nuclear elastic scattering experiments has been to construct an equivalent potential.[1] One thereby reduces the unmanageable many-body problem to that of a single particle being scattered by an optical potential which both describes the nuclear target and relates the nuclear scattering to the elementary projectile-nucleon amplitudes. A particularly clear discussion of this problem, appropriate to high-energy scattering, was given some time ago by Glauber.[2] This theory is essentially an extension of Fraunhofer diffraction theory to composite targets and, in its simplest form, yields the well-known optical potential

$$v_0(\vec{x}) = 2m V(\vec{x}) = -4\pi A \rho(\vec{x}) f_k(0), \qquad (1)$$

where A is the number of particles in the target, $\rho(\vec{x})$ is the nuclear single-particle density (normalized to unity), and $f_k(0)$ is the projectile-nucleon forward on-shell scattering amplitude at momentum k. This potential is very appealing, since it is local (allowing a comparatively simple solution to the one-body scattering equation) and since it is expressed in terms of quantities which are directly measurable in other experiments.

The Glauber theory has proved to be remarkably successful in describing a wide variety of high-energy elastic scattering experiments, and a straightforward extension of the theory to coherent production processes (e.g., nuclear ρ^0 photoproduction) has provided the only presently feasible method for extracting information on the interactions of short-lived particles with nucleons.[3] In addition, the predictions generated by this simple optical potential for pion-nucleus cross sections in the vicinity of the 3-3 resonance[4] agree reasonably well with recent experimental data, despite the lack of any real theoretical justification for such an application. The widespread use and success of the Glauber optical potential warrants a close examination of the corrections to Eq. (1). That is, for a consistent application of the theory to experimental data, we must be able to evaluate in a systematic fashion leading corrections to the lowest-order result and to demonstrate that these are small. The correction arising from a more detailed description of the target structure has already been studied in great detail.[3] In fact, Glauber's lectures[2] included the modification of the optical potential arising from two-body correlations:

$$v(\vec{x}) = v_0(\vec{x}) \left(1 + i \frac{l_c}{2k} v_0(\vec{x})\right), \tag{2}$$

where v_0 is given above and l_c is the nucleon-nucleon correlation length. Our goal here is to obtain a second correction to the lowest-order potential, one which accounts for the off-energy-shell scattering of the projectile during intermediate stages of the multiple-scattering process. We shall aim for a local correction and expect a form similar to Eq. (2); i.e., since both the correlation and off-shell effects enter only when the projectile can scatter at least twice, the leading correction should be proportional to $v_0{}^2$ in both cases. We shall work within the framework of potential scattering, and, in particular, assume that the projectile-nucleon interaction is described by an arbitrary sum of separable potentials in each partial wave. This is both very general and easy to work with (the fully-off-shell amplitude can be explicitly written down). We shall find that at very high energies, the potentials can be entirely eliminated in favor of the energy derivative of the *on-shell* elementary amplitude (the conditions under which this result holds, independent of the potential model, will be discussed). The Glauber optical potential is known to work very well in this regime, and indeed we find that the off-shell correction, like the correlation correction, is quite small. We then consider what happens as we lower the energy and enter the region where there may be resonances between the incident and target particles (specifically, we have in mind pion scattering, since there are several prominent high-energy resonances in the πN cross section). Here, we specialize to the case of a single separable potential and, through the solution to the inverse scattering problem, express our correction directly in terms of the projectile-nucleon complex phase shifts. Quantitative results are presented for two model problems, those of absorptive and elastic projectile-nucleon resonances. We stress that we are computing a correction to $v_0(\vec{x})$ and that, if this becomes large, application of the Glauber formalism must be seriously questioned.

Previous work on off-shell effects in very-high-energy nuclear scattering (i.e., in the "Glauber region") has been somewhat limited. Harrington[5] and Eisenberg[6] have demonstrated that the Watson multiple-scattering series, which contains an infinite number of scattering events and off-shell elementary amplitudes, reduces at very high energies to the Glauber series, which contains a maximum of A scatterings and strictly on-shell amplitudes. This implies that the off-shell correction to the lowest-order optical potential vanishes at infinite projectile momentum but does not give the size of the correction at finite energies. More quantitative work on off-shell effects has centered upon pion scattering below and in the vicinity of the 3-3 resonance. We shall discuss the work of Ericson and Hufner[7] in some detail. They assume a simple analytic form for the off-shell πN forward-scattering amplitude close to the 3-3 resonance and use this to solve for the refractive index in nuclear matter. Their results differ appreciably from ours and the relation between the two approaches will be shown. The new features of the work presented here are, first, that the leading off-shell effects are incorporated directly into the high-energy optical potential and, second, that this correction is expressed in terms of the on-shell projectile-nucleon interaction.

II. OFF-SHELL CORRECTION

We start with the idea that the many-body scattering problem has been replaced by that of projectile scattering in a nonlocal optical potential $v_k(\vec{x}, \vec{y}) = 2mV$. The connection between this potential and the original nuclear scattering problem lies in the identification

$$v(\vec{p}, \vec{q}; k) = -4\pi A\rho(\vec{p}-\vec{q}) f(\vec{p}, \vec{q}; k), \qquad (3)$$

where $\rho(\vec{p}-\vec{q})$ is the Fourier transform of the nuclear single-particle density, $f(\vec{p}, \vec{q}; k)$ is the off-shell projectile nucleon scattering amplitude, and the "off-shell potential" is defined by

$$v(\vec{p}, \vec{q}; k) \equiv \int d\vec{x}\, d\vec{y}\; e^{-i\vec{p}\cdot\vec{x}}\, v_k(\vec{x}, \vec{y}) e^{i\vec{q}\cdot\vec{y}}. \qquad (4)$$

Equation (3) is widely used and provides our starting point. We give a brief derivation of this result in order to indicate its theoretical foundations.

Assuming that: (i) The projectile-nucleon scattering matrix is equated with that for scattering on unbound nucleons (i.e., the impulse approximation), and that (ii) the energy transferred to the target at any stage during the multiple-scattering process is negligible compared to the incident energy, the Watson multiple-scattering series[1] gives for the nuclear elastic scattering amplitude

$$F(\vec{k}', \vec{k}) = \int d\vec{x}_1 \cdots d\vec{x}_A |\Psi_0(\vec{x}_1 \cdots \vec{x}_A)|^2 \tilde{F}(\vec{k}', \vec{k}; \vec{x}_1 \cdots \vec{x}_A),$$

$$\begin{aligned}\tilde{F}(\vec{k}', \vec{k}; \vec{x}_1 \cdots \vec{x}_A) = \sum_{i=1}^{A} e^{-i\vec{k}'\cdot\vec{x}_i} \Bigg\{ & f_i(\vec{k}', \vec{k}; k) \\ & + \sum_{j\neq i} \int \frac{d\vec{p}}{(2\pi)^3} f_i(\vec{k}', \vec{p}; k) \left[\frac{-4\pi e^{i\vec{p}\cdot(\vec{x}_i-\vec{x}_j)}}{k^2-p^2+i\eta}\right] f_j(\vec{p}, \vec{k}; k) \\ & + \sum_{j\neq i}\sum_{k\neq j} \int \frac{d\vec{p}_1}{(2\pi)^3}\frac{d\vec{p}_2}{(2\pi)^3} f_i(\vec{k}', \vec{p}_1; k) \left[\frac{-4\pi e^{i\vec{p}_1\cdot(\vec{x}_i-\vec{x}_j)}}{k^2-p_1^{\,2}+i\eta}\right] f_j(\vec{p}_1, \vec{p}_2; k) \\ & \times \left[\frac{-4\pi e^{i\vec{p}_2\cdot(\vec{x}_j-\vec{x}_k)}}{k^2-p_2^{\,2}+i\eta}\right] f_k(\vec{p}_2, \vec{k}; k) + \cdots \Bigg\} e^{i\vec{k}\cdot\vec{x}_i}, \end{aligned} \qquad (5)$$

where Ψ_0 is the nuclear ground-state wave function and f_i is the off-shell amplitude on the ith nucleon (from now on, we drop this index and assume all nucleons are alike). We require three additional assumptions: (iii) The independent-particle model is employed; (iv) the projectile is allowed to scatter at most once from each target nucleon; (v) $N_{max} \ll A$, where N_{max} is the maximum number of scatterings which must be retained in the Watson series in order to reach a desired accuracy in the nuclear scattering amplitude. With these, Eq. (5) simplifies greatly:

$$\begin{aligned}-4\pi F(\vec{k}', \vec{k}) = & -4\pi A\rho(\vec{k}'-\vec{k}) f(\vec{k}', \vec{k}; k) \\ & + \int \frac{d\vec{p}}{(2\pi)^3} [-4\pi A\rho(\vec{k}'-\vec{p}) f(\vec{k}', \vec{p}; k)] \frac{1}{k^2-p^2+i\eta} [-4\pi A\rho(\vec{p}-\vec{k}) f(\vec{p}, \vec{k}; k)] \\ & + \int \frac{d\vec{p}_1}{(2\pi)^3}\frac{d\vec{p}_2}{(2\pi)^3} [-4\pi A\rho(\vec{k}'-\vec{p}_1) f(\vec{k}', \vec{p}_1; k)] \frac{1}{k^2-p_1^{\,2}+i\eta} [-4\pi A\rho(\vec{p}_1-\vec{p}_2) f(\vec{p}_1, \vec{p}_2; k)] \\ & \times \frac{1}{k^2-p_2^{\,2}+i\eta} [-4\pi A\rho(\vec{p}_2-\vec{k}) f(\vec{p}_2, \vec{k}; k)] + \cdots . \end{aligned} \qquad (6)$$

This can now be recognized as the Born series for elastic scattering of the projectile in the potential $v(\vec{p}, \vec{q}; k)$ given by Eq. (3). Finally, we point out that Foldy and Walecka[8] derived this result directly from the many-particle Schrödinger equation for a model of separable interactions and that $v_0(\vec{x})$ is obtained by evaluating v on the energy shell.

In principle, we could stop at this point and, given a model for the two-body scattering matrix, compute the optical potential. However, we know that the much simpler "on-shell" potential $v_0(\vec{x})$ describes high-energy scattering experiments quite well, and our philosophy is to treat the off-shell effects as a local correction to the lowest-order result. To this end, we define the average and relative coordinates and momenta as $\vec{r} = \frac{1}{2}(\vec{x}+\vec{y})$, $\vec{\rho} = (\vec{x}-\vec{y})$, $\vec{P} = \frac{1}{2}(\vec{p}+\vec{q})$, and $\vec{\Delta} = (\vec{p}-\vec{q})$. Then, in the large nucleus approximation [i.e., sharply peaked $\rho(\vec{q})$], Eq. (3) yields the configuration space optical potential

$$\begin{aligned} v_k(\vec{r}, \vec{\rho}) &= \int \frac{d\vec{P}}{(2\pi)^3}\frac{d\vec{\Delta}}{(2\pi)^3} e^{i\vec{P}\cdot\vec{\rho}} e^{i\vec{\Delta}\cdot\vec{r}} v(\vec{P}, \vec{\Delta}; k) \\ &= v_0(\vec{r})\eta_k(\vec{\rho}), \end{aligned} \qquad (7)$$

where $v_0(\vec{r}) = -4\pi A\rho(\vec{r}) f_k(0)$ is just the Glauber potential and

$$\eta_k(\vec{\rho}) = \frac{1}{f_k(0)} \int \frac{d\vec{q}}{(2\pi)^3} e^{i\vec{q}\cdot\vec{\rho}} f(\vec{q},\vec{q};k) . \tag{8}$$

This is a nonlocal potential with η_k playing the role of a nonlocal smearing function. Clearly, if the forward off-shell amplitude is momentum-independent, then $\eta_k(\vec{\rho}) = \delta^{(3)}(\vec{\rho})$ and we return to the standard optical potential. Our task now is to extract an equivalent local potential[9] while allowing for off-shell variation of the scattering amplitude.

The Schrödinger equation for scattering from the nonlocal potential Eq. (7) can be written as

$$2i\vec{k}\cdot\vec{\nabla}\phi(\vec{r}) + \nabla^2\phi(\vec{r}) = v(\vec{r})\phi(\vec{r}) , \tag{9}$$

where we have removed the rapidly varying phase factor from the wave function $\psi(\vec{r}) \equiv e^{i\vec{k}\cdot\vec{r}}\phi(\vec{r})$, and where the equivalent potential is defined as

$$v(\vec{r}) = v_0(\vec{r}) \frac{\int d\vec{\rho} e^{i\vec{k}\cdot\vec{\rho}} \eta_k(\vec{\rho})\phi(\vec{r}-\vec{\rho})}{\phi(\vec{r})} . \tag{10}$$

The simplicity of Eq. (9) is of course misleading, since we are still required to find the solution $\phi(\vec{r})$ for the full nonlocal potential $v_k(\vec{r},\vec{\rho})$ in order to find the equivalent potential. We start with the assumption that the nonlocality will be short ranged at very high energies, an assumption which can always be checked for any given form of the interaction by directly solving Eqs. (8)–(10). This is extremely difficult in practice, and our assumption is that the size of our simple off-shell correction will provide a consistency check.

We proceed by expanding $\phi(\vec{r}-\vec{\rho})$ in Eq. (10) about the point $\vec{r}$. The first observation is that the lowest-order term is exactly[10] $v_0(\vec{r})$, because

$$\int d\vec{\rho}\, e^{-i\vec{k}\cdot\vec{\rho}} \eta_k(\vec{\rho}) \equiv 1 .$$

Retaining the second term in the expansion, and defining the dimensionless quantity γ according to

$$\int d\vec{\rho}\, e^{-i\vec{k}\cdot\vec{\rho}} \vec{\rho}\eta_k(\vec{\rho}) \equiv i\gamma\vec{k}/k^2 , \tag{11}$$

we obtain for the equivalent potential

$$v(\vec{r}) = v_0(\vec{r}) \left[1 - i\frac{\gamma}{k^2} \frac{\vec{k}\cdot\vec{\nabla}\phi}{\phi} \right]. \tag{12}$$

We further assume that in evaluating the correction term above, we may take $\phi(\vec{r})$ as the solution to the Schrödinger equation in the lowest-order potential [i.e., Eq. (9) with $v = v_0$]. At high energies, the Laplacian term is very small (the eikonal approximation corresponds to dropping it) and Eq. (12) becomes just

$$v(\vec{r}) = v_0(\vec{r}) \left[1 - \frac{\gamma}{2k^2} v_0(\vec{r}) \right] . \tag{13}$$

We now have the leading off-shell correction in a form similar to the correlation correction of Eq. (2), its magnitude being determined by the projectile-nucleon dynamics through the quantity γ. Simple manipulation of Eqs. (8) and (11) results in the expression

$$\gamma = \frac{k}{f_k(0)} \left[\frac{\partial}{\partial q} f(q;k) \right]_{q=k} , \tag{14}$$

where $f(q;k) \equiv f(\vec{q},\vec{q};k)$. This directly relates the leading correction to the optical potential to the off-shell variation of the two-body amplitude.

It will prove useful to write γ for a specific model of the elementary interaction. We consider an arbitrary sum of separable potentials in each partial wave:

$$u(\vec{x},\vec{y}) = \sum_l u_l(x,y)(2l+1)P_l(\hat{x}\cdot\hat{y}) ,$$
$$u_l(x,y) = \sum_n \lambda_{nl} u_{nl}(x) u_{nl}(y) . \tag{15}$$

This interaction is both quite general and easy to work with, since the full off-shell projectile-nucleon scattering amplitude can be explicitly written down:

$$f(\vec{p},\vec{q};k) = \sum_l f_l(p,q;k)(2l+1)P_l(\hat{p}\cdot\hat{q}) ,$$
$$f_l(p,q;k) = -\sum_{nn'} \frac{\lambda_{nl}}{4\pi} u_{nl}(p) R^l_{nn'}(k) u_{n'l}(q) \tag{16}$$

with the matrix R defined according to

$$\sum_{n'} R^l_{nn'}(k) \left[\delta_{n'n''} + \lambda_{n''l} \int \frac{d\vec{t}}{(2\pi)^3} \frac{u_{n'l}(t) u_{n''l}(t)}{t^2 - k^2 - i\eta} \right] = \delta_{nn''} . \tag{17}$$

Note that the k-dependence is isolated in R. Equation (14) now gives

$$\gamma = \frac{k}{f_k(0)} \sum_l (2l+1) \sum_{nn'} \left(-\frac{\lambda_{nl}}{4\pi} \right) R^l_{nn'}(k) \times \frac{\partial}{\partial k} \left[u_{nl}(k) u_{n'l}(k) \right] \tag{18a}$$

which, for a single separable potential, further reduces to

$$\gamma = \frac{k}{f_k(0)} \sum_l (2l+1) f_l(k) \frac{\partial}{\partial k} \ln[u_l(k)]^2 . \tag{18b}$$

Given the interaction form factors u_{nl}, Eqs. (13) and (18) provide a straightforward manner in which to evaluate the off-shell modification of the optical potential. Nevertheless, we shall attempt to go further and relate γ to measurable quantities.

III. VERY HIGH ENERGY

We first consider the description of the elementary interaction via an arbitrary sum of separable potentials in each partial wave. Examination of Eqs. (16) and (17) reveals that, as long as the partial wave amplitudes $f_l(k)$ vanish faster than k^{-1} as $k\to\infty$, the projectile-nucleon interaction is described by the Born approximation at very high energy

$$R^l_{nn'}(k)\underset{k\to\infty}{\longrightarrow}\delta_{nn'}\ .$$

The expression for γ now simplifies greatly, since the energy derivative can be moved outside the summation sign in Eq. (18a):

$$\gamma=\frac{k}{f_k(0)}\frac{\partial}{\partial k}f_k(0)=k\frac{\partial}{\partial k}\ln f_k(0)\,. \tag{19}$$

This is the main result in this section and expresses the leading off-shell correction entirely in terms of the *on-shell* forward amplitude.

Elementary hadron-hadron total cross sections become constant at high energy. This behavior is associated with large absorption cross sections and cannot be obtained from an energy-independent potential. However, we stress that we have used a very general potential[11] and that the final result depends entirely upon the scattering amplitude. With this as justification, we evaluate Eq. (19) with *empirical* laboratory elastic scattering amplitudes. For a purely imaginary high-energy forward amplitude and a constant total cross section, we have $\gamma=1$, giving

$$v(\vec{r})=v_0(\vec{r})\left(1-\frac{1}{2k^2}v_0(\vec{r})\right)=v_0(\vec{r})\left(1+i\frac{A\sigma_T\rho(\vec{r})}{2k}\right). \tag{20}$$

The correction term adds a real part to the optical potential and vanishes like k^{-1} at very high energy. For nuclear matter densities and an elementary cross section of 40 mb, the ratio of real to imaginary parts of the optical potential above is just (1 GeV/15k). This is very small for energies where the Born approximation has any chance to be valid and the Glauber potential is therefore expected to work quite well in this energy range.

Finally, we remark that the simple result in Eq. (19) can be obtained immediately from Eq. (14) without resorting to the potential model for the elementary interaction. Clearly, the condition we must impose upon the off-shell amplitude is that

$$\left(\frac{\partial}{\partial q}f(q;k)\right)_{q=k}\gg\left(\frac{\partial}{\partial k}f(q;k)\right)_{q=k}.$$

However, the question of how far down in energy we can go while satisfying this condition can be answered only within the framework of a specific model for the projectile-nucleon dynamics.

IV. RESONANCE REGION

We now consider what happens as the incident energy is lowered into the region where there are projectile-nucleon resonances. Specialize to the case where the interaction is described by a single separable potential in each partial wave. Equations (16) and (17), taken on shell, define the inverse scattering problem which must be solved in order to obtain potentials which reproduce the elementary, partial wave scattering amplitudes:

$$f_l(k)=\frac{\eta_l(k)e^{2i\delta_l(k)}-1}{2ik}\equiv\frac{e^{2i\tau_l(k)}-1}{2ik}=-\frac{\lambda_l}{4\pi}\,\frac{[u_l(k)]^2}{1-\lambda_l\int[d\vec{t}/(2\pi)^3][u_l(t)]^2/(k^2-t^2+i\eta)}, \tag{21}$$

where $\tau_l(k)$ is the complex phase shift. Inversion of this equation[12-14] follows with trivial modification the standard procedure used in the case of real phase shifts, and we simply state the result:

$$\frac{\lambda_l}{4\pi}[u_l(k)]^2=ik^{-1}\sinh(i\tau_l)\exp\left(\frac{1}{\pi}\text{P.V.}\int_0^\infty dx\frac{\tau_l(x)}{k^2-x}\right). \tag{22}$$

Finally, we insert this into Eq. (18b) and obtain γ directly in terms of the phase shifts:

$$\gamma\equiv\frac{1}{f_k(0)}\sum_l(2l+1)f_l(k)\gamma_l\,,$$

$$\gamma_l=\frac{2}{\pi}\tau_l-1+\frac{k}{2}\frac{\partial}{\partial k}\ln\left(\frac{[2ikf_l(k)]^2}{1+2ikf_l(k)}\right)-\frac{2k^2}{\pi}\text{P.V.}\int_0^\infty dx\frac{\tau_l(x)-\tau_l(k^2)}{(x-k^2)^2}\ . \tag{23}$$

Equation (23) is the main result of this section. We stress that although γ can be computed only in terms of measurable quantities, this result is in no sense model-independent, but rather depends upon the specific method used for the off-shell extrapolation. That is, the assumption of a single separable potential in each partial wave defines the off-shell scattering amplitude in terms of the on shell, and we have simply eliminated the intermediate step of explicitly computing the potential. One could instead choose to construct phase-equivalent interactions with several separable potentials in each partial wave and then insert these into Eq. (18a); this is a fairly simply way to test the sensitivity of the off-shell correction to different continuations off shell. Here, we restrict ourselves to consideration of two physically interesting model problems within the framework of a single separable potential.

A. Absorptive Resonance

Our first model problem is motivated by the idea that we are working down in energy into the region of πN resonances. The first significant structure which we encounter in the $T=\frac{3}{2}$ and $T=\frac{1}{2}$ channels are highly absorptive F-wave resonances [the $F_{37}(1940)$ and $F_{15}(1960)$, respectively]. Consequently, we shall consider the situation in which the projectile-nucleon amplitude is dominated by an absorptive, resonant F-wave interaction (taking the nucleon as spinless). We take the real part of the phase shift to be zero, and the absorption is given by

$$\eta_3(k^2)=\exp\left(\frac{-a(k^2/k_R^{\,2})\Gamma^4}{(k^2-k_R^{\,2})^2+\Gamma^4}\right) \tag{24}$$

with $a=2$, $k_R=1$ GeV, and $\Gamma=250$ MeV. This form was chosen for simplicity, since it allows us to evaluate easily the integral in Eq. (23) (the integral is not sensitive to the precise threshold behavior). Figure 1 shows the real and imaginary parts of γ as a function of incident momentum; also shown, in order that the rapidity of the variation of γ can be seen, is the quantity $(1-\eta)$, drawn to arbitrary scale. To understand these results, let $\eta=e^{-g(k)}$, where $g(k)$ is positive definite and peaks at resonance. Equation (22) now gives

$$\operatorname{Im}\gamma=\frac{1}{\pi}\,g(k)+\frac{k^2}{\pi}\,\text{P.V.}\int_0^\infty dx\,\frac{g(k^2)-g(x)}{(k^2-x)^2}\;,$$

$$\operatorname{Re}\gamma=-1+\frac{k}{2}\,\frac{1+\exp[-g(k)]}{1-\exp[-g(k)]}\,g'(k)\;,$$

and, since the elementary amplitude is pure imaginary,

$$v=-i\,|v_0|\left(1-\frac{\operatorname{Im}\gamma}{2k^2}\,|v_0|\right)+\frac{\operatorname{Re}\gamma}{2k^2}\,|v_0|^2\,.$$

Note that, at resonance, $\operatorname{Im}\gamma$ is positive and reaches a maximum, while $\operatorname{Re}\gamma=-1$ because of the vanishing derivative. In addition, we see that $\operatorname{Im}\gamma$ scales with $g(k)$, so that we expect a significant reduction in the imaginary part of the optical potential for a highly absorptive resonance. This is seen explicitly in Figure 2, where we display

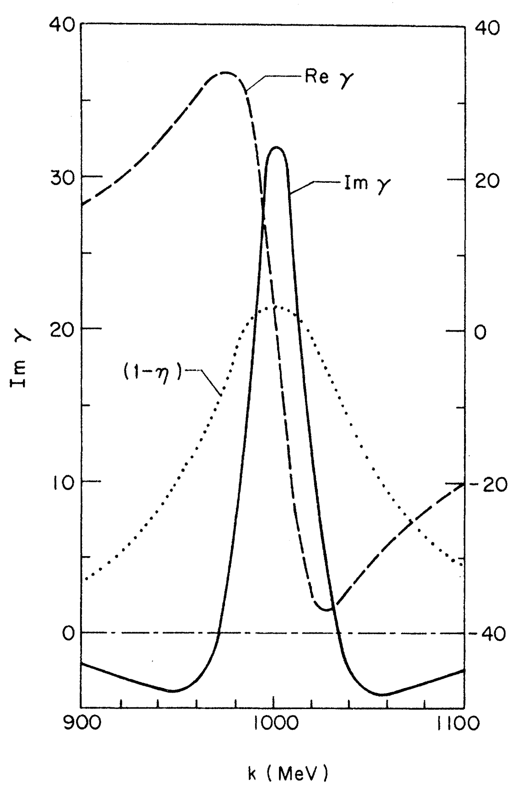

FIG. 1. Real and imaginary parts of γ as a function of incident momentum for the projectile-nucleon interaction described by Eq. (24). Dotted line is $(1-\eta)$ drawn to arbitrary scale.

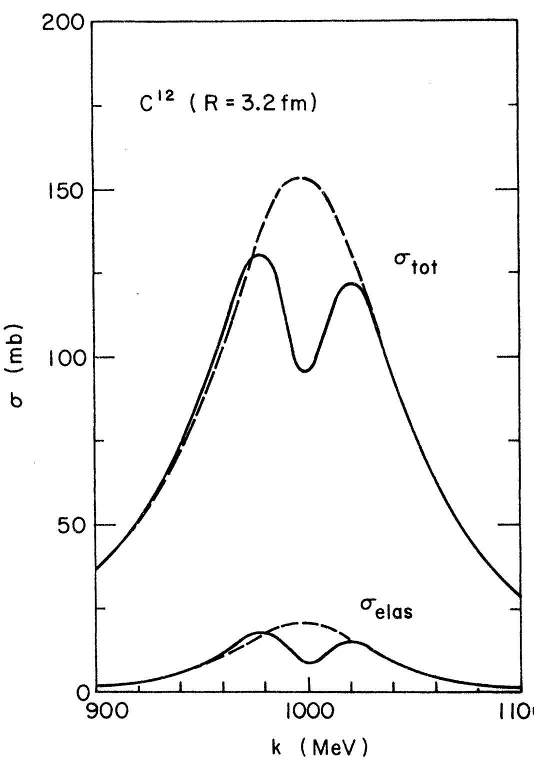

FIG. 2. Total and elastic cross sections for scattering on ^{12}C (uniform sphere with $R=3.2$ fm) both with (solid line) and without (dashed line) the off-shell correction corresponding to Fig. 1.

the total and elastic cross sections as a function of incident momentum for scattering on ^{12}C both with (solid line) and without (dashed line) the off-shell correction; the nucleus was assumed to have a uniform density with radius $R = 3.2$ fm. The elastic cross section is very small compared to the total because of the purely absorptive projectile-nucleon amplitude. The interesting feature is of course the pronounced dip in the cross section in a fairly small region around the resonance energy. This is certainly an appreciably larger effect than that due to nucleon-nucleon correlations,[3] but we caution once again that a specific off-shell extrapolation has been assumed. The sensitivity of this result to other reasonable methods of continuing off shell should certainly be investigated.

B. Elastic Resonances

Next we turn to the case of a real phase shift.

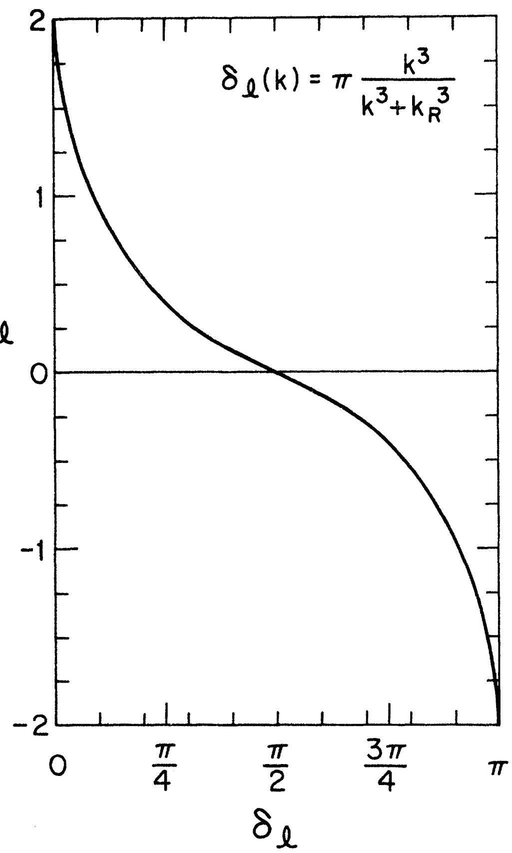

FIG. 3. γ_l as a function of the phase shift given by Eq. (25) with $n = 3$, corresponding to dominance of the projectile-nucleon amplitude by a p-wave elastic resonance.

Our expression for γ_l simplifies to

$$\gamma_l(k) = \left(\frac{2}{\pi}\delta_l - 1\right) + k\frac{\partial}{\partial k}\ln(\sin\delta_l) + \xi_l(k),$$
$$\xi_l(k) = -\frac{2k^2}{\pi} PV\int_0^\infty dx\, \frac{\delta_l(x) - \delta_l(k^2)}{(x-k^2)^2}. \tag{25}$$

Note the threshold behavior $\gamma_l \to 2l$ as $k \to 0$; this will lead to an interesting comparison between our results and those of Ericson and Hufner.[7]

We consider the case of an elastic resonant partial wave, by which we mean one with the phase shift rising smoothly from zero to π (we can assume that δ falls back to zero at some arbitrarily large energy), such that $\delta(k_R) = \frac{1}{2}\pi$, where k_R is the position of the resonance (e.g., the $T = \frac{3}{2}$, $J = \frac{3}{2}$ partial wave in πN scattering). Our first observation is that the first two terms in $\gamma_l(k)$ *vanish* at the resonance. The last term is more complicated but, at the resonance, we can rewrite it as

$$\xi_l(k_R) = -\frac{2}{\pi}\int_0^1 \frac{dy}{(1-y)^2}\left[\delta_l(y) + \delta_l\left(\frac{1}{y}\right) - \pi\right], \tag{26}$$

where we have let $\delta_l \to \delta_l$ $(y = k^2/k_R^2)$. It is clear that the particular combination of phase shifts given in the integrand above tends to cancel for a resonant phase shift. More quantitatively, consider the phase shift

$$\delta_l = \pi\frac{k^n}{k^n + k_R^n} \tag{27}$$

with $n > 0$. To obtain the correct threshold behavior, we must choose $n = 2l + 1$, but we take the attitude that, since the precise threshold behavior is not important in computing $\xi_l(k)$ close to k_R, n can be varied to fit the resonance width. The

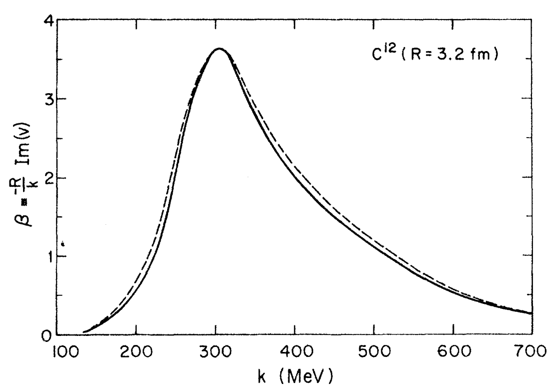

FIG. 4. Imaginary part of the pion optical potential near the 3-3 resonance, with (solid line) and without (dashed line) the off-shell correction. The target nucleus is ^{12}C ($R = 3.2$ fm).

point is that the integrand in Eq. (26) vanishes identically for *any* $n>0$, causing the entire leading order off-shell effect to vanish at resonance. Needless to say, there are many other possible forms for the phase shift besides those described by Eq. (27), and in general these will not cause $\xi_l(k_R)$ to vanish. Nevertheless, the arguments above indicate that, if the off-shell extrapolation is performed according to a separable potential which reproduces the phase shift at all energies, then the leading off-shell correction to the optical potential is "anomalously" small at a "classic" elastic resonance.

We can evaluate γ_l analytically for $n=3$, which in fact gives the correct p-wave threshold behavior:

$$\gamma_l(\eta=3)=\left(\frac{2}{\pi}\delta_l-1\right)+\frac{3}{\pi}\delta_l(\pi-\delta_l)\cot\delta_l+\frac{4\delta_l}{\pi-\delta_l}(\beta_1+\beta_2)\,,$$

$$\beta_1=\frac{2\pi(\pi-\delta_l)^2(\delta_l-\frac{1}{2}\pi)-2\delta_l(\pi-\delta_l)^3\ln\left[\delta_l/(\pi-\delta_l)\right]}{4\pi^2(\delta_l-\frac{1}{2}\pi)^2}\,,$$

$$\beta_2=\frac{2\pi}{3\sqrt{3}}\left[\frac{(\pi-\delta_l)^5}{\delta_l}\right]^{1/3}\times\frac{\delta_l^{4/3}-(\pi-\delta_l)^{4/3}}{\left[\delta_l^{4/3}+\delta_l^{2/3}(\pi-\delta_l)^{2/3}+(\pi-\delta_l)^{4/3}\right]^2}\,. \qquad (28)$$

This is plotted in Fig. 3 as a function of the phase shift $\delta_l=\pi k^3/(k^3+k_R^3)$. Note that γ_l varies rapidly away from threshold and that in the vicinity of the resonance $\frac{1}{4}\pi<\delta_l<\frac{3}{4}\pi$, we have $|\gamma_l|<0.4$. To get an idea as to how significantly this affects the optical potential in a problem of interest, we use Eq. (28) in computing the optical potential for scattering on ^{12}C ($R=3.2$ fm) close to the 3-3 resonance. In other words, we take Eqs. (13) and (28) with the measured $T=\frac{3}{2}$, $J=\frac{3}{2}$ πN phase shifts. This is clearly not a consistent procedure, but

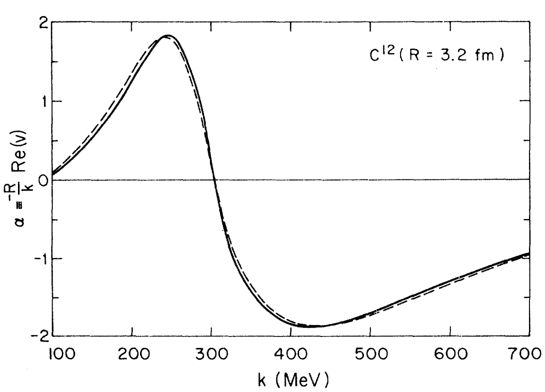

FIG. 5. Real part of the pion optical potential corresponding to Fig. 4.

Figs. 4 and 5 show that the modification of the imaginary and real parts of the optical potential is quite small. Our previous arguments imply that this cannot change very much[15] when a detailed numerical calculation of γ_l is performed with the experimental phase shifts.

The smallness of the off-shell correction at the 3-3 resonance is certainly very surprising when compared to the large effects computed by Ericson and Hufner (EH). They start with an expression for the energy-dependent index of refraction $n(E)$ for a pion traversing infinite nuclear matter:

$$n(E)=\kappa(E)/k\,,$$
$$\kappa^2-4\pi\rho f^{\pi N}(\kappa,E)=E^2-m_\pi^{\;2}=k^2\,. \qquad (29)$$

Here, k is the free-space momentum of a pion with energy E, ρ is the nuclear matter density, $\kappa(E)$ is the pion wave number inside nuclear matter, and $f^{\pi N}(\kappa,E)$ is the forward off-shell scattering amplitude. Given the functional dependence of $f^{\pi N}$ on momentum and energy, we can solve Eq. (29) for the optical potential $v(E)=2k^2[1-n(E)]$. EH choose the simple analytic form

$$f^{\pi N}(k,E)=\frac{Ck^2}{E-E_R+\frac{1}{2}i\,\Gamma(E)}\,, \qquad (30)$$

where C and $\Gamma(E)$ are fixed to give the correct total cross section. They argue that this form is suggested by the Chew-Low model and by p-wave scattering from a short-range separable potential, since the amplitude is of the form $u(k)^2/D(E)$ with $u(k)\sim k$ giving the proper threshold behavior. Equation (29) is then solved algebraically, and EH find significant corrections to v_0. We shall "solve" Eq. (29) in a different way, allowing us to show the origin of the difference between our results and those of EH.

As with our derivation of the off-shell correction, we assume that the leading correction is small and expand[16] $\kappa(E)$ around the on-shell momentum k. Retaining only the leading correction, we easily arrive at the expression

$$n(E)-1=-\frac{v_0}{2k^2}\left(1-\frac{\gamma}{2k^2}v_0\right)\,,$$
$$v_0=-4\pi\rho f_k(0)\,,\qquad \gamma=\frac{k}{f_k(0)}\frac{\partial}{\partial k}f(k,E)\,. \qquad (31)$$

This is immediately recognized as the same expression we have used in evaluating the off-shell effect, although we feel that our approach makes clearer the underlying assumptions. However, EH employ the scattering amplitude Eq. (30), yielding $\gamma=2$. Note that this comes entirely from the *threshold behavior* and agrees with our result $\gamma_l(0)=2l$ [see Eq. (25) and Fig. 3]. In contrast to this, we have noted that the off-shell extrapolation can really be performed according to a separable potential simply by writing $\gamma = k(\partial/\partial k) \times \ln[u(k)]^2$ and then replacing $u(k)$ by the solution to the inverse-scattering problem. We have already seen that this leads to roughly an order of magnitude reduction of the off-shell correction as we go from threshold to resonance. This is consistent with the observation of Silbar and Sternheim,[17] who solved Eq. (29) numerically for various "reasonable" off-shell continuations and concluded that the correction was very sensitive to the extrapolation procedure. The separable potential simply gives results very different from those of Ericson-Hufner and Silbar-Sternheim. Our result is also consistent with the calculation of Landau and Tabakin,[14] who computed the form factor $u(k)$ for the 3-3 partial wave by feeding the experimental πN phase shifts into Eq. (22). Examination of their Fig. 9 reveals that $u(k)$ reaches a maximum very close to the resonance energy.

A final, bothersome point in connection with the comparison to EH is that their amplitude is motivated by the very successful Chew-Low model and that this seems to give results so different from those of the separable potential approach. Again, the difficulty is that EH have retained only the threshold momentum dependence. The separable potential implied by the static theory[18] is $u(k) \sim kg(k)/(m_\pi^2 + k^2)^{1/4}$, where $g(k)$ is the cutoff function corresponding to an extended source distribution (u^2 can be obtained directly by taking the P_{33} projection of the crossed πN Born diagram). A good fit to the πN phase shifts is obtained[19] with a Yukawa source function $g(k) = (1 + k^2/a^2)^{-1}$ of range $a^{-1} = 0.38$ fm. This "Chew-Low potential" predicts an off-shell correction similar to that in Fig. 3: At resonance, $\gamma(k)$ has fallen substantially away from the threshold value $\gamma(0) = 2$ and asymptotically approaches $\gamma(\infty) = -3$. In other words, the Chew-Low potential and the separable potential approach described above yield qualitatively similar results.

V. DISCUSSION AND CONCLUSIONS

We have obtained the leading correction to the lowest-order high-energy optical potential arising from off-shell scattering of the projectile during intermediate stages of the multiple-scattering process. Under the set of assumptions listed in Sec. II, the optical potential can be written

$$v(\vec{r}) = v_0(\vec{r})\left(1 - \frac{\gamma}{2k^2}\, v_0(\vec{r})\right),$$

$$v_0(\vec{r}) = -4\pi A \rho(\vec{r}) f_k(0).$$

The size of the off-shell correction is determined by γ, which is in turn related to the off-shell variation of the projectile-nucleon scattering amplitude

$$\gamma = k\left(\frac{\partial}{\partial q} \ln f(q; k)\right)_{q=k}. \tag{14}$$

Assuming a very general form for the elementary interaction (i.e., an arbitrary sum of separable potentials in each partial wave), this reduces at very high energies to

$$\gamma = k\,\frac{\partial}{\partial k}\, \ln f_k(0) \tag{19}$$

which is independent of the details of the interaction and depends only upon the on-shell amplitude.

At lower energies, in the region of projectile-nucleon resonances, we specialized to the case of a single separable interaction in each partial wave. This allows us to compute the off-shell amplitude directly from on-shell information:

$$\gamma = \frac{1}{f_k(0)} \sum_l (2l+1) f_l(k) \gamma_l\,,$$

$$\gamma_l = \frac{2}{\pi}\tau_l - 1 + \frac{k}{2}\frac{\partial}{\partial k} \ln\left(\frac{[2ikf_l(k)]^2}{1 + 2ikf_l(k)}\right) - \frac{2k^2}{\pi}\, PV\int_0^\infty dx\, \frac{\tau_l(x) - \tau_l(k^2)}{(x-k^2)^2}\,, \tag{23}$$

where $\tau_l(k)$ is the projectile-nucleon complex phase shift. However, we now point to an inconsistency in our formalism: The phase shifts give us the scattering amplitudes in the projectile-nucleon c.m. frame, while Eq. (14) is to be evaluated in the laboratory frame[20] (we shall equate the laboratory and π-nucleus c.m. frames, since we are considering only large nuclei). The simplest remedy is to assume a transformation law for the off-shell amplitude based upon that used for on-shell amplitudes. Considering the case of an incident pion, our prescription will be to compute the πN c.m. frame off-shell amplitude according to Eq. (16) and then to identify

$$\frac{1}{q} f(q; k) = \frac{1}{q'} f(q'; k'),$$

where primes are used to denote quantities in the πN c.m. frame and

$$q' = qm[m^2 + \mu^2 + 2m(\mu^2 + q^2)^{1/2}]^{-1/2}$$

for μ and m the pion and nucleon masses, respectively. Inserting this into our expression for γ, we have finally

$$\gamma = \alpha\gamma_{(0)} + \gamma_{(1)}, \tag{31}$$

where $\gamma_{(0)}$ is to be computed according to Eq. (23) (with the appropriate πN c.m. variables), and where

$$\gamma_{(1)} = 1 - \alpha ,$$
$$\alpha = \frac{(\mu^2 + m^2)(\mu^2 + k^2)^{1/2} + 2m\mu^2 + mk^2}{(\mu^2 + k^2)^{1/2}[\mu^2 + m^2 + 2m(\mu^2 + k^2)^{1/2}]} \ . \tag{32}$$

Equations (31) and (32) should be used in any comparison to experimental data. Note that $\alpha \to 1$ as $k \to 0$, so that we return to our original result in this limit. Also, one can easily verify that α does not differ from unity very much in the region of πN resonances. This prescription for transforming the off-shell forward amplitude is not unique, but it is encouraging that this transformation effect does not appreciably alter the results which we obtained above and now summarize.

At very high energies, we found that the size of the off-shell correction was very small, justifying use of the "on-shell" or lowest-order optical potential. In the region of projectile-nucleon resonances, we examined two model problems by considering absorptive and elastic resonant phase shifts. We found a substantial off-shell correction when scattering in the vicinity of an absorptive resonance, manifesting itself in a sharp reduction of the imaginary part of the optical potential in a narrow region around the resonance energy. On the other hand, we found an "anomalously" small effect in the region of an elastic resonance, with the leading off-shell correction actually vanishing at the resonance energy for a certain class of phase shifts. Comparing this with the work of Ericson and Hufner[7] near the 3-3 resonance, we conclude that the correction is quite sensitive to the off-shell extrapolation. We extrapolated according to a single separable potential, which is hopefully a rational way to describe the two-body interaction close to resonance. Nevertheless, in view of the sensitivity of the results, one should certainly investigate the consequences of continuing off-shell through other dynamical models.

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[9]An equivalent local potential is defined as one which produces the same scattering amplitude as does the nonlocal potential.

[10]Note that, although the basic assumptions listed after Eq. (5) may indeed by better satisfied at high energy and/or small angles, the only assumption required to this point which *explicitly* depends upon either of these conditions is (ii).

[11]Foldy and Walecka (Ref. 8) have sketched how an arbitrary spherically symmetric potential can in many cases be represented by a sum of separable potentials. See Appendix B of Ref. 8.

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[15]For example, consider the effects of modifying the high-energy behavior of the phase shift. Write $\delta_l(k) = \delta_l^{(0)} k + \Delta\delta_l(k)$, where $\delta_l^{(0)}$ represents the phase shift given, say, by Eq. (25), and where $\Delta\delta_l$ will be constructed to damp the high-energy part of the phase shift to zero (i.e., $\Delta\delta_l \equiv 0$ for $k < k_0$, where $k_0 \geq 2$ GeV is the maximum energy for which πN phase shifts are available, and $\Delta\delta_l \to -\pi$ for $k \to \infty$). If we assume that the resonant phase shift satisfies $0 < \delta_l < \pi$, then

$$0 < \Delta\gamma_l \ (k < k_0) < 2k^2 \int_{k_0^2}^{\infty} \frac{dx}{(x - k_0^2)^2} = 2\frac{k^2}{k_0^2 - k^2} \ .$$

This is extremely small in the vicinity of the resonance, implying that the off-shell correction close to resonance is very insensitive to the high-energy extrapolation of the phase shift.

[16]This is really the limit in which the index of refraction equation is valid. See C. Dover, J. Hufner, and R. Lemmer, Ann. Phys. (N.Y.) 66, 248 (1971) for an interesting discussion of pion propagation in nuclear matter which relates to this point.

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