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Astrophysical β-Decay Formulas and Superheavy Elements*

Legesse Senbetu†
Institute for Pure & Applied Physical Sciences and Department of Physics, University of California at San Diego, La Jolla, California 92037
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We comment on currently used β-decay formulas in astrophysical and superheavy-element applications. A new formula of comparable simplicity but significantly improved accuracy is presented which predicts β-decay rates that increase more strongly with energy than do conventional extrapolations in the neutron-rich superheavy-element region.

In β-decay formulas currently used in astrophysical applications and the calculations of half-lives of superheavy elements, the dependence on Z is completely neglected[1] or taken into account "on average" by a constant factor, $10^{1.5}$.[2] Thus, the β-decay half-life of such formulas depends on the energy only. In this paper, we report a formula in which the effect of the Coulomb interaction of the β particle by the nuclear charge on the half-life is explicitly taken into account. The exponent of the energy term in our expression for the half-life, instead of being a constant integer, normally 6 (see Refs. 1 and 2), is replaced by a Z-dependent number. For example, increasing Z from $\lesssim 10$ to 118 changes the exponent in the new formula from 6 to 5. As we show below, the currently used formula is a good approximation for small Z and large decay energy only.

To derive the new formula, we start with the Coulomb correction factor,[3]

$$F(Z,w)=2(1+s)(2PR)^{2s-2}e^{\pi\eta}\left|\frac{\Gamma(s+i\eta)}{\Gamma(1+2s)}\right|^2, \qquad (1)$$

where

w is the total energy of β particle in units of m_ec^2,

R is the radius of the residual nucleus in units of $\hbar/m_ec$,

$P=(w^2-1)^{1/2}$,

$S=[1-(\alpha Z)^2]^{1/2}$,

$\eta=\alpha Zw/P$, and α is the fine-structure constant.

By comparing with numerical calculations, we found that we can take

$$|\Gamma(s+i\eta)|^2=\frac{2[1+C(\alpha Z)^2]}{2+C(\alpha Z)^2}\,\frac{2\pi\eta}{e^{\pi\eta}-e^{-\pi\eta}} \qquad (2)$$

to within 5% accuracy in the area of interest. (C = Euler's constant.)

Following Feenberg and Trigg,[4] we use the approximation

$$\frac{1}{1-e^{-2\pi\eta}}=\frac{e^{2\pi\eta}}{2\pi\gamma}u\left[1-u\left(\frac{1}{2w^2}+\frac{3}{8w^4}\right)+\frac{u^2}{4w^4}\right], \qquad (3)$$

where

$$\gamma=\alpha Z \quad \text{and} \quad u=\frac{2\pi\gamma}{e^{2\pi\gamma}-1}.$$

Thus, $F(Z,w)$ can be written as

$$F(Z,w)=B(Z)\left[1-u\left(\frac{1}{2w^2}+\frac{3}{8w^4}\right)+\frac{u^2}{4w^4}\right](w^2-1)^{s-3/2}w \tag{4}$$

and

$$B(Z)=\frac{2^{2s}(1+s)}{(2s)!^2}R^{2s-2}\frac{1+C\gamma^2}{2+C\gamma^2}ue^{2\pi\gamma}. \tag{5}$$

The Fermi integrated function, $f(Z,w_0)$, is given by[5]

$$f(Z,w_0)=\int_1^{w_0}F(Z,w)(w^2-1)^{1/2}(w_0-w)^2 w\,dw\,. \tag{6}$$

Substituting for $F(w,Z)$, we make a change of variables to obtain[6]

$$\int_0^a \frac{\chi^{\mu-1}}{(1+\beta x)^\nu}dx=\frac{a^\mu}{\mu}\,{}_2F_1(\nu,\,\mu;\,1+\mu;\,-\beta a), \tag{7}$$

where ${}_2F_1$ is the generalized hypergeometric function.[7] Using this, Eq. (6) becomes:

$$\begin{aligned}f(w,Z)=B(Z)\frac{(w^2-1)^s}{2s}\Bigg\{&\frac{w(w^2-1)}{2(s+1)}+\tfrac{1}{2}wu+(1-y)\Bigg[\frac{3(w^2-1)^2}{4(s+1)(s+2)}\,{}_2F_1(1,-(1+s);s+3;y)\\&+\left(\frac{w^2-1}{4(s+1)}u-\frac{w^2(w^2-1)}{2(s+1)}\right){}_2F_1(1,-s;s+2;y)-(\tfrac{1}{2}w^2+\tfrac{3}{8})u\,{}_2F_1(1,1-s;1+s;y)\Bigg]\\&+(1-x)[(u-u^2)2w\,{}_2F_1(1,1;1+s;x)+(1-x)^{1/2}(u^2-\tfrac{3}{2}u)\tfrac{1}{4}w^2\,{}_2F_1(\tfrac{3}{2},1;1+s;x)]\Bigg\},\end{aligned} \tag{8}$$

where

$$y=\frac{w-1}{w+1} \quad \text{and} \quad x=\frac{w^2-1}{w^2}.$$

For $w>1$, keeping only the highest-order term in w, the above expression reduces to our final approximate formula for the Fermi-integrated function:

$$f(w,Z)=B(Z)\frac{w^{3+2s}}{(s+1)(1+2s)(3+2s)}. \tag{9}$$

For small Z, i.e., $s\simeq 1$ and $\beta(Z)\simeq 1$, we get

$$f(w,Z)=\frac{w^5}{30}, \tag{10}$$

which is the current approximation.[1,2]

To calculate the half-life, we follow the method of Nix and Fiset.[2] Since

$$w=\frac{w_\beta}{m_ec^2}=\frac{Q_\beta}{m_ec^2}+1\,, \tag{11}$$

where w_β is the total maximum energy of the emitted β particle, we can write $f(w_\beta,Z)$ as

$$f(w_\beta,Z)=\frac{B(Z)}{(s+1)(1+2s)(3+2s)}\frac{1}{(m_ec^2)^{3+2s}}w_\beta^{3+2s}. \tag{12}$$

So the average value of $f(w_\beta,Z)$ is given by[1]

$$\begin{aligned}\bar f(w_\beta,Z)&=\frac{B(Z)}{(s+1)(1+2s)(3+2s)}\frac{1}{(m_ec^2)^{3+2s}}\\&\quad\times\int_0^{Q_\beta}(w_\beta-E)^{3+2s}\tfrac{1}{3}\rho\,dE\\&=\frac{B(Z)}{(s+1)(s+2)(1+2s)(3+2s)}\left(\frac{1}{m_ec^2}\right)^{3+2s}\\&\quad\times\tfrac{1}{6}\rho[w_\beta^{4+2s}-(m_ec^2)^{4+2s}],\end{aligned} \tag{13}$$

where ρ is the average density of states in the daughter nucleus. One uses the empirical result given by Fowler, Seeger, and Clayton[1] for ρ. The inverse half-life[2] is then, in our approximation,

$$\begin{aligned}\frac{1}{\tau_\beta}=\frac{1}{ft}&\frac{B(Z)}{(s+1)(s+2)(1+2s)(3+2s)}\left(\frac{1}{m_ec^2}\right)^{3+2s}\\&\times\tfrac{1}{6}\rho[w_\beta^{4+2s}-(m_ec^2)^{4+2s}],\end{aligned} \tag{14}$$

where ft is the comparative half-life.[8] For small Z this reduces to

$$\frac{1}{\tau_\beta}=\frac{1}{ft}\frac{1}{540}\frac{\rho}{(m_ec^2)^5}[w_\beta^{6}-(m_ec^2)^6] \tag{15}$$

which is the formula used by Nix and Fiset[2] except for the factor of $c(Z_d)=10^{1.5}$.

Keeping common factors as constant parameters, the half-lives obtained by the new formula were compared with those predicted from a formula

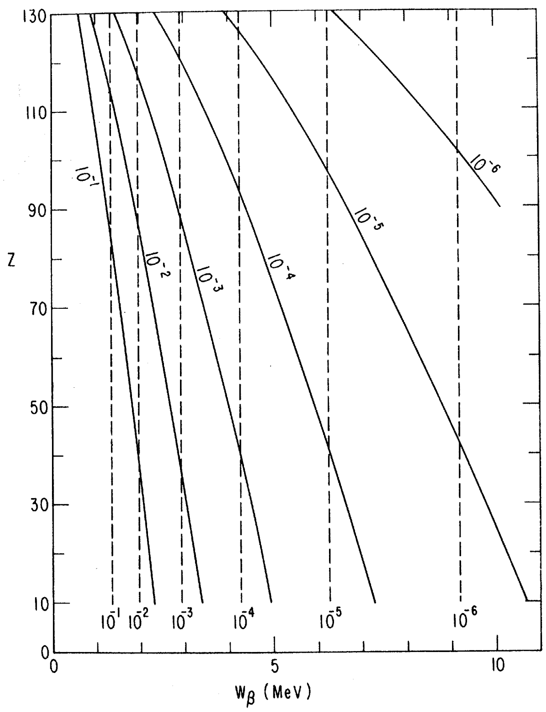

FIG. 1. A plot of Z versus w_β for various values of $\tau_\beta(p)/ft$. This common factor is given by the numbers next to corresponding curves. The solid lines result from the new formula and the straight, dashed lines from the conventional formula (Ref. 2). We set $R = 1.2(2.5Z)^{1/3} m_e c/\hbar$ in Eq. (5) although use of a constant average value, e.g., $R = 0.022$, leads to results which are practically indistinguishable in this plot, i.e., most of the curvature in the solid lines is due to other Z dependence in the new formula.

used by Nix and Fiset (Fig. 1). We find that:
(a) For $Z > 100$ and $1 < w_\beta < 10$ MeV, the half-lives predicted by the new formula are shorter;
(b) for $Z < 90$, the new formula predicts longer half-lives. A comparison with experimental data for $w_\beta \gtrsim 2$ MeV in the lead region[2] supports this trend of our formula.

Applying the standard estimate[2] for the error in τ_β resulting from an error in Q_β, we found that the error in our τ_β is less than $10^{\pm(2+s)}$, $s = [1-(\alpha Z)^2]^{1/2}$. The error in Ref. 2 is $10^{\pm 3}$. We observe that for large Z the new formula reduces the error by an order of magnitude.

$\text{Log}_{10} f(Z, w)$ was calculated using the final approximate value we derived and compared to the almost exact value obtained by numerical calculation. It is found that:
(a) For $w_\beta \geq 2$ MeV, the error was less than 0.08 in absolute values for all $Z \leq 130$;
(b) the error decreases as one considers larger Z and w_β values. For example, for $Z = 80$ and $w_\beta = 15$ MeV, the error is less than 0.02 in absolute value, while that of the current approximation, $f(Z, w) = w^5/30$, is approximately 1[1];
(c) for $w_\beta \geq 1$, the error is less than 0.5 for all Z values.

We conclude that the new formula, Eq. (14), for the half-life has comparable simplicity but considerably improved accuracy. The conventional formula can therefore be replaced in all its current applications. Also, the small error in $\log_{10} f(w, Z)$ suggests that Eq. (9) provides a good approximation to calculate the ft values for allowed transitions. After this paper was submitted, the new formula was used to calculate r-process cycle times.[9] It gave results that are about 1 order of magnitude shorter than previous values.

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