Unified Description of Phenomenological Models of Ground-State Bands in Even-Even Nuclei

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A generalized stretching model for the description of the ground band levels in even-even nuclei is developed under the provision that the higher-order cranking corrections and Coriolis-antipairing effects, as well as the corrections due to quadrupole centrifugal stretching, are all effectively included in the stretching term expressible in terms of a "generalized stretching variable" t. A basic set of equations is derived by writing the nuclear moment of inertia $\mathcal{I}$ as a function of t as $\mathcal{I}_0 f(t)$. It is shown that practically all the successful two- and three-parameter models advanced during the past few years can be obtained from this basic set of equations through appropriate choices for $f(t)$. For example, a linear cutoff in the Taylor's expansion for $f(t)$ gives the equations of the variable moment-of-inertia model. The interrelationships, and the correspondence of respective parameters, in various models are established. Further, it is shown that, for models in which $f(t)$ is also an explicit function of the nuclear spin I, the effective moment of inertia for excited states is renormalized from the value $\mathcal{I}_0 f(t)$ by an additional spin-dependent term.

1. INTRODUCTION

During recent years the development of high-precision detection equipment coupled with the progress in heavy-ion reactions has resulted in identification of high-angular-momentum states in ground-band sequence of levels in practically all even-even nuclei removed from the magic regions. This, in turn, has led to a closer scrutiny of the pronounced deviations from the $I(I+1)$ dependence of the spacings between the energy levels of such bands. Attempts aimed at a satisfactory description of these energy levels have been made over the years in the form of several two-parameter[1-10] and lately three-parameter[11-14] models based on semiempirical, phenomenological, or semimicroscopic considerations. Judged by the "internal" criterion of looking at the rms deviations of the excitation energies calculated by employing the model parameters evaluated through a least-squares fit to *all* the known energy levels in comparison with the relative uncertainties of the experimental data, many of these models appear to offer practically the same degree of acceptability.

In microscopic theories these deviations from the rigid-rotator formula are attributed mainly to inclusion of higher-order cranking effects, centrifugal-stretching effects, and Coriolis-antipairing (CAP) effects.[15-23] The question has been raised[23] as to "why the different parametrizations, taking into account only one specific correction term among others, are able to account for the deviations from the $I(I+1)$ rule." While trying to examine this question, Ma and Rasmussen[23] arrived at the conclusion that if the direct coupling between the centrifugal-stretching effect and the CAP effect is weak, then the fourth-order cranking correction and the CAP effect can be treated as modes of a generalized vibration in the same way as the centrifugal stretching. Thus the problem reduces to a several-mode vibration-rotation problem and the energy expression (we use the unit $\hbar=1$) can be written as

$$E_I=\frac{I(I+1)}{2\mathcal{I}(x_i)}+\frac{1}{2}\sum_{i=1}^{4} C_i(x_i-x_{i0})^2 \tag{1}$$

with the variational condition

$$\frac{\partial E_I}{\partial x_i}=0, \quad i=1,\ 2,\ 3,\ 4 \text{ at a fixed } I, \tag{2}$$

where

$$x_i\equiv\{x_1,\ x_2,\ x_3,\ x_4\}\equiv\{\beta,\ \nu_p,\ \nu_n,\ \omega^2\} \tag{3}$$

represent various parameters on which the moment of inertia $\mathcal{I}$ depends; β represents quadrupole deformation of the nucleus, ν_p and ν_n represent pairing parameters for protons and neutrons, respectively, ω is the angular velocity, and x_{i0} are the values of x_i in the ground state ($I=0$) with $x_{40}=0$. Constants C_i are

$$C_i\equiv\{C_1,\ C_2,\ C_3,\ C_4\}\equiv\{C_\beta,\ C_{\nu_p},\ C_{\nu_n},\ C_{\omega^2}\} \tag{4}$$

and represent, respectively, spring constant, pairing stiffness for protons and neutrons, respectively, and adiabatic parameter appearing in cranking-model formulation.

Ma and Rasmussen[23] showed that *Eqs. (1) and (2) may be reduced by algebraic substitution to a* *one-dimensional equation such that the total energy of the system (rotating nucleus) can be expressed in a form resembling that of a centrifugal stretching model*[1,2,4] with the provision that the correction term in the stretching model is looked upon in a much broader sense than just being due to quadrupole centrifugal stretching; it effectively includes the CAP and the higher-order cranking effects, as well, such that the problem is in effect reduced to one mode only expressible in terms of the "generalized stretching variable."

In this paper we develop this generalized approach and discuss its applications. Section 2 includes an account of our mathematical formulation leading to the basic set of energy equations of our unified description. This is done by writing down the dependence of the moment of inertia as $\mathcal{I} = \mathcal{I}_0 f(t)$, where $\mathcal{I}_0$ is the ground-state moment of inertia, and the function $f(t)$ is a function of the "generalized stretching variable" t. In the basic set of equations the functional dependence $f(t)$ of the moment of inertia is left undefined. In the following two sections it is shown how specific choices for $f(t)$ lead to the mathematical expressions of practically every "successful" two- and three-parameter model developed during the past few years for description of the energy levels in the ground-state bands of even-even nuclei. In particular it is shown that a linear cutoff in the Taylor's expansion for $f(t)$ gives the equations for the "variable-moment-of-inertia" (VMI) model, as well as those in the two-parameter cranking model of Harris (Sec. 3B 1). A slight modification of the above choice of $f(t)$ leads to the results of the centrifugal-stretching (CS) model (Sec. 3B 2). In these cases the energy is given through two parametric equations. Other models, which explicitly retain $I(I+1)$ dependence in analytical expressions for energy, are derivable by choosing $f(t)$ to be explicitly dependent on the angular momentum I as well (Sec. 3C). These include, in addition to the earlier Bohr-Mottelson two-parameter formula including $I^2(I+1)^2$ term, the semiempirical approaches of Ejiri (Sec. 3C 2), Sood (Sec. 3C 3), Holmberg-Lipas (Sec. 3C 4), semimicroscopic result of Warke-Khadkikar (Sec. 3C 5), and nuclear-softness model of Gupta (Sec. 3C 6). The concluding subsection (3D) in Sec. 3 describes certain other choices for $f(t)$ which result in only the limiting cases and hence not of practical interest. Section 4 follows the same procedure in arriving at the formulas of the various three- (and more-) parameter models through suitable choices for $f(t, I)$. These include the extension of the parametric equations of the stretching and cranking models (Sec. 4A), the $I(I+1)$ power series expansion (Sec. 4B 1), the "shape-fluctuation" model of Satpathy and Satpathy, and also the "anharmonic-vibration" model of Das, Dreizler, and Klein (Sec. 4B 2) and "nuclear-softness" model (Sec. 4B 3). The last subsection (4C) points out an extension of the generalized model by inclusion of next-order correction term in the potential energy. In Sec. 5 we bring out an essentially unexplored, but significantly important, feature of all such formulations based on the relation $\mathcal{I} = \mathcal{I}_0 f(t)$. Naïvely, one assumes $\mathcal{I}$ to denote the moment of inertia. However, as mentioned above, most of the models involve an explicit angular momentum dependence of f as well; in all such cases the effective moment of inertia also includes, in addition to $\mathcal{I} = \mathcal{I}_0 f$, a term arising from the explicit dependence of f on I. This modified definition of the moment of inertia may lead to a straightforward description of certain otherwise puzzling features of rotational spectra; e.g., negative isomer shifts, back-bending behavior in $\mathcal{I} - \omega^2$ plot, etc., which will be discussed in a separate communication.

2. MATHEMATICAL FORMULATION AND BASIC SET OF EQUATIONS

In view of the considerations outlined above the energy of a rotational state in the generalized vibration-rotation concept can be written as

$$E_I = \frac{I(I+1)}{2\mathcal{I}(s_I)} + \tfrac{1}{2} C' (s_I - s_0)^2 , \qquad (5)$$

where $\mathcal{I}(s_I)$ is the moment of inertia depending on a "generalized variable" s_I (ground-state value being s_0). Constant C' depends on the C_i s and has the dimensions of energy. The equilibrium value of the variable s_I for each spin value I is determined by the minimization condition

$$\frac{\partial E_I}{\partial s_I} = 0 . \qquad (6)$$

The generalized stretching variable t_I, which is a measure of the deviation of generalized variable s_I from its ground-state value, is accordingly defined as

$$t_I = s_I - s_0 . \qquad (7)$$

In what follows we drop the spin subscript I. Substitution of (7) in (5) gives

$$E = \frac{I(I+1)}{2\mathcal{I}(t)} + \tfrac{1}{2} C' t^2 \qquad (8)$$

and the application of minimization condition (6) gives the parametric equations

$$E = \frac{I(I+1)}{2\mathcal{I}} \left(1 + t \frac{\partial \mathcal{I}}{\partial t} / 2\mathcal{I}\right) , \qquad (9)$$

$$C' t = \frac{I(I+1)}{2\mathcal{I}^2} \frac{\partial \mathcal{I}}{\partial t} . \qquad (10)$$

At this stage we shall introduce the semiclassical relation between moment of inertia $\mathcal{J}$, angular momentum I, and angular velocity ω:

$$\omega\mathcal{J} = [I(I+1)]^{1/2} \tag{11}$$

which enables us to write Eq. (10) as follows:

$$\frac{t}{\partial\mathcal{J}/\partial t} = \frac{\omega^2}{2C'} . \tag{12}$$

One may explicitly write the dependence of moment of inertia on t as follows:

$$\mathcal{J} = \mathcal{J}_0 f(t) \tag{13}$$

such that

$$f(t)|_{t=0} = 1 , \tag{14}$$

gives $\mathcal{J}_0$ as the ground-state moment of inertia. Equations (9), (10), and (12) can be rewritten in terms of $f(t)$ as follows:

$$E = \frac{I(I+1)}{2\mathcal{J}_0}\frac{(2f + t\,\partial f/\partial t)}{2f^2} , \tag{15}$$

$$\frac{tf^2}{\partial f/\partial t} = \frac{I(I+1)}{2C'\mathcal{J}_0} , \tag{16}$$

and

$$\frac{t}{\partial f/\partial t} = \frac{\mathcal{J}_0\omega^2}{2C'} . \tag{17}$$

Equations (15)–(17) constitute the basic set of equations of our formulation. We show in the following that practically all the successful phenomenological energy expressions can be obtained by taking suitable choices of the function $f(t)$.

3. TWO-PARAMETER APPROACHES

A. Possible Choice of $f(t)$ and Its Special Cases

In the absence of any knowledge about the functional dependence of moment of inertia on the generalized stretching variable t, we make Taylor's expansion for $\mathcal{J}(t)$ and get

$$\mathcal{J}(t) = \mathcal{J}_0 + t\left(\frac{\partial\mathcal{J}}{\partial t}\right)_{t=0} + \frac{t^2}{2!}\left(\frac{\partial^2\mathcal{J}}{\partial t^2}\right)_{t=0} + \cdots$$

$$= \mathcal{J}_0(1 + b_1t + b_2t^2 + b_3t^3 + \cdots) , \tag{18}$$

where

$$b_n = \frac{1}{\mathcal{J}_0}\frac{1}{n!}\left(\frac{\partial^n\mathcal{J}}{\partial t^n}\right)_{t=0} .$$

Thus

$$f(t) = 1 + b_1t + b_2t^2 + b_3t^3 + \cdots . \tag{19}$$

Retaining the first two terms in the expansion (19) we have the linear dependence for the moment of inertia on t. However, we take the following somewhat more general dependence of $f(t)$ on t:

$$f(t) = (1 + b_1 t)^n , \tag{20}$$

where n is some positive integer or fraction. A positive n is required to ensure that the moment of inertia increases with spin.

It may appear that we have a three-parameter ($\mathcal{J}_0$, C', and b_1) formulation for energy. However, the structure of our basic equation (8) is such that we can always scale the stretching variable t, i.e., if we define a new stretching variable

$$u = b_1 t , \tag{21}$$

this will not change the structure of any of the equations [(8), (15)–(17)]. The scale factor b_1 can always be absorbed in the parameter C' by redefining it as

$$\frac{C'}{b_1^2} \to C . \tag{22}$$

This is always permissible so long as the parameters are "free" which are determined by fitting the experimental energy levels. Thus we write

$$\mathcal{J} = \mathcal{J}_0 f(u) , \tag{23}$$

where

$$f(u) = (1+u)^n . \tag{24}$$

With this choice energy equations are

$$E = \frac{I(I+1)}{2\mathcal{J}_0}\left[\frac{1 + u(1+\frac{1}{2}n)}{(1+u)^{n+1}}\right] \tag{25}$$

and

$$u(1+u)^{n+1} = \frac{nI(I+1)}{2\mathcal{J}_0 C} . \tag{26}$$

In principle, u may be eliminated leaving one equation for energy as a function of spin I. The solutions depend parametrically on the coefficients $\mathcal{J}_0$ and C. Combining equations (25)–(26) we get an alternative expression for energy:

$$E = \frac{Cu}{n}\,[1 + u(1+\tfrac{1}{2}n)] . \tag{27}$$

Relation (17) between the stretching variable and frequency becomes

$$\frac{u}{(1+u)^{n-1}} = \frac{n\mathcal{J}_0\omega^2}{2C} . \tag{28}$$

The relative increase of moment of inertia with angular momentum I can be derived from Eqs. (24) and (26):

$$\mathcal{J}^{-1}\frac{d\mathcal{J}}{dI} = f^{-1}\frac{\partial f}{\partial u}\frac{\partial u}{\partial I}$$

$$= \frac{(2I+1)n^2}{(1+u)^{n+1}(1+2u+nu)(2\mathcal{J}_0 C)} \tag{29}$$

and for the particular case $I=0$ we get

$$\left[\mathcal{I}^{-1}\left(\frac{d\mathcal{I}}{dI}\right)\right]_{I=0}=\frac{n^2}{2\mathcal{I}_0 C}=n^2\sigma\,, \tag{30}$$

where we have defined the "softness" parameter

$$\sigma=\frac{1}{2\mathcal{I}_0 C}\,. \tag{31}$$

Equation (26), which determines stretching for each spin state I for a given set of parameters $\mathcal{I}_0$ and C, can also be written in terms of the softness parameter as follows:

$$u(1+u)^{n+1}=\sigma n I(I+1)\,. \tag{32}$$

B. Range of Validity of the Model

The two limits of this generalized stretching model are obtained by considering the limiting cases for the softness parameter σ.

In the adiabatic limit, we have $\sigma=0$ and hence $u=0$ through Eq. (32). In this limit, energy expression (25) assumes the simple form

$$E_I(\sigma\to 0)=I(I+1)/2\mathcal{I}_0\,, \tag{33}$$

giving the energy ratios for a rigid rotator

$$R_I(\sigma\to 0)=E_I/E_2|_{\sigma\to 0}=\tfrac{1}{6}I(I+1)\,. \tag{34}$$

On the other hand, in the limit of very soft nuclei, we have $\sigma\to\infty$, and from Eq. (32) we obtain

$$u\sim[n\sigma I(I+1)]^{1/n+2}\,. \tag{35}$$

Equation (25) then becomes

$$E_I(\sigma\to\infty)=\frac{I(I+1)}{2\mathcal{I}}(1+\tfrac{1}{2}n),$$

which leads to the following expression for the energy ratios R_I in this limit:

$$R_I(\sigma\to\infty)=[\tfrac{1}{6}I(I+1)]^{2/n+2}\,. \tag{36}$$

Equations (34) and (36) define the range of validity of Eqs. (25) and (26) in terms of the energy ratios as follows:

$$[\tfrac{1}{6}I(I+1)]^{2/n+2}\leq R_I\leq\tfrac{1}{6}I(I+1)\,, \tag{37}$$

which in the case $I=4$ becomes

$$(\tfrac{10}{3})^{2/n+2}\leq R_4\leq\tfrac{10}{3}\,. \tag{38}$$

One can see that the lower limit for R_4 depends on the value of n in Eq. (24). At this stage we may mention that Draper, McCauley, and Smith[10] employed a general relation $\mathcal{I}\approx\beta^n$, where n is a noninteger parameter to be determined for each nucleus. Their results show that the best n values range from 0.7 to 2.8. In all the cases, the numerical least-squares fit to the experimental energy levels are good, showing that the energy levels are not very sensitive to the value of n in the β^n dependence of the moment of inertia [this is clearly related to our choice $f(u)=(1+u)^n$] provided the other parameters are adjusted accordingly.

Treating n as a free parameter to be determined for each nucleus, we essentially have a three-parameter description. However, two of the successful phenomenological models adopt constant values of n equal to 1 and 2, respectively, and we discuss them below.

1. $n=1$ Case and VMI and Harris Models

Defining relation for the moment of inertia for $n=1$ becomes

$$\mathcal{I}=\mathcal{I}_0(1+u)\,. \tag{39}$$

The corresponding energy equations can be obtained by putting $n=1$ in Eqs. (25)–(27). We now show that the equations thus obtained are VMI model[9] equations written in simplified form and are also equivalent to Harris equations.[3] Equations (25) and (26) with $n=1$ and substitution for u from Eq. (39) become

$$\mathcal{I}^3-\mathcal{I}^2\mathcal{I}_0-\frac{I(I+1)\mathcal{I}_0^{\,2}}{2C}=0\,, \tag{40}$$

and

$$E=\frac{I(I+1)}{2\mathcal{I}}\left(1+\frac{I(I+1)\mathcal{I}_0^{\,2}}{4\mathcal{I}^3 C}\right)\,. \tag{41}$$

Similarly Eqs. (26) and (27) become after substitution for u from Eq. (28)

$$E=\tfrac{1}{2}\omega^2\left(\mathcal{I}_0+\frac{3\omega^2\mathcal{I}_0^{\,2}}{4C}\right)\,, \tag{42}$$

and

$$[I(I+1)]^{1/2}=\omega\left(\mathcal{I}_0+\frac{\omega^2\mathcal{I}_0^{\,2}}{2C}\right)\,. \tag{43}$$

Comparison of Eqs. (40) and (41) with Eqs. (8) and (9) of Ref. 9 and of Eq. (42) and (43) with two-parameter equations (24′) and (25′) of Ref. 3 shows their exact equivalence with the following correspondence of parameters:

Our description:	$\mathcal{I}_0$	$C/\mathcal{I}_0^{\,2}$	σ	u
VMI model:	$\mathcal{I}_0$	C	σ	$(\mathcal{I}-\mathcal{I}_0)/\mathcal{I}_0$
Harris model:	$\mathcal{I}_0$	$1/(4C_H)$	$\cdots$	$2C_H\omega^2/\mathcal{I}_0$

Equation (38), which defines the range of validity of the energy expressions, gives for $n=1$ the following interval for R_4:

$$(\tfrac{10}{3})^{2/3}\leq R_4\leq\tfrac{10}{3}\,. \tag{44}$$

The limitation of the above formulas is easily seen in our approach. The linear approximation for $f(u)$ cannot be a satisfactory choice for the cases wherein u is large (i.e., when the relative change in the moment of inertia with respect to that in the ground state is large). In such cases one should consider next term in the Taylor's expansion (18). Now this situation arises at high-spin states for deformed nuclei; while for "soft" nuclei, this is true even at low-spin values. This explains why in the VMI model the fit is not as good for the transitional and nearly spherical nuclei as that for the well-deformed nuclei; even for the latter the deviations are large at high-spin values.

2. n=2 Case and CS Model

This choice of n gives centrifugal stretching (CS) model of Moszkowski[4] and Sood.[8] Defining relation for the moment of inertia is

$$\mathcal{I}=\mathcal{I}_0(1+u)^2\,. \tag{45}$$

Energy equations obtained by putting $n=2$ in Eqs. (25)–(27) are equivalent to those obtained in the CS model with the following correspondence of parameters:

Our description:	$\mathcal{I}_0$	$C\mathcal{I}_0$	σ	u
CS model:	$\mathcal{I}_0$	D	$1/(2D)$	$v/(1-v)$

We next show[8] that the expressions given by the CS model (our description: $n=2$) are equivalent in the "restricted" sense to those given by Harris model. Equations (26)–(28) with $n=2$ can be put in the form:

$$E=\tfrac{1}{2}\mathcal{I}_0\omega^2\left[1+\sum_{n=1}^{\infty}(1+2n)\omega^{2n}\left(\frac{\mathcal{I}_0}{C}\right)^n\right], \tag{46}$$

and

$$[I(I+1)]^{1/2}=\mathcal{I}_0\omega\sum_{n=0}^{\infty}(1+n)\,\omega^{2n}\left(\frac{\mathcal{I}_0}{C}\right)^n. \tag{47}$$

Comparison of these equations with Eqs. (24′)–(25′) of generalized Harris model[3] shows their equivalence with the following correspondence:

Our:	$\mathcal{I}_0$	$\mathcal{I}_0\left(\frac{\mathcal{I}_0}{C}\right)$	$\mathcal{I}_0\left(\frac{\mathcal{I}_0}{C}\right)^2$	$\mathcal{I}_0\left(\frac{\mathcal{I}_0}{C}\right)^3$	···
Harris:	$\mathcal{I}_0$	C_H	D	F	···

It is clear from the above what we mean by "restricted equivalence." In our expressions, terms to all orders in ω^2 are included in a two-parameter description, whereas the Harris formulation needs an infinite parameter set for the purpose.

For $n=2$ the range of validity (38) of the energy equations is given by

$$(\tfrac{10}{3})^{1/2}\leqslant R_4\leqslant\tfrac{10}{3}\,. \tag{48}$$

The interval defined by Eq. (48) is thus larger than that for the VMI model. This deficiency of the VMI model has later been removed[9] by allowing the ground-state moment of inertia to have negative values.

C. Other Acceptable Forms for $f(t)$ and Corresponding Energy Expressions

Function $f(t)$ appearing in the defining relation (13) for the moment of inertia is normally expected to satisfy the following two basic requirements:

$$\begin{aligned}&(1)\ [f(t)]_{t=0}=1;\\&(2)\ \text{it is an increasing function of } t.\end{aligned} \tag{49}$$

Earlier we have discussed how in the absence of any criterion for the choice of $f(t)$ one may use the Taylor's expansion and obtain good agreement with experimental data by retaining the linear term (and its slightly generalized form). Now we investigate some other forms of $f(t)$ which are consistent with Eq. (49) and which give some of the other successful phenomenological models.

1. Bohr-Mottelson Formula

Consider the choice

$$f(t)=(1-at)^{-1}, \tag{50}$$

where a is some parameter which may, or may not, be spin dependent. Equation (16) which determines t for a given spin I becomes

$$t=\frac{aI(I+1)}{2\mathcal{I}_0C'} \tag{51}$$

and the energy expression is no longer parametric. In fact, one gets from Eq. (15)

$$\begin{aligned}E&=\frac{I(I+1)}{2\mathcal{I}_0}\,(1-\tfrac{1}{2}at)\\&=\frac{I(I+1)}{2\mathcal{I}_0}-\frac{a^2I^2(I+1)^2}{8\mathcal{I}_0{}^2C'}\,.\end{aligned} \tag{52}$$

Equation (52) is the well-known equation

$$E=AI(I+1)-BI^2(I+1)^2 \tag{53}$$

of Bohr and Mottelson with the correspondence of parameters

$$A=\frac{1}{2\mathcal{I}_0},\quad B=\frac{a^2}{8\mathcal{I}_0{}^2C'}\,. \tag{54}$$

This equation constitutes the earliest prescription for explaining deviations from rigid-rotator spectrum. The inadequacy of this equation for high-spin states is easily seen from Eq. (51) which brings in too rapid an increase in stretching parameter with spin resulting in an overcorrection. *Only way to damp this rise is to assume that "a" is spin dependent.* Now several choices are possible and a few of them leading to existing successful phenomenological models are discussed below.

2. *Ejiri Formula*

The choice

$$a^2 = \frac{1}{(I+1)^2} \tag{55}$$

gives the following expression for the energy

$$E = \frac{I(I+1)}{2\mathcal{G}_0} - \frac{I^2}{8\mathcal{G}_0^2 C'}, \tag{56}$$

which can be identified with Ejiri formula[5]

$$E = kI(I+1) + qI \tag{57}$$

with the correspondence of parameters

$$k + q = \frac{1}{2\mathcal{G}_0}, \quad q = \frac{1}{8\mathcal{G}_0^2 C'}. \tag{58}$$

3. *Sood Formula*

The choice

$$a^2 = \frac{8C'^2\mathcal{G}_0^2}{1 + N(2C'\mathcal{G}_0)I(I+1)} \tag{59}$$

leads to the following expression for energy

$$E = AI(I+1)\left(1 - \frac{(B/A)I(I+1)}{1 + N(B/A)I(I+1)}\right), \tag{60}$$

where

$$A = \frac{1}{2\mathcal{G}_0}, \quad \frac{B}{A} = 2\mathcal{G}_0 C'. \tag{61}$$

Equation (60) is the semiempirical formula of Sood[6] obtained by summing an infinite series in $I(I+1)$ and is one of the most successful descriptions for nuclei in the deformed region with the choice $N = 2.85 - 0.05I$.

4. *Holmberg-Lipas Formula*

Next we take the following spin dependence for a^2:

$$a^2 = \frac{8\mathcal{G}_0^2 C'^2}{I^2(I+1)^2}\left[1 + \frac{I(I+1)}{2\mathcal{G}_0 C'} - \left(1 + \frac{I(I+1)}{\mathcal{G}_0 C'}\right)^{1/2}\right]. \tag{62}$$

It may be noted that, for $I=0$, a^2 has the limiting value equal to unity. Energy expression (52) with the choice (62) for a^2 becomes

$$E = C'\left[\left(1 + \frac{I(I+1)}{\mathcal{G}_0 C'}\right)^{1/2} - 1\right] \tag{63}$$

which may be compared with the energy equation of Holmberg and Lipas[7]

$$E = d\{[1 + bI(I+1)]^{1/2} - 1\}, \tag{64}$$

with the identification of parameters

$$d = C', \quad b = \frac{1}{\mathcal{G}_0 C'}. \tag{65}$$

5. *Warke-Khadkikar Formula*

Next we take the following functional dependence

$$f(t) = (1 - at)^{-2}. \tag{66}$$

From Eq. (16) we get after simplification

$$t = \frac{aI(I+1)/C'\mathcal{G}_0}{1 + a^2 I(I+1)/C'\mathcal{G}_0}. \tag{67}$$

Energy expression (15) is given by

$$E = \frac{I(I+1)}{2\mathcal{G}_0}(1 - at) = \frac{AI(I+1)}{1 + (B/A)I(I+1)}, \tag{68}$$

where

$$A = \frac{1}{2\mathcal{G}_0}, \quad B = \frac{a^2}{2C'\mathcal{G}_0^2}. \tag{69}$$

Equation (68) is the one derived by Warke and Khadkikar[22] from a microscopic approach.

6. *Gupta's Nuclear-Softness Model*

In the relation (66) for $f(t)$, assuming a simple spin dependence for a^2 of the type

$$a^2 = \frac{1}{I+1}, \tag{70}$$

results in the following energy expression:

$$E = \frac{AI(I+1)}{1 + \sigma_1 I}, \tag{71}$$

where

$$A = \frac{1}{2\mathcal{G}_0}, \quad \sigma_1 = \frac{1}{\mathcal{G}_0 C'}. \tag{72}$$

Equation (71) is the two-parameter nuclear-softness model NS(2) of Gupta.[14]

D. Nonacceptable Choices for $f(t)$

Although several choices of $f(t)$ can be made satisfying the requirements (49), not all of them lead to "physical" spectra of practical interest. This can be seen by taking the following functional form:

$$f(t) = (1 - at^2)^{-1}. \tag{73}$$

Substituting Eq. (73) in Eq. (15) results in the energy expression for the limiting rigid-rotator case only:

$$E=\frac{I(I+1)}{2\mathcal{I}_0}. \tag{74}$$

This strange result can be understood starting from the basic equation (16). With the choice (73) and the application of minimization condition we get the following expression for the determination of t:

$$t\left(\frac{aI(I+1)}{\mathcal{I}_0}-C'\right)=0 \tag{75}$$

which, for a constant C', can be satisfied only if the stretching variable is zero for all spins, i.e., there is no variation in the moment of inertia with spin and hence the occurrence of the rigid-rotator spectrum.

Another such "unphysical" spectrum results from the choice

$$f(t)=1+at^2. \tag{76}$$

Energy expression (15) becomes

$$E=\frac{I(I+1)}{2\mathcal{I}_0}\left(\frac{1+2at^2}{(1+at^2)^2}\right), \tag{77}$$

where t is to be determined from Eq. (16), which becomes

$$(1+at^2)^2=\frac{aI(I+1)}{C'\mathcal{I}_0},\quad I\neq 0. \tag{78}$$

Equations (77) and (78) can be combined to give the following equation for energy:

$$E=A[I(I+1)]^{1/2}-B,\quad I\neq 0$$
$$=0,\qquad I=0, \tag{79}$$

where we have put

$$A=\left(\frac{C'}{a\mathcal{I}_0}\right)^{1/2},\quad B=\frac{C'}{2a}. \tag{80}$$

Equation (79) also corresponds to a limiting spectrum only and cannot describe energies of a wide range of nuclei, because it implies that the ratio $(E_I-E_2)/(E_{I+2}-E_2)$ is the same for all nuclei under consideration which certainly is not true.

4. THREE-PARAMETER APPROACHES

In Sec. 3 we have seen how several forms for $f(t)$ in the defining relation (13) for the moment of inertia give various two-parameter descriptions for the energies of ground-state band levels in even-even nuclei. Even if we had a two-parameter description which gave agreement to the same accuracy for nuclei in various regions (e.g. well-deformed, transitional, nearly spherical, etc.) and for very high spin states as well – which certainly has not been achieved so far – there is an inherent shortcoming with all the two-parameter approaches. For all of these models, energy ratios E_I/E_2 depend on one parameter only; the other parameter simply fixes the energy scale. Thus for energy ratios we have a one-parameter description in each case and a plot of the energy ratios E_I/E_2 as functions of E_4/E_2 gives smooth curves in each of these descriptions. However, the corresponding experimental points do not fall on a single smooth curve. Thus one cannot simultaneously obtain exact numerical agreement for all nuclei (having same E_4/E_2 but not the same E_6/E_2, E_8/E_2, ..., etc.) using any two-parameter description. One has to introduce further parameters (extra freedom) to achieve this. We now discuss the various three-parameter approaches in the following.

A. Extension of VMI and CS Equations

We have seen earlier that a linear cutoff in the Taylor's expansion for $f(t)$ gives the equations for VMI model which constitutes one of the more successful descriptions available so far. It is therefore natural to expect that the inclusion of the next-order term in the expansion may improve the quality of fits. Thus we take

$$f(t)=1+b_1t+b_2t^2. \tag{81}$$

Since b_1 and b_2 are spin independent, we, as before, scale the stretching variable t [Eq. (21)] and take the function $f(u)$ [Eq. (23)] to be of the form

$$f(u)=1+u+bu^2. \tag{82}$$

Thus $b=b_2/b_1^2$, and instead of C' we shall have C in our equations. Now with the choice (82) energy expression (15) becomes

$$E=\frac{I(I+1)}{2\mathcal{I}_0}\left[\frac{2+3u+4bu^2}{2(1+u+bu^2)^2}\right], \tag{83}$$

where u is to be determined from the equation

$$\frac{u(1+u+bu^2)^2}{1+2bu}=\frac{I(I+1)}{2\mathcal{I}_0 C}. \tag{84}$$

Using Eqs. (82)–(83) we get the relative increase of moment of inertia with angular momentum

$$\mathcal{I}^{-1}\frac{d\mathcal{I}}{dI}=f^{-1}\frac{\partial f}{\partial u}\frac{\partial u}{\partial I}$$
$$=\frac{1}{2\mathcal{I}_0C}\,\frac{(1+2I)(1+2bu)^3}{(1+u+bu^2)^2(1+3u+9bu^2+8b^2u^3)}, \tag{85}$$

which for the particular case $I=0$ becomes

$$\left(\mathcal{I}^{-1}\frac{\partial\mathcal{I}}{\partial I}\right)_{I=0}=\frac{1}{2C\mathcal{I}_0}=\sigma, \tag{86}$$

where σ is the softness parameter defined earlier through Eq. (31).

Thus energy is given by the parametric equations (83)–(84) in terms of three parameters $\mathcal{G}_0$, C, and b or equivalently $\mathcal{G}_0$, σ, and b. For a given nucleus (characterized by the set $\mathcal{G}_0$, σ, and b) one can calculate stretching variable u for each spin state I making use of Eq. (84) and the excitation energies are then determined through Eq. (83). The ground-state moment-of-inertia parameter $\mathcal{G}_0$ serves as a scale factor, but now the energy ratios depend on the remaining two parameters b and σ. Thus nuclei having same E_4/E_2 but different ratios E_I/E_2 $(I>4)$ will be characterized by different sets of parameters b and σ.

1. Harris Formula Including Higher-Order Corrections

With the choice (82) for $f(u)$ we get from Eq. (17) the following relation between the generalized stretching u and angular velocity ω:

$$\frac{u}{1+2bu}=\frac{\mathcal{G}_0\omega^2}{2C}, \tag{87}$$

which can be rewritten in the form

$$u=\frac{\mathcal{G}_0\omega^2}{2(C-b\mathcal{G}_0\omega^2)}. \tag{88}$$

Energy equations (83) and (84) using Eq. (88) become after simplification

$$E=\frac{\mathcal{G}_0\omega^2}{2}\left[1+\sum_{n=1}^{\infty}\frac{1+2n}{4b}\,\omega^{2n}\left(\frac{b\mathcal{G}_0}{C}\right)^n\right] \tag{89}$$

and

$$[I(I+1)]^{1/2}=\mathcal{G}_0\omega\left[1+\sum_{n=1}^{\infty}\frac{1+n}{4b}\,\omega^{2n}\left(\frac{b\mathcal{G}_0}{C}\right)^n\right]. \tag{90}$$

For $b=0$ one gets only two terms [Eqs. (42) and (43)] instead of the infinite series [Eqs. (89) and (90)]. These expressions may now be compared with those obtained by Harris [Eqs. (24′) and (25′) of Ref. (3)]. It is seen that the equations in the two descriptions are identical with the following correspondence of the respective parameters:

Extended model:	$\mathcal{G}_0$	$\frac{\mathcal{G}_0^2}{4C}$	$\frac{\mathcal{G}_0}{4b}\left(\frac{b\mathcal{G}_0}{C}\right)^2$	$\frac{\mathcal{G}_0}{4b}\left(\frac{b\mathcal{G}_0}{C}\right)^3$	$\cdots$
Harris model:	$\mathcal{G}_0$	C_H	D	F	$\cdots$

Thus our expressions include terms up to all powers in ω^2, but ratio of various successive higher-order terms $(\omega^{2n},\ n\geq 3)$ is simply related to the ratio of first $-(n=1)$ and second $-(n=2)$ order contribution. Considering only the first three terms in the infinite series the two models yield identical description. However, when considering terms to all orders in ω^2, our formulation gives a three-parameter expression, whereas Harris approach presents an infinite parameter set up with the inherent necessity of an arbitrary cutoff.

It is obvious that $b=0$ gives results of VMI model and its equivalent two-parameter Harris model. Furthermore, Eq. (82) with the choice $b=0.25$ becomes

$$f(u)=(1+\tfrac{1}{2}u)^2, \tag{91}$$

which, except for the factor of $\frac{1}{2}$, is the defining relation (45) for $f(u)$ in the CS model. This scale factor of $\frac{1}{2}$ therefore will renormalize the parameter C and σ(which depends on $1/C$) by a factor of 4. Thus the extended equation with the choice

$$\mathcal{G}_0=\mathcal{G}_0^{\rm CS},\quad \sigma=4\sigma^{\rm CS},\quad b=0.25 \tag{92}$$

will reproduce the CS model results.

B. Other Three-Parameter Formulas

In Sec. 3 we have seen that taking suitable spin dependence for the parameter a appearing in the function $f(t)$ leads to various successful two-parameter phenomenological models. We now show that this approach can be extended in the sense that the spin dependence for a containing one (or more) additional parameter results in three (or many) parameters expressions for energy.

1. Bohr-Mottelson Series Formula

Consider the functional dependence (50) for $f(t)$:

$$f(t)=(1-at)^{-1}. \tag{50}$$

Taking

$$a^2=1+bI(I+1) \tag{93}$$

and its substitution in Eq. (52) gives the following equation for energy:

$$E=AI(I+1)-BI^2(I+1)^2+CI^3(I+1)^3, \tag{94}$$

where we have put

$$A=\frac{1}{2\mathcal{G}_0},\quad B=\frac{1}{8\mathcal{G}_0{}^2C'},\quad C=\frac{-b}{8\mathcal{G}_0{}^2C'}. \tag{95}$$

It is obvious that one can get the many-term power series in $I(I+1)$ for energy by taking another power series in $I(I+1)$ for a^2. As shown by Sood[6] such a power series in E may give improved agreement for "nearly rigid" nuclei for which the series may be convergent, but as one goes to "softer" nuclei the corrections overshoot the mark quite rapidly and the calculated results become even worse than those for the rigid rotator. Semiempirical formula (60) of Sood[6] adopted a summation procedure of the alternating power series [extension of Eq. (94)] under the plausible assumption that successive coefficients of higher-order terms bear a constant ratio. It is not difficult to see that the functional form (59) for a^2 is obtained from the obvious extension of Eq. (93) on exactly the same consideration.

2. *Shape-Fluctuation Model and Anharmonic-Vibration Model*

We next consider the following spin dependence for a^2:

$$a^2 = \frac{b+I}{(I+1)^2} \tag{96}$$

in the functional dependence (50) for $f(t)$. Substituting Eq. (96) in Eq. (52) gives the following energy expression:

$$E = \frac{I(I+1)}{2\mathscr{I}_0} - \frac{b}{8\mathscr{I}_0^2 C'} I^2 - \frac{1}{8\mathscr{I}_0^2 C'} I^3 , \tag{97}$$

which can be compared with the energy equation

$$E = B_0 I(I+1) + \Phi' E' I + \Phi' B' I^2 (I+1) \tag{98}$$

in the shape-fluctuation (SF) model of Satpathy and Satpathy[13] with the correspondence of parameters:

$$B_0 = \frac{1}{2\mathscr{I}_0} - \frac{1}{8C'\mathscr{I}_0^2}(b-1) ,$$

$$\Phi' E' = \frac{1}{8C'\mathscr{I}_0^2}(b-1) , \tag{99}$$

$$\Phi' B' = -\frac{1}{8\mathscr{I}_0^2 C'} .$$

Furthermore, Eq. (97) is also equivalent to the energy equation

$$E = d_1 I + d_2 I(I-2) + d_3 I(I-2)(I-4) \tag{100}$$

in the anharmonic-vibration (AV) model of Das, Dreizler, and Klein[12] with the following correspondence of parameters:

$$d_1 = \frac{3}{2\mathscr{I}_0} - \frac{1}{8\mathscr{I}_0^2 C'}(2b+4) ,$$

$$d_2 = \frac{1}{2\mathscr{I}_0} - \frac{1}{8\mathscr{I}_0^2 C'}(b+6) , \tag{101}$$

$$d_3 = -\frac{1}{8\mathscr{I}_0^2 C'} .$$

It is easy to verify that

$$d_1 - 2d_2 + 8d_3 = \frac{1}{2\mathscr{I}_0} . \tag{102}$$

3. *Nuclear-Softness Model*

Consider the choice (66) for $f(t)$. Extending the spin dependence (70) for a^2 as follows:

$$a^2 = \frac{1+s_2 I}{(I+1)} , \tag{103}$$

we get the following energy expression:

$$E = \frac{AI(I+1)}{1+\sigma_1 I + \sigma_2 I^2} , \tag{104}$$

where

$$A = \frac{1}{2\mathscr{I}_0} , \quad \sigma_1 = \frac{1}{\mathscr{I}_0 C'} , \quad \sigma_2 = \frac{s_2}{\mathscr{I}_0 C'} . \tag{105}$$

Equation (104) is the three-parameter nuclear-softness model NS(3) of Gupta.[14] It is easily seen that the many-parameter nuclear-softness model can be obtained by obvious generalization of the spin dependence (103) of a^2.

C. Extension by the Inclusion of Higher-Order Perturbation Term

Another possible three-parameter extension of our "generalized-stretching-model" equations can be made by adding the anharmonic term to the potential energy. Thus we write the energy as

$$E = \frac{I(I+1)}{2\mathscr{I}(t)} + \tfrac{1}{2} C' t^2 + \tfrac{1}{6} D t^3 , \tag{106}$$

with the minimization condition

$$\frac{\partial E}{\partial t} = 0 . \tag{107}$$

The parameter C' must be positive as a condition for stability. As before we explicitly write the dependence of the moment of inertia as

$$\mathscr{I}(t) = \mathscr{I}_0 f(t) . \tag{13}$$

Equation (106) combined with Eqs. (107) and (13) gives a pair of coupled equations:

$$E = \frac{I(I+1)}{2\mathscr{I}_0}\left(\frac{2f + t\,\partial f/\partial t}{2f^2}\right) - \frac{Dt^3}{12} , \tag{108}$$

and

$$\frac{tf^2[1+(D/2C')t]}{\partial f/\partial t} = \frac{I(I+1)}{2C'\mathscr{I}_0} . \tag{109}$$

In principle, t may be eliminated leaving one equation for E depending parametrically on the coefficients $\mathscr{I}_0$, C', and D.

5. EFFECTIVE MOMENT OF INERTIA FROM THE GENERALIZED STRETCHING MODEL

We have seen in earlier sections that taking suitable choices for the function $f(t)$ in the defining relation (13) for the moment of inertia one gets practically equations of *all* the successful phenomenological models of even-even nuclei from the basic set of equations (15)–(17) of the generalized stretching model. It has also been seen that many of the models involve an explicit angular momentum (spin) dependence of f. Thus, in general, we can write Eq. (13) as

$$\mathcal{G} = \mathcal{G}_0 f(t,I)\,. \tag{110}$$

We now show that in such a formulation, the effective moment of inertia $\mathcal{G}^{\text{eff}}$ includes, in addition to $\mathcal{G}_0 f$, a term arising from the explicit functional dependence of f on spin. From the canonical relation for an axially symmetric rotator

$$\omega = \frac{dE}{d[I(I+1)]^{1/2}}\,, \tag{111}$$

and Eq. (11) which gives the relation between moment of inertia $\mathcal{G}$, angular velocity ω, and angular momentum I, we obtain the following expressions:

$$\mathcal{G}^{\text{eff}} = \frac{1}{2}\left[\frac{dE}{dI(I+1)}\right]^{-1} \tag{112}$$

and

$$\omega^2 = 4I(I+1)\left(\frac{dE}{dI(I+1)}\right)^2. \tag{113}$$

Thus effective moment of inertia and angular velocity are both related to the spin derivative of the energy. Let us use Eq. (112) to write down the expressions for the effective moment of inertia in the generalized stretching model. Differentiating Eqs. (15) and (16) with respect to $I(I+1)$ and combining the resulting equations we get

$$\frac{dE}{dI(I+1)} = \frac{1}{2\mathcal{G}^{\text{eff}}} = \frac{1}{2\mathcal{G}_0 f} + \frac{I(I+1)}{2\mathcal{G}_0}\,\frac{\partial}{\partial I(I+1)}\left(\frac{1}{f(t,I)}\right). \tag{114}$$

In second term on the right in Eq. (114) differentiation is done for the explicit spin dependence of f (which in our formulation appears through the spin dependence of the parameter a). Equation (114) shows that the relation $\mathcal{G} = \mathcal{G}_0 f(t)$ gives the complete spin dependence for the moment of inertia for only those models for which the function $f(t)$ does not contain explicit spin dependence, e.g., the VMI and the CS models, etc. For other models (having explicit spin dependence in f) an extra term contributes to the moment of inertia, thus modifying it. It may be noted that the ground-state moment of inertia $\mathcal{G}_0$ is not effected by this "renormalization." Equation (114) can also be written as

$$\frac{dE}{dI(I+1)} = \frac{1}{2\mathcal{G}^{\text{eff}}} = \frac{\partial}{\partial I(I+1)}\left(\frac{I(I+1)}{2\mathcal{G}_0 f(t,I)}\right) \tag{115}$$

which clearly brings out the distinction between effective (renormalized) moment of inertia $\mathcal{G}^{\text{eff}}$ and $\mathcal{G} = \mathcal{G}_0 f$. It is to be noted that *in comparing the inertial parameters of different models one should compare the respective effective moment of inertia and not simply* $\mathcal{G}_0 f$.

We illustrate the use of Eq. (115) for one particular case, say, Ejiri model. From Eqs. (50), (51), and (55) we get

$$\frac{1}{2\mathcal{G}^{\text{eff}}} = \frac{\partial}{\partial I(I+1)}\left(\frac{I(I+1)(1-t/I+1)}{2\mathcal{G}_0}\right)$$
$$= \frac{1}{2\mathcal{G}_0} - \frac{t}{2\mathcal{G}_0(2I+1)}\,. \tag{116}$$

Substitution for t from Eqs. (51) and (55) and making use of the correspondence (58) of parameters we get

$$\frac{1}{2\mathcal{G}^{\text{eff}}} = (k+q) - \frac{2Iq}{(2I+1)}\,, \tag{117}$$

such that

$$\frac{1}{2\mathcal{G}_0} = k+q\,. \tag{118}$$

On the other hand, using Eq. (110) we obtain

$$\frac{1}{2\mathcal{G}} = (k+q) - \frac{2Iq}{(I+1)}\,. \tag{119}$$

Thus we find that, while the ground-state moment of inertia $\mathcal{G}_0$ remains the same whether we use Eq. (110) or Eq. (115), the moment of inertia for higher-spin states is not the same from the two equations. The effective moment-of-inertia expression includes the additional term specified in Eq. (114).

6. CONCLUSIONS

In this paper we have presented a unified formulation which reduces to any particular two-, three- (or more) parameter model with an appropriate choice of the functional dependence $f(t,I)$ of the nuclear moment of inertia $\mathcal{G}$. The interrelationships, as well as the correspondence of model parameters in various successful models, as summarized in Table I, is clearly brought out in such a presentation. Naturally, we have not discussed any quantitative data or numerical results which already have been included in respective studies; nor have we attempted to sit in judgement trying to determine the relative merits or shortcoming of the particular models. In fact our study brings out that, notwithstanding the claims and counterclaims of the proponents of the various phenomenological models, there is, as yet, very little to choose between them on physical grounds. The respective models may, with due caution and with certain established credibility, be used to guide the experimentalists in their search for as

TABLE I. Summary of the inter-relationships of various phenomenological models as brought out by the generalized stretching formulation and the correspondence of the respective model parameters. We have put $1/(8\mathscr{I}_0^2 C') = g$.

Model	Functional dependence	Explicit spin dependence in f	Correspondence of model parameters
VMI	$1+b_1 t$		$\mathscr{I}_0=\mathscr{I}_0$; $\sigma=\frac{b_1^2}{2C'\mathscr{I}_0}$
Harris	$1+b_1 t$		$\mathscr{I}_0=\mathscr{I}_0$; $C_H=\frac{\mathscr{I}_0^2 b_1^2}{4C'}$
CS	$(1+b_1 t)^2$		$\mathscr{I}_0=\mathscr{I}_0$; $D=\frac{C'\mathscr{I}_0}{b_1^2}$
Bohr and Mottelson	$(1-at)^{-1}$	$a^2=1$	$A=\frac{1}{2\mathscr{I}_0}$; $B=g$
Ejiri	$(1-at)^{-1}$	$a^2=\frac{1}{(I+1)^2}$	$k=\frac{1}{2\mathscr{I}_0}-g$; $q=g$
Sood	$(1-at)^{-1}$	$a^2=\frac{8\mathscr{I}_0^2 C'^2}{1+N(2\mathscr{I}_0 C')I(I+1)}$	$A=\frac{1}{2\mathscr{I}_0}$; $B=C'$
Holmberg and Lipas	$(1-at)^{-1}$	$a^2=\frac{8\mathscr{I}_0^2 C'^2}{I^2(I+1)^2}\left[1+\frac{I(I+1)}{2\mathscr{I}_0 C'}-\sqrt{1+\frac{I(I+1)}{\mathscr{I}_0 C'}}\right]$	$d=C'$; $b=\frac{1}{\mathscr{I}_0 C'}$
Power series in $I(I+1)$	$(1-at)^{-1}$	$a^2=1+bI(I+1)+\ldots$	$A=\frac{1}{2\mathscr{I}_0}$, $B=g$; $C=-gb$; ...
Shape fluctuation	$(1-at)^{-1}$	$a^2=\frac{b+I}{(I+1)^2}$	$B_0=\frac{1}{2\mathscr{I}_0}-(b-1)g$ $B'\Phi'=(b-1)g$ $E'\Phi'=-g$
Anharmonic vibration	$(1-at)^{-1}$	$a^2=\frac{b+I}{(I+1)^2}$	$d_1=\frac{3}{2\mathscr{I}_0}-(2b+4)g$ $d_2=\frac{1}{2\mathscr{I}_0}-(b+6)g$ $d_3=-g$
Warke and Khadkikar	$(1-at)^{-2}$	$a^2=1$	$A=\frac{1}{2\mathscr{I}_0}$; $B=\frac{1}{2\mathscr{I}_0^2 C'}$
Nuclear softness NS(2)	$(1-at)^{-2}$	$a^2=\frac{1}{I+1}$	$A=\frac{1}{2\mathscr{I}_0}$; $\sigma_1=\frac{1}{\mathscr{I}_0 C'}$
Nuclear softness NS(3)	$(1-at)^{-2}$	$a^2=\frac{1+s_2 I}{I+1}$	$A=\frac{1}{2\mathscr{I}_0}$; $\sigma_1=\frac{1}{\mathscr{I}_0 C'}$; $\sigma_2=\frac{s_2}{\mathscr{I}_0 C'}$
Rigid rotator	$(1-at^2)^{-1}$		$A=\frac{1}{2\mathscr{I}_0}$
Vibrator	$(1+at^2)$	$a^2=1$	$A=\left(\frac{C'}{\mathscr{I}_0}\right)^{1/2}$; $B=\frac{C'}{2}$

yet unidentified levels in nuclei in various regions. In addition, and perhaps more significantly, the models may, as suggested above, result in specifying the functional dependence of the nuclear moments of inertia on angular momentum and on the stretching variable for guidance of the pure theorists in their attempt to develop a microscopic theory of nuclear rotation. Of course one has to always remember that this generalized stretching is an "effective" concept whose description will not adhere to the hydrodynamical or irrotational flow motions, since it effectively includes also the higher-order cranking, as well as the Coriolis antipairing effects.

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