

Relation between the deuteron form factor at high momentum transfer and the high energy neutron-proton scattering amplitude

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A nonrelativistic potential-model version of the factorization assumption, used in perturbative QCD calculations of hadronic form factors, is used, along with the Born approximation valid at high energies, to derive a remarkably simple relationship between the impulse approximation contribution to the deuteron form factor at high-momentum transfer and the high-energy neutron-proton scattering amplitude. The relation states that the form factor at a given value of Q^2 is proportional to the scattering amplitude at a specific energy and scattering angle. This suggests that an accurate computation of the form factors at large Q^2 requires a simultaneous description of the phase shifts at a related energy, a statement that seems reasonable regardless of any derivation. Our form factor-scattering amplitude relation is shown to be accurate for some examples. However, if the potential consists of a strong short distance repulsive term and a strong longer-ranged attractive term, as typically occurs in many realistic potentials, the relation is found to be accurate only for ridiculously large values of Q . More general arguments, using only the Schrödinger equation, suggest a strong, but complicated, relationship between the form factor and scattering amplitude. Furthermore, the use of recently obtained soft potentials, along with an appropriate current operator, may allow calculations of form factors that are consistent with the necessary phase shifts.

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I. INTRODUCTION

The deuteron form factor $A(Q^2)$ has been measured at Jefferson Laboratory and four momentum transfers up to $Q^2=6$ GeV² [1], improving the SLAC measurements [2], and measurements at larger Q^2 are planned. These efforts have caused much interest on improving calculations of the form factors at higher values of Q^2 up to about 11 GeV². The best calculations are elegant in their use of the very latest realistic, high-precision nucleon-nucleon potentials. These potentials are based on using detailed knowledge of the long- and medium-range parts of the potentials and on using artful modeling of the short-distance physics. Typically, the parameters of the potentials are tuned to obtain an accurate reproduction of the measured phase shifts up to 300 MeV laboratory kinetic energy, T . Increasing the range of energies of the validity of the potential should increase the ability of the potential to describe those aspects of the deuteron wave function which enter at high-momentum transfer [3].

Indeed it seems reasonable to expect that describing a form factor at a given momentum transfer Q^2 would require a reproduction of the large angle nucleon-nucleon scattering amplitude at

$$T \sim \frac{Q^2}{2m_N}, \quad (1)$$

where m_N is the nucleon mass. Qualitatively this is because the relative momentum, which dominates the overlap integral for the form factor is $\sim Q/2$. The implication of Eq. (1) is that for $Q^2 > 1$ GeV² one needs to use a potential that

successfully describes the neutron-proton phase shifts for $T > 500$ MeV. To our knowledge, no such consistency check of the potential used to compute the deuteron wave function has been made.

One might think that a simple kinematic relationship such as Eq. (1) might not apply because the nucleons in the deuteron are bound. Therefore we need to demonstrate explicitly that there is a strong relationship between the form factor and the scattering amplitude at large energies and large angles. Indeed, the specific result [see Eqs. (15) and (17) below] differs quantitatively from Eq. (1). This is obtained using the simplest possible dynamics: nonrelativistic spin-less nucleons interacting with an energy-independent local potential. Clearly, it is not our purpose to be realistic. Instead we merely wish to point out that, under certain conditions, the form factor can be proportional to the scattering amplitude.

The derivation of the relation between the form factor and the scattering amplitude proceeds in Sec. II by applying the factorization approximation commonly used in perturbative QCD derivations of hadronic form factors along with the first Born approximation expected to be valid at high energies. The requirements for each approximation are investigated and correction terms obtained. The accuracy of the approximations and the resulting form factor scattering amplitude relation are studied using simple interactions: attractive Coulomb potential and sum of attractive and repulsive square wells in Sec. III, and some implications for other models are discussed. A detailed numerical study using the Malfliet-Tjon potential is made in Sec. IV. The specific approximations used in Sec. II appear to be marginally successful, if the potential obtains a total weak attraction by combining strong

repulsion at short distances with strong attraction at larger distances. Therefore, a more general argument, using only the Schrödinger equation, is presented in Sec. V. A discussion of the implications of our results is presented in Sec. VI.

The reader may immediately question the use of nonrelativistic (NR) dynamics, because many specific relativistic effects are known. For a recent review, see Ref. [4]. However, NR dynamics are not irrelevant at $T=500$ MeV. Furthermore, the use of relativistic light front dynamics shows that relativistic dynamics is not very different from nonrelativistic dynamics: two nucleons dominate, there is a wave equation, and the specific relativistic effects in the deuteron are not very large unless Q^2 is very high. The specific differences between the nonrelativistic and light-front approaches are relatively well understood and lead to a small easing of the constraint (1) for $Q^2 \geq m_d^2$. Therefore, we turn to the necessary derivation without further apology.

II. BASIC IDEA

The deuteron wave function ψ is defined by the Schrödinger equation:

$$\left(\frac{p^2}{2\mu} + V\right)\psi = -\epsilon_B\psi, \quad (2)$$

where $\mu = m_N/2$. The form factor is given in terms of the momentum-space wave function as

$$F(q) = \int d^3p \psi(p) \psi(|\mathbf{p} + \mathbf{q}/2|). \quad (3)$$

In the widely employed Breit frame $\mathbf{q}^2 = Q^2$. For large enough values of q , and for a potential (expressed in momentum space) that decreases as a power of q , the integral may be simplified because q can be much larger than the typical values of p for which the wave function is near its largest value. The deuteron wave function is known to have a limited momentum content, being essentially 0 for momenta greater than about 600 MeV/ c .

We aim for a simplification of Eq. (3) that is valid at large momentum transfer. The basic assumption is that, for large enough values of q , one can regard $q \gg p$ even though the integral extends over all values of p . This leads to the factorization approximation commonly used in perturbative QCD calculations of hadronic form factors. A pedagogic discussion of the technique we employ is presented in Ref. [5]. There are two regions for which the integrand of Eq. (3) is largest: $p \approx 0$ for which the first wave function is large, and $\mathbf{p} \approx -\mathbf{q}/2$ for which the second wave function is large. The regions contribute equally so that we may say

$$F(Q^2) \approx F_a(Q^2) = 2\psi(q/2) \int d^3p \psi(p), \quad (4)$$

with the integral over all momentum being proportional to the coordinate space wave function evaluated at the origin:

$$\psi(r=0) = \int \frac{d^3p}{(2\pi)^{3/2}} \psi(p). \quad (5)$$

One may also derive Eq. (4) using coordinate space arguments and determine the leading correction term. By regarding $p \ll q/2$ one finds [6] the approximate form factor and its leading correction term:

$$\begin{aligned} F(q) &= \int d^3p \psi(p) \int \frac{d^3r}{(2\pi)^{3/2}} \psi(r) \exp[i(\mathbf{p} + \mathbf{q}/2) \cdot \mathbf{r}] \\ &\approx \int d^3p \psi(p) \int \frac{d^3r}{(2\pi)^{3/2}} \psi(r) (1 - p^2 r^2/6) \exp(i\mathbf{q}/2 \cdot \mathbf{r}) \\ &\approx F_a(Q^2) - (2\pi)^{3/2} \nabla^2 \psi(r=0) \frac{\nabla^2}{6} \psi(q). \end{aligned} \quad (6)$$

If the integrals exist, and if the wave functions fall as a power of q , the ratio of the second to the first terms of Eq. (6) is proportional to $[\int p^2 dp \psi(p) p^2 / \int p^2 dp \psi(p)] / q^2$ which vanishes for sufficiently large values of q^2 .

According to Eq. (4), the momentum transfer dependence of the form factor is obtained from the wave function $\psi(q/2)$, for sufficiently large values of Q^2 . Thus we attempt to obtain $\psi(q)$ from the momentum-space version of Eq. (2):

$$\psi(q/2) = \frac{1}{-\epsilon_B - \frac{q^2}{8\mu}} \int d^3p \langle \mathbf{q}/2 | V | \mathbf{p} \rangle \psi(p). \quad (7)$$

We again use the idea that q can be much greater than p , so that the wave function $\psi(p)$ can be approximated as

$$\begin{aligned} \psi(q/2) &\approx \frac{-8\mu}{q^2} \langle \mathbf{q}/2 | V | \mathbf{0} \rangle \int d^3p \psi(p) \\ &= \frac{-8\mu}{q^2} \langle \mathbf{q}/2 | V | \mathbf{0} \rangle (2\pi)^{3/2} \psi(r=0). \end{aligned} \quad (8)$$

This approximation seems very natural, if one is used to the ideas of perturbative QCD. In particular, the entire momentum transfer is taken up by a single action of the potential, so that the important positions in coordinate space region are those for which the potential has its greatest variation.

It is useful to examine Eq. (8) from the view of coordinate space. One has

$$\begin{aligned} &\int d^3p \langle \mathbf{q}/2 | V | \mathbf{p} \rangle \psi(p) \psi(p) \\ &= \int \frac{d^3r}{(2\pi)^{3/2}} \exp(-i\mathbf{q}/2 \cdot \mathbf{r}) V(r) \psi(r). \end{aligned} \quad (9)$$

One may now observe that the approximation, Eq. (8), relies on replacing the product $V(r)\psi(r)$ by $V(r)\psi(0)$. This is allowed only if the potential has a much more significant variation than the wave function for positions near the origin.

The expression, Eq. (8), can be used to obtain the form factor from Eq. (3). The result is

$$F_b(Q^2) \approx \left(\frac{-16\mu}{q^2} \right) \langle \mathbf{q}/2 | V | \mathbf{0} \rangle \Phi^2, \quad (10)$$

$$\Phi \equiv \int d^3p \psi(p). \quad (11)$$

The approximation (10) has been verified numerically in Ref. [7] for the case of the semirealistic Hulthén model.

The content of Eq. (10) is that the form factor is proportional to the square of the bound state wave function at the origin times the Fourier transform of the potential:

$$\begin{aligned} \langle \mathbf{q}/2 | V | \mathbf{0} \rangle &= \frac{1}{(2\pi)^3} \int d^3r e^{(i/2)\mathbf{q}\cdot\mathbf{r}} V(r) \\ &= \frac{4\pi}{(2\pi)^3} \int r^2 dr \frac{\sin \frac{1}{2}qr}{\frac{1}{2}qr} V(r). \end{aligned} \quad (12)$$

The desired relation between the form factor and the scattering amplitude can be obtained by realizing that at energies E much greater than the characteristic strength of the potential, the first Born approximation is valid and the scattering amplitude for a c.m. angle θ , $f_E(\theta) = f_E(\mathbf{k}, \mathbf{k}')$, $E = k^2/2\mu = k'^2/2\mu$, $\mathbf{k} \cdot \mathbf{k}' = k^2 \cos \theta$ is also a Fourier transform of the potential. The validity of the first Born approximation, at high energies, for energy-dependent local potentials, is well verified in many quantum mechanics textbooks. In that case

$$f_E(\theta) \approx - \frac{4\pi^2\mu}{(2\pi)^3} \int d^3r e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} V(r). \quad (14)$$

The relation between $f_E(\theta)$ and $\langle \mathbf{q}/2 | V | \mathbf{0} \rangle$ is obtained by specifying

$$\frac{1}{4}q^2 = 2k^2(1 - \cos \theta) = 4\mu E(1 - \cos \theta) = Tm_N(1 - \cos \theta). \quad (15)$$

Imposing the relation (15) immediately yields

$$f_E(\theta) = -4\pi^2\mu \langle \mathbf{q}/2 | V | \mathbf{0} \rangle, \quad (16)$$

and with Eq. (10)

$$F(Q^2) = \frac{1}{\pi^2 q^2} f_E(\theta) \Phi^2. \quad (17)$$

This is the result we have been seeking. The key point is that the form factor is expressed as the high-energy scattering amplitude times well-defined factors, implying that a correct calculation for the form factor can only be achieved in models in which the scattering amplitude is accurately reproduced.

III. TOY MODELS

The arguments used in the preceding section are analogous to those presented in derivations of QCD factorization

theorems. However, these have not been used often for nuclear targets. We therefore discuss two simple examples.

A. Coulomb binding

We take the potential to be

$$V(r) = -\frac{g^2}{r}, \quad (18)$$

with the exact wave function given by

$$\psi(r) = Ne^{-r/a}, \quad (19)$$

with $a = 1/(\mu g^2)$. The form factor $F(q)$ of Eq. (3) is given by

$$F(q^2) = \frac{1}{(a^2 q^2 + 16)^2}. \quad (20)$$

The wave function in momentum space is

$$\psi(p) = \frac{4\pi N}{(2\pi)^{3/2}} \frac{2}{a \left(q^2 + \frac{1}{a^2} \right)^2}, \quad (21)$$

so that the approximation of Eq. (4) yields

$$F_a(q^2) = \frac{1}{(a^2 q^2 + 4)}, \quad (22)$$

F_a is close to F for $q^2 a^2 \gg 3$. Using $\epsilon_B = a^2/2\mu$ and the deuteron binding energy and $\mu = M_N/2$ means that all that is required is $q \gg 75 \text{ MeV}/c$. The approximation of Eq. (10) yields

$$F_b(q^2) = \frac{1}{a^4 q^4}, \quad (23)$$

which is valid under similar conditions.

The next task is to determine the conditions that the first Born approximation, Eq. (14), be valid. It is well known that, for the Coulomb interaction, this approximation reproduces the exact scattering cross section. However, the correct scattering amplitude is complex, while the approximation, Eq. (14), gives a real result. Thus the condition that Eq. (14) be valid is the condition that the s -wave scattering phase shift be small. This is the condition that $g^2 M_N / (2k) \ll 1$, or $ka \gg 1$. Since $a = 1/45 \text{ MeV}/c$, one needs only $k \gg 45 \text{ MeV}/c$.

Thus the approximations of the preceding section are easily satisfied for dynamics defined by the Coulomb potential. More generally, it is reasonable to expect that, if the potential is local, purely attractive with a significant gradient at the origin and magnitude determined by the very small deuteron binding energy (2.2 MeV), the approximations needed to reach the form factor scattering amplitude relation (17) are well satisfied. In such cases, the potential varies more rapidly than the wave function for positions near the origin, and the potential is weak enough so that the first Born approximation can be accurate for reasonable momentum transfers.

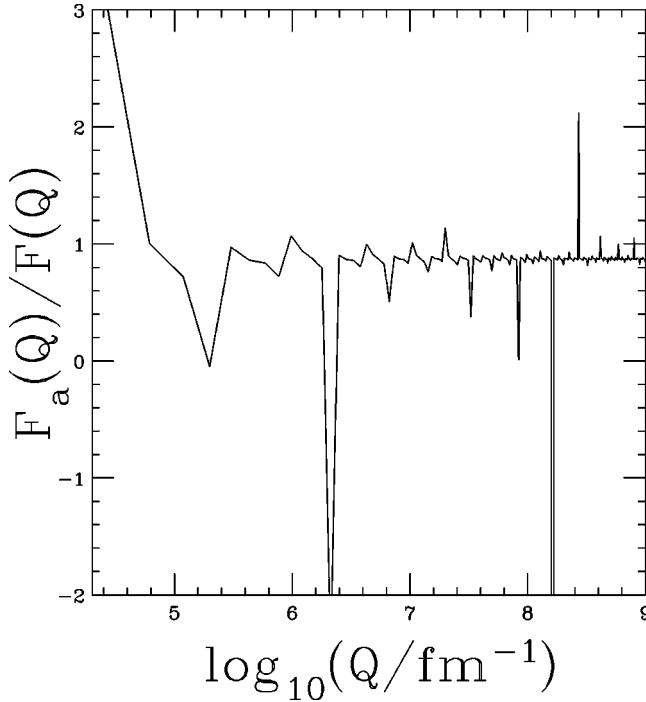


FIG. 1. Ratio of approximate to exact form factor.

B. Short-ranged repulsive square well plus long-ranged attractive square well

The nuclear force is repulsive at short distances and attractive at long distances between nucleons. Thus we consider the model defined by

$$V(r) = V_0\theta(R_0 - r) - V_1\theta(R_1 - r)\theta(r - R_0), \quad V_{0,1} > 0. \quad (24)$$

We take $R_0 = 0.4$ fm and $R_1 = 1.5$ fm. Then the observed deuteron binding energy is reproduced using $V_0 = 0.302$ fm⁻¹, $V_1 = 29.18$ fm⁻¹ = 5.757 GeV. This corresponds to a hard core repulsion. The wave function $\psi(r)$ is given (up to an overall normalization constant) by

$$\begin{aligned} \psi(r) &= \frac{\sin KR_0 \sinh \gamma r}{\sin KR_1 \sinh \gamma R_0}, \quad r < R_0 \\ &= \frac{\sin Kr}{\sin KR_1}, \quad R_0 \leq r < R_1 \\ &= \frac{\exp(-r/a)}{\exp(-R_1/a)}, \quad r \geq R_1. \end{aligned} \quad (25)$$

The exact $[F(q)]$ and approximate $[F_a(q)]$ Eq. (4) form factors are compared by displaying their ratio in Fig. 1. There are some wild fluctuations arising from nodes of $F(Q)$, but these are not the most interesting feature. The true difficulty is that the approximation becomes valid only for extremely large values of $q > 100$ fm⁻¹. Furthermore, the ratio $F_a(q)/F(q)$ is 0.87 instead of unity. That difficulties exist can be seen immediately by comparing Eqs. (4) and (25). The coordinate-space wave function vanishes at the origin, so the leading approximation vanishes. This signals a breakdown of

the derivation. The point is that the factorization, Eq. (4), depends on the potential varying more rapidly near the origin than the wave function.

One might then be amazed that the approximate form factor is even close to the exact one. That this occurs can be understood from the discussion below, Eq. (6). The action of ∇_q^2 does not introduce an extra factor of $1/Q^2$ because the momentum-space wave function contains terms $\sim \sin QR_{0,1}/Q^2$.

This section contains two simple models. The approximations work magnificently if the Coulomb potential is used, but fail miserably with the nucleon-force-motivated model of Eq. (24). Let us examine a more realistic model to see if our approximations are relevant for understanding deuteron physics.

IV. MALFLIET-TJON POTENTIAL

The short-distance repulsion and longer-ranged attraction are simulated via potentials of the Yukawa form in the Malfliet-Tjon potential [8]:

$$V(r) = -\lambda_A \frac{e^{-\mu_A r}}{r} + \lambda_R \frac{e^{-\mu_R r}}{r}. \quad (26)$$

The parameters were chosen to reproduce the deuteron binding energy, scattering length, effective range, and phase shifts up to a laboratory energy of 300 MeV. This potential is much smoother than the square wells of Eq. (24).

Our aim is to study the approximations given in Eqs. (4) and (8) and the Born approximation of Eq. (14). The comparison between the exact form factor $F(Q^2)$ and the approximation $F_a(Q^2)$ of Eq. (4) is displayed in Fig. 2.

The approximation does work, but only for huge values of Q . The value of $Q \sim 140$ fm⁻¹ or $Q^2 \approx 900$ (GeV/c)² for its natural logarithm to reach the value of 5. One may search for the cause of this bizarre limit by examining the correction term shown of Eq. (6). This is governed by the kinetic energy operator acting on the wave function at the origin. The Schrödinger equation says that this is essentially the potential times the wave function, with a product that varies as $(\lambda_R - \lambda_A)$ near the origin. The strong nature of the repulsive term $\lambda_R = 7.41$ causes the expansion shown in Eq. (6) to converge very slowly.

One may gain further insight by studying the relationship between the exact and approximate wave function of Eq. (8). As shown in Fig. 3, the approximation attains validity only at supremely large values of the relative momentum. Here accuracy requires $V(r)\psi(r) \approx V(r)\psi(0)$ for values of r near the origin. For the Malfliet-Tjon potential $V(r) \sim 1/r$ and $\psi(r)$ approaches 0, in contrast to the Coulomb wave function. From this, the failure of the approximation seems natural. Moreover, many realistic potentials behave in a similar fashion.

The final requirement needed to achieve the result (17) is the validity of the Born approximation. Our comparison of the exact and Born approximation to the forward scattering amplitude is shown in Fig. 4. The partial wave expansion is used unless the energy is large enough (lab energy greater

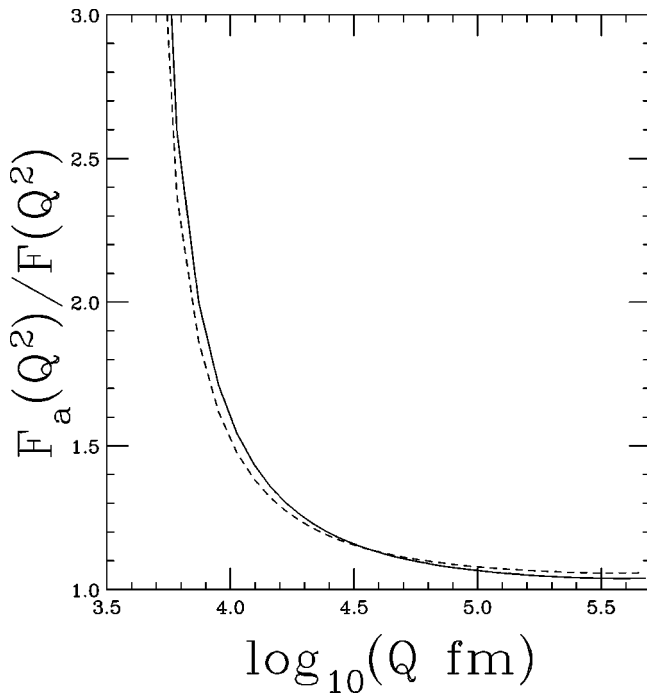


FIG. 2. Ratio of approximate to exact form factor for the Malfliet-Tjon deuteron wave function. The solid curve compares the approximation of Eq. (17) with the exact form factor. The dashed curve compares the approximation of Eq. (4) with the exact form factor.

than 1 or 2 GeV) for the eikonal approximation to reproduce the exact scattering amplitude. Figure 4 shows that the Born approximation does become valid, but only at absurdly high energies. This figure shows results only for forward scatter-

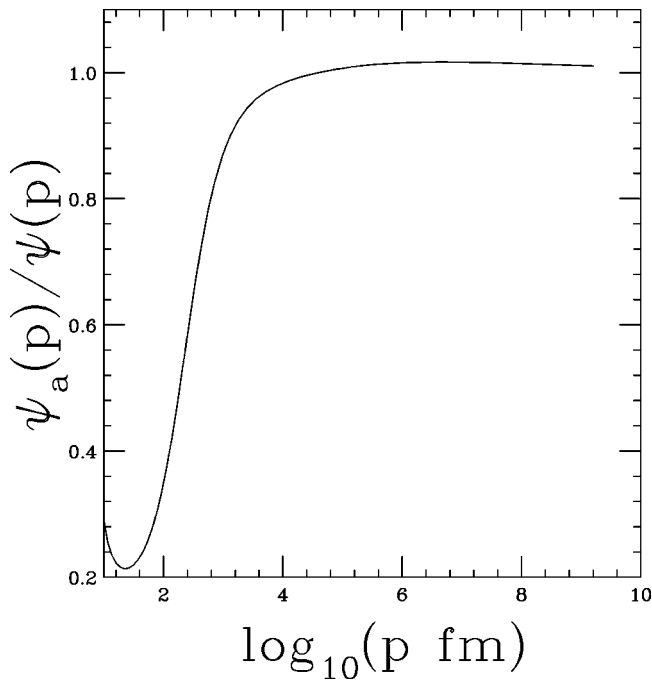


FIG. 3. Ratio of approximate to exact wave function for the Malfliet-Tjon deuteron wave function.

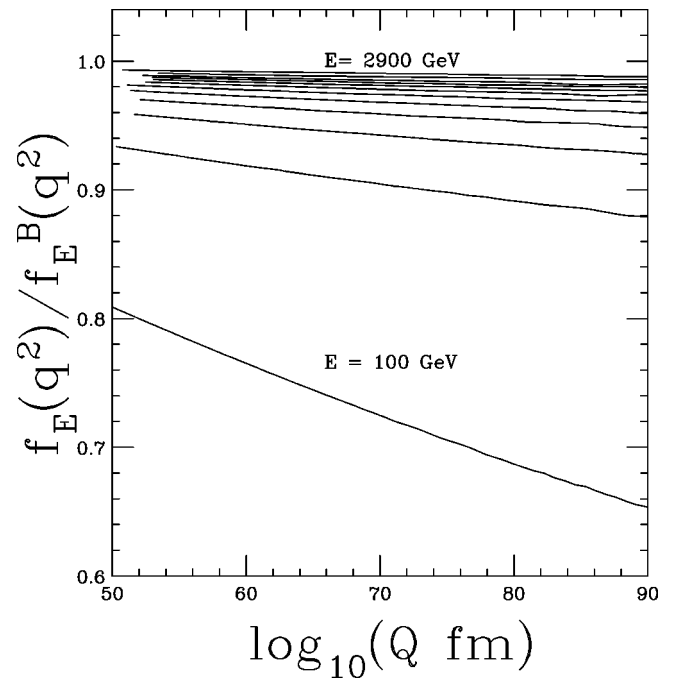


FIG. 4. Ratio of approximate to the real part of the exact forward scattering amplitude for the Malfliet-Tjon potential. E is a laboratory energy.

ing, whereas our relation (17) requires high-momentum transfer. The convergence to the Born approximation occurs for higher energies for large scattering angles, if the Malfliet-Tjon potential is used.

The slow approach to the Born approximation can be understood from the eikonal formalism in which the scattering amplitude is expressed in terms of an integral

$$f_E(\theta) = -ik \int_0^\infty b db J_0[2kb \sin(\theta/2)] (e^{i\chi(b)} - 1), \quad (27)$$

with the phase shift function $\chi(b)$ given as

$$\chi(b) = -\frac{\mu}{k} \int_0^\infty dz V(\sqrt{b^2 + z^2}). \quad (28)$$

For the Malfliet-Tjon potential, $\chi(b)$ is given by

$$\chi(b) = \frac{m}{k} [\lambda_R K_0(b\mu_R) - \lambda_A K_0(b\mu_A)], \quad (29)$$

which is small only for very large values of the relative momentum k . While the overall potential is able to produce only one weakly bound state, the weak attraction arises from the cancellation of two very strong terms of opposite sign that generally causes large values of $\chi(b)$.

The factorization approximations for Eqs. (4) and (8) and the Born approximation of Eq. (14) work for sufficiently large values of the transferred momenta. A reasonable reader might examine the Figs. 2–4 and conclude that “sufficiently large” means too large to be relevant for real experiments. However, success at asymptotically high momenta indicates that the key assumption Eqs. (4) and (8) that the form factor

is proportional to the wave function at the origin is a very stringent requirement that is especially difficult to satisfy for a wave function that vanishes at the origin. Furthermore, high-momentum transfer scattering may proceed via a set of small momentum transfer processes under certain circumstances. Thus the condition that the Born approximation (scattering by a single action of the potential) be valid is also a very strong one, as is spectacularly manifest in Fig. 4. Truly magnificently high energies are required for the Born approximation to be valid.

While the relation, Eq. (17), suggests a close connection between the form factor and the scattering amplitude it is worthwhile to ask if one can derive a connection between the form factor and the scattering amplitude that does not directly involve the wave function at the origin and single-scattering assumptions. The desired connection is displayed in the next section.

V. GENERAL APPROACH

The central idea is that the wave function is dominated by its low-momentum components and that high-momentum components are obtained by at most one high-momentum transfer operation. We intend to manipulate the Schrödinger equation to derive an explicit relation between the form factor and the scattering amplitude. Projection operators are used to develop the necessary formalism. Let

$$Q \equiv \int d^3p \theta(\Lambda - p) |\mathbf{p}\rangle \langle \mathbf{p}|, \quad P \equiv \int d^3p \theta(p - \Lambda) |\mathbf{p}\rangle \langle \mathbf{p}|, \quad (30)$$

with P a projection operator on high-momentum states and Λ a parameter, approximately 600 MeV/ c or higher, denoting the separation between the high- and low-momentum transfer regions. Then the Schrödinger equation can be expressed in terms of low, $Q|\psi\rangle \equiv |\psi\rangle_Q$, and high, $P|\psi\rangle \equiv |\psi\rangle_P$, momentum components:

$$|\psi\rangle_P = \frac{1}{-\epsilon_B - T_{PP}} (V_{PP}|\psi\rangle_P + V_{PQ}|\psi\rangle_Q), \quad (31)$$

$$|\psi\rangle_Q = \frac{1}{-\epsilon_B - T_{QQ}} (V_{QQ}|\psi\rangle_Q + V_{QP}|\psi\rangle_P), \quad (32)$$

where the notation $PTP = T_{PP}$, $PVP = V_{PP}$, etc. is used for the kinetic energy T and potential energy operators. Substitute Eq. (32) into Eq. (31) to obtain

$$|\psi\rangle_P = \frac{1}{-\epsilon_B - T_{PP}} \left(V_{PP} + V_{PQ} \frac{1}{-\epsilon_B - T_{QQ} - V_{QQ}} V_{QP} \right) |\psi\rangle_P. \quad (33)$$

Consider the complete eigenstates of the Hermitian, energy-independent operator $T_{QQ} + V_{QQ}$:

$$E_k |\phi_k\rangle^{(-)} = (T_{QQ} + V_{QQ}) |\phi_k\rangle^{(-)}, \quad (34)$$

$$E_n |\phi_n\rangle = (T_{QQ} + V_{QQ}) |\phi_n\rangle, \quad (35)$$

in which $|\phi_k\rangle^{(-)}$ is a scattering state with incoming boundary conditions, $E_k = k^2/2\mu$, and $E_n \neq -\epsilon_B$, $|\phi_n\rangle$ are the energies and wave functions of any bound states that may exist. The use of completeness in Eq. (33) leads to

$$|\psi\rangle_P = \frac{1}{-\epsilon_B - T_{PP}} \left(V_{PP} + \int d^3k \frac{V_{PQ} |\phi_k\rangle^{(-)} \langle \phi_k| V_{QP}}{-\epsilon_B - E_k} + \sum_n \frac{V_{PQ} |\phi_n\rangle \langle \phi_n| V_{QP}}{-\epsilon_B - E_n} \right) |\psi\rangle_P. \quad (36)$$

The form factor depends on the high-momentum components of the deuteron wave function, so consider the quantity $\langle \mathbf{q}/2 | \psi \rangle_P$ for $Q/2 > \Lambda$. Then $\langle \mathbf{q}/2 | \psi \rangle_P = \langle \mathbf{q}/2 | \psi \rangle$. The second term on the right-hand side of Eq. (36) is determined by the matrix elements $\langle \mathbf{q}/2 | V | \phi_k \rangle^{(-)}$, which are scattering amplitudes for off-energy-shell kinematics [9]. These matrix elements are the terms which contain the relation between the form factor and the scattering amplitude. To see this, take the parameter Λ to be just a bit less than $q/2$. The largest of the matrix elements $\langle \mathbf{q}/2 | V | \phi_k \rangle^{(-)}$ will be those for which k is just a bit less than Λ . In this case, the kinematics are nearly on shell. Transition matrix elements are continuous, so that it is reasonable to expect that reproducing the observed scattering amplitudes at high energy is important for reproducing the important features of the high-momentum components of bound state wave functions.

VI. DISCUSSION

The use of two factorization approximations, Eqs. (4) and (8), combined with the first Born approximation, Eq. (14), lead to a statement (17) that the form factor is proportional to the scattering amplitude times the square of the coordinate space wave function at the origin divided by Q^2 . A relation very similar to Eq. (17) was derived long ago [10] using a scale-invariant, six-quark model. It is also true that the relation between the deuteron form factor and the scattering amplitude has also been the subject of Ref. [11] in which dispersion relations are used to compute the deuteron charge form factor with experimental phase shifts as the essential input. Thus one sees the close relation between $F(Q^2)$ and f_E from a variety of different approaches: nonrelativistic dynamics, light-front dynamics, quark models, and dispersion relations.

But the validity of Eq. (17) depends on the use of factorization approximations and the Born approximation that might not be valid for realistic nucleon-nucleon potentials. The analysis of toy models, Sec. II, and the Malfliet-Tjon potential, Sec. III, suggests that only a soft potential can be expected to satisfy the conditions necessary to obtain the result (17). Until recently, the idea that a soft potential could also be realistic was only a faint hope. But recent work, using effective field theory have introduced a set of soft-realistic potentials [12–14]. It is possible that these potentials, which do not contain those features of the Malfliet-Tjon potential that violate the necessary approximations, could satisfy Eq. (17).

It is worthwhile to suppose that Eq. (17) could be accurate for some soft realistic potential. Then we may estimate the kinetic energies required for reproducing the phase shifts. Since we are dealing with the S -wave deuteron wave function we can take $\cos \theta=0$, to correspond to the maximal momentum transfer for identical particles. In this case, Eq. (15) gives

$$Q^2 = 4m_N T. \quad (37)$$

The use of this is simple. Suppose the phase shifts are well described up to about $T=350$ MeV, which is the standard stated upper limit. Then one can calculate the form factor up to only $Q^2 \approx 1.4$ GeV². But modern high-precision NN potentials actually describe the data up to laboratory energies of about 1 GeV [15,16]. Thus one may compute the form factors up to $Q^2 \approx 4$ GeV². However, present measurements reach $Q^2=6$ GeV², and there are plans to reach higher values. Thus the constraints we present may be relevant for present and future measurements.

An additional worry is that the analysis of the previous paragraph might require a relativistic treatment. If we instead were to use light cone models as in Refs. [17,18] we would find that up to $Q^2 \sim m_d^2$ the light cone fractions of the nucleons are approximately equal and the relation, Eq. (15), holds, but with $T=2E+E^2/2m_N$. At larger values of Q^2 , the increase of the effective invariant energy with Q^2 decreases somewhat. In any case, one can see that for $Q^2 \geq 2$ GeV² one reaches the region where masses in the intermediate state exceed 3 GeV and the legitimacy of the two nucleon approximation becomes highly questionable, as discussed in

Ref. [16]. Then one would need to include either new hadronic degrees of freedom in the deuteron wave function: $\pi NN, \Delta\Delta, \dots$, or to account explicitly for quark-gluon degrees of freedom.

The implications of our result (17) is that potentials used to obtain the deuteron wave function should be tested by computing the corresponding phase shifts. If one wants an accurate calculation, the phase shifts need to be correctly obtained up to kinetic energies given by Eq. (37). This requirement appears, according to the arguments of Sec. V, to be more general [see Eq. (36)] than the accuracy of the asymptotic relation between the scattering amplitude and the form factor, Eq. (17). The slow onset of asymptotia occurs, in part, because the wave function at the origin is severely suppressed by strong short distance repulsion, and because single hard scattering is not strongly dominant over the possibility of achieving a high-momentum transfer as a result of several small momentum transfer processes. Thus there are cancellations between the contribution dominating in the asymptotic region and the other contribution. All together this suggests that potentials currently employed to compute deuteron form factors be tested for consistency with high-energy phase shifts. A soft potential [11–13] and an appropriately derived current operator should also be used.

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