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Wick's limit and a new method for estimating neutron reaction cross sections

F. S. Dietrich, J. D. Anderson, and R. W. Bauer
Lawrence Livermore National Laboratory, Livermore, California 94550, USA

S. M. Grimes
Ohio University, Athens, Ohio 45701, USA
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We construct an analytic model to demonstrate qualitatively the correspondence between the measured neutron total cross section and the regions where Wick's limit is actually an equality. This model does not give sufficiently accurate quantitative results, so we extend our calculations by using the nuclear optical model with both local and global parameters. We then demonstrate how Wick's limit can be used to give useful reaction cross-section information.

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I. INTRODUCTION

Wick's limit [1] has been used by neutron physicists to estimate zero-degree neutron elastic-scattering cross sections.[1] The fact that this limit was nearly an equality has been known for over four decades [2,3]. We point out that there exists an analytic model for actually estimating the accuracy of this near equality and that for fast neutrons (neutrons in the MeV energy range) it is probably accurate to within a few percent for $A>27$. Although useful in providing an intuitive understanding of how the near equality comes about, the analytic model is not sufficiently accurate to be useful in evaluating modern neutron data where the precision of the experimental data is in the neighborhood of 1%.

We begin by defining Wick's limit. We then describe and apply the simple analytic model to illustrate the connection between Wick's limit and the measured total cross sections. Since our analytic model is not sufficiently accurate we examine several optical-model calculations for neutrons incident on ^{208}Pb and note the quality of fits to the total cross-section data and the corresponding reliability of the Wick's limit derived from these calculations. Using a recent global optical potential [4] we delineate the regions of incident energy and target mass for which the approximate equality holds. Finally, we demonstrate a new method of deducing reaction cross sections using Wick's limit.

II. DEFINITION OF WICK'S LIMIT

Wick's limit [1] is derived from the optical theorem which relates the imaginary part of the zero-degree scattering amplitude $f(0°)$ to the total cross section σ_{tot}, i.e.,

$$\mathrm{Im}\, f(0°) = \frac{k}{4\pi}\sigma_{tot}, \qquad (1)$$

where k is the center-of-mass wave number. The zero-degree differential elastic cross section σ_0 is given by

$$\sigma_0 = [\mathrm{Re} f(0°)]^2 + [\mathrm{Im}\, f(0°)]^2 \geqslant [\mathrm{Im}\, f(0°)]^2. \qquad (2)$$

Thus the zero-degree differential elastic cross section σ_0 must be equal to or exceed Wick's limit which is

$$\sigma_0^W \equiv \left(\frac{k}{4\pi}\sigma_{tot}\right)^2. \qquad (3)$$

It is convenient to define a fractional deviation of the zero-degree differential cross section from Wick's limit as

$$\eta = \frac{\sigma_0 - \sigma_0^W}{\sigma_0^W}. \qquad (4)$$

Wick's limit is a useful concept when η is small. The conditions for which this is true will be discussed in the following sections.

III. ANALYTIC MODEL

By invoking a simple model of the total cross section (the nuclear Ramsauer model [5–9]) one can obtain an analytic estimate of the real part of the scattering amplitude from neutron total cross-section data. Thus, within the model we can estimate the validity of Wick's limit as well as the first-order correction term.

In describing the total cross section we use a combination of formulations by Peterson [5] and by Bohr and Mottelson [6] as follows. We assume a zero-degree scattering amplitude of the form

$$f(0°) = ik(R + \lambda\!\!\!^{-})^2(1 - \alpha e^{i\beta})/2, \qquad (5)$$

where $\lambda\!\!\!^{-}=1/k$ is the usual reduced wavelength. By using Eq. (1) one then obtains

$$\sigma_{tot} = 2\pi(R + \lambda\!\!\!^{-})^2(1 - \alpha \cos \beta). \qquad (6)$$

The average behavior of the total cross section can be described by $\sigma_{tot}=2\pi(R+\lambda\!\!\!^{-})^2$ (the "black nucleus" approximation) and the effect of the coherent nuclear Ramsauer effect is reflected in the $(1-\alpha \cos \beta)$ term. We note that the average cross-section behavior is well fit [5] by $R=1.35A^{1/3}$ fm. The argument of the periodic term β can be

[1] Magnetic moment (Mott-Schwinger) scattering yields a divergent cross section at zero degrees, but is not relevant in the present context because its effects are largely confined to angles much lower than those in measured angular distributions. This is discussed briefly in Sec. VI and the Appendix.

understood in terms of the difference in phase between the wave passing outside the nuclear surface and the wave passing through a medium with an index of refraction, i.e.,

$$\beta = (4/3)\psi R(k_{nucl} - k_{inc}), \quad (7)$$

where $(4/3)R$ is the average chord length of the neutron passing through the nucleus and ψ is the index of refraction given by

$$\psi = \frac{E+V}{E}\left[1 - \left(\frac{V}{E+V}\right)^{3/2}\right]. \quad (8)$$

In the above expression $k_{nucl}=[2m(E+V)/\hbar^2]^{1/2}$ where the positive quantity V is the depth of the average real optical potential, E is the neutron energy, and $k_{inc}=(2mE/\hbar^2)^{1/2}$.

The coefficient α represents the absorption of the incident wave and in this crude model is given by

$$\alpha = \exp\left[-\frac{LW}{\hbar}\left(\frac{2m}{E+V}\right)^{1/2}\right], \quad (9)$$

where the positive quantity W is the strength of the average absorption (imaginary) potential and L is an appropriate nuclear dimension. Although we will use only the empirical parametrizations of α and β determined from Ref. [8] in the numerical results shown in the present work, it is important to have some idea of the energy dependence of these parameters to understand the limits on the applicability of our procedure. Given that our assumed scattering amplitude, Eq. (5), does yield a total cross-section description [Eq. (6)] that provides an adequate fit to the total cross-section data, we may use it to evaluate the zero-degree differential cross section via Eq. (2). Thus we obtain

$$\sigma_0 = \left[\frac{k}{2}(R+\lambda\!\!\!^{-})^2\right]^2[(1-\alpha\cos\beta)^2 + \alpha^2\sin^2\beta], \quad (10)$$

where $\alpha^2 \sin^2 \beta$ is the additional contribution to the zero-degree cross section from the real part of the scattering amplitude. This implies that we can estimate the accuracy of the use of Wick's limit as an equality via the relation

$$\sigma_0 = \sigma_0^W\left[1 + \frac{\alpha^2\sin^2\beta}{(1-\alpha\cos\beta)^2}\right], \quad (11)$$

in which we note that the correction term is proportional to α^2. This expression may be rearranged to yield the fractional deviation η in the analytic model as

$$\eta = \frac{\alpha^2\sin^2\beta}{(1-\alpha\cos\beta)^2}. \quad (12)$$

IV. EVALUATION OF THE ANALYTIC MODEL

We now present as an example the case of lead (Pb). The total cross section has been very accurately measured over the energy range 5–540 MeV [10]. We first divide the measured cross section by $2\pi(R+\lambda\!\!\!^{-})^2$ where $R=1.35A^{1/3}$ fm following the procedure of Ref. [8]. We then deduce from the plot of the normalized cross section versus $\sqrt{E}$ shown in Fig. 1 that these data are well described by

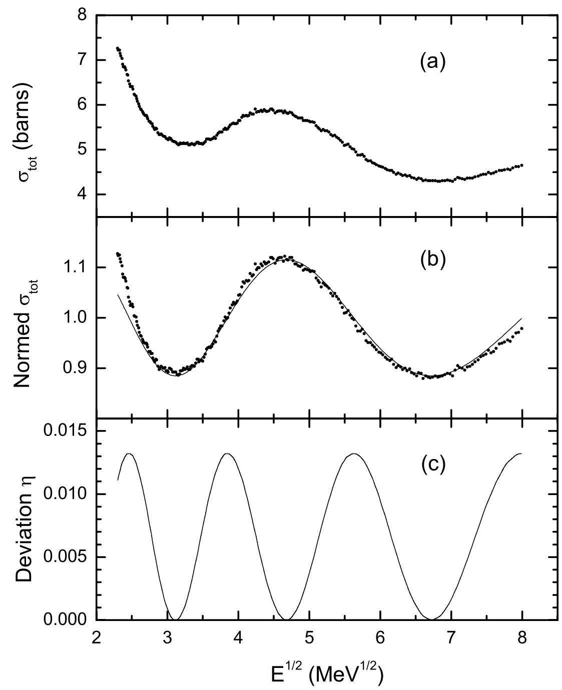

FIG. 1. (a) Total neutron cross section σ_{tot} of natural Pb measured by Abfalterer *et al.* [10] plotted vs. the square root of the neutron energy over the energy range 5.3–64 MeV. (b) The normalized total cross section $\sigma_{tot}/2\pi(R+\lambda\!\!\!^{-})^2$ defined in Eq. (13). (c) The fractional deviation η of the zero-degree differential cross section from Wick's limit as calculated in the analytic model [Eq. (12)].

$$\frac{\sigma_{tot}}{2\pi(R+\lambda\!\!\!^{-})^2} = 1 - \alpha\cos\beta, \quad (13)$$

where we have used the globally fitted parameters of Table II in Ref. [8] to determine β.

It is clear that in the neutron energy region from 6 to 60 MeV the measurements can be well fitted with a fixed value of α, namely, α=0.115. This implies a maximum correction to "Wick's equality" of just over 1%. The accuracy of this correction depends on α^2 but we note that varying α from 0.09 to 0.12, the extremes allowed by the data in this region, changes this correction only from 0.8% to 1.4%. An estimate of the energy dependence of the correction term can also be obtained by fitting the energy dependence of β. The fractional deviation of the zero-degree cross section from Wick's limit, as evaluated from Eq. (12) for this simple analytic model, is also plotted in Fig. 1.

To show the connection between Wick's limit, the true zero-degree cross section, and the total cross section, it is useful to plot the behavior of the quantity $S=\alpha\exp(i\beta)$ as a function of energy. S is determined from the scattering amplitude via Eq. (5). It is defined so that in the low-energy limit (s waves only) it corresponds to the collision matrix element for $l=0$. It also represents the common value of the collision matrix element in a single phase shift approximation; i.e., the assumption that the complex phase shift is the

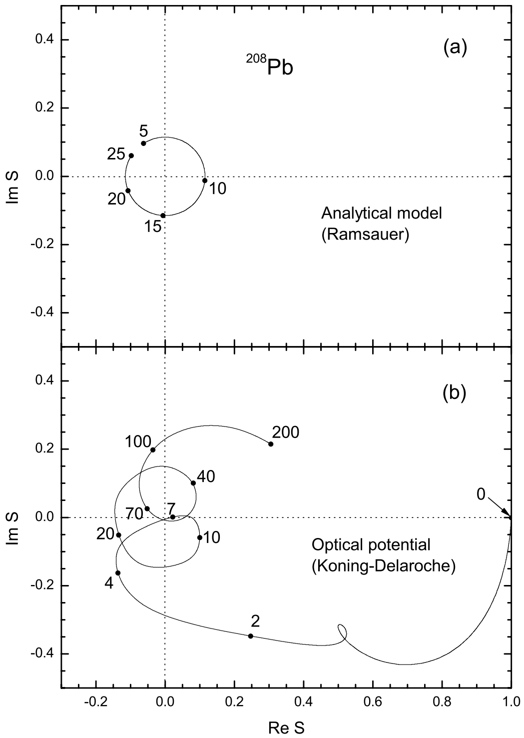

FIG. 2. Behavior of the quantity $S=\alpha\,\exp(i\beta)$ as defined in Eq. (5) for ^{208}Pb, employing (a) the simple analytic model and (b) an optical model calculation using the Koning-Delaroche potential [4]. The labels on the trajectories are the incident neutron energies in MeV.

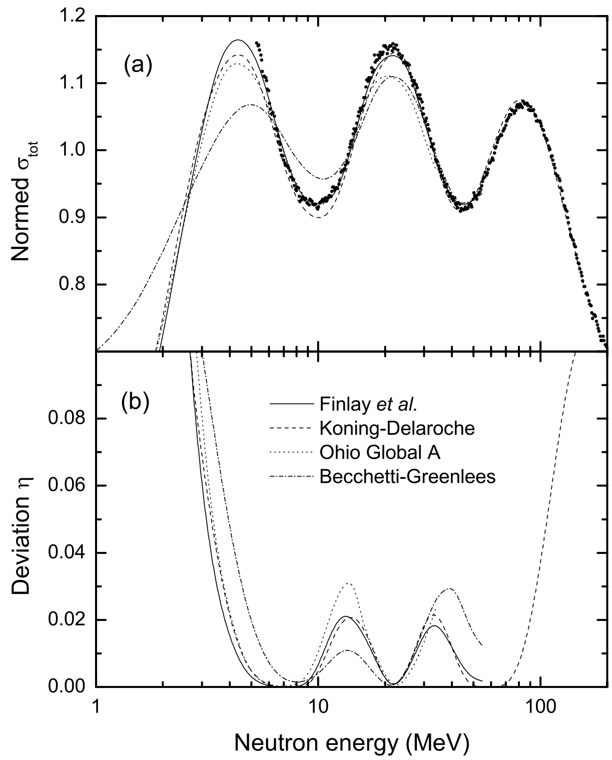

FIG. 3. Results of four optical-model calculations as described in the text. (a) Total neutron cross sections divided by $2\pi(R+\lambda\!\!\!^{-})^2$ for ^{208}Pb compared with experiment on natural Pb measured by Abfalterer *et al.* [10]. (b) The fractional deviation η of the zero-degree differential cross section from Wick's limit as defined in Eq. (4).

same in all partial waves. This last assumption is critical for the usefulness of the Ramsauer model, and its justification has been discussed in Ref. [11]. In terms of $f(0°)$ and S, the fractional deviation from Wick's limit may be expressed as

$$\eta=\left(\frac{\mathrm{Re}f(0°)}{\mathrm{Im}f(0°)}\right)^2=\left(\frac{\mathrm{Im}S}{1-\mathrm{Re}S}\right)^2. \qquad (14)$$

In the simple analytic model with an energy-independent value of α, all values of S lie on a circle in the complex plane as indicated in the upper part of Fig. 2, as shown for energies between 5 and 25 MeV. Wick's limit is an equality whenever the trajectory of S crosses the real axis. This occurs for $\beta=n\pi$, where n is an integer, and these points correspond to the maxima and minima in the total cross section. The maximum deviations from Wick's limit occur when the trajectory crosses the imaginary axis, which happens for $\beta=m\pi/2$, where m is an odd integer. These points correspond to the inflection points in the normalized total cross section, as can be seen in Fig. 1.

The Ramsauer model is equivalent to a square well potential and thus neglects the nuclear surface. From Fig. 8 of Ref. [9] we find that the addition of a surface term, although not affecting the maxima in the total cross section, significantly modifies the behavior of S near the minima in the total cross section. Thus the minima in the deviation of the zero-degree cross section from Wick's limit do not come exactly at the minima in the total cross section. Optical-model calculations also support these conclusions (see Figs. 7 and 8 in Ref. [9]).

Thus we find a significant value of ImS at the cross section minima, and only at the maxima in the total cross-sections does the Wick equality hold. We also note that realistic optical-model calculations do not yield a constant value of $|S|$, unlike the simple model. Consequently the analytic model, while providing useful guidance on the behavior of the deviation from Wick's limit, does not correctly estimate the maximum deviation from equality. Therefore we now investigate optical-model calculations.

V. OPTICAL-MODEL CALCULATIONS

For our sensitivity studies we have chosen four different sets of optical-model parameters. The first set is the neutron potential of Becchetti and Greenlees [12], which was chosen because it was one of the first attempts to fit a large number of nuclei over a significant energy range. The second parametrization is the Ohio global A set [13]. This set allows for significant energy variations of the parameters, but covers a smaller energy range. The third set was due to Finlay *et al.* [14], which covered a large energy range and was developed to fit a wide variety of data for neutrons incident on ^{208}Pb. The fourth set, due to Koning and Delaroche [4], is a recent global potential covering a wide mass and energy range.

Since we originally considered using the analytic model because of its excellent representation of the total cross section, we begin by comparing our various optical-model calculations (Fig. 3) with the same high-precision total cross-section measurements [10] shown in Fig. 1. We also plot the fractional difference $(\sigma_0-\sigma_0^W)/\sigma_0^W$ between the calculated zero-degree cross section and the Wick's limit. It is immediately obvious that the Becchetti-Greenlees potential deviates by more than 5% from the measured data and does not meet our criteria for a good fit. This should not be surprising since the high-precision data were not available at the time of Becchetti and Greenlees's work. The latest global parametrization of Koning and Delaroche as well the potential of Finlay *et al.* both meet the criteria of fitting the total cross-sections at the 1–2% level. The Ohio Global parameterization is an excellent fit over part of the energy range, but deviates by about 5% at the cross section maximum around 20 MeV. Since the slope of the cross section versus energy is different from the other potentials, we also see a significant difference at about 15 MeV in the magnitude of the fractional deviation from Wick's limit.

We may conclude from the above discussion that the two most satisfactory optical-model parameter sets (as judged by comparison with the total cross section data) give satisfactory agreement with the measured total cross section and produce very similar results for the deviation from Wick's limit. The comparison of these calculations with the results of the analytic model shows that the minimum deviation, corresponding to the cross-section maxima, is correctly given, but that the minimum, corresponding to the minimum cross section, is displaced in energy. We further note that the maximum deviation given by our optical-model calculations is almost a factor of 2 larger than that from our analytic model. We also conclude that our correction term, which varies from 7% at 3 MeV to 4% at 100 MeV, may be a useful concept for Pb over this energy range.

Further insight into the relation between the values of Wick's limit given by the analytic model and the optical model can be gained by examining the quantity S introduced in the preceding section, as shown in Fig. 2. The forward scattering amplitude $f(0°)$ was calculated from the Koning-Delaroche potential [4], and S was obtained by inversion of Eq. (5) as

$$S = 1 + \frac{2if(0°)}{k(R+\lambda\!\!\!^{-})^2}, \qquad (15)$$

where $R=1.35A^{1/3}$ fm. As noted earlier, S is a single phase shift parametrization of the solution to the scattering problem. As the incident energy approaches zero, S must go to 1 as shown in the lower portion of the figure, since S is exactly the s-wave collision matrix element in this limit. When the incident energy is sufficiently large the trajectory of S executes loops in the neighborhood of the origin, resembling the behavior of the analytic model shown in the upper portion. This is the energy region for which the single phase shift parametrization is a useful representation of the scattering problem, and for which Wick's limit is an equality when the trajectory crosses the real axis. At higher energies the single phase shift picture is invalidated because the refractive effects that tend to equalize the phase shifts in the various partial waves are reduced, and the trajectory no longer circles in the neighborhood of the origin. This picture shows that there is a specific energy region in which Wick's limit is close to the exact zero-degree cross section and is therefore useful; in the case of ^{208}Pb this region is approximately 4–80 MeV.

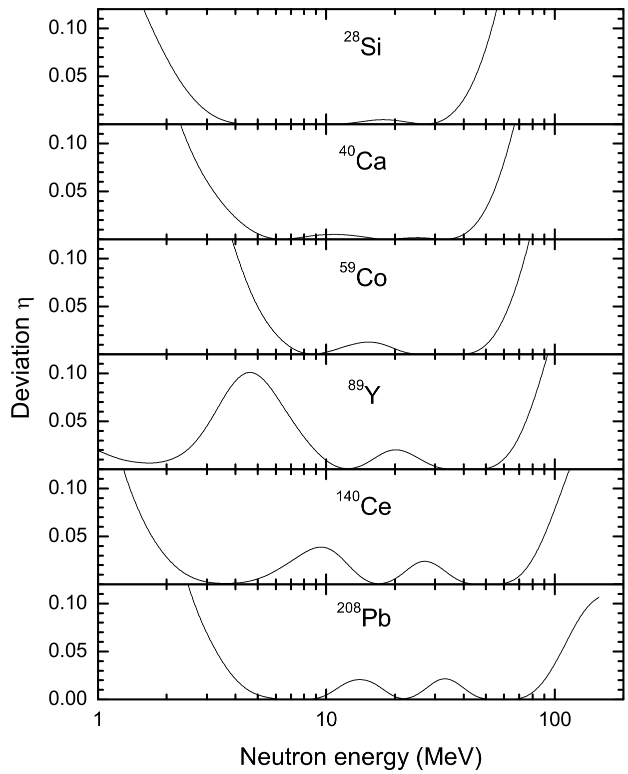

FIG. 4. Optical-model calculations of the fractional deviation $\eta=(\sigma_0-\sigma_0^W)/\sigma_0^W$ of the true zero-degree differential cross section from Wick's limit, using the potential of Koning and Delaroche [4]. These calculations show that the deviation is small over a wide range of incident energies and target masses.

To explore the mass and energy ranges for which Wick's limit is an approximate equality, we have performed optical-model calculations for several nuclei from ^{28}Si to ^{208}Pb using the Koning-Delaroche potential [4]. The fractional deviations η are shown in Fig. 4. In all cases there is a wide energy range over which η does not exceed a few percent. Below and above this range the deviation grows to a large value. The trajectories of S (not shown) characterize this behavior in a manner similar to that shown in Fig. 2 for ^{208}Pb. The trajectory loops near the origin twice for the nuclei heavier than $A\approx 90$, and once for lighter nuclei. The intermediate nucleus ^{89}Y shows an anomalously small deviation at low energies in the neighborhood of 1–2 MeV. This corresponds to a small loop in the trajectory at low energies before the approximate single phase shift behavior is well established.

VI. USING WICK'S LIMIT TO DETERMINE REACTION CROSS SECTIONS

In this section we use Wick's limit to introduce a new method for determining neutron reaction cross sections. This quantity, sometimes called the nonelastic cross section, is the difference between the total cross section and the angle-integrated elastic cross section:

$$\sigma_{reac} = \sigma_{tot} - \sigma_{elas}. \tag{16}$$

Direct measurements of σ_{reac} are difficult. Most of them have been made by measurements of the attenuation of neutrons in a spherical shell of the sample material (see, e.g., Refs. [15–17]). Such measurements are sparse and are subject to systematic errors that must be carefully evaluated. Determining σ_{reac} by subtracting independent measurements of σ_{tot} and σ_{elas} is also difficult, because the subtraction of two large quantities magnifies the resultant error.

We now show that using Wick's limit to relate the two quantities on the right of Eq. (16) allows us to obtain an expression in which the errors in these quantities are correlated in a manner that greatly reduces the resultant error in σ_{reac}. We will also show that the model dependence introduced by this procedure is very small when the deviation between Wick's limit and the exact zero-degree elastic differential cross section is small. The conditions for which this is true were discussed in a previous section.

To proceed, we define a quantity determined entirely by experiment,

$$F = \frac{\sigma_{elas}}{\sigma_0} = \frac{1}{\sigma_0}\int d\Omega \frac{d\sigma_{elas}}{d\Omega}, \tag{17}$$

which is the ratio of the integral over solid angle of a measured elastic angular distribution to its value at zero degrees. It is important to note that F does not require knowledge of the absolute value of the elastic differential cross section. We may now express the original expression for σ_{reac}, Eq. (16), as

$$\sigma_{reac} = \sigma_{tot} - \sigma_0 F, \tag{18}$$

which in turn may be expressed as

$$\sigma_{reac} = \sigma_{tot} - (1+\eta)F\left(\frac{k}{4\pi}\right)^2 \sigma_{tot}^2, \tag{19}$$

where we have used the definition of Wick's limit and its fractional deviation from the true zero-degree cross section [Eqs. (3) and (4)]. In the last expression we identify two independent experimental quantities σ_{tot} and F and a calculated quantity, η, which is determined from an optical-model calculation. The error in σ_{reac} is found by adding the contributions from these three independent quantities in quadrature. This interpretation assumes that compound elastic scattering is negligible. Modifications to Eq. (19) when compound elastic scattering is present are discussed in the Appendix. An additional effect, the scattering of the neutron's magnetic moment from the Coulomb field (Mott-Schwinger scattering) [18–20] is largely confined to very small angles and does not play a significant role in the extrapolation to zero degrees of currently available measured angular distributions, for which the minimum angle is in the range 12°–20°. This issue is also discussed briefly in the Appendix.

The error in σ_{reac} due to the error $\Delta\sigma_{tot}$ in the measured value of the total cross section is easily found to be

$$\Delta\sigma_{reac}^{(1)} = \left| 1 - 2(1+\eta)F\left(\frac{k}{4\pi}\right)^2 \sigma_{tot} \right| \Delta\sigma_{tot}. \tag{20}$$

This expression is the difference of two positive terms, which significantly reduces the error in σ_{reac}. In fact, the cancellation of the two terms can lead to a very small contribution to the error in σ_{reac} due to the error in σ_{tot} in practical cases. To see this, we use Eq. (19) to eliminate F and thereby rewrite $\Delta\sigma_{reac}^{(1)}$ as

$$\frac{\Delta\sigma_{reac}^{(1)}}{\sigma_{reac}} = \left| 2 - \frac{\sigma_{tot}}{\sigma_{reac}} \right| \frac{\Delta\sigma_{tot}}{\sigma_{tot}}. \tag{21}$$

The Ramsauer model, supported by realistic optical-model calculations, shows that over a wide target-mass and energy ranges, σ_{tot} oscillates about $2\sigma_{reac}$ as a function of energy with an amplitude approximately 10% of the value of σ_{tot}. Therefore, the cancellation in the two terms of Eq. (21) is strong and at certain energies is exact.

The error contribution from η is

$$\Delta\sigma_{reac}^{(2)} = F\left(\frac{k}{4\pi}\right)^2 \sigma_{tot}^2 \Delta\eta. \tag{22}$$

Again using Eq. (19), this may be expressed as

$$\frac{\Delta\sigma_{reac}^{(2)}}{\sigma_{reac}} = \left(\frac{\sigma_{tot}}{\sigma_{reac}} - 1\right)\frac{\eta}{1+\eta}\;\frac{\Delta\eta}{\eta}. \tag{23}$$

Under the conditions noted above in the discussion of $\Delta\sigma_{reac}^{(1)}$ the expression in parentheses is close to 1, and the maximum value of η is a few percent. Therefore a rather large value of the fractional error $\Delta\eta/\eta$ will lead to a small fractional error contribution to σ_{reac}.

The final error contribution, due to F, is

$$\Delta\sigma_{reac}^{(3)} = (1+\eta)\left(\frac{k}{4\pi}\right)^2 \sigma_{tot}^2\, \Delta F, \tag{24}$$

which may be transformed to

$$\frac{\Delta\sigma_{reac}^{(3)}}{\sigma_{reac}} = \left(\frac{\sigma_{tot}}{\sigma_{reac}} - 1\right)\frac{\Delta F}{F}. \tag{25}$$

Again, the parenthesized expression is close to 1 in favorable circumstances.

We have not yet considered the effects of compound elastic scattering. This will be small in most cases where the deviation from Wick's limit is small, but must be treated correctly if it is present. It is shown in the Appendix that compound elastic scattering adds an additional term to Eq. (19) that requires an estimate of the compound contribution to the zero-degree elastic differential cross section. For rotational nuclei we have the complication that the elastic angular distribution is rarely measured alone for incident neutrons because of insufficient energy resolution. When the angular distributions include a well-defined set of excited states in the ground-state band it is straightforward to extend the present treatment to this case with the help of results from coupled-channel calculations.

TABLE I. Values of reaction cross sections for neutrons on ^{208}Pb calculated from Eq. (19). The total cross sections and constant-geometry optical potential used to calculate η were taken from Ref. [14]. Uncertainties in the reaction cross section were taken from the results shown in Table II.

E(MeV)	σ_{tot}(b)	η	F	σ_{reac}(b)
7	5.78	3.22×10^{-7}	0.4768	2.393±0.045
20	5.85	2.13×10^{-3}	0.1608	2.477±0.046
22	5.79	3.27×10^{-5}	0.1440	2.538±0.044
24	5.67	2.44×10^{-3}	0.1359	2.450±0.044

In summary, Wick's limit allows the original subtraction expression, Eq. (16) to be transformed so that the error due to the uncertainty in the total cross-section can be made very small and an absolute cross-section measurement of the elastic cross section is absent. There is no error amplification due to the subtraction of large quantities. The model dependence introduced by the use of Wick's limit is quantified via Eqs. (22) and (23) and is small when Wick's limit is close to the exact value. In these circumstances the dominant error contribution is due to F, the ratio of the solid-angle integral of the elastic angular distribution to its value at zero degrees. Since the zero-degree cross section must be obtained by extrapolation, considerable care must be taken in the analysis of the angular distribution if F is to be well determined. An example of how this can be done will be shown next.

Very precise measurements of neutron elastic-scattering angular distributions on ^{208}Pb were reported in Ref. [14] at incident energies 7, 20, 22, and 24 MeV. In that work the observation was made that Wick's limit was a near equality at these energies. This fact was used to confirm the estimate of the absolute normalization of the angular distributions (2% at 7 MeV; 3% at the higher energies) by showing that the extrapolated zero-degree cross section was consistent with Wick's limit. Reaction cross sections were also obtained by subtracting the angle-integrated elastic angular distributions from total cross sections. To illustrate the usefulness of the technique presented here, we calculate the reaction cross sections using Eq. (18) and compare them with those obtained by subtraction. Results are shown in Table I, and the uncertainties as well as their components are shown in Table II. The extracted cross sections are shown in Fig. 5, along with the results of the direct subtraction procedure taken from Ref. [14].

To find the quantity F we employ the method of Ref. [14], as shown in Fig. 2 of that paper for the 22-MeV data. The

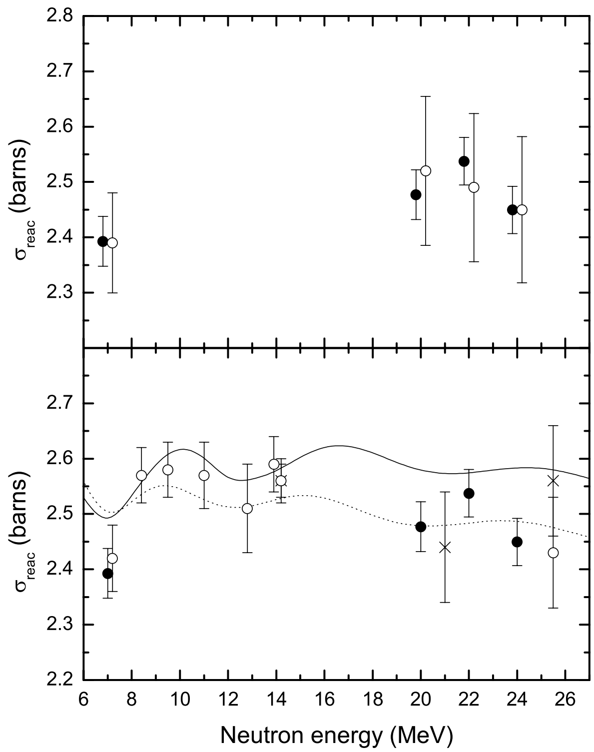

FIG. 5. Upper portion: reaction cross sections for neutrons on ^{208}Pb at 7, 20, 22, and 24 MeV inferred by the method discussed herein from total cross sections and elastic angular distributions in Ref. [14] (closed circles). These are compared with reaction cross section obtained by direct subtraction of the total and elastic cross sections (open circles). The points are displaced slightly in energy to show the uncertainties. Lower portion: The present results (closed circles) compared with sphere-transmission measurements from Refs. [15–17] on natural Pb (crosses) and ^{209}Bi (open circles). The solid curve is obtained from the constant-geometry optical model of Ref. [14], and the dotted curve from that of Ref. [4].

values of χ^2 per degree of freedom, the integrated elastic cross section, and the zero-degree differential cross section are calculated as a function of L_{max}, the maximum order of a Legendre polynomial fit. For well-behaved data the χ^2 per degree of freedom shows a distinct knee at a critical value of L_{max}, beyond which its value is roughly constant. The elastic and zero-degree cross sections show the same behavior, but eventually become erratic for higher L_{max} because the fitting function is not adequately constrained by the data. We obtain the desired cross sections from the stable region just above the knee. We have repeated the Legendre fits of Ref. [14] to verify them and to estimate the uncertainty in F, which we estimate as 1.3%.

TABLE II. Uncertainty estimates for reaction cross sections shown in Table I as calculated from Eqs. (21), (23), and (25). The last column is the final fractional error obtained by adding the three previous columns in quadrature.

E(MeV)	$\Delta\sigma_{tot}/\sigma_{tot}$	$\Delta\eta/\eta$	$\Delta F/F$	$\Delta\sigma^{(1)}_{reac}/\sigma_{reac}$	$\Delta\sigma^{(2)}_{reac}/\sigma_{reac}$	$\Delta\sigma^{(3)}_{reac}/\sigma_{reac}$	$\Delta\sigma_{reac}/\sigma_{reac}$
7	0.010	0.5	0.013	0.0043	2.3×10^{-7}	0.0184	0.0189
20	0.015	0.5	0.013	0.0056	1.5×10^{-3}	0.0177	0.0186
22	0.016	0.5	0.013	0.0044	2.1×10^{-5}	0.0167	0.0172
24	0.016	0.5	0.013	0.0050	1.6×10^{-3}	0.0171	0.0179

The values of the total cross sections were those used in Ref. [14]. More recent measurements [21,10] on natural Pb, ^{208}Pb, and Bi are available, but they are consistent with those used in Ref. [14] within stated uncertainties. As shown in Table II (column $\Delta\sigma^{(1)}_{reac}/\sigma_{reac}$) the error from the total cross section is small compared to that from $F(\Delta\sigma^{(3)}_{reac}/\sigma_{reac})$.

Values of η were calculated from the constant-geometry optical model in Ref. [14]. The error contribution $\Delta\sigma^{(2)}_{reac}/\sigma_{reac}$ is negligible, even with the assumed 50% error in this quantity. Of course, this result is somewhat artificial since Wick's limit is a near equality at the chosen energies.

As can be seen in the upper portion of Fig. 5 the results are consistent with the direct subtraction procedure but with much smaller errors, which are dominated by the uncertainties in F. In the bottom portion of the figure we show the new results in comparison with sphere-transmission measurements on natural Pb and ^{209}Bi from Refs. [15–17]. We have also shown cross sections calculated from two optical potentials, the constant-geometry potential from Ref. [14] (solid line), and the recent potential of Koning and Delaroche [4] (dotted line).

Finally, we summarize the conditions for which the new method for finding reaction cross sections is expected to be reliable. It is important that the model dependence introduced by the use of Wick's limit should be weak. This requires the deviation of Wick's limit from an equality to be sufficiently well determined by optical-model calculations so that an uncertainty may be assigned to it that does not dominate the other error contributions; see Eq. (23) and discussion following it. Figure 4 shows that the deviation is small over a wide mass range and over an energy range that extends from a few MeV to several tens of MeV depending on the target mass. Assigning a 50% error to the fractional deviation $\Delta\eta/\eta$, as suggested by the variation in the optical-model calculations shown in Fig. 3, leads to an uncertainty contribution in the range (0–2)%. Outside this energy range the increased error contribution makes the method less useful, even though it is valid in principle. Since the optical model yields only energy-averaged observables, the method should be used only when the energy spread in the beam used to measure the angular distributions is sufficiently large to achieve a corresponding energy average over possible resonant structure. This is normally the case for medium and heavy nuclei, but the method may be inaccurate for light nuclei where this condition is often not satisfied. We repeat the caution that systematic and statistical errors in the elastic angular distribution measurement must be sufficiently under control to ensure the reliability of the extrapolation to zero degrees required for the determination of F [Eq. (17)]. There are currently scant neutron elastic angular distribution data available at energies above the regions shown in Fig. 4 where the deviation from Wick's limit is small, which is an additional reason for limiting application of the method to the region where the optical-model results are favorable.

VII. SUMMARY

We have applied an analytic model, the nuclear Ramsauer model, to demonstrate that Wick's limit is an equality at several energies and that these energies are well correlated with energy modulations in the total cross section. We have also demonstrated the inadequacies of this model and presented realistic optical-model calculations to obtain quantitative results. Furthermore we demonstrated that only two of the optical models examined gave sufficiently adequate representation of the high-precision total neutron cross-section measurements (with less than 2% error) so that we considered them reliable for generating estimates of the deviation from the Wick's limit equality. From this analysis we conclude that small values ($<4\%$) of the deviation of Wick's limit from an equality are found in the energy range 6–60 MeV for ^{208}Pb, with larger uncertainties outside this energy range. We also performed optical model calculations showing that the similar behavior is found over a wide mass range from light to heavy nuclei.

Having established reliable estimates of the deviation of the Wick's limit from equality, we developed a method of reducing the error on reaction cross sections determined from total and elastic-scattering cross sections. The usual procedure of obtaining the reaction cross section from the difference between the total cross section and the elastic-scattering cross section normally results in uncertainties of over 5% in the reaction cross section. Using our new procedure, when Wick's limit is nearly an equality, we are able to reduce this error significantly. Since the energy dependence of the reaction cross section is very slow, just a few data points are adequate to delineate this quantity. We presented data for ^{208}Pb to demonstrate the usefulness of this technique, showing that the error in the cross section was reduced by more than a factor of 2 from that given by a simple subtraction. Conditions for the usefulness of the method are noted at the end of the preceding section. If elastic angular distributions of sufficient quality are available, the method should be useful over a wide mass range for neutron energies from a few MeV to approximately 50 MeV.

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APPENDIX

In this appendix we give a more careful derivation of the expression for calculating reaction cross sections using Wick's limit and elastic angular distributions. In particular, we wish to show the effects of correctly including compound elastic scattering. We also briefly discuss the role of Mott-Schwinger scattering in the context of the present work.

The treatment previously given implicitly assumed that an energy average over an incident-energy interval large enough to smooth out cross-section fluctuations due to resonances had been carried out. This is a necessary condition for the use of the optical model in calculating Wick's limit. Following standard procedures, we divide the forward scattering amplitude $f(0°)$ into an average part $\bar{f}(0°)$ and a fluctuating part $f^{fl}(0°)$, defined so that the energy average of $f^{fl}(0°)$ vanishes. By using the complete $f(0°)$ in the expressions for the optical theorem and for the zero-degree differential cross section, and then taking an energy average (indicated by a bar), we have

$$\bar{\sigma}_{tot} = (4\pi/k)\mathrm{Im}\bar{f}(0°), \tag{A1}$$

$$\bar{\sigma}_0 = \overline{|f(0°)|^2} = \sigma_0^{shape} + \sigma_0^{cmpd}, \tag{A2}$$

where $\sigma_0^{shape} = |\bar{f}(0°)|^2$ and $\sigma_0^{cmpd} = \overline{|f^{fl}(0°)|^2}$. This expression explicitly indicates the separation of the observable differential elastic cross section into a shape-elastic part that can be estimated from an optical model and a compound part that requires a Hauser-Feshbach treatment with width fluctuations.

Similar to what was done in Sec. VI we express the energy-averaged reaction cross section as the difference between the energy-averaged total and integrated elastic cross sections,

$$\bar{\sigma}_{reac} = \bar{\sigma}_{tot} - \bar{\sigma}_{elas} = \bar{\sigma}_{tot} - \bar{\sigma}_0 \bar{F}, \tag{A3}$$

where

$$\bar{F} = \frac{\bar{\sigma}_{elas}}{\bar{\sigma}_0} = \frac{1}{\bar{\sigma}_0}\int d\Omega \frac{d\bar{\sigma}_{elas}}{d\Omega}. \tag{A4}$$

We define a Wick's limit related to the average total cross section by

$$\bar{\sigma}_0^W = [\mathrm{Im}\bar{f}(0°)]^2 = \left(\frac{k\bar{\sigma}_{tot}}{4\pi}\right)^2, \tag{A5}$$

and a fractional deviation of the shape-elastic cross section from this energy-averaged version of Wick's limit by

$$\bar{\eta} = \frac{\sigma_0^{shape} - \bar{\sigma}_0^W}{\bar{\sigma}_0^W} = \left[\frac{\mathrm{Re}\,\bar{f}(0°)}{\mathrm{Im}\bar{f}(0°)}\right]^2. \tag{A6}$$

With the above definitions, Eq. (A3) may be recast as

$$\bar{\sigma}_{reac} = \bar{\sigma}_{tot} - (1+\bar{\eta})\bar{F}\left(\frac{k}{4\pi}\right)^2 \bar{\sigma}_{tot}^2 - \bar{F}\sigma_0^{cmpd}. \tag{A7}$$

This expression is identical to that in Eq. (19), except for an extra term containing the compound elastic cross section. This term must be calculated from a reaction model and its uncertainty estimated. Since the shape-elastic scattering is strongly forward peaked, and the compound elastic is only slightly anisotropic, a rough upper limit on the compound elastic contribution may be obtained by looking at the deepest minimum in the elastic angular distribution.

The effect of magnetic moment (Mott-Schwinger) scattering on neutron differential elastic cross sections and polarizations has been calculated by several groups over the last few decades [18–20]. All of these treatments agree that the main effects on the cross sections are confined to very small angles (less than approximately 2°). Thus the effects of Mott-Schwinger scattering can be ignored in a consistent treatment in which the minimum angle at which measurements are made is large enough so that the Mott-Schwinger effect is negligible, and in which the optical-model analysis does not include the effect. Minimum angles in currently available angular distribution data are typically in the 12°−20° range. A simple estimate based on the results in Ref. [20] for 24 -MeV neutrons on Bi shows that the singular (Born approximation) part of the Mott-Schwinger cross section arising from the long-range $1/r^3$ interaction, which has angular dependence proportional to $\cot^2(\theta/2)$, is much less that 1% of the nuclear cross section at 12°, and can therefore be neglected. The most recent treatment of the Mott-Schwinger effect [20] showed that corrections to the Born approximation lead to a component in the Mott-Schwinger cross section that falls much less rapidly with angle than the Born term but can be seen in the minima of the elastic diffraction pattern at large angles. This small component may appropriately be assumed to be covered up by the phenomenological optical potential. Moreover, plots of the extrapolated zero-degree cross section as a function of maximum order of the Legendre polynomial (see Sec. VI and Ref. [14]) show no evidence for an increase of the cross section beyond a value $L_{max} \approx kR$. This would not be the case if there were a significant contribution from the long-range Mott-Schwinger interaction. However, we caution that if angular distribution data become available with minimum angles significantly smaller than those in currently available data, the relevance of the Mott-Schwinger cross section should be revisited.

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