Dilute nuclear matter in chiral perturbation theory

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We use chiral perturbation theory to compute the effective nucleon propagator in an expansion about low density in the chiral limit. We neglect four-nucleon interactions and focus on pion exchange. Evaluating the nucleon self-energy on its mass shell to leading order, we show that the effective nucleon mass increases by a small amount. We discuss the relevance of our results to the structure of compact stars.

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As nuclear matter is compressed, eventually a transition to quark-gluon matter occurs. By asymptotic freedom, at very high densities the equation of state can be computed in perturbation theory [1–3]. This can be extended to moderate densities by various approximation schemes [2,3]. At low densities, the conventional approach is to use phenomenological potentials to fit observed properties of nuclear matter [4] and then extrapolate up in density.

How the nuclear equation of state matches onto that for quark matter is of great significance for astrophysics [3,5]. The standard expectation, such as in quantum hadrodynamics, for example [6], is that hadronic pressure rises quickly to a value near that for an ideal Fermi gas of quarks and even exceeds it at densities above normal nuclear density. In this case there is only one type of hadronic star which might have a (small) quark core. If the hadronic pressure is small relative to that of ideal quarks, though, then there can be two classes of hadronic stars. There are "ordinary" neutron stars, which are mainly composed of nucleons. In addition, there are stars with a large quark core; their mass and radius are (approximately) half that of ordinary neutron stars.

Thus it is imperative to understand the equation of state for nuclear matter. In Ref. [7], Savage and Wise compute mass shifts for the baryon octet using chiral perturbation theory, including all operators which contribute to leading order in the density. Due to four-nucleon interactions, they find that all masses decrease with increasing density; extrapolating to nuclear matter densities, the shifts are considerable. The self-energies were computed at zero momentum, though, while the physical point is on the mass shell. In the presence of a Fermi sea, the mass shell changes, in a way which is easily computed. To leading order in the density, the difference in mass shell only affects exchange, and not contact, terms. In this paper we compute the nucleon self-energy on its mass shell, from the diagram for pion exchange. In this case, unlike Ref. [7], the usual logarithms of chiral perturbation theory appear on the mass shell. We note that the shift in the nucleon mass is dominated by contact terms, not single pion exchange, so the following exercise is a minor point of principle.

We assume that nucleons, of mass m, are heavy, and that pions, with mass m_π, are very light. The interaction of pions and nucleons is determined by the spontaneous breaking of chiral symmetry [6,8–12]. For light pions, to leading order in chiral perturbation theory, the only parameter which enters is the pion decay constant, $f_\pi \approx 93$ MeV.

The proceeding calculation is elementary, and, besides those of Savage and Wise [7], it is similar to computations by Horowitz and Serot [9], by Bernard, Kaiser, and Meißner [10], and by Meißner, Oller, and Wirzba [11].

The parameters which enter can be understood without explicit computation. A Fermi gas of nucleons is characterized by a Fermi momentum p_f up to and including nuclear matter densities, $p_f \ll m$. In an expansion about low densities, the natural parameter which enters into the nucleon propagator is just the density, $n_{\mathrm{nucl}} \sim p_f^3$. We would like a dimensionless parameter to characterize the expansion. At one loop order, chiral perturbation theory brings in two powers of $1/f_\pi$. The only other parameter in the problem is the nucleon mass m (at least for vanishing pion mass). Thus to leading order, the corrections to the nucleon propagator are proportional to

$$\frac{p_f^3}{m f_\pi^2}, \tag{1}$$

which we now compute.

To leading order in chiral perturbation theory, we take the nucleon Lagrangian to be

$$L = \overline{\psi}\left(\partial\!\!\!/ - \gamma^0 \mu + m - i\frac{g_A}{f_\pi}\partial\!\!\!/ \pi \gamma_5\right)\psi, \tag{2}$$

where $\mu = \sqrt{p_f^2 + m^2}$ is the chemical potential, $g_A \approx 1.2$ is the axial vector coupling constant, and $\pi = \pi^a \sigma^a/2$, where the σ^a's are Pauli matrices in SU(2) flavor. Other interactions, such as between two nucleons and more than two pions, involve more powers of $1/f_\pi$, and so enter beyond leading order in the density. There are tadpole contributions, with a pion in the loop, but like the tadpoles from four-nucleon interactions, these are independent of the external momentum and so of the choice of mass shell.

Thus we consider single pion exchange, which contributes to the nucleon self-energy Σ as

$$\Sigma(P) = -\frac{3g_A^2}{4f_\pi^2}\int \frac{d^4K}{(2\pi)^4}\frac{1}{(P-K)^2+m_\pi^2} \times \gamma_5(\not{P}-\not{K})\frac{1}{-i\not{K}-\gamma^0\mu+m}\gamma_5(\not{P}-\not{K}), \qquad (3)$$

$P=(p^0,\vec{p})$ is the four-momentum of the nucleon. The diagrams are evaluated using the imaginary time formalism.

In general, the nucleon self-energy $\Sigma(P)$ is a rather complicated function of p^0 and $\vec{p}$ [9,10]. Here we shall only compute the nucleon self-energy at a special point, on its mass shell:

$$p^0 = p^0_{ms} = i(\mu - E_p) \approx i\frac{(p_f^2-p^2)}{2m}, \qquad (4)$$

E_p is the energy of a nucleon with momentum p, so $\mu=E_{p_f}$, and $E_p \approx m+p^2/(2m)+\cdots$. We also assume that the momentum p is on the order of the Fermi momentum, but it need not be especially near p_f.

Working on the mass shell allows us to greatly simplify the calculation. As we are working at nonzero fermion density, it is convenient to do the integral over k^0 first and then integrate over $\vec{k}$. In the imaginary time formalism, we first compute the diagram for real p^0 and then analytically continue to imaginary values of p^0, as in Eq. (4).

In the integrand, there are four poles: two from the pion propagator, at $k^0=p^0\pm i\sqrt{(\vec{k}-\vec{p})^2+m_\pi^2}$, and two from the nucleon propagator, at $k_0=i(\mu\pm E_k)$. Closing the contour in imaginary p^0 plane, only those poles in the upper half-plane contribute.

All we are interested in, though, are the density dependent effects. Of the four poles in the one loop diagram for the nucleon propagator, clearly one is special. The pole at which $k_0=i(\mu-E_k)$ is in the upper half-plane when $k<p_f$ and moves into the lower half-plane when $k>p_f$. For the other three poles, the sign of their imaginary part does not change with k.

All of the density dependent effects in the one loop diagram for the nucleon propagator are due to the shift in this one pole. To see this, note that the chemical potential μ only enters by changing $ip^0 \to ip^0+\mu$. If we work on the mass shell, however, then $ip^0+\mu=+E_p$; while p^0 changes with μ, $ip^0+\mu$ does not. Thus if no poles switched the sign of their imaginary part, then we would find that there were no density dependent effects in the propagator at one loop order. For example, there is wave-function renormalization for the nucleon field, but given that μ enters just as a shift in p^0, this is standard; there is no new wave-function renormalization associated with $\mu\gamma^0$, separate from $\not{p}$.

The contribution of the pole at $k^0=i(\mu-E_k)$ is simple to include: one only integrates over momentum below the Fermi surface. A similar result is found, rather more immediately, using the real time formalism. There, the nucleon propagator is the sum of two terms, one is the same as in the vacuum plus a density dependent term.

To pick up the contribution of just this one pole, we take

$$\int \frac{dk^0}{2\pi}\frac{1}{-i\not{K}-\gamma^0\mu+m} = -\frac{1}{2E_k}(i\not{K}+\gamma^0\mu+m). \qquad (5)$$

On the right-hand side, $k_0=i(\mu-E_k)$, and only $|k|<p_f$ contribute to the integral over $\vec{k}$.

Now we need to sandwich the inverse nucleon propagator in this expression between the $\gamma_5(\not{P}-\not{K})$'s from the pion vertices. There are three types of terms which contribute. One is from the term $\sim m$ in the nucleon propagator,

$$c_m = \gamma_5(\not{P}-\not{K})m\gamma_5(\not{P}-\not{K}) = -m(P-K)^2, \qquad (6)$$

one from the term $\sim\mu\gamma^0$,

$$c_\mu = \gamma_5(\not{P}-\not{K})\mu\gamma^0\gamma_5(\not{P}-\not{K}) = \gamma^0\mu[(p^0-k^0)^2-(\vec{p}-\vec{k})^2] + 2\mu(p^0-k^0)(\vec{\not{p}}-\vec{\not{k}}), \qquad (7)$$

and one from the term $\sim\not{K}$,

$$c_p = \gamma_5(\not{P}-\not{K})i\not{K}\gamma_5(\not{P}-\not{K}) = i[2(P-K)\cdot K\not{P}-(P^2-K^2)\not{K}]. \qquad (8)$$

To compute the leading terms about small density, we can greatly simplify these expressions. For example, for the nucleon energy, we can replace $E_k\approx m$, since corrections are down by $(p_f/m)^2$. Further, as we are computing on the mass shell, the energy p^0 is small relative to the spatial momentum; in magnitude, as $p^0\sim p^2/m$, p^0 is down by p_f/m relative to p. This means that in the pion propagator, and in c_m, Eq. (6), we can replace $(P-K)^2\approx(\vec{p}-\vec{k})^2$.

For the other contributions, one must be careful to keep track of relatively small terms, $\sim\vec{\not{p}}$, and also $\sim p^0\gamma^0$. For c_μ, Eq. (7), for the piece γ^0 we can drop $(p^0-k^0)^2$ relative to $(\vec{p}-\vec{k})^2$ and take $\mu\approx m$. However, for the piece $\sim\mu(p^0-k^0)$, we have to keep track of the subdominant term, so

$$c_\mu \approx -m\gamma^0(\vec{p}-\vec{k})^2 + i(p^2-k^2)(\vec{\not{p}}-\vec{\not{k}}). \qquad (9)$$

For the last term, c_p in Eq. (8), we can approximate

$$c_p \approx i[2(\vec{p}-\vec{k})\cdot\vec{k}\not{P}-(p^2-k^2)\not{K}]. \qquad (10)$$

We keep the terms $\sim\gamma^0$, which are nominally down by p_f/m, in order to extract the term $\sim p^0\gamma^0$.

Adding all of these terms together, we find a remarkable simplification

$$\Sigma(p^0_{ms},p) \approx -[(ip^0_{ms}+\mu)\gamma^0 + i\vec{\not{p}} + m]\Sigma_0(p), \qquad (11)$$

where

$$\Sigma_0(p) = \frac{3g_A^2}{8mf_\pi^2}\int_{k\leq p_f}\frac{d^3k}{(2\pi)^3}\left(\frac{(\vec{p}-\vec{k})^2}{(\vec{p}-\vec{k})^2+m_\pi^2}\right). \tag{12}$$

We also checked that the same result is found using the real time formalism.

This form is illuminating, because it is obvious that in the chiral limit, when $m_\pi=0$, the function Σ_0 is independent of momentum:

$$\Sigma_0(p) = +\frac{g_A^2}{16\pi^2}\frac{p_f^3}{mf_\pi^2} = +\frac{3g_A^2}{32}\frac{n_{\text{nucl}}}{mf_\pi^2}, \tag{13}$$

where $n_{\text{nucl}}=2p_f^3/(3\pi^2)$ is the density of nucleons.

Away from the chiral limit, $m_\pi\neq 0$, Σ_0 is momentum dependent:

$$\Sigma_0(p) = \frac{g_A^2}{16\pi^2}\frac{p_f^3}{mf_\pi^2}\left(1+\frac{3m_\pi^2}{2p_f^2}\delta\Sigma_0(p)\right), \tag{14}$$

$$\delta\Sigma_0(p) = \left(\frac{p_f^2-p^2+m_\pi^2}{4pp_f}\right)\ln\left(\frac{(p_f-p)^2+m_\pi^2}{(p_f+p)^2+m_\pi^2}\right) + \frac{m_\pi}{p_f}\left[\arctan\left(\frac{p_f+p}{m_\pi}\right)+\arctan\left(\frac{p_f-p}{m_\pi}\right)\right]-1. \tag{15}$$

At zero momentum, $p=0$, this agrees with Savage and Wise [7]. We see that chiral logarithms appear when $p\neq 0$, although there is an $\arctan(p_f/m)$ at $p=0$. These chiral logarithms are standard [8], and relatively innocuous. Even at the Fermi surface, $p=p_f$, they vanish like $m_\pi^2\ln(m_\pi)$ as $m_\pi\to 0$.

One can compute Σ_0 as a function of m_π. For illustration, consider its value at the Fermi surface. Then one can show that increasing the pion mass tends to decrease the value of Σ_0; as $m_\pi\to\infty$, Σ_0 vanishes like $\approx p_f^3/(mf_\pi^2)(p_f^2/m_\pi^2)$. That Σ_0 vanishes like $\approx 1/m_\pi^2$ at large m_π is evident from the integral representation, Eq. (12).

We can use these results to compute the nature of nucleon quasiparticles. Adding the self-energy, the effective nucleon inverse propagator is

$$\begin{aligned}\Delta_{eff}^{-1}(p_{ms}^0,\vec{p}) &= \Delta_{bare}^{-1}-\Sigma\\ &= -[(ip_{ms}^0+\mu)\gamma^0+i\vec{\not{p}}](1-\Sigma_0)+m(1+\Sigma_0).\end{aligned} \tag{16}$$

The change in the position of the pole in the nucleon propagator is easy to compute. In particular, the mass of the nucleon is shifted *up*:

$$m_{eff} = m\left(\frac{1+\Sigma_0}{1-\Sigma_0}\right)\approx m(1+2\Sigma_0). \tag{17}$$

This expression holds in the chiral limit. Half of the mass shift arises from the shift in the term $\sim m$ and half from what can be viewed as wave-function renormalization.

Away from the chiral limit, where Σ_0 is a function of momentum, the change in the mass cannot be read off so immediately. In that case, one has to define the effective mass by other means, as in Eq. (11.66) of Ref. [13].

This increase in the effective nucleon mass is in contrast to what happens at zero density, but nonzero temperature. To leading order in an expansion about zero temperature, in the chiral limit the nucleon mass does not shift to $\sim T^2$ [14].

At normal nuclear matter density, $p_f\approx 270$ MeV. The correction which we computed, from single pion exchange, is tiny, $2\Sigma_0\approx 0.04$. This suggests that chiral perturbation theory might be a reasonable guide to the properties of nucleons, even at nuclear matter densities.

This conclusion is premature. While the corrections to the nucleon propagator are very small, corrections to the pion propagator are large. For most momentum, such as near the pion mass shell, the corrections to the pion propagator are like those of the nucleon, proportional to the density, $\sim p_f^3/(mf_\pi^2)$, Eq. (1). If the pion is far off its mass shell, though, with an energy $\omega\sim p_f^2/m$, it can scatter into a nucleon particle-hole pair. For such nearly static pions, the pion self-energy is enhanced by a factor of $m/\omega\sim m^2/p_f^2$. The correct expansion parameter for the pion propagator is then not $p_f^3/(mf_\pi^2)$, but

$$\frac{p_f^3}{mf_\pi^2}\frac{m^2}{p_f^2}\sim\left(\frac{g_A^2}{2\pi^2}\right)\frac{mp_f}{f_\pi^2}. \tag{18}$$

Numerically, this parameter is *much* larger than Σ_0 in Eq. (13). In fact, as we are dealing with a nonrelativistic system, this enhancement of the pion propagator is well known from condensed matter physics, and represents the need to resum the nearly static pion propagator through the random phase approximation (RPA) [13]. Indeed, the factor of $g_A^2/(2\pi^2)$ arises from an explicit calculation in the RPA limit, from Eq. (4.21) of Meißner, Oller, and Wirzba [11]. For normal nuclear matter density, the parameter of the RPA pion propagator in Eq. (18) is ≈ 2 at nuclear matter densities. Since this parameter is only linear in the Fermi momentum, if we require that this parameter be less than, say, 1/2, this means that we can use chiral perturbation theory to compute the nuclear equation of state only up to $p_f\sim 70$ MeV. This corresponds to densities which are $(1/4)^3=1/64$ those of normal nuclear matter.

This restriction on the use of chiral perturbation theory is not that surprising. In computing the free energy, the typical pion momentum is of order $\sim p_f$, with energies $\sim p_f^2/m$. To use a chiral Lagrangian, the pion momentum should be small relative to f_π, which is similar to the condition derived from Eq. (18). What is not evident is while there is a factor of $1/(2\pi^2)$ from chiral perturbation theory in Eq. (18), this is compensated by the factor of the nucleon mass in the numerator.

Nevertheless, such computations [10,11] are manifestly of interest, so as to gain a more general understanding of the nuclear equation of state. Carrying out such calculations beyond leading order is technically very challenging. Using a RPA corrected propagator for the nearly static pion is straightforward. What is difficult is knowing how to separate diagrams with two pion exchange from other effects. In quantum hadrodynamics [6], one must separate two pion exchange from that of heavier mesons, such as the σ and the ω. In "pionless" effective theories [15], two pion exchange contributes to pointlike interactions between four or more nucleons.

We can draw some tentative conclusions about the hadronic pressure, which motivated this study. To leading order, the nonideal terms in the pressure are proportional to Σ_0, which is very small [10]. At higher order, even if corrections to the pion propagator are large, their effect on the nucleon propagator, and the free energy, can still be small, as a large correction to a small number. Thus the possibility of a hadronic phase with a small pressure, required for a new class of quark stars, remains viable.

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