PHYSICAL REVIEW C

NUCLEAR PHYSICS

THIRD SERIES, VOL. 7, NO. 1 January 1973

Distorted-Wave Theory of Multistep Processes*

D. Robson†

Physics Department, Princeton University, Princeton, New Jersey 08540

(Received 18 August 1972)

The theory of multistep processes based upon the distorted-wave Born approximation as a leading term is investigated. Iterative procedures in which the many-body Green's operator for the complete system is replaced by a non-Hermitian or optical Green's operator are shown to have some unappealing features. Alternative procedures using a nonlinear form of the distorted transition matrix equation, corresponding to a dispersion-theory approach, are discussed and appear to provide a more consistent iterative procedure. A generalized optical theorem is derived and used to sum the multistep amplitudes which conserve energy. The summed result shows that the normal distorted-wave matrix element should be replaced by a modified leading term unless the absorption in initial and final states is weak. Difficulties with the theory involving multistep amplitudes off the energy shell and a new calculable form for two-step processes are suggested.

I. INTRODUCTION

The possibility that multistep processes may be important in direct nuclear reaction studies is a topic of considerable current interest. It is interesting from a fundamental point of view, since the basis of "one-step" processes in terms of distorted-wave Born-approximation (DWBA) matrix elements has yet to receive a firmer basis than its present semiempirical success. The calculation of higher-order terms is consequently important in order to know whether or not the DWBA method is really the first term in a divergent series. More correctly we need to find out just which series, if any, the DWBA belongs to, particularly in the case of rearrangement reactions.

In the next section two basic methods of iterating are considered which produce two-step processes in terms of DWBA matrix elements. The first method involves the relatively obvious procedure of replacing the total Green's operator by an appropriately chosen distorted Green's operator. The second method is based on nonlinear equations for the distorted T matrix and yields an alternative iteration procedure. It also points out some possible inadequacies of the first method as an iterative procedure. The second method is then rearranged so that a better understanding of higher-order terms can be obtained. In particular all those multistep processes which conserve energy can be summed using a generalized optical theorem and leads to simple results in the limit of strong absorption. Difficulties with the remaining "off-shell" multistep processes are discussed in Sec. IV.

II. TWO-STEP APPROXIMATIONS

We begin from the usual two-potential theory[1] which allows the T matrix for a given transition from channel α to channel β to be expressed using a relatively obvious and standard notation as

$$T_{\beta\alpha} = \langle \beta |\, V_\beta + V_\beta G^{(+)} V_\alpha |\, \alpha \rangle = \hat{T}_{\beta\alpha} + t_{\beta\alpha} , \qquad (1)$$

where

$$\hat{T}_{\beta\alpha} = \langle x_\beta^{(-)} | \hat{V}_\beta + \hat{V}_\beta G^{(+)} \hat{V}_\alpha | x_\alpha^{(+)} \rangle \tag{2}$$

is the distorted-wave T matrix with $\hat{V} = V - U$ and for $E_\alpha = E_\beta = E$,

$$t_{\beta\alpha} = \operatorname*{Lim}_{\eta \to 0^+} (i\eta) \langle x_\beta^{(-)} | U_\beta (E + i\eta - H_\beta)^{-1} | \alpha \rangle$$

yields the "optical" t matrix. The distorting potentials U_α, U_β are usually chosen to connect only to a limited class of states, $\alpha(U_\alpha)$ and $\beta(U_\beta)$, respectively; e.g. the simple potential $U_\alpha(r_\alpha)$ will only connect the state $|\alpha\rangle$ to states $|\alpha'\rangle$ which differ in their relative motion, i.e., "elastic" scattering states. With the further assumption that the parameters of U_β, U_α are chosen so that $t_{\beta\beta'}$, $t_{\alpha\alpha'}$ describe the observed scattering within the limited classes of states $\beta(U_\beta)$, $\alpha(U_\alpha)$, respectively, and that these two classes are each "fragmentation conserving" then

$$t_{\beta\alpha} = \langle x_\beta^{(-)} | U_\beta | \alpha \rangle \delta_{\alpha(U_\alpha), \beta(U_\beta)}, \tag{3}$$

$$\hat{T}_{\beta\alpha} = \langle x_\beta^{(-)} | \hat{V}_\beta + \hat{V}_\beta G^{(+)} \hat{V}_\alpha | x_\alpha^{(+)} \rangle (1 - \delta_{\alpha(U_\alpha), \beta(U_\beta)}). \tag{4}$$

This result gives a complete separation of the T matrix into a t matrix which describes the scattering between the limited set of channels $\alpha(U_\alpha)$ or $\beta(U_\beta)$ and the distorted $\hat{T}$ matrix which describes the scattering between all other channels α, β. In most applications $t_{\beta\alpha}$ arises from optical-model potentials U_β or U_α and describes the elastic scattering in each channel, respectively. The distorted $\hat{T}$ matrix in this case describes all nonelastic scatterings and has a leading term

$$\hat{T}_{\beta\alpha}^{\mathrm{DWBA}} = \langle x_\beta^{(-)} | \hat{V}_\beta | x_\alpha^{(+)} \rangle,$$

which is conventional DWBA. Recently there have been calculations based on generalized optical potentials which allow inelastic scattering to a small number of target (or residual) nucleus states. In this case $t_{\beta\beta'}, t_{\alpha\alpha'}$ represent coupled-channel calculations for elastic and inelastic scattering in the subspaces $\beta(U_\beta)$, $\alpha(U_\alpha)$, respectively, and the distorted $\hat{T}$ matrix describes all other transitions usually via its leading term [which is termed a coupled-channels Born approximation (CCBA)].

The result expressed by Eq. (4) involves a sum of two terms,

$$\hat{T}_{\beta\alpha} = \hat{T}_{\beta\alpha}^{(1)} + \hat{T}_{\beta\alpha}^{(M)},$$

in which $\hat{T}_{\beta\alpha}^{(1)}$ is usually $\hat{T}_{\beta\alpha}^{\mathrm{DWBA}}$. Presumably we are to associate

$$\hat{T}_{\beta\alpha}^{(M)} = \langle x_\beta^{(-)} | \hat{V}_\beta G^{(+)} \hat{V}_\alpha | x_\alpha^{(+)} \rangle, \quad \beta \neq \alpha(U_\alpha), \tag{5}$$

with multistep processes and to expect the "lowest-order" terms to contain only two-step processes; hopefully via intermediate states other than those of class $\alpha(U_\alpha)$ or $\beta(U_\beta)$.

The problem then is to find a consistent scheme for obtaining $\hat{T}_{\beta\alpha}^{(2)}$, $\hat{T}_{\beta\alpha}^{(3)}$, etc. At the same time we have to keep in mind the problem posed by Greider and Dodd,[2] namely that the iteration of $G^{(+)}$ via relations of the form

$$G^{(+)} = G_\gamma^{(+)} + G_\gamma^{(+)} \hat{V}_\gamma G^{(+)},$$

with

$$G^{(+)} = \operatorname*{Lim}_{\eta \to 0^+} (E + i\eta - H)^{-1},$$

$$G_\gamma^{(+)} = \operatorname*{Lim}_{\eta \to 0^+} (E + i\eta - H_\gamma - U_\gamma)^{-1},$$

and

$$H = H_\gamma + V_\gamma = H_\alpha + V_\alpha = H_\beta + V_\beta,$$

can easily lead to divergent series so that $\hat{T}_{\beta\alpha}^{(2)}$, $\hat{T}_{\beta\alpha}^{(3)}$, etc., may be of similar magnitude and a calculation of $\hat{T}_{\beta\alpha}^{(2)}$ would be meaningless.

It is instructive however to discuss two-step approximations via such iterated Green's operators because calculations using this method are being made[3] and the results obtained sometimes yield rather large magnitudes for two-step processes. Since the large contributions obtained for two-step processes in reactions such as (h, t) between analog states is very surprising to some of us, the approximation $G^{(+)} \simeq G_\gamma^{(+)}$ is not above suspicion. We therefore also study a more exact treatment of two-step processes which can be compared even at a formal level with the approximation $G^{(+)} \simeq G_\gamma^{(+)}$. We discuss each method in turn.

A. Iteration of Green's Operators

The iteration

$$G^{(+)} = G_\gamma^{(+)} + G_\gamma^{(+)} \hat{V}_\gamma G^{(+)}$$

naturally leads to a set of multistep processes, viz.

$$G^{(+)} = G_\gamma^{(+)} + G_\gamma^{(+)} \hat{V}_\gamma G_\delta^{(+)} + G_\gamma^{(+)} \hat{V}_\gamma G_\delta^{(+)} \hat{V}_\delta G_\epsilon^{(+)} + \cdots$$

and our physics intuition is invoked to decide just which two-body fragmentations we regard as most relevant to the intermediate states labeled here by γ, δ, ϵ. The first term presumably yields a two-step mechanism via all intermediate states appropriate to the fragmentation labeled γ, the relative motion of the fragments being described by the optical potential U_γ, i.e.,

$$\hat{T}_{\beta\alpha}^{(2)}(\gamma) = \langle x_\beta^{(-)} | \hat{V}_\beta G_\gamma^{(+)} \hat{V}_\alpha | x_\alpha^{(+)} \rangle. \tag{6}$$

All we have to do according to Eq. (6) is to calculate a second-order DWBA integral, provided we only consider bound pairs of fragments in the

intermediate states γ. Under these conditions and the simplifying assumption of a central potential for U_γ, a typical intermediate state has the form,

$$\xi_\gamma^{(+)} = \phi_{\gamma_1}\phi_{\gamma_2}u_\gamma^{(+)} = \phi_{\gamma_{12}}u_\gamma^{(+)},$$

where ϕ_{γ_1}, ϕ_{γ_2} represent the internal states of the two bound fragments and $u_\gamma^{(+)}$ represents the relative motion. The function u_γ is a solution of the equation

$$(K_\gamma + U_\gamma - \epsilon_\gamma)u_\gamma^{(+)} = 0$$

with relative motion energy ϵ_γ, and K_γ is the kinetic energy operator.

A typical two-step process is then given by

$$\hat{T}^{(2)}_{\beta\alpha}(\gamma_{12}) = \langle x_\beta^{(-)}|\ \hat{V}_\beta|\ \phi_{\gamma_{12}}\rangle (E - E_{\gamma_{12}} - K_\gamma - U_\gamma + i\eta)^{-1} \times \langle \phi_{\gamma_{12}}|\ \hat{V}_\alpha|\ x_\alpha^{(+)}\rangle , \tag{7}$$

where the propagator needs special emphasis because it involves a non-Hermitian (complex) potential U_γ. For such potentials the relative motion functions $u_\gamma^{(+)}$ are not orthogonal and it is necessary[4] to use biorthogonal sets of states, i.e., sets defined by

$$(K_\gamma + U_\gamma^\dagger - \epsilon_\gamma)\tilde{u}_\gamma^{(+)} = 0,$$
$$(K_\gamma + U_\gamma - \epsilon_\gamma)u_\gamma^{(+)} = 0,$$

and

$$\tilde{u}_\gamma^{(+)} \sim C_\gamma[e^{i\vec{k}_\gamma\cdot\vec{r}_\gamma} + \tilde{f}(\theta_\gamma)e^{ik_\gamma r_\gamma}r_\gamma^{-1}],$$
$$u_\gamma^{(+)} \sim C_\gamma[e^{i\vec{k}_\gamma\cdot\vec{r}_\gamma} + f(\theta_\gamma)e^{ik_\gamma r_\gamma}r_\gamma^{-1}].$$

Clearly if $u_\gamma^{(+)}$ is a solution of an absorptive potential ($\mathrm{Im}\,U_\gamma < 0$) then the conjugate solution $\tilde{u}_\gamma^{(+)}$ is a solution of a "creative" potential ($\mathrm{Im}\,U_\gamma > 0$). The solutions $u_\gamma^{(-)}$ are solutions of the same equation as $\tilde{u}_\gamma^{(+)}$ but satisfy different boundary conditions,[5]

$$u_\gamma^{(-)} \sim C_\gamma^*[e^{i\vec{k}_\gamma\cdot\vec{r}_\gamma} + f^*(\pi - \theta_\gamma)e^{-ik_\gamma r_\gamma}r_\gamma^{-1}].$$

The conjugate solution to $u_\gamma^{(-)}$ is $\tilde{u}_\gamma^{(-)}$, and it satisfies the same equation as $u_\gamma^{(+)}$ but different boundary conditions:

$$\tilde{u}_\gamma^{(-)} \sim C_\gamma^*[e^{i\vec{k}_\gamma\cdot\vec{r}_\gamma} + \tilde{f}^*(\pi - \theta_\gamma)e^{-ik_\gamma r_\gamma}r_\gamma^{-1}].$$

The orthogonality and completeness relations are then

$$\sum_\gamma |\ u_\gamma^{(\pm)}\rangle\langle\tilde{u}_\gamma^{(\pm)}| = 1 = \sum_\gamma |\ \tilde{u}_\gamma^{(\pm)}\rangle\langle u_\gamma^{(\pm)}|$$

in which

$$\langle\tilde{u}_\gamma^{(\pm)}(\epsilon_\gamma)|u_{\gamma'}^{(\pm)}(\epsilon_{\gamma'})\rangle = \delta_{\gamma\gamma'}\delta_{\epsilon_\gamma\epsilon'_\gamma},$$

for bound states and

$$\langle\tilde{u}_\gamma^{(\pm)}(\epsilon_\gamma)|u_{\gamma'}^{(\pm)}(\epsilon_{\gamma'})\rangle = \delta_{\gamma\gamma'}\delta(\epsilon_\gamma - \epsilon_{\gamma'})$$

for continuum states. Consequently, the optical propagator in Eq. (7) can be expanded as an integral over all real continuum energies plus a sum over discrete states necessary to provide the complete set:

$$(E - E_{\gamma_{12}} - K_\gamma - U_\gamma + i\eta)^{-1} = \sum_{\gamma_{12}} |\ u_\gamma^{(+)}\rangle (E - E_\gamma + i\eta)^{-1}\langle\tilde{u}_\gamma^{(+)}|$$

with $E_\gamma = E_{\gamma_{12}} + \epsilon_\gamma$.

Putting $\tilde{\xi}_\gamma^{(+)}$ in place of $\phi_{\gamma_{12}}\tilde{u}_\gamma^{(+)}$ and assuming $\tilde{x}_\gamma^{(+)}$ is related to $\tilde{\xi}_\gamma^{(+)}$ by

$$\tilde{\xi}_\gamma^{(+)} = C_\gamma\tilde{x}_\gamma^{(+)},$$

we have

$$\hat{T}^{(2)}_{\beta\alpha}(\gamma_{12}) = \sum_{\gamma_{12}}\rho_\gamma\langle x_\beta^{(-)}|\ \hat{V}_\beta|\ x_\gamma^{(+)}\rangle (E - E_\gamma + i\eta)^{-1} \times \langle\tilde{x}_\gamma^{(+)}|\ \hat{V}_\alpha|\ x_\alpha^{(+)}\rangle$$
$$= \sum_{\gamma_{12}}\rho_\gamma\hat{T}^{(1)}_{\beta\gamma}(E - E_\gamma + i\eta)^{-1}\hat{Q}^{(1)}_{\gamma\alpha}, \tag{8}$$

in which $\rho_\gamma = |C_\gamma|^2$ is the density of states. Notice, however, the occurrence of the operator $\hat{Q}^{(1)}$ rather than $\hat{T}^{(1)}$ which are not equal for a nonzero value of U_γ. The state $\tilde{x}^{(+)}$ is not equal to $x^{(-)}$ even for a real potential. For a real potential $\tilde{\xi}^{(+)}$ and $\xi^{(-)}$ are related by a unitary transformation, but for a complex potential the transformation is nonunitary. Using our completeness relations and assuming C_γ is real hereafter, we deduce that

$$\hat{Q}^{(1)}_{\gamma\alpha} = \sum_{\gamma'(U_\gamma)}\langle\tilde{u}_\gamma^{(+)}|\tilde{u}_{\gamma'}^{(-)}\rangle\hat{T}^{(1)}_{\gamma'\alpha} = \sum_{\gamma'(U_\gamma)}\tilde{s}^\dagger_{\gamma\gamma'}\hat{T}^{(1)}_{\gamma'\alpha},$$

where

$$\tilde{s}_{\gamma'\gamma} = \langle\tilde{u}_{\gamma'}^{(-)}|\tilde{u}_\gamma^{(+)}\rangle = \langle\tilde{\xi}_{\gamma'}^{(-)}|\tilde{\xi}_\gamma^{(+)}\rangle$$

is the scattering (s) matrix for the optical potential U_γ^*. Since U_γ^* is a creative operator then we expect

$$[\underline{\tilde{s}}^\dagger\underline{\tilde{s}}]_{\gamma\gamma} > 1$$

will occur and consequently $\hat{Q}^{(1)}$ can have a magnitude which exceeds the magnitude of $\hat{T}^{(1)}$ by a large factor if the U_γ potential is strongly absorptive.

In order to see this creation process more clearly it is instructive to use a formalism where $\tilde{s}$ is diagonal, e.g. eigenchannels or more obviously

partial waves for spinless fragments. A given partial wave of angular momentum l is studied via the expansions in Legendre polynomials:

$$u^{(\pm)} = \sum_l (2l+1) i^l f_l^{(\pm)}(r) P_l(\hat{k}\cdot\hat{r}),$$

$$\tilde{u}^{(\pm)} = \sum_l (2l+1) i^l \tilde{f}_l^{(\pm)}(r) P_l(\hat{k}\cdot\hat{r}),$$

where

$$f_l^{(-)} = [f_l^{(+)}]^*, \quad \tilde{f}_l^{(-)} = [\tilde{f}_l^{(+)}]^*,$$

according to the known time-reversal relations. The states $\tilde{f}_l^{(-)}$ and $f_l^{(+)}$ are both regular solutions of the same radial equation and consequently they can only differ by a complex constant which arises because $\tilde{f}_l^{(-)}$ and $f_l^{(+)}$ satisfy different boundary conditions asymptotically. Using the relations

$$f_l^{(+)} \sim y_l [h_l^* - s_l h_l],$$

$$\tilde{f}_l^{(+)} \sim \tilde{y}_l [h_l^* - \tilde{s}_l h_l],$$

and

$$\tilde{f}_l^{(-)} = \tilde{f}_l^{(+)*} = c_l f_l^{(+)},$$

where h_l^* and h_l are ingoing and outgoing spherical Bessel functions, one easily deduces the relations

$$y_l = \tilde{y}_l^*,$$

$$s_l^{-1} = \tilde{s}_l^* = c_l .$$

Evaluating a partial wave amplitude $\hat{Q}_l^{(1)}$ thus involves the substitution

$$\hat{Q}_l^{(1)} = \tilde{s}_l^* \hat{T}_l^{(1)} = s_l^{-1} \hat{T}_l^{(1)} .$$

Expressing s_l in terms of its complex phase $\delta_l = \lambda_l + i\mu_l$ (with $\mu_l > 0$ for $\mathrm{Im}\, U_\gamma < 0$) yields

$$s_l = e^{2i\delta_l} = e^{2i\lambda_l} e^{-2\mu_l}$$

and

$$s_l^{-1} = \tilde{s}_l^* = e^{-2i\delta_l} = e^{-2i\lambda_l} e^{+2\mu_l}$$

corresponding to a creative process via the term $e^{+2\mu_l}$.

It is interesting to ask why the leading term $\hat{T}_{\beta\alpha}^{(1)}$ itself does not suffer from the "creativity" problem because after all the distorted wave is related to the initial unperturbed system by an integral equation, e.g.

$$|\, x_\alpha^{(+)}\rangle = |\, \alpha\rangle + (E + i\eta - H_\alpha - U_\alpha)^{-1} U_\alpha |\, \alpha\rangle$$

involving an optical Green's operator. The leading term then involves

$$\hat{T}_{\beta\alpha}^{(1)} = \langle x_\beta^{(-)}|\, \hat{V}_\beta |\, \alpha\rangle + \langle x_\beta^{(-)}|\, \hat{V}_\beta (E + i\eta - H_\alpha - U_\alpha)^{-1} U_\alpha |\, \alpha\rangle$$

$$= \sum_{\alpha'} \sum_{\alpha''} \left[\langle x_\beta^{(-)}|\, \hat{V}_\beta |\, \xi_{\alpha'}^{(+)}\rangle \tilde{s}^\dagger_{\alpha'\alpha''} \langle \xi_{\alpha''}^{(-)}|\, \alpha\rangle + \langle x_\beta^{(-)}|\, \hat{V}_\beta |\, \xi_{\alpha'}^{(+)}\rangle \frac{\tilde{s}^\dagger_{\alpha'\alpha''}}{E + i\eta - E_{\alpha''}} \langle \xi_{\alpha''}^{(-)}|\, U_\alpha |\, \alpha\rangle \right],$$

in which the second term accounts for all the multistep processes via the U_α interaction. In this case however the creativity via $\tilde{s}^\dagger$ and the matrix element $\langle \xi_{\alpha''}^{(-)}|\, U_\alpha |\, \alpha\rangle$ correspond to the *same* scattering potential. Moreover the term $\langle x_\beta^{(-)}|\, \hat{V}_\beta |\, \alpha\rangle$ belongs to the same unperturbed Hamiltonian H_α as the term involving the optical operator in the asymptotic region. This self-consistency is best expressed using the original integral equation for $|\, x_\alpha^{(+)}\rangle$ to give

$$\langle \xi_{\alpha''}^{(-)}|\, \alpha\rangle = \langle \xi_{\alpha''}^{(-)}|\, x_\alpha^{(+)}\rangle - (E + i\eta - E_{\alpha''})^{-1} \langle \xi_{\alpha''}^{(-)}|\, U_\alpha |\, \alpha\rangle$$

so that all the multistep scattering terms are exactly canceled off and we are left with

$$\hat{T}_{\beta\alpha}^{(1)} = \sum_{\alpha'} \sum_{\alpha''} \langle x_\beta^{(-)}|\, \hat{V}_\beta |\, x_{\alpha'}^{(+)}\rangle \tilde{s}^\dagger_{\alpha'\alpha''} s_{\alpha''\alpha}$$

$$= \langle x_\beta^{(-)}|\, V_\beta |\, x_\alpha^{(+)}\rangle ,$$

where $s_{\alpha''\alpha} = \langle \xi_{\alpha''}^{(-)}|\, \xi_\alpha^{(+)}\rangle = C_\alpha \langle \xi_{\alpha''}^{(-)}|\, x_\alpha^{(+)}\rangle$. Since clearly we also have $\underline{\tilde{s}}^\dagger \underline{s} = \underline{1}$ we see that the creativity is exactly nullified by an equal absorptivity.

The coupled channels Born approximation $\hat{T}_{\beta\alpha}^{\mathrm{CCBA}}$ is just a special case of $\hat{T}_{\beta\alpha}^{(1)}$, consequently calculations[6] involving "multisteps" due to *inelastic* scattering (in either the initial or final distortions) will not suffer from creativity in the intermediate states. It is not clear, however, that a perturbative treatment of the inelastic scattering distortions such as envisaged by Bindal and Koshel[3] will avoid the creativity problem. The point is that when inelastic scattering is treated by perturbation theory the self-consistency discussed above will no longer be valid and consequently the cancellation effects may not occur with sufficient accuracy.

The iterative method based on the replacement of $G^{(+)}$ by $G_\gamma^{(+)}$ leads to two-step processes of the form

$$\hat{T}_{\beta\alpha}^{(2)}(\gamma) = \sum_\gamma \sum_{\gamma'} \hat{T}_{\beta\gamma}^{(1)} (E + i\eta - E_\gamma)^{-1} \tilde{s}^\dagger_{\gamma\gamma'} \hat{T}_{\gamma'\alpha}^{(1)} C_\gamma C_{\gamma'} \tag{9}$$

and in general the occurrence of a creative inter-

mediate stage with no self-consistency, since $\gamma \neq \alpha(U_\alpha)$ or $\beta(U_\beta)$ which leads us to seriously question the accuracy of such an iterative procedure. We now investigate these suspicions more quantitatively by the use of an alternative scheme which appears to be more accurate.

B. Nonlinear Equations

The nonlinear equations for the T matrix are well known when plane-wave bases $|\alpha\rangle$, $|\beta\rangle$ are used,[1] but the equivalent equations for $\hat{T}$ do not appear to have been seriously discussed as yet. The appropriate nonlinear equations are obtained by the introduction[1] of complete "orthonormal" sets:

$$\mathbf{S}_{\lambda,\gamma} |\Psi^{(\pm)}_{\lambda\gamma}\rangle \langle\Psi^{(\pm)}_{\lambda\gamma}| = 1,$$

$$\langle\Psi^{(\pm)}_{\lambda\gamma}|\Psi^{(\pm)}_{\lambda'\gamma'}\rangle = \delta_{\gamma\gamma'}\delta(E_\lambda - E_{\lambda'}),$$

where the states $\Psi^{(\pm)}_{\lambda\gamma}$ are eigenstates of the total Hamiltonian H with eigenvalues E_λ, γ indicates the particular channel containing incident waves and $\mathbf{S}_\gamma$ runs over *all* open channels at a given energy E_λ. The completeness and "orthonormality" of these states are fully discussed by Goldberger and Watson and need not be repeated here. In general the sets $\Psi^{(\pm)}_{\lambda\gamma}$ will include some bound states Φ_λ which we formally include above via the $\mathbf{S}$ sign. Of course Φ_λ is common to both the (+) and (−) sets and is independent of the sign.

The multistep term in Eq. (5) above can therefore be written as

$$\hat{T}^{(M)}_{\beta\alpha} = \mathbf{S}_{\lambda,\gamma} \langle x^{(-)}_\beta| \hat{V}_\beta |\Psi^{(+)}_{\lambda\gamma}\rangle (E - E_\lambda + i\eta)^{-1} \times \langle\Psi^{(+)}_{\lambda\gamma}| \hat{V}_\alpha |x^{(+)}_\alpha\rangle .$$

The S matrix for the complete system at energy E_λ is defined[1] by

$$S_{\gamma\delta} = \langle\Psi^{(-)}_{\lambda'\gamma}|\Psi^{(+)}_{\lambda\delta}\rangle = \delta(E_{\lambda'} - E_\lambda)[\delta_{\gamma\delta} - 2\pi i T_{\lambda\delta} C_\gamma C_\delta],$$

where

$$T_{\gamma\delta} = \langle\gamma| V_\gamma |\Psi^{(+)}_{\lambda\delta}\rangle C_\delta^{-1},$$

is the full T matrix. Relating $\langle\Psi^{(+)}|$ to $\langle\Psi^{(-)}|$ now involves a unitary transformation via $S^\dagger$ and we find

$$\hat{T}^{(M)}_{\beta\alpha} = \mathbf{S}_\gamma \mathbf{S}_\delta \hat{T}_{\beta,\lambda\gamma}(E - E_\lambda + i\eta)^{-1} S^\dagger_{\gamma\delta}(E_\lambda) \hat{T}_{\lambda\delta,\alpha} C_\gamma C_\delta . \tag{10}$$

This equation can be iterated by substitutions $\hat{T} = \hat{T}^{(1)}$ which at least ordinary DWBA phenomenology suggests is a good guess. Picking off a two-step term is then possible by assuming δ belongs to the same class as $\gamma(U_\gamma)$ which corresponds to a particular class of intermediate states as before. The first iteration of our nonlinear equation above yields then the two-step approximation:

$$\hat{T}^{(2)}_{\beta\alpha}(\gamma_{12}) = \mathbf{SS}_{\gamma_{12}(U_\gamma),\,\gamma'_{12}(U_\gamma)} \hat{T}^{(1)}_{\beta\gamma}(E - E_\gamma + i\eta)^{-1} s^\dagger_{\gamma\gamma'} \hat{T}^{(1)}_{\gamma'\alpha} \times C_\gamma C_{\gamma'} , \tag{11}$$

where $S^\dagger$ is replaced by $s^\dagger$, since the potential U_γ is assumed to have the same scattering matrix as the many-body interaction V_γ *within* the subspace of channels $\gamma(U_\gamma)$.

A comparison of Eq. (11) with Eq. (9) immediately shows that for continuum energies ϵ_γ in the sum the only difference[7] is in the replacement of $\bar{s}$ by s. In Eq. (9) there is also a sum over bound states ($\epsilon_\gamma < 0$) which does not occur in Eq. (11) but at high energies corresponding to "direct reactions" we expect such differences to be unimportant. In any event there is no reason to believe the bound state contribution in Eq. (9) will in any way compensate for the replacement of s by $\bar{s}$ in the continuum part.

The replacement of s by $\bar{s}$ is a large effect for certain partial waves. In particular for a central potential U_γ and spinless particles we obtain

$$\bar{s}_l^* = e^{4i\mu_l} s_l^* = (1 - \mathcal{T}_l)^{-1} s_l^*$$

with $\mathcal{T}_l$ being the optical transmission coefficient. For strongly absorbed particles transmissions very close to unity are often obtained so that partial wave cross sections obtained from Eq. (9) and Eq. (11) may differ by several orders of magnitude. It is perhaps worth pointing out that although $S^\dagger_{\gamma\gamma'}$ can be replaced by $[\underline{S}^{-1}]_{\gamma\gamma'}$ via unitarity of the full S matrix it does not follow that $[\underline{S}^{-1}]_{\gamma\gamma}$ (corresponding to a definite intermediate state) is at all related to $S_{\gamma\gamma}{}^{-1} = s_{\gamma\gamma}{}^{-1} = \bar{s}^*_{\gamma\gamma}$ unless S is diagonal corresponding to zero absorption. In the case of strong absorption S goes to zero along the diagonal and such a replacement is clearly invalid.

The iteration of the nonlinear equations appears to have several redeeming features:

(i) The substitutions $\Psi^{(\pm)} \simeq \xi^{(\pm)}$ corresponding to $\hat{T} \simeq \hat{T}^{(1)}$ in $\hat{T}^{(M)}$ are the same as that which yielded $\hat{T}^{(1)}$ as the leading term.

(ii) The S matrix $S^\dagger_{\gamma',\gamma}$ is automatically given by $s^\dagger_{\gamma',\gamma}$ for the optical potential U_γ if the states γ, γ' belong to the same subset $\gamma(U_\gamma)$ because $S_{\gamma'\gamma}$ and $s_{\gamma'\gamma}$ are made "equal" by parameter adjustment.

(iii) No creative processes occur in the intermediate states so that the iterated series is more likely to converge.

(iv) In the sum over γ in Eq. (10) *all* possible fragmentations occur so that all two-step mechanisms can be calculated as a coherent sum over a set of two-step matrix elements derived from the same propagator.

There is however a nontrivial difficulty associated with all the results discussed as yet. It has often been argued that the use of DWBA in place of PWBA (plane-wave Born approximation) corresponds to summing all the multistep processes involving elastic scattering in either the initial or final states. A careful examination of Eq. (10) shows that such a philosophy is simply not correct "off-the-energy-shell" because, as we shall show, $\hat{T}$ can no longer be related to the full T matrix via Eqs. (1)–(4). Even worse the "on-shell" multistep process can be summed to all orders using unitarity relations and in the case of strong absorption is directly proportional to the $\hat{T}$ matrix element itself. A full discussion of these problems is given in the next sections and need not concern us in comparing Methods A and B as reasonable iterative procedures because both methods are subject to these general criticisms.

Returning to the question of the validity of the iterated Green's operator method we can make three further arguments which suggest that it is not an accurate procedure. First the substitution $G^{(+)} \simeq G_\gamma^{(+)}$ is not consistent with DWBA itself. The plane-wave expression [Eq. (1)] with the substitution $G^{(+)} \simeq G_\alpha^{(+)}$ or $G^{(+)} \simeq G_\beta^{(+)}$ yields,

$$T_{\beta\alpha} \simeq \langle \beta |\, V_\beta + V_\beta G_\alpha^{(+)} V_\alpha |\, \alpha \rangle$$

or

$$\langle \beta |\, V_\beta + V_\beta G_\beta^{(+)} V_\alpha |\, \alpha \rangle \,,$$

which does not yield $\hat{T}_{\beta\alpha}^{\mathrm{DWBA}}$ as the approximation for the leading one-step process. Indeed DWBA philosophy relies on treating channels α and β on an equivalent basis. All substitutions of the type $G^{(+)} \cong G_\gamma^{(+)}$ are therefore contrary to this philosophy, since they select a definite channel fragmentation γ.

Second if one looks at the expansions of $G^{(+)}$ and $G_\gamma^{(+)}$ and compares terms of the same type then one is effectively substituting

$$\langle \Psi_{\lambda\gamma}^{(+)} | \rightarrow \langle \tilde{\xi}_\gamma^{(+)} | = \sum_{\gamma'} \bar{s}_{\gamma\gamma'}^{\dagger} \langle \xi_{\gamma'}^{(-)} | \,,$$

whereas the integral equation for $\langle \Psi_{\lambda\gamma}^{(-)} |$ and its relation to $\langle \Psi_{\lambda\gamma}^{(+)} |$ suggests the substitution

$$\langle \Psi_{\lambda\gamma}^{(+)} | \simeq \sum_{\gamma'} S_{\gamma\gamma'}^{\dagger} \langle \xi_{\gamma'}^{(-)} | \,.$$

Of course it could be argued that the term $\langle \tilde{\xi}_\gamma^{(+)} |$ is supposed to "globally" account for all fragmentations other than γ and its creative nature is in accordance with this requirement. The problem which arises however is that the "creativity" induced by U_γ^* represents only the coupling between the state γ and all other open channels δ (or compound states if an energy average is performed). It does not follow then that U_γ^* is in anyway representative of the coupling between the missing intermediate states δ and the states of real interest (α and β). In particular the number of intermediate states δ which will *simultaneously* connect strongly to states α and β ($\alpha \neq \beta$) is expected to be considerably less than the number of states δ which connect strongly to a single state γ. Consequently we expect $\bar{s}^{\dagger}$ will overestimate the creativity for a two-step process $\alpha \rightarrow \gamma \rightarrow \beta$ and presumably the error gets worse as the strength of the absorption potential is increased.

Finally we point out that the method of approximation based on $G^{(+)} \simeq G_\gamma^{(+)}$ is likely to fail because the $\bar{s}$ matrix in the limit of strong creativity is very poorly determined by elastic scattering. The point is that elastic scattering involves the linear combination $(1 - s)$, or some similar combination, and for $|s| < \epsilon$, where ϵ is some small value, the value of s could equally well be replaced by zero (as indeed it was in some of the earlier strong absorption theories). Unfortunately, the two-step processes with $G^{(+)} \simeq G_\gamma^{(+)}$ involve s^{-1} and the difference between ϵ and 0 becomes a large number ϵ^{-1} versus infinity which is catastrophic.

Some argument could perhaps be presented that in the strong absorption case $\hat{T}^{(1)}$ in Eq. (9) occurs quadratically and becomes zero at a rate which more than cancels the creativity in $\bar{s}^{\dagger}$. In conventional optical-model calculations this simply is not a possibility because in most cases of interest the DWBA matrix elements have nonnegligible contributions from radial distances beyond the range of the optical potential. Beyond the optical potential the distorted partial waves approach a fixed nonzero solution independently of the optical potential, i.e.,

$$f_l^{(+)} \sim y_l(h_l^* - s_l h_l) \rightarrow y_l h_l^* \,.$$

Clearly then, $\hat{T}^{(1)}$ also approaches a constant finite value in conventional DWBA calculations no matter how strong the absorption gets in the optical-model region. Consequently, the product matrix $\hat{T}_{\beta\gamma}^{(1)} \bar{s}_{\gamma\gamma'}^{\dagger} \hat{T}_{\gamma'\alpha}^{(1)}$ is ill-conditioned in the limit of strong absorption because $\bar{s}^{\dagger}$ is singular in a poorly determined manner which is essentially unrelated to the channels α, β of physical interest.

III. ON-SHELL MULTISTEP PROCESSES

It is important to try and understand the importance of multistep processes for those components where summation of the series is feasible. The "on-shell" part of the multistep processes is defined by

$$\hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}) = -i\pi \sum_{\gamma,\gamma'} \delta(E - E_\lambda)\hat{T}_{\beta\gamma}S^{\dagger}_{\gamma\gamma'}\hat{T}_{\gamma'\alpha}C_\gamma C_{\gamma'} .$$

This is easily summed using the "unitarity" relation

$$\hat{T}^{(+)}_{\beta\alpha} - \hat{T}^{(-)}_{\beta\alpha} = \langle x^{(-)}_\beta |\ \hat{V}_\beta[G^{(+)} - G^{(-)}]\hat{V}_\alpha |\ x^{(+)}_\alpha\rangle = 2\hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}),$$

which yields after a little algebra the result

$$\hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}) = \tfrac{1}{2}\hat{T}^{(+)}_{\beta\alpha} - \tfrac{1}{2}[C_\beta C_\alpha]^{-1} \times \sum_{\beta',\alpha'} C_{\beta'}C_{\alpha'}s_{\beta\beta'}s_{\alpha\alpha'}[\hat{T}^{(+)}_{\alpha'\beta'}]^* . \qquad (12)$$

In the simplest case of spinless particles and partial waves with a central potential we have

$$\hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}) = \tfrac{1}{2}\hat{T}^{(+)}_{\beta\alpha} - \tfrac{1}{2}s_\beta s_\alpha[\hat{T}^{(+)}_{\alpha\beta}]^* \qquad (13)$$

which is to be compared with the full T-matrix relation

$$T^{(M)}_{\beta\alpha}(\mathrm{ON}) = \tfrac{1}{2}T^{(+)}_{\beta\alpha} - \tfrac{1}{2}[T^{(+)}_{\alpha\beta}]^* .$$

In the limit that U_β and/or U_α involves strong absorption so that $|s_\beta|$ and/or $|s_\alpha| \ll 1$ occurs then

$$\hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}) \xrightarrow[|s|\to 0]{} \tfrac{1}{2}\hat{T}^{(+)}_{\beta\alpha} , \qquad (14)$$

which points out the vital importance of including multistep processes in the limit of strongly absorbing projectiles being used.

Equation (12) allows us in principle to eliminate the multistep "on-shell" contribution entirely. For simplicity we shall indicate the method only for Eq. (13) wherein the sets $\beta(U_\beta)$ and $\alpha(U_\alpha)$ are single-valued as in conventional DWBA. We find

$$\hat{T}_{\beta\alpha} = 2(1 - |s_\beta|^2|s_\alpha|^2)^{-1} \times [\hat{T}_{\beta\alpha} - \hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}) - s_\beta s_\alpha(\hat{T}_{\alpha\beta} - \hat{T}^{(M)}_{\alpha\beta}(\mathrm{ON}))^*],$$

which has a "leading term"

$$\hat{T}^{(L)}_{\beta\alpha} = 2(1 - |s_\beta|^2|s_\alpha|^2)^{-1}[\hat{T}^{(1)}_{\beta\alpha} - s_\beta s_\alpha \hat{T}^{(1)*}_{\alpha\beta}] \qquad (15)$$

corresponding to a "renormalized DWBA." The usefulness of Eq. (15) is limited to situations where multistep off-shell scattering is weak and the absorption is nonnegligible in either channel α or channel β. [Clearly when $|s_\beta| = |s_\alpha| = 1$ one is trying to calculate zero over zero in Eq. (15) and the procedure is meaningless. Presumably when absorptions become weak enough the standard DWBA should be used as a leading term.]

Equation (13) represents a sort of generalized unitarity relation or "optical theorem" which can sometimes be used as an estimate of the multistep scattering contribution. In particular for charge exchange reactions where isospin is taken as a good quantum number the states β and α are related by vector coupling coefficients and in the total isospin representation one would obtain

$$\hat{T}^{(M)}_{\beta\alpha}(\mathrm{ON}) = \tfrac{1}{2}(\hat{T}_{\beta\alpha} - s_\beta s_\alpha \hat{T}^*_{\beta\alpha}) \simeq \tfrac{1}{2}(\hat{T}^{(1)}_{\beta\alpha} - s_\beta s_\alpha \hat{T}^{(1)*}_{\beta\alpha})$$

so that one expects $|T^{(M)}_{\beta\alpha}(\mathrm{ON})|^2$ cannot exceed $|\hat{T}^{(1)}_{\beta\alpha}|^2$ unless $T^{(M)}_{\beta\alpha}(\mathrm{OFF})$ (corresponding to "off-shell" multistep scattering) is important.

IV. OFF-SHELL MULTISTEP AMPLITUDES

In general the off-shell amplitude involves a sum over bound states and an integral over continuum states:

$$\hat{T}^{(M)}_{\beta\alpha}(\mathrm{OFF}) = \sum_\lambda (E - E_\lambda)^{-1}\langle x^{(-)}_\beta |\hat{V}_\beta|\Phi_\lambda\rangle\langle\Phi_\lambda|\hat{V}_\alpha|x^{(+)}_\alpha\rangle + \sum_\gamma\sum_{\gamma'}\int dE_{\lambda\gamma}\frac{P}{E - E_{\lambda\gamma}}\hat{T}_{\beta\gamma}(E_{\lambda\gamma})S^{\dagger}_{\gamma\gamma'}(E_{\lambda\gamma})\hat{T}_{\gamma'\alpha}(E_{\lambda\gamma})C_\gamma C_{\gamma'} \qquad (16)$$

in which P indicates that the principal value of the integral is to be taken. The energy variable $E_{\lambda\gamma}$ is assumed to be continuous over the range beginning at the threshold for the particular channel fragmentation (contained in the γ label) to infinity. The matrix elements $\hat{T}_{\beta\gamma}$, $\hat{T}_{\gamma'\alpha}$ are off-shell, since $E_{\lambda\gamma} \neq E$

(although $E = E_\alpha = E_\beta$ is assumed) and are given by

$$\hat{T}_{\beta\gamma}(E_{\lambda\gamma}) = \langle x_\beta^{(-)} | \hat{V}_\beta | \Psi_{\lambda\gamma}^{(+)} \rangle$$
$$= \langle x_\beta^{(-)} | \hat{V}_\beta + \hat{V}_\beta (E_{\lambda\gamma} + i\eta - H)^{-1} \hat{V}_\gamma | x_\gamma^{(+)} \rangle$$

and

$$\hat{T}_{\gamma'\alpha}(E_{\lambda\gamma}) = \langle \Psi_{\lambda\gamma'}^{(-)} | \hat{V}_\alpha | x_\alpha^{(+)} \rangle$$
$$= \langle x_{\gamma'}^{(-)} | \hat{V}_\alpha + \hat{V}_{\gamma'} (E_{\lambda\gamma} + i\eta - H)^{-1} \hat{V}_\alpha | x_\alpha^{(+)} \rangle \, .$$

Notice in these expressions that $x_\gamma^{(+)}$, $x_{\gamma'}^{(-)}$ are eigenfunctions of their respective Hamiltonians with eigenvalues $E_{\lambda\gamma}$, whereas $x_\beta^{(-)}$, $x_\alpha^{(+)}$ are eigenfunctions (of different Hamiltonians in general) with eigenvalues $E_\beta, E_\alpha (= E)$, respectively.

It is worth emphasizing that the usual two potential formula relating $\hat{T}$ to the full T matrix, i.e.,

$$\underline{\hat{T}} = \underline{T} - \underline{t}$$

no longer holds in a useful way off the energy shell. We find instead the relations,

$$\hat{T}_{\beta\gamma}(E_{\lambda\gamma}) = T_{\beta\gamma}(E_{\lambda\gamma}) - \langle x_\beta^{(-)} | U_\beta | \gamma \rangle \delta_{\gamma(U_\gamma),\beta(U_\beta)}$$
$$-(E_\beta - E_{\lambda\gamma}) \langle x_\beta^{(-)} | U_\beta (E_\beta + i\eta - H_\beta)^{-1} (E_{\lambda\gamma} + i\eta - H_\beta)^{-1} V_\beta | \Psi_{\lambda\gamma}^{(+)} \rangle$$

and

$$\hat{T}_{\gamma'\alpha}(E_{\lambda\gamma}) = T_{\gamma'\alpha}(E_{\lambda\gamma}) - \langle \gamma' | U_\alpha | x_\alpha^{(+)} \rangle \delta_{\gamma'(U_{\gamma'}),\alpha(U_\alpha)}$$
$$-(E_\alpha - E_{\lambda\gamma}) \langle \Psi_{\lambda\gamma'}^{(-)} | V_{\gamma'} (E_{\lambda\gamma} + i\eta - H_\alpha)^{-1} (E_\alpha + i\eta - H_\alpha)^{-1} U_\alpha | x_\alpha^{(+)} \rangle \, ,$$

in which

$$T_{\beta\gamma}(E_{\lambda\gamma}) = \langle \beta | V_\beta + V_\beta (E_{\lambda\gamma} + i\eta - H)^{-1} V_\gamma | \gamma \rangle \, ,$$
$$T_{\gamma'\alpha}(E_{\lambda\gamma}) = \langle \gamma' | V_\alpha + V_{\gamma'} (E_{\lambda\gamma} + i\eta - H)^{-1} V_\alpha | \alpha \rangle$$

defines the corresponding T matrix "half-off-the-energy-shell."

In the above equations the terms $\langle x_\beta^{(-)} | U_\beta | \gamma \rangle$, $\langle \gamma' | U_\alpha | x_\alpha^{(+)} \rangle$ appear to be the appropriate continuations of the optical t matrix off-the-energy-shell. However the terms involving $(E_\beta - E_{\lambda\gamma})$ or $(E_\alpha - E_{\lambda\gamma})$ are new contributions which, as far as we can see, need not be small. Another related facet of the off-shell amplitudes is the breakdown of the conventional postprior relations,[1] i.e., for rearrangement collisions:

$$\langle \beta | V_\beta - V_\gamma | \gamma \rangle = (E_\gamma - E_\beta) \langle \beta | \gamma \rangle \, ,$$

$$\langle x_\beta^{(-)} | \hat{V}_\beta - \hat{V}_\gamma | x_\gamma^{(+)} \rangle = (E_\gamma - E_\beta) \langle x_\beta^{(-)} | x_\gamma^{(+)} \rangle \, ,$$

which yield zeroes explicitly only when $E_\beta = E_\gamma$ because the various states are not orthogonal.

We return now to the evaluation of Eq. (16) via iterative procedures. The sum over λ corresponding to bound eigenstates of H is expected to lead to small contributions at high energies corresponding to direct reactions. It can in many cases be calculated using a model Hamiltonian H^M which approximately reproduces the properties of the physically observed bound states (we ignore electromagnetic interactions in this discussion). The continuum contribution is more difficult to deal with since at high energies there are many possible channels (γ) open and the principal value integral necessitates a wide range of energies being used. Clearly we must invoke our physical intuition.

The most appropriate definition of a two-step process via the continuum appears to be to replace $\hat{T}$ in Eq. (16) by $\hat{T}^{(1)}$ and to use the relation

$$\underline{S}^\dagger = \underline{1} + 2\pi i \underline{\delta}(E - E')(\underline{\hat{T}} + \underline{t})^\dagger = \underline{s}^\dagger + 2\pi i \underline{\delta}(E - E') \underline{\hat{T}}^\dagger$$

in which C is suppressed and the relation $\underline{T} = \underline{\hat{T}} + \underline{t}$ is valid because S is always on-the-energy-shell. Terms involving $\hat{T}\hat{T}^\dagger\hat{T}$ are third-order in $\hat{T}$ and consequently should be ignored to yield the sum over all continuum "two-step" processes as

$$\hat{T}_{\beta\alpha}^{(2)}(\mathrm{OFF}) = \sum_\gamma \sum_{\gamma'} \int dE_{\lambda\gamma} \frac{P}{E - E_{\lambda\gamma}} \hat{T}_{\beta\gamma}^{(1)}(E_{\lambda\gamma})$$
$$\times s_{\gamma\gamma'}^\dagger(E_{\lambda\gamma}) \hat{T}_{\gamma'\alpha}(E_{\lambda\gamma}) C_\gamma C_{\gamma'} \, . \qquad (17)$$

Equation (17) appears to be the most tractable definition of the two-step mechanisms and has the nice feature that $s^{+}(E_{\lambda\gamma})$ vanishes as $E_{\lambda\gamma}\to\infty$ due to strong absorption occurring as the energy is increased. Since $\hat{T}^{(1)}$ is expected to also vanish at high energies due to poor overlaps occurring the range of energies required is reasonably finite.

Unfortunately (and unlike the on-shell situation) performing the sums over γ,γ' via a "generalized optical theorem" for the off-shell amplitudes does not lead to a useful iterative procedure because the first-order term $\hat{T}^{(1)}_{\beta\alpha}$ is independent of the energy variables in the principal value integral. It appears therefore that we are forced to consider the evaluation of Eq. (17) as the next step in the DWBA "series." The question of whether such an approach will have convergence properties is unknown and no doubt the initial thinking will have to be based on empirical successes of the numerical calculations.

SUMMARY

In the foregoing sections we have attempted to understand the role of multistep processes relative to the usual one-step DWBA process. We have shown that the most obvious method[3] of iterating to produce two-step processes is very likely to be misleading due to the "creativity" and divergence properties of the approximation $G^{(+)}\to G^{(+)}_{\gamma}$. A better treatment allows the energy conserving multistep processes to be summed and leads us to suggest the use of a "renormalized DWBA." Calculations using this method will be investigated and reported on in a later communication. In the situation where off-shell multistep scattering is important we have obtained a "calculable form" [Eq. (17)] for two-step processes.

Until now the use of DWBA (or CCBA) has been justified by phenomenological successes. The possibility of testing its validity by calculating in a consistent manner the next order approximations is a formidable but an exciting one. Hopefully the present work will provide the necessary stimulation for correct calculations aimed in this direction.

ACKNOWLEDGMENTS

The author gratefully acknowledges the hospitality and support of the Princeton University Physics Department and Cyclotron Laboratory. Particular thanks are due Professor G. T. Garvey for reading the early version of this manuscript, and Dr. J. R. Comfort for many discussions and for directing my attention to the recent work on two-step mechanisms. A critical reading of the present manuscript by Professor R. Koshel and Professor E. Rost is much appreciated.

---

*Work supported in part by National Science Foundation Grants Nos. GP-15855 and GP-32884.

†Alfred P. Sloan Research Fellow.

‡On leave from Physics Department, Florida State University.

[1]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1967).

[2]K. Greider and L. Dodd, Phys. Rev. 146, 671 (1966).

[3]M. Toyama, Phys. Letters 38B, 147 (1972); R. F. Schaefer and G. F. Bertsch, *ibid.* 38B, 159 (1972); P. D. Kunz and E. Rost, Bull. Am. Phys. Soc. 17, 509 (1972); P. K. Bindal and R. D. Koshel, to be published.

[4]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Vol. 1, p. 884.

[5]N. F. Mott and H. S. M. Massey, *The Theory of Atomic Collisions* (Oxford: University Press, 1949), Chap. VIII.

[6]See, for example, R. J. Ascuitto and N. K. Glendenning, Phys. Rev. 181, 1396 (1969).

[7]To be more precise there is also a difference between the continuum states $\chi^{(\pm)}_{\gamma}$ in Eqs. (9) and (11). The states in Eq. (9) are all solutions of the same optical-model potential which describes the scattering matrix at the given energy E. The states in Eq. (11) are solutions of the optical potential at each energy E_{γ} so that $S^{\dagger}_{\gamma\gamma'}=s^{\dagger}_{\gamma\gamma'}$ for all E_{γ}. To the extent that the optical operators are slow functions of energy and due to the qualitative nature of the discussion it appears reasonable to ignore this difference.