Relativistic harmonic oscillator with spin symmetry

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The eigenfunctions and eigenenergies for a Dirac Hamiltonian with equal scalar and vector harmonic oscillator potentials are derived. Equal scalar and vector potentials may be applicable to the spectrum of an antinucleon embedded in a nucleus. Triaxial, axially deformed, and spherical oscillator potentials are considered. The spectrum has a spin symmetry for all cases and, for the spherical harmonic oscillator potential, a higher symmetry analogous to the SU(3) symmetry of the nonrelativistic harmonic oscillator is discussed.

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I. INTRODUCTION

Recent theoretical investigations have suggested the possibility that the lifetime of an antinucleon inside a nucleus could be significantly enhanced [1]. The relativistic mean fields of antinucleons calculated in a self-consistent Hartree approximation of a nuclear field theory [2,3] indicate that the scalar $V_S(\vec{r})$ and vector potentials $V_V(\vec{r})$ are approximately equal. This implies that the antinucleon spectrum will have an approximate spin symmetry [4,5]. This conclusion is consistent with the fact that the nucleon spectrum has an approximate pseudospin symmetry [6,7] because the scalar and vector mean field potentials of a nucleon are approximately equal but opposite in sign and the vector potential changes sign under charge conjugation [8]. In fact, the negative energy states of the nucleon do show a strong spin symmetry [9]. Of course, whether such states could be observable can only be reliably estimated if the antinucleon annihilation potential is included in the mean field calculations.

In this paper we shall solve for the eigenfunctions and eigenenergies of the triaxial, axially deformed, and spherical relativistic harmonic oscillators for equal scalar and vector potentials with the expectation that these results could be helpful in drawing conclusions about the feasibility of observing the spectrum of an antinucleon in a nuclear environment. The spherical relativistic harmonic oscillators with spin symmetry [10–15] and pseudospin symmetry [16,17] have been studied previously, but in this paper we derive the eigenfunctions and eigenenergies for the triaxial and axially deformed harmonic oscillators as well.

II. THE DIRAC HAMILTONIAN AND SPIN SYMMETRY

The Dirac Hamiltonian H with an external scalar $V_S(\vec{r})$ and vector $V_V(\vec{r})$ potential is given by

$$H = \boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta[M + V_S(\vec{r})] + V_V(\vec{r}), \tag{1}$$

where $\boldsymbol{\alpha},\beta$ are the usual Dirac matrices, M is the nucleon mass, and we set $c=1$. The Dirac Hamiltonian is invariant under a SU(2) algebra for two limits: $V_S(\vec{r})=V_V(\vec{r})+C_s$ and $V_S(\vec{r})=-V_V(\vec{r})+C_{ps}$ where C_s, C_{ps} are constants [5]. The former limit has application to the spectrum of mesons for which the spin-orbit splitting is small [18] and for the spectrum of an antinucleon in the mean field of nucleons [8,9]. The latter limit leads to pseudospin symmetry in nuclei [6]. This symmetry occurs independent of the shape of the nucleus: spherical, axial deformed, or triaxial.

A. Spin symmetry generators

The generators for the spin SU(2) algebra, S_q, which commute with the Dirac Hamiltonian, $[H_s,S_q]=0$, for the spin symmetry limit $V_S(\vec{r})=V_V(\vec{r})+C_s$, are given by [5]

$$S_q = \begin{pmatrix} s_q & 0 \\ 0 & \tilde{s}_q \end{pmatrix} = \begin{pmatrix} s_q & 0 \\ 0 & U_p s_q U_p \end{pmatrix}, \tag{2}$$

where $S_q=\sigma_q/2$ are the usual spin generators, σ_q the Pauli matrices, and $U_p=\boldsymbol{\sigma}\cdot\boldsymbol{p}/p$ is the momentum-helicity unitary operator [19]. Thus the operators S_i generate an SU(2) invariant symmetry of H_s. Therefore each eigenstate of the Dirac Hamiltonian has a partner with the same energy,

$$H_s\Phi^s_{k,\mu}(\vec{r}) = E_k\Phi^s_{k,\mu}(\vec{r}), \tag{3}$$

where k are the other quantum numbers and $\mu=\pm\frac{1}{2}$ is the eigenvalue of S_z,

$$S_z\Phi^s_{k,\mu}(\vec{r}) = \mu\Phi^s_{k,\mu}(\vec{r}). \tag{4}$$

The eigenstates in the doublet will be connected by the generators $S_\pm$,

$$S_\pm\Phi^s_{k,\mu}(\vec{r}) = \sqrt{\left(\frac{1}{2}\mp\tilde{\mu}\right)\left(\frac{3}{2}\pm\tilde{\mu}\right)}\Phi^s_{k,\mu\pm1}(\vec{r}). \tag{5}$$

The fact that Dirac eigenfunctions belong to the spinor representation of the spin SU(2), as given in Eqs. (4) and (5), leads to conditions on the Dirac amplitudes [20–23].

B. Dirac eigenfunctions and spin symmetry

The Dirac eigenfunction can be written as a four dimensional vector

$$\Phi_{k,\mu}(\vec{r}) = \begin{pmatrix} g^{+}_{k,\mu}(\vec{r}) \\ g^{-}_{k,\mu}(\vec{r}) \\ if^{+}_{k,\mu}(\vec{r}) \\ if^{-}_{k,\mu}(\vec{r}) \end{pmatrix}, \tag{6}$$

where $g^{+}_{k,\mu}(\vec{r})$ are the "upper Dirac components" where + indicates spin up and − spin down and $f^{+}_{k,\mu}(\vec{r})$ are the "lower Dirac components" where + indicates spin up and − spin down. However, spin symmetry imposes conditions on these eigenfunctions [22,23] which are derived from Eqs. (4) and (5):

$$g^{+}_{k,1/2}(\vec{r}) = g^{-}_{k,-1/2}(\vec{r}) = g_k(\vec{r}), \tag{7a}$$

$$g^{+}_{k,-1/2}(\vec{r}) = g^{-}_{k,1/2}(\vec{r}) = 0, \tag{7b}$$

$$f^{+}_{k,1/2}(\vec{r}) = -f^{-}_{k,-1/2}(\vec{r}) = f_k(\vec{r}), \tag{7c}$$

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial x}\right) f^{+}_{k,-1/2}(\vec{r}) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial x}\right) f^{-}_{k,1/2}(\vec{r}), \tag{7d}$$

$$\frac{\partial}{\partial z} f^{\pm}_{k,\mp 1/2}(\vec{r}) = \pm\left(\frac{\partial}{\partial x} \mp i\frac{\partial}{\partial x}\right) f^{\pm}_{k,\pm 1/2}(\vec{r}). \tag{7e}$$

Thus for spin symmetry the Dirac spin doublets are

$$\Phi^{s}_{k,1/2}(\vec{r}) = \begin{pmatrix} g_k(\vec{r}) \\ 0 \\ if_k(\vec{r}) \\ if^{-}_{k,1/2}(\vec{r}) \end{pmatrix}, \quad \Phi^{s}_{k,-1/2}(\vec{r}) = \begin{pmatrix} 0 \\ g_k(\vec{r}) \\ if^{+}_{k,-1/2}(\vec{r}) \\ -if_k(\vec{r}) \end{pmatrix}. \tag{8}$$

C. Second order differential equation for the eigenfunctions

The Dirac Hamiltonian, Eq. (1), gives first order differential relations between $g_k(\vec{r})$ and $f_k(\vec{r})$ and $f^{\pm}_{k,\mp 1/2}(\vec{r})$. In the usual way we turn these equations into a second order differential relation for the upper component. In the limit of spin symmetry this second order equation becomes

$$[p^2 + 2(\tilde{E}_k + \tilde{M})V(\vec{r}) - \tilde{E}_k^2 + \tilde{M}^2]g_k(\vec{r}) = 0, \tag{9}$$

where $V_S(\vec{r}) = V(\vec{r}) + V_S^0$, $V_V(\vec{r}) = V(\vec{r}) + V_V^0$, $\tilde{M} = M + V_S^0$, and $\tilde{E}_k = E_k - V_V^0$. From Eq. (1) and setting $\hbar = 1$, the lower components become

$$f_k(\vec{r}) = \frac{-1}{(\tilde{M} + \tilde{E})}\frac{\partial}{\partial z} g_k(\vec{r}), \tag{10a}$$

$$f^{\pm}_{k,\mp 1/2}(\vec{r}) = \frac{-1}{(\tilde{M} + \tilde{E})}\left(\frac{\partial}{\partial x} \mp i\frac{\partial}{\partial y}\right) g_k(\vec{r}). \tag{10b}$$

These relations are consistent with the conditions on the eigenfunctions imposed by spin symmetry in Eq. (7).

Equation (9) is basically the energy-dependent Schrödinger equation without any spin dependence. Hence any potential, which can be solved analytically with the nonrelativistic Schrödinger equation, is solvable in the spin limit, but the energy spectrum will be different because of the nonlinear dependence on the energy. Hence the Coulomb and harmonic oscillator are solvable. The Coulomb potential has been solved (for general scalar and vector potentials) [6] and applied to meson spectroscopy [18]. More appropriate for nuclei is the harmonic oscillator.

III. HARMONIC OSCILLATOR

First we shall discuss the triaxial harmonic oscillator, then the axially symmetric harmonic oscillator, and finally the spherical harmonic oscillator.

A. Triaxial harmonic oscillator

For the potential $V(\vec{r}) = (\tilde{M}/2)(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2)$, the second order differential equation (9) becomes

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - (\tilde{E}_{n_1,n_2,n_3} + \tilde{M})\tilde{M}(\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2) + \tilde{E}^2_{n_1,n_2,n_3} - \tilde{M}^2\right] g_{n_1,n_2,n_3}(\vec{r}) = 0. \tag{11}$$

1. Eigenfunctions

Introducing the product ansatz for the eigenfunction $g_{n_1,n_2,n_3}(\vec{r}) \propto g_{n_1}(x) g_{n_2}(y) g n_3(z)$, we derive the three equations

$$\left[\frac{\partial^2}{\partial x_i^2} - x_i^2 + 2n_i + 1\right] g_{n_i}(x_i) = 0, \tag{12}$$

where $x_1 = \lambda_1 x$, $x_2 = \lambda_2 y$, $x_3 = \lambda_3 z$, and

$$\lambda_i = [(\tilde{E}_{n_1,n_2,n_3} + \tilde{M})\tilde{M}\omega_i^2]^{1/4}, \tag{13a}$$

$$\tilde{E}^2_{n_1,n_2,n_3} - \tilde{M}^2 = 2\sum_i \lambda_i^2\left(n_i + \frac{1}{2}\right). \tag{13b}$$

The bound eigenstates are given by

$$g_{n_1,n_2,n_3}(\vec{r}) = \mathcal{N}(E_{n_1,n_2,n_3}) g_{n_1}(x_1) g_{n_2}(x_2) g_{n_3}(x_3), \tag{14}$$

where

$$g_n(x_i) = \sqrt{\frac{\lambda_i}{\sqrt{\pi}2^{n_i}n_i!}}e^{-x_i^2/2}H_{n_i}(x_i), \tag{15}$$

and $H_{n_i}(x_i)$ is the Hermite polynomial which means that $g_{n_1,n_2,n_3}(\vec{r})$ has n_1, n_2, n_3 nodes in the x, y, and z directions, respectively. $\mathcal{N}(E_{n_1,n_2,n_3})$ is the normalization determined by $\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz[|g_{n_1,n_2,n_3}(\vec{r})|^2+|f^{+}_{n_1,n_2,n_3,\mu}(\vec{r})|^2 + |f^{-}_{n_1,n_2,n_3,\mu}(\vec{r})|^2]=1$.

From Eq. (10) the lower components are

$$f_{n_1,n_2,n_3}(\vec{r}) = \mathcal{N}(E_{n_1,n_2,n_3})g_{n_1}(x_1)g_{n_2}(x_2)f_{n_3+1}(x_3), \tag{16a}$$

$$f^{(\pm)}_{n_1,n_2,n_3,\mp 1/2}(\vec{r}) = \mathcal{N}(E_{n_1,n_2,n_3})[f_{n_1+1}(x_1)g_{n_2}(x_2) \mp ig_{n_1}(x_1)f_{n_2+1}(x_2)]g_{n_3}(x_3), \tag{16b}$$

where

$$f_{n_i+1}(x_i) = \lambda_i\sqrt{\frac{\lambda_i}{\sqrt{\pi}2^{n_i}n_i!}}e^{-x_i^2}\left(\frac{H_{n_i+1}(x_i)}{2} - nH_{n_i-1}(x_i)\right). \tag{17}$$

Clearly the function $f_{n+1}(x)$ is a polynomial of order $n+1$. Evaluation for low n demonstrates that it has $n+1$ nodes so we assume that it has $n+1$ nodes for all n. This means that $f_{n_1,n_2,n_3}(\vec{r})$ has one more node in the z direction than $g_{n_1,n_2,n_3}(\vec{r})$ and the same number of nodes in the x and y directions as $g_{n_1,n_2,n_3}(\vec{r})$. On the other hand the amplitudes $f^{(\pm)}_{n_1,n_2,n_3,\mp 1/2}(\vec{r})$ have the same number of nodes in the z direction as $g_{n_1,n_2,n_3}(\vec{r})$.

Using these amplitudes the normalization becomes

$$\mathcal{N}(E) = \sqrt{\frac{2(\tilde{E}+\tilde{M})}{3\tilde{E}+\tilde{M}}}. \tag{18}$$

2. Eigenenergies

From Eq. (13) the eigenvalue equation is

$$\sqrt{\mathcal{E}_{n_1,n_2,n_3}+1}(\mathcal{E}_{n_1,n_2,n_3}-1) = \sum_{i=1}^{3}C_i\left(n_i+\frac{1}{2}\right), \tag{19}$$

where $\mathcal{E}_{n_1,n_2,n_3}=\tilde{E}_{n_1,n_2,n_3}/\tilde{M}$, $C_i=2\omega_i/\tilde{M}$, and $n_i=0,1,\ldots$. Thus both the eigenfunctions and the eigenenergies are independent of spin.

This eigenvalue equation is solved on MATHEMATICA,

$$E_{n_1,n_2,n_3} = \tilde{M}\left[B(A_{n_1,n_2,n_3}) + \frac{1}{3} + \frac{4}{9B(A_{n_1,n_2,n_3})}\right] + V_V^0, \tag{20}$$

where

$$B(A_{n_1,n_2,n_3}) = \left[A_{n_1,n_2,n_3} + \sqrt{A^2_{n_1,n_2,n_3} - \frac{32}{27}}\right]^{2/3}, \tag{21}$$

and $A_{n_1,n_2,n_3}=\Sigma_{i=1}^{3}C_i\left(n_i+\frac{1}{2}\right)$.

The eigenvalues E_{n_1,n_2,n_3} are real for all values of n_1, n_2, n_3 as long as $C_i, V_{S,V}^0$ are real. Although true it is not obvious because $B(A_{n_1,n_2,n_3})$ is not real for all A_{n_1,n_2,n_3} real. From Eq. (21), $B(A_{n_1,n_2,n_3})$ is clearly complex for $A_{n_1,n_2,n_3}<\sqrt{32/27}$. However, we now show analytically that E_{n_1,n_2,n_3} will still be real even if $B(A_{n_1,n_2,n_3})$ is complex as long as $|B(A_{n_1,n_2,n_3})| = 2/3$.

The imaginary part of E_{n_1,n_2,n_3} is

$$\mathrm{Im}\ E_{n_1,n_2,n_3} = \frac{\left[B(A_{n_1,n_2,n_3}) + \frac{1}{3} + \frac{4}{9B(A_{n_1,n_2,n_3})}\right] - \left[B(A_{n_1,n_2,n_3}) + \frac{1}{3} + \frac{4}{9B(A_{n_1,n_2,n_3})}\right]^*}{2i}. \tag{22}$$

Writing $B(A_{n_1,n_2,n_3})=|B(A_{n_1,n_2,n_3})|e^{i\psi}$,

$$\mathrm{Im}\ E_{n_1,n_2,n_3} = \left(|B(A_{n_1,n_2,n_3})| - \frac{4}{9|B(A_{n_1,n_2,n_3})|}\right)\sin\psi, \tag{23}$$

and therefore E_{n_1,n_2,n_3} is real if $|B(A_{n_1,n_2,n_3})|=2/3$ independent of ψ. One can show numerically that $|B(A_{n_1,n_2,n_3})| = 2/3$ for *all* A_{n_1,n_2,n_3} in the range from zero to $\sqrt{32/27}$, $0 \leqslant A_{n_1,n_2,n_3} \leqslant \sqrt{32/27}$. For $A_{n_1,n_2,n_3} \geqslant \sqrt{32/27}, B(A_{n_1,n_2,n_3})$ is clearly real and hence E_{n_1,n_2,n_3} is real.

The spectrum is nonlinear in contrast to the nonrelativistic harmonic oscillator. However, for small A_{n_1,n_2,n_3},

$$E_{n_1,n_2,n_3} \approx \tilde{M}\left(1 + \frac{A_{n_1,n_2,n_3}}{\sqrt{2}} + \cdots\right) + V_V^0 \tag{24}$$

and therefore the binding energy, $E_{n_1,n_2,n_3}-\tilde{M}\approx\Sigma_{i=1}^{3}\omega_i\left(n_i+\frac{1}{2}\right)+V_V^0$, in agreement with the nonrelativistic harmonic oscillator. For large A_{n_1,n_2,n_3} the spectrum goes as

$$E_{n_1,n_2,n_3} \approx \tilde{M}\left(A^{2/3}_{n_1,n_2,n_3} + \frac{1}{3} + \cdots\right) + V_V^0, \tag{25}$$

which, in lowest order, agrees with the spectrum for $\tilde{M}\to 0$ [11].

B. Axially symmetric harmonic oscillator

For the axially symmetric harmonic oscillator $\omega_1=\omega_2=\omega_\perp$ and, hence, the potential depends only on $r_\perp$ and not the azimuthal angle ϕ, $V(\vec{r})=(\tilde{M}/2)(\omega_\perp^2 r_\perp^2+\omega_3^2 z^2)$, where $x=r_\perp\cos\phi, y=r_\perp\sin\phi$. This independence of the potential on ϕ implies that the Dirac Hamiltonian is invariant under rotations about the z axis $[H_s,L_z]=0$, where

$$L_z=\begin{pmatrix}\ell_z & 0\\ 0 & \tilde{\ell}_z\end{pmatrix}, \tag{26}$$

and $\vec{\ell}=\vec{r}\times\vec{p}$ and $\tilde{\vec{\ell}}=U_p\vec{\ell}U_p$. The Dirac eigenstates will then be an eigenfunction of L_z and $J_z=L_z+S_z$.

$$L_z\Phi^s_{N,n_3,\Lambda,\mu}(\vec{r})=\Lambda\Phi^s_{N,n_3,\Lambda,\mu}(\vec{r}), \tag{27a}$$

$$J_z\Phi_{N,n_3,\Lambda,\mu}(\vec{r})=\Omega\Phi^s_{N,n_3,\Lambda,\mu}(\vec{r}),\quad \Omega=\Lambda+\mu, \tag{27b}$$

where N is the total harmonic oscillator quantum number and n_3 is the number of harmonic oscillator quanta in the z direction, and are discussed in more detail below.

1. Eigenfunctions

Since the potential has no ϕ dependence the second order differential equation (9) separates into an equation for x_3 and an equation for ρ,

$$\left[\frac{\partial^2}{\partial x_3^2}-x_3^2+2n_3+1\right]g_{n_3}(x_3)=0, \tag{28a}$$

$$\left[\frac{\partial^2}{\partial\rho^2}+\frac{1-4\Lambda^2}{4\rho^2}-\rho^2+2n_\perp+2\right]\sqrt{\rho}g_{n_\rho}(\rho)=0, \tag{28b}$$

where $\rho=\lambda_\perp r_\perp$, $n_\perp=2n_\rho+\Lambda$, and

$$g_{N,n_3,\Lambda}(\vec{r})=g_{n_\rho}(\rho)g_{n_3}(z)e^{i\Lambda\phi}, \tag{29a}$$

$$\lambda_\perp=[(\tilde{E}_{N,n_3}+\tilde{M})\tilde{M}\omega_\perp^2]^{1/4}, \tag{29b}$$

$$\tilde{E}^2_{N,n_3}-\tilde{M}^2=2\left[\lambda_\perp^2(n_\perp+1)+\lambda_3^2\left(n_3+\frac{1}{2}\right)\right]. \tag{29c}$$

The quantum numbers $N,n_\perp,n_3,\Lambda$ are the "asymptotic" quantum numbers [24], where $N=n_\perp+n_3$ is the total number of oscillator quanta, $N=0,1,\ldots,n_\perp$ is the number of oscillator quanta in the $r_\perp$ direction, $n_\perp=0,1,\ldots,N$, n_3 is the number of oscillator quanta in the z direction, $n_3=0,1,\ldots,N$, and Λ is the angular momentum along the z axis, $\Lambda=\pm n_\perp,\pm(n_\perp-2),\ldots,\pm1,0$. The upper components of the eigenstates are given by

$$g_{N,n_3,\Lambda}(\vec{r})=\mathcal{N}(E_{N,n_3})\lambda_\rho\sqrt{\frac{\lambda_3 n_\rho!}{\pi\sqrt{\pi}2^{n_3}(n_\rho+\Lambda)!\,n_3!}}\times e^{-(x_3^2+\rho^2)/2}\rho^\Lambda L^{(\Lambda)}_{n_\rho}(\rho^2)e^{i\Lambda\phi}H_{n_3}(x_3), \tag{30}$$

where $L^{(\Lambda)}_{n_\rho}(\rho^2)$ is the Laguerre polynomial which means that $g_{N,n_3,\Lambda}(\vec{r})$ has n_ρ,n_3 nodes in the $r_\perp$ and z directions, respectively. $\mathcal{N}(E_{N,n_3})$ is the normalization determined by $\int_0^{2\pi}d\phi\int_0^\infty r_\perp dr_\perp\int_{-\infty}^\infty dz[|g_{N,n_3,\Lambda}(\vec{r})|^2+|f^+_{N,n_3,\Lambda,\mu}(\vec{r})|^2+|f^-_{N,n_3,\Lambda,\mu}(\vec{r})|^2]=1$ and is the same function as given in Eq. (18). These upper components do not depend on the orientation of the spin.

The lower components are

$$f_{N,n_3,\Lambda}(\vec{r})=-\frac{\mathcal{N}(E_{N,n_3,\Lambda})}{\tilde{M}+\tilde{E}_{N,n_3}}\lambda_\rho^2\sqrt{\frac{n_\rho!}{\pi(n_\rho+\Lambda)!}}e^{-\rho^2/2}\rho^\Lambda\times L^{(\Lambda)}_{n_\rho}(\rho^2)e^{i\Lambda\phi}f_{n_3+1}(x_3), \tag{31a}$$

$$f^+_{N,n_3,\Lambda,-1/2}(\vec{r})=-\frac{\mathcal{N}(E_{N,n_3})}{\tilde{M}+\tilde{E}_{N,n_3}}\lambda_\rho^2\sqrt{\frac{n_\rho!}{\pi(n_\rho+\Lambda)!}}e^{-\rho^2/2}\rho^{\Lambda-1}\times[(n_\rho+1)L^{(\Lambda-1)}_{n_\rho+1}(\rho^2)+(n_\rho+\Lambda)\times L^{(\Lambda-1)}_{n_\rho}(\rho^2)]e^{i(\Lambda-1)\phi}g_{n_3}(x_3), \tag{31b}$$

$$f^-_{N,n_3,\Lambda,1/2}(\vec{r})=-\frac{\mathcal{N}(E_{N,n_3})}{\tilde{M}+\tilde{E}_{N,n_3}}\lambda_\rho^2\sqrt{\frac{n_\rho!}{\pi(n_\rho+\Lambda)!}}e^{-\rho^2/2}\rho^{\Lambda+1}\times[L^{(\Lambda+1)}_{n}(\rho^2)+L^{(\Lambda+1)}_{n_\rho-1}(\rho^2)]e^{i(\Lambda+1)\phi}g_{n_3}(x_3), \tag{31c}$$

where $g_{n_3}(x_3)$, $f_{n_3}(x_3)$ are defined in Eqs. (15) and (17). Clearly the function $f_{N,n_3,\Lambda}(\vec{r})$ has n_3+1 nodes in the z direction, one more node than the upper component, but the same number of nodes in the $r_\perp$ direction. The amplitude $f^+_{N,n_3,\Lambda,-1/2}(\vec{r})$ has the same number of nodes in the z direction as the upper component. However, it has $n_\rho+1$ nodes in the $r_\perp$ direction for low n_ρ and we assume it has $n_\rho+1$ node for all n_ρ; that is, one more node in the $r_\perp$ direction than the upper component. On the other hand the amplitude $f^-_{N,n_3,\Lambda,1/2}(\vec{r})$ has the same number of nodes in the z direction and the $r_\perp$ direction as the upper component.

2. Eigenenergies

From Eq. (29) the eigenvalue equation is

$$\sqrt{\mathcal{E}_{N,n_3}+1}(\mathcal{E}_{N,n_3}-1)=C_\perp(n_\perp+1)+C_3\left(n_3+\tfrac{1}{2}\right), \tag{32}$$

where $\mathcal{E}_{N,n_3}=\tilde{E}_{N,n_3}/\tilde{M}, C_\perp=2\omega_\perp/\tilde{M}$, and $n_\rho,n_3=0,1,\ldots$. Thus the eigenenergies not only have a degeneracy due to spin symmetry but they have an additional degeneracy in that they only depend on N and n_3 and not on Λ.

The discussion about eigenenergies is the same as for triaxial nuclei and the energy spectrum is given by

$$E_{N,n_3}=\tilde{M}\left[B(A_{n_\perp,n_3})+\frac{1}{3}+\frac{4}{9B(A_{n_\perp,n_3})}\right]+V_V^0, \tag{33}$$

where $A_{n_\perp,n_3}=C_\perp(n_\perp+1)+C_3\left(n_3+\frac{1}{2}\right)$.

C. Spherical harmonic oscillator

For a spherical harmonic oscillator $\omega_i=\omega$ and hence the potential depends only on the radial coordinate, $r=\sqrt{x^2+y^2+z^2}$, and is independent of the polar angle, θ, $z=r\cos(\theta)$, as well as ϕ. The Dirac Hamiltonian will be invariant with respect to rotations about all three axes, $[L_i,H_s]=0$ where

$$\vec{L}=\begin{pmatrix}\vec{\ell} & 0\\ 0 & \vec{\ell}\end{pmatrix}, \tag{34}$$

and hence invariant with respect to a $\mathrm{SU}_L(2)\times\mathrm{SU}(2)$ group where $\mathrm{SU}_L(2)$ is generated by the orbital angular momentum operators $\vec{L}$. Since the total angular momentum, $\vec{J}=\vec{L}+\vec{S}$, is also conserved, rather than using the four row basis for this eigenfunction, it is more convenient to introduce the spin function χ_μ explicitly. The states that are a degenerate doublet are then the states with $j=\ell\pm\frac{1}{2}$ and they have the two row form [22]:

$$\Psi^s_{n_r,\ell,j,M}(\vec{r})=\begin{pmatrix} g_{n_r,\ell}(r)[Y^{(\ell)}(\theta,\phi)\chi]^{(j)}_M \\ if_{n_r,\ell,j}(r)[Y^{(\ell_j)}(\theta,\phi)\chi]^{(j)}_M\end{pmatrix}, \tag{35}$$

where $\ell_j=\ell\pm1$ for $j=\ell\pm\frac{1}{2}$, $Y^{(\ell)}_m(\theta,\phi)$ is the spherical harmonic of order ℓ, n_r is the number of radial nodes of the upper amplitude, and $[Y^{(\ell)}(\theta,\phi)\chi]^{(j)}_M$ is the coupled amplitude $\Sigma_{m\mu}C^{\ell(1/2)j}_{m\mu M}Y^{(\ell)}_m(\theta,\phi)\chi_\mu$. Thus the spherical symmetry reduces the number of amplitudes in the doublet even further from four to three.

The Dirac eigenstates will then be an eigenfunction of $\vec{L}\cdot\vec{L}$, $\vec{J}\cdot\vec{J}$, and J_z,

$$\vec{J}\cdot\vec{J}\Psi^s_{n_r,\ell,j,M}(\vec{r})=j(j+1)\Psi^s_{n_r,\ell,j,M}(\vec{r}), \tag{36a}$$

$$\vec{L}\cdot\vec{L}\Psi^s_{n_r,\ell,j,M}(\vec{r})=\ell(\ell+1)\Psi^s_{n_r,\ell,j,M}(\vec{r}), \tag{36b}$$

$$J_z\Psi^s_{n_r,\ell,j,M}(\vec{r})=M\Psi^s_{n_r,\ell,j,M}(\vec{r}). \tag{36c}$$

The differential equation for $g_{n_r,\ell}(r)$ becomes

$$\left[\frac{d^2}{dx^2}-\frac{\ell(\ell+1)}{x^2}-x^2+\frac{(\tilde{E}_N^2-\tilde{M}^2)}{\left[(\tilde{E}_N+\tilde{M})\dfrac{\tilde{M}\omega^2}{2}\right]^{1/2}}\right]rg_{n_r,\ell}(r)=0, \tag{37}$$

where $\tilde{E}_N=E_N-V^0_V$, $\tilde{M}=M+V^0_S$, $x=\lambda r$, and

$$\lambda=[(\tilde{E}_N+\tilde{M})\tilde{M}\omega^2]^{1/4}, \tag{38a}$$

$$\tilde{E}_N^2-\tilde{M}^2=2\lambda^2\left(N+\frac{3}{2}\right). \tag{38b}$$

1. Eigenfunctions

The solutions to this differential equation are well known and lead to the upper amplitudes of the eigenfunctions,

$$g_{n_r,\ell}(r)=\mathcal{N}(E_{N,\ell})\sqrt{\frac{2\lambda^3 n!}{\Gamma\left(\ell+n+\dfrac{3}{2}\right)}}e^{-x^2/2}x^\ell L^{(\ell+1/2)}_{n_r}(x^2), \tag{39}$$

where $\mathcal{N}(E_N)$ is the normalization determined by $\int_0^{2\pi}d\phi\int_{-\pi}^{\pi}\sin(\theta)d\theta\int_0^\infty r^2dr[|g_{n_r,\ell}(\vec{r})|^2+|f_{n_r,\ell,j}(\vec{r})|^2]=1$ and is the same function as given in Eq. (18). Clearly $g_{n_r,\ell}(r)$ has n_r nodes in the radial direction.

The lower components are

$$f_{n_r,\ell,j=\ell-1/2}(\vec{r})=-\frac{\mathcal{N}(E_N)}{\tilde{M}+\tilde{E}_N}\sqrt{\frac{2\lambda^5 n!}{\Gamma(\ell+n_r+\frac{3}{2})}}e^{-x^2/2}x^{\ell-1}[(n_r+1)\times L^{(\ell-1/2)}_{n_r+1}(x^2)+(n_r+\ell+\tfrac{1}{2})L^{(\ell-1/2)}_{n_r}(x^2)], \tag{40}$$

$$f_{n_r,\ell,j=\ell+1/2}(\vec{r})=-\frac{\mathcal{N}(E_N)}{\tilde{M}+\tilde{E}_N}\sqrt{\frac{2\lambda^5 n!}{\Gamma(\ell+n_r+\frac{3}{2})}}e^{-x^2/2}x^{\ell+1}\times[L^{(\ell+3/2)}_{n_r}(x^2)+L^{(\ell+3/2)}_{n_r-1}(x^2)]. \tag{41}$$

Clearly the function $f_{n_r,\ell,j=\ell-1/2}(\vec{r})$ has n_r+1 nodes, one more node than the upper component. The amplitude $f_{n_r,\ell,j=\ell-1/2}(\vec{r})$ has the same number of nodes as the upper component. This agrees with the general theorem relating the number of radial nodes of the lower components to the number of radial nodes of the upper component [25].

2. Energy eigenvalues

The eigenvalue equation is

$$\sqrt{\mathcal{E}_N+1}(\mathcal{E}_N-1)=C(N+\tfrac{3}{2}), \tag{42}$$

where $\mathcal{E}_N=\tilde{E}_N/\tilde{M}$, $C=2\omega/\tilde{M}$, and N is the total oscillator quantum number, $N=2n+\ell=0,1,\ldots$. We note that there is not only a degeneracy due to spin symmetry but there is also the usual degeneracy of the nonrelativistic harmonic oscillator; namely, that the energy depends only on the total harmonic oscillator quantum number and the states with orbital angular momentum $\ell=N,N-2,\ldots,0$ or 1 and angular momentum projection $m=\ell,\ell-1,\ldots,\ell$ are all degenerate.

Again the eigenvalue is

$$E_N=\tilde{M}\left[B(A_N)+\frac{1}{3}+\frac{4}{9B(A_N)}\right]+V^0_V, \tag{43}$$

and $A_N=C(N+\frac{3}{2})$.

D. Energy spectrum

In Fig. 1 we plot the spherical harmonic oscillator Dirac binding energies $E_D=E_N-M$ with E_N given in Eq. (43), the solid curve, as a function of N. We chose the parameters to fit the lowest eigenenergies of the spectrum of an antiproton

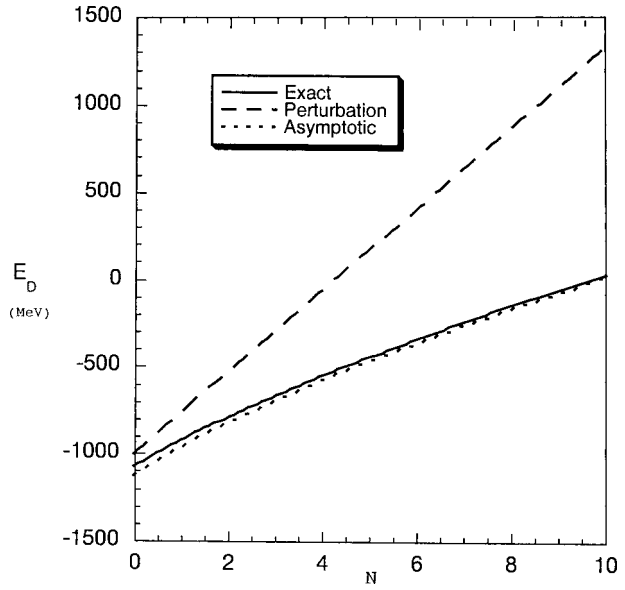

FIG. 1. The Dirac binding energies E_D for the spherical harmonic oscillator as a function of N. The exact energies are the solid line, the perturbation approximation is the dashed line, and the asymptotic approximation is the short dashed line.

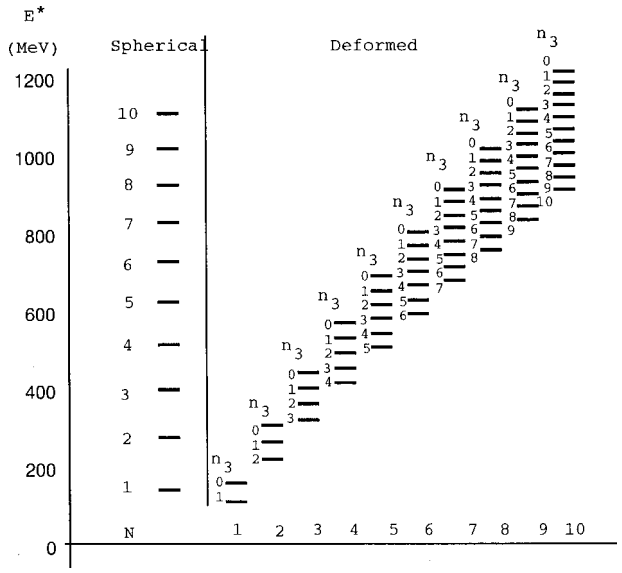

FIG. 2. On the left the excitation energies E^* for the spherical harmonic oscillator are plotted for each N. Each level has a $(N+2)(N+1)$ degeneracy because of spin symmetry and because the allowed orbital angular momenta are $\ell=N,N-2,\ldots,0$ or 1 and the allowed orbital angular momentum projections are $m=\ell,\ell-1,\ldots,-\ell$. On the right the excitation energies E^* for the deformed harmonic oscillator are plotted in staggered groups for each N. Each group contains the levels for $n_3=0,1,\ldots,N$. Each level has a $2(N-n_3)+1$ degeneracy for $(N-n_3)$ even and a $2(N-n_3+1)$ degeneracy for $(N-n_3)$ odd because of spin symmetry and because the allowed orbital angular momentum projections are $\Lambda=\pm(N-n_3),\pm(N-n_3-2),\ldots,\pm1$ or 0.

outside of ^{16}O in the relativistic mean field approximation [1] and they are $C=1.33$, $\tilde{M}=252$ MeV, and $V_V^0=-677$ MeV.

The dashed curve is E_D using the pertubation approximation of E_N given in Eq. (24). The short-dashed curve is E_D using the asymptotic limit of E_N given in Eq. (25). Clearly the eigenenergies are in the relativistic asymptotic regime and not the linear regime of the nonrelativistic harmonic oscillator.

In Fig. 2 we plot the spherical harmonic oscillator excitation energies $E_N^*=E_N-E_0$ for different N on the far left. Each level has a $(N+2)(N+1)$ degeneracy because of spin symmetry and because the allowed orbital angular momenta are $\ell=N,N-2,\ldots,0$ or 1 and the allowed orbital angular momentum projections are $m=\ell,\ell-1,\ldots,-\ell$. In the right of Fig. 2 we plot the deformed excitation energies $E_{N,n_3}^*=E_{N,n_3}-E_{0,0}$. The deformed excitation energies are staggered in groups of each N and each group contains the levels of $n_3=0,1,\ldots,N$ with the excitation energy increasing with decreasing n_3. The dimensionless oscillator strengths are determined by $C^3=C_\perp^2 C_3$ and assuming a deformation $\delta=0.33$ which leads to $C_\perp=1.49, C_3=1.05$ [24]. Each level has a $2(N-n_3)+1$ degeneracy for $(N-n_3)$ even and a $2(N-n_3+1)$ degeneracy for $(N-n_3)$ odd because of spin symmetry and because the allowed orbital angular momenturm projections are $\Lambda=\pm(N-n_3),\pm(N-n_3-2),\ldots,\pm1$ or 0. The splitting of the levels within each N appears to be approximately linear with n_3.

E. Relativistic contribution

The normalization $\mathcal{N}(E)$ has the same functional form independent of whether the harmonic oscillator is triaxial, axially deformed, or spherical. This normalization has also been calculated independently by using $\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz[|f_\mu^+(\vec{r})|^2+|f_\mu^-(\vec{r})|^2]=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\ g(\vec{r})^*p^2/(\tilde{E}+\tilde{M})^2 g(\vec{r})$ and we find agreement between the two different ways of calculating $\mathcal{N}(E)$.

This also tells us that the probability of the lower component to the upper component is given by

$$R_k=\frac{\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz[|f_{k,\mu}^+(\vec{r})|^2+|f_{k,\mu}^-(\vec{r})|^2]}{\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz|g_k(\vec{r})|^2}=\frac{\tilde{E}_k-\tilde{M}}{2(\tilde{E}_k+\tilde{M})}. \tag{44}$$

Thus for $\tilde{E}_k\approx\tilde{M}$ the system is not very relativistic and the contribution of the lower components is small. For $\tilde{E}_k\gg\tilde{M}$, this ratio approaches $\frac{1}{2}$. For free particles this ratio approaches unity which indicates that the harmonic oscillator reduces the relativistic effect.

In Fig. 3 we plot this ratio for the spherical harmonic oscillator, R_N, as a function of N. Even for the most bound states this probability is about 20% and thus the antinucleon bound inside the nucleus is much more relativistic than a nucleon inside a nucleus for which this probability is about 1%.

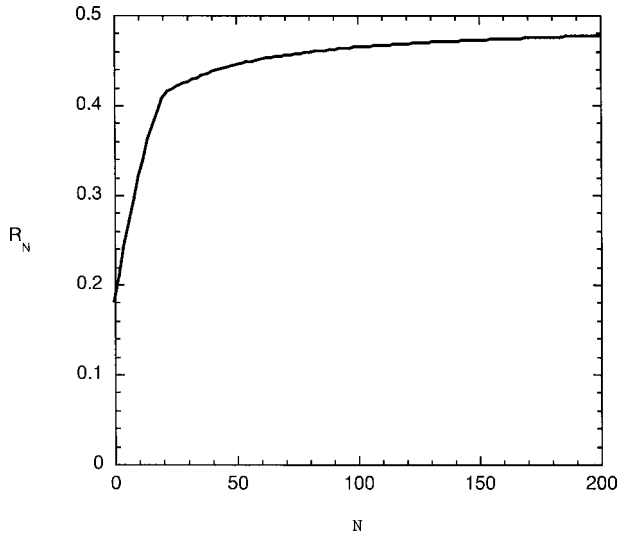

FIG. 3. The ratio of the probability of the lower components to the upper components, R_N, for the spherical harmonic oscillator as a function of N.

IV. HIGHER ORDER SYMMETRY

The nonrelativistic spherical harmonic oscillator has an SU(3) symmetry [26]. This symmetry is generated by the orbital angular momentum xsoperators $\tilde{\ell}$ and the quadrupole operators

$$q_m = [rr]^{(2)}_m \lambda^2_{NR} + \frac{[pp]^{(2)}_m}{\lambda^2_{NR}}, \tag{45}$$

Where $[rr]^{(2)}_m$ means coupled to angular momentum rank 2 and projection m and $\lambda_{NR}=\sqrt{M\omega}$. This quadrupole operator is then a function of the dimensionless variable $\vec{x}_{NR}=\lambda_{NR}\vec{r}$. These generators connect the degenerate states of the harmonic oscillator.

The same degeneracy that appears in the nonrelativistic spectrum appears in the relativistic spectrum. The upper component of the relativistic eigenfunction given in Eq. (35) has the same form as the nonrelativistic harmonic oscillator eigenfunction except that it is function of the relativistic dimensionless variable $\vec{x}=\lambda\vec{r}$. Therefore if the generators are written in terms of the relativistic dimensionless variable they will connect the upper components of all the degenerate states of the relativistic harmonic oscillator in the same manner as the nonrelativistic quadrupole in Eq. (45) connects the degenerate eigenstates of the nonrelativistic harmonic oscillator. Likewise, since the lower components are proportional to U_p operating on the upper components, the dimensionless quadrupole operator transformed by U_p connects the lower components of the degenerate states in the same manner as the nonrelativistic quadrupole operator in Eq. (45) connects the degenerate eigenstates of the nonrelativistic harmonic oscillator. However, λ depends on the energy [see Eq. (38a)] and therefore the relativistic quadrupole generator is

$$Q_m = \begin{pmatrix} [rr]^{(2)}_m & 0 \\ 0 & U_p[rr]^{(2)}_m U_p \end{pmatrix} \omega\sqrt{\tilde{M}(H_s - V^0_V + \tilde{M})1} + [pp]^{(2)}_m \frac{1}{\omega\sqrt{\tilde{M}(H_s - V^0_V + \tilde{M})}}, \tag{46}$$

since U_p commutes with $\vec{p}$. These generators along with the orbital angular momentum $\vec{L}$, given by Eq. (34), connect the degenerate states with each other. Work on the algebra is in progress. For $M\to\infty, Q_m \to \overline{Q}_m$,

$$\overline{Q}_m = \begin{pmatrix} [rr]^{(2)}_m & 0 \\ 0 & U_p[rr]^{(2)}_m U_p \end{pmatrix} \omega M + \frac{[pp]^{(2)}_m}{\omega M}, \tag{47}$$

which forms an SU(3) algebra with $\vec{L}$.

V. SUMMARY AND CONCLUSIONS

We have derived the eigenfunctions and eigenenergies for a Dirac Hamiltonian with triaxial, axially deformed, and spherical harmonic oscillator potentials and with equal scalar and vector potentials. In all cases the Dirac Hamiltonian is invariant with respect to the SU(2) spin symmetry and thus the eigenenergies are independent of spin. For axially symmetric potentials the Dirac Hamiltonian is invariant with respect to a $SU(2)\times U(1)$ group and the eigenenergies are degenerate with respect to the orbital angular momentum projection along the z axis which generates the U(1). For the spherical oscillator the eigenenergies are degenerate with respect to the orbital angular mometum and hence invariant under the $SU_L(2)\times SU(2)$ group. These energies also have a higher degeneracy which is the same as the nonrelativistic harmonic oscillator; that is, they depend only on the total harmonic oscillator quantum number. The generators that connect these degenerate states have been derived but a larger symmetry group analogous to SU(3) symmetry has yet to be identified. However, for infinite mass, the spherical relativistic harmonic oscillator is invariant with respect to an $SU(3)\times SU(2)$ group.

The eigenenergies have the same functional form for triaxial, axially deformed, and spherical potentials and depend on one variable which is a linear combination of the oscillator quanta in a given direction weighted by the strength of the oscillator potential in that direction. The spectrum is infinite and the eigenenergies and are linear in the harmonic oscillator quanta for small oscillator strength but increase slower than linear for large oscillator strength.

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