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Pion electroproduction, partially conserved axial-vector current, chiral Ward identities, and the axial form factor revisited

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We reinvestigate Adler's partially conserved axial-vector current relation in the presence of an external electromagnetic field within the framework of QCD coupled to external fields. We discuss pion electroproduction within a tree-level approximation to chiral perturbation theory and explicitly verify a chiral Ward identity referred to as the Adler-Gilman relation. We critically examine soft-momentum techniques and point out how inadmissable approximations may lead to results incompatible with chiral symmetry. As a result we confirm that threshold pion electroproduction is indeed a tool to obtain information on the axial form factor of the nucleon.

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I. INTRODUCTION

Pion photo- and electroproduction both have a long tradition as a tool to obtain information on strong as well as electroweak properties of the pion and the nucleon. For example, as early as 1954 Kroll and Ruderman [1], in their famous low-energy theorem, discussed the possibility of extracting the renormalized pion-nucleon coupling constant $g_{\pi N}$ from charged pion photoproduction at threshold. Besides Lorentz covariance, the essential ingredient entering the derivation of the Kroll-Ruderman theorem was the application of the Ward identity [2] resulting from gauge invariance.

The algebra of currents and the hypothesis of a partially conserved axial-vector current (PCAC) resulted in additional constraints such as the theorem of Fubini, Furlan, and Rossetti [3] establishing a sum rule for pion photoproduction in terms of the anomalous isovector and isoscalar magnetic moments of the nucleon. The potential of investigating the axial-vector form factor of the nucleon through the electroproduction of charged pions was first realized by Nambu and Shrauner in the framework of the chirality formalism [4,5]. Subsequently, their result has been recovered and extended within various approaches [6–10] (for an overview, see Ref. [11]).

The current-algebra and PCAC approaches of the 1960s (see, e.g., Refs. [12–14]) had in common that they made no explicit reference to the dynamical origin of the underlying symmetry. In our present understanding, the symmetry currents originate in a global chiral $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ invariance of QCD for N_f massless quark flavors. As already pointed out by Gell-Mann [15], even if a continuous symmetry is violated by large effects, it will still have some physical consequences, which can be studied if the symmetry breaking pattern is explicitly known. In the present context, the symmetry breaking in question is associated with the finite quark masses. It is rather straightforward to derive the so-called chiral Ward identities among QCD Green functions implied by the symmetry currents and the symmetry-breaking pattern, while it is more difficult to actually satisfy these constraints in practical calculations. However, in the framework of effective field theory, the chiral Ward identities will automatically be satisfied if the underlying chiral symmetry (and its breaking pattern) is systematically mapped onto the most general effective Lagrangian in terms of the relevant experimentally observed degrees of freedom [16–19]. Turning this mapping into useful consequences requires a method which allows for a rigorous analysis of a particular contribution to a Green function in terms of some expansion scheme. This is provided by Weinberg's power counting [16,20] which makes use of the special role played by the pion as the approximate Goldstone boson of spontaneous chiral symmetry breaking. Its weak coupling to other hadrons in the low-energy limit, in combination with its small mass, allow for an analysis of the low-energy structure of QCD Green functions in the framework of chiral perturbation theory (ChPT) [16–19]. In the single-nucleon sector a consistent power counting has been developed for both the so-called heavy-baryon formulation [21,22] and, more recently, also for the relativistic approach [23–25].

The present work aims at shedding additional light on the importance of chiral Ward identities in the context of extracting the axial form factor of the nucleon from pion electroproduction experiments [26–31]. We will first show that, within the framework of QCD coupled to external fields, the PCAC relation for a particular choice of the pion interpolating field is of the same form as the one originally obtained by Adler [32] through minimal substitution. This is due to the fact that, within QCD, the quark fields entering the symmetry currents are fundamental, i.e., pointlike degrees of freedom.

Our subsequent discussion of the chiral Ward identities will be performed in the framework of a tree-level approximation to chiral perturbation theory in order to keep the line of arguments as transparent as possible. In terms of a loop expansion such tree-level diagrams may be understood as the leading order in an expansion in terms of $\hbar$ [33,34]. Moreover, chiral Ward identities are expected to be satisfied order by order in the loop expansion [24,25]. Of course, ChPT also allows one to systematically evaluate corrections to the tree-level results. In the context of pion photo- and electroproduction this was done in a series of papers by Bernard *et al.* [35–38].

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In the effective-field-theory approach we will point out the distinction between the chiral Ward identities and the so-called electromagnetic Ward-Takahashi identities [2,39,40] applying to the effective degrees of freedom. The origin of these additional identities resides in the fact that the effective hadronic degrees of freedom, namely pions and nucleons, are carriers of U(1) representations. As a consequence, the building blocks of a calculation in the effective theory also have to satisfy these identities.

Our approach will allow us to clarify a discussion triggered by a paper of Haberzettl [41], where it was argued that in the case of pion electroproduction PCAC does not provide any additional constraints beyond the Goldberger-Treiman relation. We will explicitly point out which step in the discussion of Ref. [41] is problematic (see also Refs. [42–45] for additional discussion).

Our paper is organized as follows. In Sec. II we first rederive Adler's PCAC relation in the framework of the QCD Lagrangian coupled to external fields. We then define the relevant Green functions and establish a chiral Ward identity (Adler-Gilman relation [6]) entering pion photo- and electroproduction. In Sec. III we introduce the relevant parts of the effective Lagrangians used. In Sec. IV we discuss various matrix elements involving the axial-vector current and the pseudoscalar density in order to illustrate the simplest chiral Ward identity, namely the PCAC relation, at work. Those readers familiar with the concepts of chiral perturbation theory are invited to immediately move forward to Sec. V containing the central piece of this work, namely, a discussion of the Adler-Gilman relation and its relation to the pion production amplitude. We will study the traditional soft-momentum limit and comment on the method of Ref. [41]. Finally, we will explain how, in terms of the effective degrees of freedom, the constraints due to the chiral symmetry of QCD are translated into relations among the vertices of the effective theory. General conclusions are presented in Sec. VI.

II. THE PCAC RELATION IN THE PRESENCE OF EXTERNAL FIELDS

For N_f massless quarks, the QCD Hamiltonian is invariant under the operation of the chiral group $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ on the left- and right-handed quark fields [17]. Associated with this invariance are $2(N_f^2-1)$ symmetry currents. Here we will restrict ourselves to a discussion of the approximate $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry in the sector of two light flavors. The finite u- and d-quark masses result in explicit divergences of the symmetry currents. However, as first pointed out by Gell-Mann, the equal-time commutation relations still play an important role even if the symmetry is explicitly broken [15]. In general, the symmetry currents will lead to chiral Ward identities relating various QCD Green functions with each other.

A. The PCAC relation in the presence of external fields

The starting point of our discussion is a derivation of the SU(2) PCAC relation from the QCD Lagrangian in the presence of general external fields. To that end, we consider the two-flavor QCD Lagrangian coupled to external c-number fields $v_\mu(x)$, $v_\mu^{(s)}(x)$, $a_\mu(x)$, $s(x)$, and $p(x)$ [17]:

$$\mathcal{L} = \mathcal{L}^0_{\mathrm{QCD}} + \mathcal{L}_{\mathrm{ext}} = \mathcal{L}^0_{\mathrm{QCD}} + \bar{q}\gamma^\mu\left(v_\mu + \tfrac{1}{3}v_\mu^{(s)} + \gamma_5 a_\mu\right)q - \bar{q}(s - i\gamma_5 p)q, \tag{2.1}$$

where L^0_{QCD} refers to the QCD Lagrangian for massless u and d quarks.[1] The external fields are color neutral, Hermitian 2×2 matrices, where we parametrize the matrix character, with respect to the (suppressed) flavor indices u and d of the quark fields, as [17]

$$v_\mu(x) = \frac{1}{2}[r_\mu(x) + l_\mu(x)] = \sum_{i=1}^{3} \frac{\tau_i}{2} v^i_\mu(x), \tag{2.2a}$$

$$s(x) = 1_{2\times 2} s_0(x) + \sum_{i=1}^{3} \tau_i s_i(x), \tag{2.2b}$$

$$a_\mu(x) = \frac{1}{2}[r_\mu(x) - l_\mu(x)] = \sum_{i=1}^{3} \frac{\tau_i}{2} a^i_\mu(x), \tag{2.2c}$$

$$p(x) = 1_{2\times 2} p_0(x) + \sum_{i=1}^{3} \tau_i p_i(x). \tag{2.2d}$$

Here, we do not consider a coupling to an external axial-vector singlet field, because the corresponding singlet axial-vector current has an anomaly such that the Green functions involving the axial-vector singlet current are related to Green functions containing the contraction of the gluon field-strength tensor with its dual. The ordinary two-flavor QCD Lagrangian is recovered by setting $v_\mu = v_\mu^{(s)} = a_\mu = p = 0$ and $s = \mathrm{diag}(m_u, m_d)$ in Eq. (2.1). For simplicity, we will disregard the mass difference $m_u - m_d$ and consider the isospin-symmetric case $m_u = m_d = \hat{m}$.

The Lagrangian of Eq. (2.1) is invariant under *local* $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ transformations of the left-handed and right-handed quark fields,

$$q_L(x) \equiv \frac{1}{2}(1-\gamma_5)q(x) \mapsto \exp\left(-i\frac{\theta(x)}{3}\right) V_L(x) q_L(x), \tag{2.3a}$$

$$q_R(x) \equiv \frac{1}{2}(1+\gamma_5)q(x) \mapsto \exp\left(-i\frac{\theta(x)}{3}\right) V_R(x) q_R(x), \tag{2.3b}$$

where

$$V_{L/R}(x) = \exp\left(-i\sum_{i=1}^{3} \frac{\tau_i}{2} \theta^i_{L/R}(x)\right), \tag{2.4}$$

provided the external fields transform as

[1] The remaining flavors $s, \dots, t$ appear with their respective mass terms.

$$r_\mu \mapsto V_R r_\mu V_R^\dagger + iV_R \partial_\mu V_R^\dagger, \tag{2.5a}$$

$$l_\mu \mapsto V_L l_\mu V_L^\dagger + iV_L \partial_\mu V_L^\dagger, \tag{2.5b}$$

$$v_\mu^{(s)} \mapsto v_\mu^{(s)} - \partial_\mu \theta, \tag{2.5c}$$

$$s + ip \mapsto V_R(s+ip)V_L^\dagger, \tag{2.5d}$$

$$s - ip \mapsto V_L(s-ip)V_R^\dagger. \tag{2.5e}$$

Applying the method of Gell-Mann and Lévy [46] to identify currents and their divergences by investigating the variation of the Lagrangian under local infinitesimal transformations,

$$J^\mu = \frac{\partial\, \delta\mathcal{L}}{\partial\,(\partial_\mu \epsilon)}, \tag{2.6a}$$

$$\partial_\mu J^\mu = \frac{\partial\, \delta\mathcal{L}}{\partial\, \epsilon}, \tag{2.6b}$$

leads to the following expressions for the vector and axial-vector currents:

$$V_i^\mu = \bar{q}\gamma^\mu \frac{\tau_i}{2} q, \tag{2.7a}$$

$$A_i^\mu = \bar{q}\gamma^\mu \gamma_5 \frac{\tau_i}{2} q. \tag{2.7b}$$

If the external fields are not simultaneously transformed and one considers a *global* chiral transformation only, the divergences of the currents read

$$\partial_\mu V_i^\mu = i\bar{q}\gamma^\mu \left[\frac{\tau_i}{2}, v_\mu\right] q + i\bar{q}\gamma^\mu\gamma_5 \left[\frac{\tau_i}{2}, a_\mu\right] q - i\bar{q}\left[\frac{\tau_i}{2}, s\right] q - \bar{q}\gamma_5 \left[\frac{\tau_i}{2}, p\right] q, \tag{2.8a}$$

$$\partial_\mu A_i^\mu = i\bar{q}\gamma^\mu\gamma_5 \left[\frac{\tau_i}{2}, v_\mu\right] q + i\bar{q}\gamma^\mu \left[\frac{\tau_i}{2}, a_\mu\right] q + i\bar{q}\gamma_5 \left\{\frac{\tau_i}{2}, s\right\} q + \bar{q}\left\{\frac{\tau_i}{2}, p\right\} q. \tag{2.8b}$$

In the present case we intend to consider the QCD Lagrangian for a finite light quark mass $\hat{m}$ in combination with a coupling to an external electromagnetic field $\mathcal{A}_\mu$ given by[2]

$$-eA_\mu J^\mu = -eA_\mu\left(\tfrac{2}{3}\bar{u}\gamma^\mu u - \tfrac{1}{3}\bar{d}\gamma^\mu d\right) = \tfrac{1}{3}\bar{q}\gamma^\mu v_\mu^{(s)} q + \bar{q}\gamma^\mu v_\mu q, \tag{2.9}$$

from which we conclude

$$v_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu, \quad v_\mu = -\frac{e}{2}\tau_3 \mathcal{A}_\mu, \quad a_\mu = p = 0, \quad s = \hat{m}1_{2\times 2}. \tag{2.10}$$

In this case the expressions for the divergence of the vector and axial-vector currents, respectively, read

$$\partial_\mu V_i^\mu = -\epsilon_{3ij} e A_\mu \bar{q}\gamma^\mu \frac{\tau_j}{2} q = -\epsilon_{3ij} e A_\mu V_j^\mu, \tag{2.11a}$$

$$\partial_\mu A_i^\mu = -eA_\mu \epsilon_{3ij}\bar{q}\gamma^\mu\gamma_5 \frac{\tau_j}{2} q + \hat{m} i\bar{q}\gamma_5 \tau_i q = -eA_\mu \epsilon_{3ij} A_j^\mu + \hat{m} P_i, \tag{2.11b}$$

where we have introduced the isovector pseudoscalar density

$$P_i = i\bar{q}\gamma_5 \tau_i q. \tag{2.12}$$

From Eq. (2.11b) we see that the axial-vector current is conserved, if there is no external electromagnetic field *and* if the quark mass vanishes (chiral limit). Strictly speaking, the right-hand side of Eq. (2.11b) should also involve the anomaly term contributing to $\partial_\mu A_3^\mu$ [47–49]. However, this term is of second order in the elementary charge and thus not relevant for the following discussion of pion electroproduction which will only be considered to first order in e. We note the formal similarity of Eq. (2.11b) to the (pre-QCD) PCAC relation obtained by Adler through the inclusion of the electromagnetic interactions with minimal electromagnetic coupling (see the appendix of Ref. [32]).[3] Since in QCD the quarks are taken as truly elementary, their interaction with an (external) electromagnetic field is of such a minimal type.

B. Green functions

Before investigating the consequences of Eq. (2.11b) with respect to the pion-electroproduction amplitude, we first have to define the nucleon matrix elements of the relevant quark bilinears and their time-ordered products (Green functions). The first one is the nucleon matrix element of the axial-vector current,

$$\mathcal{M}_{A,i}^\mu = \langle N(p_f)|A_i^\mu(0)|N(p_i)\rangle, \tag{2.13}$$

where the subscripts A and i refer to *axial-vector current* and isospin component i, respectively. The matrix element of the time-ordered product of the electromagnetic current J^μ and the isovector axial-vector current A_i^ν is defined as[4]

[2]We use natural units $\hbar=c=1, e>0, e^2/(4\pi)\approx 1/137$.

[3]In Adler's version, the right-hand side of Eq. (2.11b) contains a renormalized field operator creating and destroying pions.

[4]Strictly speaking one should work with the covariant time-ordered product (T^*) which, typically, differs from the ordinary time-ordered product by a noncovariant (seagull) term [50].

$$\mathcal{M}^{\mu\nu}_{JA,i} = \int d^4x\, e^{-ik\cdot x}\langle N(p_f)|T[J^\mu(x)A_i^\nu(0)]|N(p_i)\rangle \tag{2.14a}$$

$$= \int d^4x\, e^{iq\cdot x}\langle N(p_f)|T[J^\mu(0)A_i^\nu(x)]|N(p_i)\rangle. \tag{2.14b}$$

Finally, the Green function involving the electromagnetic current J^μ and the isovector pseudoscalar density P_i reads

$$\mathcal{M}^{\mu}_{JP,i} = \int d^4x\, e^{-ik\cdot x}\langle N(p_f)|T[J^\mu(x)P_i(0)]|N(p_i)\rangle \tag{2.15a}$$

$$= \int d^4x\, e^{iq\cdot x}\langle N(p_f)|T[J^\mu(0)P_i(x)]|N(p_i)\rangle. \tag{2.15b}$$

The Green functions of Eqs. (2.13)–(2.15) are related through the chiral Ward identity (for a pedagogical introduction to this topic see, e.g., Refs. [51,52])[5]

$$q_\nu \mathcal{M}^{\mu\nu}_{JA,i} = i\hat{m}\mathcal{M}^{\mu}_{JP,i} + \epsilon_{3ij}\mathcal{M}^{\mu}_{A,j}. \tag{2.16}$$

We will refer to Eq. (2.16) as the Adler-Gilman relation (see Sec. II B of Ref. [6] for their pre-QCD version). An alternative way of obtaining the analog of Eq. (2.16) consists of evaluating the PCAC relation of Adler [32],

$$(\partial_\mu \delta_{ij} + e\mathcal{A}_\mu \epsilon_{3ij})A^\mu_j = m_\pi^2 F_\pi \Phi_i, \tag{2.17}$$

between a final nucleon state and an initial state consisting of a nucleon and a (virtual) photon (see, e.g., Refs. [8–10,54–57]).

C. Interpolating field and pion electroproduction amplitude

The connection to the pion electroproduction amplitude is established by noting that the pseudoscalar density serves as an interpolating pion field. The matrix element of the pseudoscalar quark density evaluated between a single-pion state and the vacuum is defined in terms of the coupling constant G_π [17],

$$\langle 0|P_i(0)|\pi_j(q)\rangle = \delta_{ij}G_\pi. \tag{2.18}$$

With the help of this constant we can define an interpolating pion field[6]

$$\Phi_i(x) = \frac{P_i(x)}{G_\pi} = \frac{P_i(x)}{2BF} = \frac{\hat{m}P_i(x)}{m_\pi^2 F_\pi}, \tag{2.19}$$

where the second and third equality signs refer to the lowest-order result of mesonic chiral perturbation theory [17,52].

In the one-photon-exchange approximation, the invariant amplitude for pion electroproduction[7] $\gamma^*(k)+N(p_i)\to\pi^i(q)+N(p_f)$ can be written as $\mathcal{M}_i=-ie\epsilon_\mu\mathcal{M}^\mu_i$, where $\epsilon_\mu=e\bar{u}\gamma_\mu u/k^2$ is the polarization vector of the virtual photon and $\mathcal{M}^\mu_i$ the transition-current matrix element,

$$\mathcal{M}^\mu_i = \langle N(p_f), \pi^i(q)|J^\mu(0)|N(p_i)\rangle. \tag{2.20}$$

Using the reduction formalism of Lehmann, Symanzik, and Zimmermann (LSZ) [58] with the interpolating pion field of Eq. (2.19) in combination with the chiral Ward identity of Eq. (2.16), one obtains a relation for the transition-current matrix element in terms of QCD Green functions,

$$\mathcal{M}^\mu_i = -i\frac{\hat{m}}{m_\pi^2 F_\pi}\lim_{q^2\to m_\pi^2}(q^2-m_\pi^2)\mathcal{M}^\mu_{JP,i}$$
$$= \frac{1}{m_\pi^2 F_\pi}\lim_{q^2\to m_\pi^2}(q^2-m_\pi^2)(\epsilon_{3ij}\mathcal{M}^\mu_{A,j} - q_\nu\mathcal{M}^{\mu\nu}_{JA,i}). \tag{2.21}$$

This type of relation has been the starting point for studying the consequences of the PCAC hypothesis on threshold photo- and electroproduction. Note that the QCD chiral Ward identity of Eq. (2.16) holds for any value of q^2, i.e., there is no need to stay in the vicinity of the squared pion mass, $q^2\approx m_\pi^2$. However, the connection to the physical pion-production process requires $q^2=m_\pi^2$.

III. THE EFFECTIVE LAGRANGIAN

As already emphasized in the 1960s by Weinberg, phenomenological Lagrangians provide a straightforward way of obtaining the results of current algebra in the so-called phenomenological approximation, i.e., at tree level [59–62]. Furthermore, modern techniques of effective field theory allow one to also systematically calculate higher-order corrections to tree-level results in the framework of chiral perturbation theory [16–22,24,25].

In order to study the consequences of the chiral Ward identities for threshold pion electroproduction, we will start from the most general effective chiral Lagrangian up to and including $O(p^3)$ in the baryonic sector. However, for pedagogical purposes, we will make use of two (drastic) simplifications. First, we will restrict ourselves to tree-level results only, because they already reveal the main features regarding the connection between chiral Ward identities and pion production. The chiral corrections due to meson loops have been studied in detail by Bernard *et al.* [35–37] and are beyond

[5]The standard derivation in terms of the ordinary time-ordered product leads to equal-time commutators of current densities and symmetry currents. A naive application of the canonical commutation relations neglects the so-called Schwinger terms [53]. According to Feynman's conjecture these Schwinger terms cancel with the above seagull terms. For a discussion of the validity of this hypothesis, the interested reader is referred to Ref. [50].

[6]Every field $\Phi_i(x)$, which satisfies the relation $\langle 0|\Phi_i(x)|\pi_j(q)\rangle = \delta_{ij}e^{-iq\cdot x}$, can serve as an interpolating pion field.

[7]For a discussion of the relevant kinematics and formalism see, e.g., Refs. [10,11].

the scope of this work (see Ref. [38] for an overview). In particular, a discrepancy between the determination of the root-mean-square axial radius from anti-neutrino-proton scattering and charged threshold pion electroproduction, respectively, was explained in terms of such chiral pion-loop corrections [36]. Second, even at the phenomenological level, we only consider a subset of terms which give rise to nontrivial contributions to the Green functions. In the end, we will always comment on the full result at the one-loop level.

A. Mesonic Lagrangian

The most general lowest-order mesonic chiral Lagrangian can be written as [17]

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger + \chi U^\dagger + U\chi^\dagger],$$
$$U(x) = \exp\left[i\frac{\vec{\tau}\cdot\vec{\pi}(x)}{F}\right], \tag{3.1}$$

where the covariant derivative

$$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu \tag{3.2}$$

involves the external fields of Eqs. (2.5a) and (2.5b), and

$$\chi = 2B(s+ip) \tag{3.3}$$

contains the external scalar and pseudoscalar fields of Eqs. (2.5d) and (2.5e). Furthermore, F is the pion-decay constant in the chiral limit, $F_\pi = F[1+O(\hat{m})] = 92.4$ MeV, and B is related to the scalar quark condensate in the chiral limit, $\langle 0|\bar{u}u|0\rangle_0 = \langle 0|\bar{d}d|0\rangle_0 = -F^2 B$. Inserting $s = \hat{m}1_{2\times 2}$ in Eq. (3.1), one finds, at lowest order in the momentum and quark-mass expansion, the relation $m_\pi^2 = 2B\hat{m}$. Combining Eqs. (2.5) with the transformation

$$U \mapsto V_R U V_L^\dagger, \tag{3.4}$$

the effective Lagrangian of Eq. (3.1) has the same *local* $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_V$ symmetry as the QCD Lagrangian of Eq. (2.1).

According to the power counting scheme for processes involving a single nucleon [16,20], an $O(p^3)$ calculation contains mesonic contributions of $O(p^4)$. Thus, strictly speaking, we would also need the $O(p^4)$ mesonic Lagrangian of Gasser and Leutwyler [17] for a full discussion. However, we restrict ourselves to the $O(p^2)$ contributions, because, in the above sense, they already illustrate the relevant features regarding the chiral Ward identities.

B. Pion-nucleon Lagrangian

The most general lowest-order relativistic pion-nucleon chiral Lagrangian reads [19]

$$\mathcal{L}_{\pi N}^{(1)} = \bar{\Psi}\left(i\not{D} - m + \frac{\mathring{g}_A}{2}\gamma^\mu\gamma_5 u_\mu\right)\Psi, \tag{3.5}$$

where

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix} \tag{3.6}$$

is the nucleon isospin doublet. The covariant derivative is defined as

$$D_\mu\Psi = (\partial_\mu + \Gamma_\mu - i v_\mu^{(s)})\Psi \tag{3.7}$$

with

$$\Gamma_\mu = \tfrac{1}{2}[u^\dagger(\partial_\mu - i r_\mu)u + u(\partial_\mu - i l_\mu)u^\dagger] \tag{3.8}$$

and

$$u_\mu = i[u^\dagger(\partial_\mu - i r_\mu)u - u(\partial_\mu - i l_\mu)u^\dagger], \tag{3.9}$$

where $u = \sqrt{U}$. At this order the effective Lagrangian contains two (new) low-energy constants, namely, the nucleon mass m and the axial-vector coupling constant $\mathring{g}_A$ in the chiral limit, respectively. The Lagrangian of Eq. (3.5) is invariant under *local* transformations, provided

$$\Psi \mapsto \exp[-i\theta(x)](V_R U V_L^\dagger)^{-1/2} V_R \sqrt{U}\Psi, \quad U \mapsto V_R U V_L^\dagger. \tag{3.10}$$

We will neglect the next-to-leading-order pion-nucleon Lagrangian $\mathcal{L}_{\pi N}^{(2)}$ [19,63,64], except for the terms giving rise to the isoscalar and isovector anomalous magnetic moments of the nucleon, respectively,

$$\mathcal{L}_{\mathrm{eff}}^{(2)} = \frac{c_6}{2}\bar{\Psi}\sigma^{\mu\nu}f_{\mu\nu}^{+}\Psi + \frac{c_7}{2}\bar{\Psi}\sigma^{\mu\nu}\Psi(\partial_\mu v_\nu^{(s)} - \partial_\nu v_\mu^{(s)}), \tag{3.11}$$

where

$$f_{\mu\nu}^{\pm} = u f_{\mu\nu}^{L} u^\dagger \pm u^\dagger f_{\mu\nu}^{R} u, \tag{3.12a}$$

$$f_{\mu\nu}^{R} = \partial_\mu r_\nu - \partial_\nu r_\mu - i[r_\mu, r_\nu], \tag{3.12b}$$

$$f_{\mu\nu}^{L} = \partial_\mu l_\nu - \partial_\nu l_\mu - i[l_\mu, l_\nu]. \tag{3.12c}$$

Inserting for the external fields the electromagnetic case of Eq. (2.20), the constants c_6 and c_7 are given in terms of the isovector and isoscalar anomalous magnetic moments of the nucleon in the chiral limit, respectively,

$$c_6 = \frac{\mathring{\kappa}_v}{4m}, \tag{3.13a}$$

$$c_7 = \frac{\mathring{\kappa}_s}{2m}. \tag{3.13b}$$

To the order we are considering, all chiral limit values entering Eq. (3.13) can be replaced by their empirical values. The respective numerical values are $\kappa_s = \kappa_p + \kappa_n = -0.120$ and $\kappa_v = \kappa_p - \kappa_n = 3.706$.

In principle, the most general $\mathcal{L}_{\pi N}^{(2)}$ also generates quark-mass corrections of $O(\hat{m})$ to the nucleon mass and the axial-vector coupling constant, respectively. The first one is not really important for our discussion, whereas the second one

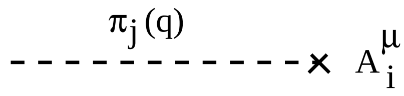

FIG. 1. Matrix element of the axial-vector current between a one-pion state and the vacuum.

will, in a similar fashion, also be provided by an $O(p^3)$ term, which for illustrative purposes we will keep.

Finally, out of the 23 terms of the third-order pion-nucleon Lagrangian [63], rewritten in relativistic notation, we only keep the following three terms:[8]

$$\mathcal{L}_{\rm eff}^{(3)} = \frac{1}{2(4\pi F)^2}\bar{\Psi}\gamma^\mu\gamma_5\{b_{17}u_\mu \mathrm{Tr}(\chi_+) + ib_{19}[D_\mu, \chi_-] + b_{23}[D^\nu, f_{-\mu\nu}]\}\Psi \tag{3.14}$$

with

$$\chi_\pm = u^\dagger \chi u^\dagger \pm u\chi^\dagger u. \tag{3.15}$$

The constants b_{17}, b_{19}, and b_{23} will be discussed below.

In order to summarize, in the following analysis we employ as the effective Lagrangian

$$\mathcal{L}_{\rm eff} = \mathcal{L}_2 + \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\rm eff}^{(2)} + \mathcal{L}_{\rm eff}^{(3)}, \tag{3.16}$$

where the explicit expressions are given in Eqs. (3.1), (3.5), (3.11), and (3.14). Within the tree-level approximation, it is consistent to replace the nucleon mass in the chiral limit, m, by the physical nucleon mass m_N.

IV. AXIAL-VECTOR-CURRENT MATRIX ELEMENTS

A. First example

As the most simple application of the PCAC relation without an external electromagnetic field, we first consider the matrix elements of the axial-vector current and the pseudoscalar density evaluated between a one-pion state and the vacuum. To that end, we insert the relevant external fields in the effective Lagrangian and identify the vertices by applying the usual Feynman rules. This example serves as the most elementary illustration of how chiral Ward identities are satisfied in our (simplified) approach and, more generally, in chiral perturbation theory.

Using Lorentz covariance and isospin symmetry, the matrix element of the axial-vector current can be parametrized as (see Fig. 1)

$$\langle 0|A_i^\mu(x)|\pi_j(q)\rangle = iq^\mu F_\pi e^{-iq\cdot x}\delta_{ij}. \tag{4.1}$$

From the Lagrangian of Eq. (3.1) one obtains at $O(p^2)$,

[8]In rewriting the heavy-baryon Lagrangian of Ref. [63] we made use of the replacement $\bar{N}_v S_\mu N_v \mapsto \bar{\Psi}\gamma_\mu\gamma_5\Psi/2$. We do not consider the b_7 and b_8 terms which generate a q^2 dependence of the electromagnetic form factors. For the discussion of the Adler-Gilman relation these terms do not create any significant new features.

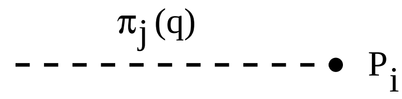

FIG. 2. Matrix element of the pseudoscalar density between a one-pion state and the vacuum.

$$\mathcal{L}_{\rm ext} = i\frac{F^2}{2}\mathrm{Tr}[(\partial^\mu U U^\dagger - \partial^\mu U^\dagger U)a_\mu] = -Fa_{\mu,i}\partial^\mu\pi_i + \cdots, \tag{4.2}$$

which results in $F_\pi = F$ at this order. For the divergence of the axial vector current one then finds

$$\begin{aligned}\langle 0|\partial_\mu A_i^\mu(x)|\pi_j(q)\rangle &= iq^\mu F_\pi \partial_\mu e^{-iq\cdot x}\delta_{ij}\\ &= m_\pi^2 F_\pi e^{-iq\cdot x}\delta_{ij}\\ &= 2\hat{m}BFe^{-iq\cdot x}\delta_{ij},\end{aligned} \tag{4.3}$$

where, in order to obtain the last equality sign, use has been made of the $O(p^2)$ predictions for F_π and m_π^2, respectively. On the other hand, the matrix element of the pseudoscalar density (see Fig. 2), Eq. (2.18), results from

$$\mathcal{L}_{\rm ext} = i\frac{F^2 B}{2}\mathrm{Tr}(pU^\dagger - Up) = 2BFp_i\pi_i + \cdots, \tag{4.4}$$

yielding $G_\pi = 2BF$. Thus we have—at $O(p^2)$—explicitly verified the relation $F_\pi m_\pi^2 = \hat{m}G_\pi$ implied by the PCAC relation. The corresponding one-loop expressions can be found in Eqs. (12.4) and (12.6) of Ref. [17].

Thus, within the framework of working in the phenomenological approximation of the effective Lagrangian of Eq. (3.16), it is consistent to replace $F \to F_\pi$ and $2\hat{m}B \to m_\pi^2$.

B. Nucleon matrix element of the pseudoscalar density

The nucleon matrix element of the pseudoscalar density (see Fig. 3) can be parametrized as

$$\hat{m}\langle N(p_f)|P_i(0)|N(p_i)\rangle = \frac{m_\pi^2 F_\pi}{m_\pi^2 - t}\, G_{\pi N}(t)\; i\bar{u}(p_f)\gamma_5\tau_i u(p_i), \qquad t = (p_f - p_i)^2. \tag{4.5}$$

Since $\hat{m}P_i(x)/(m_\pi^2 F_\pi)$ serves as an interpolating pion field [see Eq. (2.19), $G_{\pi N}$ is also referred to as the pion-nucleon form factor (for this specific choice of interpolating field). In the framework of the effective chiral Lagrangian of Eq. (3.16) one obtains two contributions with vertices from

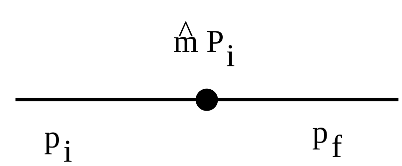

FIG. 3. Matrix element of the pseudoscalar density between one-nucleon states.

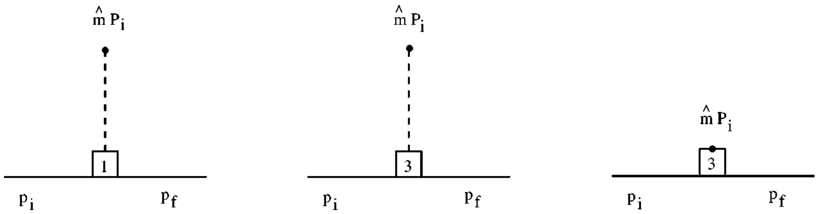

FIG. 4. Feynman diagrams contributing to the pseudoscalar density between one-nucleon states at $O(p)$ and $O(p^3)$, respectively.

$\mathcal{L}_{\pi N}^{(1)}$ and $\mathcal{L}_{\rm eff}^{(3)}$, respectively (see Fig. 4),[9]

$$G_{\pi N}(t)=\frac{m_N}{F_\pi}\left(\mathring{g}_A+\frac{b_{17}m_\pi^2}{4\pi^2F_\pi^2}-\frac{b_{19}}{8\pi^2F_\pi^2}t\right). \tag{4.6}$$

The pion-nucleon coupling constant is defined at $t=m_\pi^2$,

$$g_{\pi N}=G_{\pi N}(m_\pi^2)=\frac{m_N}{F_\pi}\left(\mathring{g}_A+\frac{(2b_{17}-b_{19})m_\pi^2}{8\pi^2F_\pi^2}\right). \tag{4.7}$$

As will be seen below, $g_A=\mathring{g}_A+b_{17}m_\pi^2/(4\pi^2F_\pi^2)$, such that the parameter b_{19} reflects the so-called Goldberger-Treiman discrepancy, i.e., the numerical violation of the Goldberger-Treiman relation,[10]

$$\Delta_{\pi N}\equiv 1-\frac{g_A m_N}{g_{\pi N}F_\pi}=-b_{19}\frac{m_N m_\pi^2}{8\pi^2 g_{\pi N}F_\pi^3}. \tag{4.8}$$

The effective Lagrangian of Eq. (3.16) reproduces exactly the same result for $\Delta_{\pi N}$ as the full one-loop calculation [see Eq. (69) of Ref. [66]].

In terms of $\Delta_{\pi N}$ and $g_{\pi N}$, the pion-nucleon form factor—at this order—can be rewritten as

$$G_{\pi N}(t)=g_{\pi N}\left(1+\Delta_{\pi N}\frac{t-m_\pi^2}{m_\pi^2}\right). \tag{4.9}$$

The full $O(p^3)$ calculation including pion loop corrections generates exactly the same functional form with all quantities replaced by their $O(p^3)$ expressions (see Sec. III 7 of Ref. [67]).

C. The pion-nucleon vertex

The pion-nucleon vertex resulting from Eq. (3.16) reads

$$\left(\mathring{g}_A+\frac{b_{17}m_\pi^2}{4\pi^2F_\pi^2}-\frac{b_{19}m_\pi^2}{8\pi^2F_\pi^2}\right)\frac{1}{F_\pi}\not{q}\gamma_5\frac{\tau_i}{2}=\frac{g_{\pi N}}{2m_N}\not{q}\gamma_5\tau_i,$$
$$q=p_i-p_f, \tag{4.10}$$

where we made use of Eq. (4.7). In particular, for $q^2\neq m_\pi^2$ the pion-nucleon vertex does *not* contain the form factor $G_{\pi N}(q^2)$ of Eq. (4.5). In general, the pion-nucleon vertex depends on the choice of the field variables in the (effective) Lagrangian. In the present case, the pion-nucleon vertex is only an auxiliary quantity, whereas the "fundamental" quantity (entering chiral Ward identities) is the QCD Green function of Eq. (4.5). Only at $q^2=m_\pi^2$, we expect the same coupling strength, since both Φ_i of Eq. (2.19) and π_i of Eq. (3.1) serve as interpolating pion fields.

D. Nucleon axial-vector-current matrix element

We now turn to the results for the axial form factors of the nucleon.[11] The matrix element of the axial-vector current evaluated between initial and final nucleon states—excluding second-class currents [68]—can be written as (see Fig. 5)

$$\langle N(p_f)|A_i^\mu(0)|N(p_i)\rangle=\bar{u}(p_f)\left[\gamma^\mu G_A(t)+\frac{(p_f-p_i)^\mu}{2m_N}G_P(t)\right]$$
$$\times\gamma_5\frac{\tau_i}{2}u(p_i), \tag{4.11}$$

where $t=(p_f-p_i)^2$, and $G_A(t)$ and $G_P(t)$ are the axial and induced pseudoscalar form factors, respectively. Within the framework of Eq. (3.16) we obtain to this order (see Fig. 6)

$$G_A(t)=g_A\left(1+\tfrac{1}{6}\langle r^2\rangle_A\, t\right), \tag{4.12a}$$

$$G_P(t)=4m_N^2\left(\frac{F_\pi g_{\pi N}}{m_N}\frac{1}{m_\pi^2-t}-\frac{1}{6}g_A\langle r^2\rangle_A\right), \tag{4.12b}$$

where

$$g_A=G_A(0)=\mathring{g}_A+\frac{b_{17}m_\pi^2}{4\pi^2F_\pi^2}, \tag{4.13a}$$

$$-\frac{1}{6}g_A\langle r^2\rangle_A=\frac{b_{23}}{(4\pi F_\pi)^2}. \tag{4.13b}$$

In the present framework, the parameter b_{17} signifies a deviation of the axial-vector coupling constant g_A from its value $\mathring{g}_A$ in the chiral limit. The parameter b_{23} is related to the axial radius. The full one-loop calculation at $O(p^3)$ [66] has the same functional form as Eqs. (4.12). In addition to the b_{17} term, the pion loops generate a further contribution to $g_A=G_A(0)$ of order $\hat{m}$ and $\hat{m}\ln(\hat{m})$. Its infinite piece is compensated by an infinity in the *bare* parameter b_{17}^0.

It is now straightforward to verify that the form factors of Eqs. (4.9) and (4.12) satisfy the relation

[9]At the order we are working it is consistent to replace $F\to F_\pi$ and $m\to m_N$.

[10]Using m_N−938.3 MeV, gA=1.267, F_π=92.4 MeV, and $g_{\pi N}$=13.21 [65], one obtains $\Delta_{\pi N}$=2.6%.

[11]For simplicity, we often refer to *both* form factors parametrizing the axial-vector-current matrix element as axial form factors.

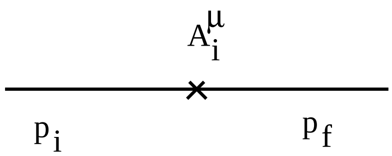

FIG. 5. Matrix element of the axial-vector current between one-nucleon states.

$$2m_N G_A(t) + \frac{t}{2m_N} G_P(t) = 2\frac{m_\pi^2 F_\pi}{m_\pi^2 - t} G_{\pi N}(t), \tag{4.14}$$

as implied by the PCAC relation, Eq. (2.11b), in the absence of an electromagnetic field $\mathcal{A}_\mu$. Evaluating Eq. (4.14) at $t=0$, one obtains

$$2m_N G_A(0) = 2F_\pi G_{\pi N}(0), \tag{4.15}$$

where we made use of the fact that $G_P(0)$ is finite for nonvanishing m_π^2. By use of $G_{\pi N}(0)=g_{\pi N}(1-\Delta_{\pi N})\approx g_{\pi N} = G_{\pi N}(m_\pi^2)$, we see that the Goldberger-Treiman relation is only approximately satisfied. Of course, in the chiral limit we recover

$$2\mathring{m}\,\mathring{G}_A(t) + \frac{t}{2\mathring{m}}\mathring{G}_P(t) = 0, \tag{4.16}$$

which also implies that the Goldberger-Treiman relation is exactly satisfied in this case.

V. PION ELECTROPRODUCTION

We will now address with the help of $\mathcal{L}_{\rm eff}$ of Eq. (3.16) how the PCAC relation enters the pion electroproduction amplitude. Neglecting isospin-symmetry-breaking effects due to different u- and d-quark masses as well as higher-order electromagnetic corrections, the amplitude for producing a pion with Cartesian isospin index i can be decomposed as [69]

$$\mathcal{M}(\pi_i) = \chi_f^\dagger(-i\epsilon_{3ij}\tau_j\mathcal{M}^{(-)} + \tau_i\mathcal{M}^{(0)} + \delta_{i3}\mathcal{M}^{(+)})\chi_i, \tag{5.1}$$

where χ_i and χ_f denote the isospinors of the initial and final nucleons, respectively, and τ_i are the ordinary Pauli matrices. With this decomposition the amplitudes for the physical processes read

$$\mathcal{M}(\gamma^* p \to n\pi^+) = \sqrt{2}(\mathcal{M}^{(0)} + \mathcal{M}^{(-)}), \tag{5.2a}$$

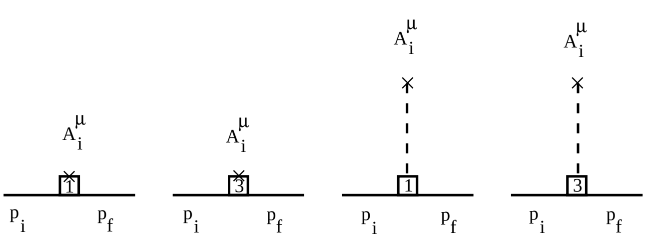

FIG. 6. Feynman diagrams contributing to the axial-vector current matrix element between one-nucleon states at $O(p)$ and $O(p^3)$.

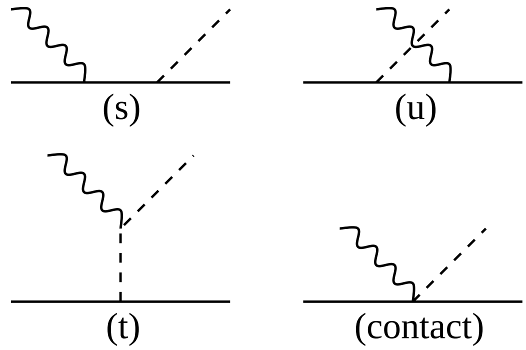

FIG. 7. Feynman diagrams contributing to pion electroproduction in the framework of $\mathcal{L}_{\rm eff}$ of Eq. (3.16).

$$\mathcal{M}(\gamma^* n \to p\pi^-) = \sqrt{2}(\mathcal{M}^{(0)} - \mathcal{M}^{(-)}), \tag{5.2b}$$

$$\mathcal{M}(\gamma^* p \to p\pi^0) = \mathcal{M}^{(+)} + \mathcal{M}^{(0)}, \tag{5.2c}$$

$$M(\gamma^* n \to n\pi^0) = M^{(+)} - M^{(0)}. \tag{5.2d}$$

A. Direct calculation

The natural way to find the pion electroproduction amplitude associated with the effective Lagrangian of Eq. (3.16) is to determine the relevant vertices involving pions, nucleons, and the electromagnetic field and to calculate the corresponding Feynman diagrams. The calculation is straightforward and involves the diagrams shown in Fig. 7:

$$\mathcal{M}_s = -e\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\left(1 - \frac{2m_N \not{q}}{s - m_N^2}\right)\gamma_5 \frac{\tau_i}{2}\epsilon\cdot\Gamma(k)u(p_i), \tag{5.3a}$$

$$\mathcal{M}_u = -e\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\epsilon\cdot\Gamma(k)\left(1 - \frac{2m_N \not{q}}{u - m_N^2}\right)\gamma_5 \frac{\tau_i}{2}u(p_i), \tag{5.3b}$$

$$\mathcal{M}_t = ieg_{\pi N}\epsilon_{3ij}\tau_j\bar{u}(p_f)\gamma_5 u(p_i)\frac{1}{t - m_\pi^2}\epsilon\cdot(2q - k), \tag{5.3c}$$

$$\begin{aligned}\mathcal{M}_{\rm contact} &= ie\epsilon_{3ij}\tau_j\frac{g_{\pi N}}{2m_N}\bar{u}(p_f)\not{\epsilon}\gamma_5 u(p_i) \\ &\quad + ie\epsilon_{3ij}\tau_j\frac{g_A}{2F_\pi}\frac{1}{6}\langle r^2\rangle_A \bar{u}(p_f)[(k-q)\cdot k\not{\epsilon} \\ &\quad - (k-q)\cdot\epsilon\not{k}]\gamma_5 u(p_i),\end{aligned} \tag{5.3d}$$

where $s=(p_i+k)^2$, $t=(p_i-p_f)^2$, and $u=(p_i-q)^2$ are the usual Mandelstam variables, satisfying $s+t+u=2m_N^2+k^2+m_\pi^2$.[12] The expressions for $g_{\pi N}$, g_A, and $\langle r^2\rangle_A$ are given in Eqs. (4.7),

[12]In the case of an off-shell pion one has to replace m_π^2 by q^2.

(4.13a), and (4.13b), respectively. Furthermore, we introduced the abbreviation

$$\Gamma^{\mu}(k) = \gamma^{\mu}\frac{1+\tau_3}{2} + i\frac{\sigma^{\mu\nu}k_{\nu}}{2m_N}\left(\frac{\kappa_s}{2} + \frac{\kappa_v}{2}\tau_3\right), \quad k = p_f - p_i, \tag{5.4}$$

for the electromagnetic vertex of the nucleon (as obtained in the framework of the effective Lagrangian). First of all, it is straightforward to check the constraints of gauge invariance for Eqs. (5.3) [70],

$$k_{\mu}\mathcal{M}_i^{\mu} = -g_{\pi N}\epsilon_{3ij}\tau_j\bar{u}(p_f)\gamma_5 u(p_i)\Delta^{-1}(q)\,\Delta\,(q-k). \tag{5.5}$$

Equation (5.5) is the electromagnetic Ward-Takahashi identity for the production of an off-shell pion consistent with the vertices and propagators obtained from $\mathcal{L}_{\text{eff}}$ [see, e.g., Eq. (4) of Ref. [71]], where the external nucleon lines are on shell. This result is not surprising, because the transformation law of Eq. (3.4) for the chiral matrix U implies for the pion field $\pi_i \mapsto \pi_i - \theta(x)\epsilon_{3ij}\pi_j$ under electromagnetic U(1) transformations. In particular, if the pion is on its mass shell, $q^2 = m_{\pi}^2$, one obtains the usual current conservation condition, $k_{\mu}\mathcal{M}_i^{\mu} = 0$, because $\Delta^{-1}(q) = 0$ in this case.

At this point we can now point out the distinction between chiral Ward identities relating QCD Green functions and (electromagnetic) Ward-Takahashi identities relating Green functions of the effective theory containing off-shell legs of the effective degrees of freedom, here pions and nucleons. The chiral Ward identities originate in the chiral symmetry of the underlying QCD Lagrangian. By considering the (most general) chiral effective Lagrangian exhibiting the same invariances as the QCD Lagrangian coupled to external fields, the constraints of the chiral Ward identities are automatically transported to the effective-Lagrangian level. On the other hand, the effective degrees of freedom are carriers of, e.g., U(1) representations resulting, in addition, in conventional Ward-Takahashi identities involving off-shell pion and nucleon vertices.[13] An example of such a Ward-Takahashi identity is given by Eq. (5.5). Note that neither the left-hand nor the right-hand side constitute QCD Green functions.

Finally, Eq. (5.3d) which, according to Eqs. (5.2), only enters charged pion production, involves the axial radius. In fact, this is not a coincidence, but will be shown to also follow from the (more complicated) application of the Adler-Gilman relation.

B. Pion electroproduction and the electromagnetic-current pseudoscalar-density Green function

According to Eq. (2.21), the pion electroproduction transition current matrix element is related to the QCD Green function involving the electromagnetic current and the pseudoscalar density. Here, we calculate $\mathcal{M}_{JP,i}^{\mu}$ of Eq. (2.15) in the framework of $\mathcal{L}_{\text{eff}}$ and explicitly verify the result for the pion electroproduction amplitude of Eqs. (5.3). The result is obtained from the seven diagrams shown in Fig. 8:

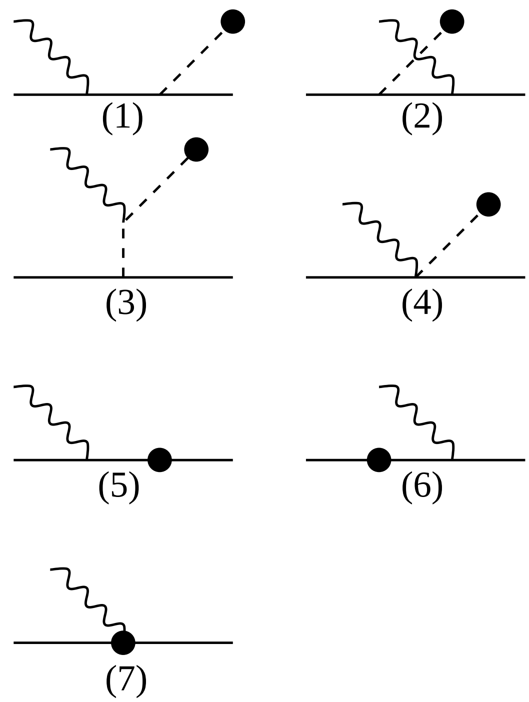

FIG. 8. Feynman diagrams contributing to the Green function involving the electromagnetic current and the pseudoscalar density. The wavy line denotes a (virtual) photon coupling to the electromagnetic current. The full circle corresponds to the pseudoscalar density.

$$\mathcal{M}_{JP,i,1}^{\mu} = \frac{2BF_{\pi}}{q^2 - m_{\pi}^2}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\left(1 - \frac{2m_N \not{q}}{s - m_N^2}\right)\gamma_5\frac{\tau_i}{2}\Gamma^{\mu}(k)u(p_i), \tag{5.6a}$$

$$\mathcal{M}_{JP,i,2}^{\mu} = \frac{2BF_{\pi}}{q^2 - m_{\pi}^2}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\Gamma^{\mu}(k)\left(1 - \frac{2m_N \not{q}}{u - m_N^2}\right)\gamma_5\frac{\tau_i}{2}u(p_i), \tag{5.6b}$$

$$\mathcal{M}_{JP,i,3}^{\mu} = -i\epsilon_{3ij}\tau_j\frac{2BF_{\pi}}{q^2 - m_{\pi}^2}\frac{1}{t - m_{\pi}^2}(2q-k)^{\mu}g_{\pi N}\bar{u}(p_f)\gamma_5 u(p_i), \tag{5.6c}$$

$$M_{JP,i,4}^{\mu} = -i\epsilon_{3ij}\tau_j\frac{2BF_{\pi}}{q^2 - m_{\pi}^2}\bar{u}(p_f)\left\{\frac{g_{\pi N}}{2m_N}\gamma^{\mu} + \frac{1}{6}\langle r^2\rangle_A\frac{g_A}{2F_{\pi}}[(k - q)k\gamma^{\mu} - (k-q)^{\mu}\not{k}]\right\}\gamma_5 u(p_i), \tag{5.6d}$$

$$\mathcal{M}_{JP,i,5}^{\mu} = \frac{2BF_{\pi}}{m_{\pi}^2}\Delta_{\pi N}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\left(1 - \frac{2m_N \not{q}}{s - m_N^2}\right)\gamma_5\frac{\tau_i}{2}\Gamma^{\mu}(k)u(p_i), \tag{5.6e}$$

---

[13]Of course, also other groups which are linearly realized on the pion and nucleon degrees of freedom, such as $SU(2)_v$, may be used to obtain consistency checks between the building blocks of the effective theory.

$$\mathcal{M}^{\mu}_{JP,i,6}=\frac{2BF_\pi}{m_\pi^2}\Delta_{\pi N}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\Gamma^\mu(k)\left(1-\frac{2m_N\not q}{u-m_N^2}\right)\gamma_5\frac{\tau_i}{2}u(p_i), \tag{5.6f}$$

$$\mathcal{M}^{\mu}_{JP,i,7}=-i\epsilon_{3ij}\tau_j\frac{2BF_\pi}{m_\pi^2}\Delta_{\pi N}\frac{g_{\pi N}}{2m_N}\bar{u}(p_f)\gamma^\mu\gamma_5 u(p_i). \tag{5.6g}$$

The expression of the Goldberger-Treiman discrepancy is given in Eq. (4.8). When multiplying Eqs. (5.6) by $-\hat{m}i$ we make use of $2\hat{m}BF_\pi=m_\pi^2F_\pi$. Second, after multiplying by $q^2-m_\pi^2$ and taking the limit $q^2\to m_\pi^2$, only those terms of Eqs. (5.6) which have a $1/(q^2-m_\pi^2)$ singularity survive. Finally, in order to obtain the invariant amplitude we have to contract the result with $-ie\epsilon_\mu$. With these replacements one easily sees the one-to-one correspondence between Eqs. (5.3a)–(5.3d) and (5.6a)–(5.6d). On the other hand Eqs. (5.6e)–(5.6g) do not contribute to pion electroproduction due to the absence of the $1/(q^2-m_\pi^2)$ pole. Thus, we have a first check of the consistency of our procedure.

As a final check of the results of Eqs. (5.6) we investigate the chiral Ward identity

$$k_\mu\mathcal{M}^{\mu}_{JP,i}=\epsilon_{3ij}\langle N(p_f)|P_j(0)|N(p_i)\rangle. \tag{5.7}$$

Contracting the first four and the final three expressions of Eqs. (5.6) with k_μ, respectively, we obtain

$$k_\mu\sum_{k=1}^{4}\mathcal{M}^{\mu}_{JP,i,k}=-i\epsilon_{3ij}\tau_j\frac{2BF_\pi}{t-m_\pi^2}g_{\pi N}\bar{u}(p_f)\gamma_5u(p_i),$$

$$k_\mu\sum_{k=5}^{7}\mathcal{M}^{\mu}_{JP,i,k}=-i\epsilon_{3ij}\tau_j\frac{2BF_\pi}{m_\pi^2}\Delta_{\pi N}g_{\pi N}\bar{u}(p_f)\gamma_5u(p_i).$$

Combining the two results we find

$$k_\mu\mathcal{M}^{\mu}_{JP,i}=-i\epsilon_{3ij}\tau_j\frac{2BF_\pi}{t-m_\pi^2}G_{\pi N}(t)\bar{u}(p_f)\gamma_5u(p_i), \tag{5.8}$$

where $G_{\pi N}(t)$ is given in Eq. (4.9). Here, we made use of the definition of Eq. (4.5) and $2\hat{m}B=m_\pi^2$. Thus, the result for the Green function $\mathcal{M}^{\mu}_{JP,i}$ is consistent with the chiral Ward identity of Eq. (5.7).

C. Adler-Gilman relation

We now turn to the explicit test of the Adler-Gilman relation, Eq. (2.16), in the framework of $\mathcal{L}_{\rm eff}$. In traditional current-algebra or soft-pion approaches, it is the right-hand side of Eq. (2.16) which serves as the starting point for the prediction of threshold pion production.

1. Electromagnetic-current axial-vector current Green function

We first need to calculate the Green function $\mathcal{M}^{\mu\nu}_{JA,i}$ of Eq. (2.14) involving the electromagnetic current and the axial-vector current (see Fig. 9):

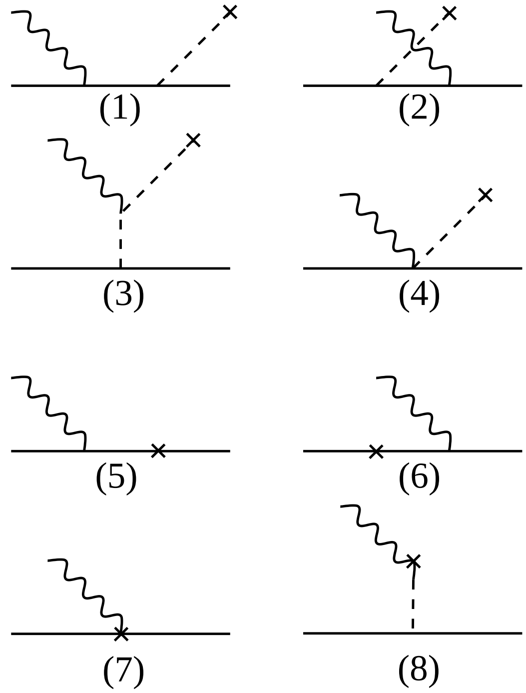

FIG. 9. Feynman diagrams contributing to the Green function involving the electromagnetic current and the axial-vector current. The wavy line denotes a (virtual) photon coupling to the electromagnetic current. The cross corresponds to the axial-vector current. The diagrams have been arranged to yield a maximal similarity with Fig. 8. Note that a diagram of type (8) is not generated in case of $\mathcal{M}^{\mu}_{JP,i}$.

$$\mathcal{M}^{\mu\nu}_{JA,i,1}=i\frac{F_\pi q^\nu}{q^2-m_\pi^2}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\left(1-\frac{2m_N\not q}{s-m_N^2}\right)\gamma_5\frac{\tau_i}{2}\Gamma^\mu(k)u(p_i), \tag{5.9a}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,2}=i\frac{F_\pi q^\nu}{q^2-m_\pi^2}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\Gamma^\mu(k)\left(1-\frac{2m_N\not q}{u-m_N^2}\right)\gamma_5\frac{\tau_i}{2}u(p_i), \tag{5.9b}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,3}=\epsilon_{3ij}\tau_j\frac{F_\pi q^\nu}{q^2-m_\pi^2}\frac{(2q-k)^\mu}{t-m_\pi^2}g_{\pi N}\bar{u}(p_f)\gamma_5u(p_i), \tag{5.9c}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,4}=\epsilon_{3ij}\tau_j\frac{F_\pi q^\nu}{q^2-m_\pi^2}\bar{u}(p_f)\left\{\frac{g_{\pi N}}{2m_N}\gamma^\mu+\frac{1}{6}\langle r^2\rangle_A\frac{g_A}{2F_\pi}[(k-q)k\gamma^\mu-(k-q)^\mu\not k]\right\}\gamma_5u(p_i), \tag{5.9d}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,5}=ig_A\frac{1}{s-m_N^2}\bar{u}(p_f)\left[\gamma^\nu\left(1+\tfrac{1}{6}\langle r^2\rangle_A\,q^2\right)-q^\nu\not q\tfrac{1}{6}\langle r^2\rangle_A\right]\gamma_5\frac{\tau_i}{2}\times(\not p_i+\not k+m_N)\Gamma^\mu(k)u(p_i), \tag{5.9e}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,6} = ig_A \frac{1}{u-m_N^2}\bar{u}(p_f)\Gamma^\mu(k)(\not{p}_i - \not{q} + m_N)$$
$$\times\left[\gamma^\nu\left(1+\tfrac{1}{6}\langle r^2\rangle_A\, q^2\right) - q^\nu \not{q}\tfrac{1}{6}\langle r^2\rangle_A\right]\gamma_5 \frac{\tau_i}{2} u(p_i), \tag{5.9f}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,7} = -\epsilon_{3ij}\tau_j g_A \tfrac{1}{6}\langle r^2\rangle_A \tfrac{1}{2}\bar{u}(p_f)[\gamma^\nu(2q-k)^\mu - \gamma^\mu(q-k)^\nu - \not{q}g^{\mu\nu}]\gamma_5 u(p_i), \tag{5.9g}$$

$$\mathcal{M}^{\mu\nu}_{JA,i,8} = -\epsilon_{3ij}\tau_j g_{\pi N} F_\pi \frac{g^{\mu\nu}}{t-m_\pi^2}\bar{u}(p_f)\gamma_5 u(p_i). \tag{5.9h}$$

Note that Eqs. (5.9a)–(5.9d) are obtained from Eqs. (5.6a)–(5.6d) by the replacement $2B \rightarrow iq^\nu$ which simply reflects the respective coupling of the external pseudoscalar and the axial-vector fields to a single pion resulting from Eq. (3.1). Moreover, the coupling to the axial-vector current provides the additional term of Eq. (5.9h) in comparison with $\mathcal{M}^\mu_{JP,i}$ of Eq. (5.6).

2. Gauge invariance

As a first test of the results of Eqs. (5.9) we will investigate electromagnetic gauge invariance by contracting $\mathcal{M}^{\mu\nu}_{JA,i}$ with the four-momentum k_μ. The corresponding chiral Ward identity reads

$$k_\mu \mathcal{M}^{\mu\nu}_{JA,i} = \epsilon_{3ij}\langle N(p_f)|A^\nu_j(0)|N(p_i)\rangle. \tag{5.10}$$

Contracting the sum of the first four and the final expressions, and the sum of the remaining three, respectively, with k_μ we obtain

$$k_\mu \sum_{k=1}^{4} \mathcal{M}^{\mu\nu}_{JA,i,k} + k_\mu \mathcal{M}^{\mu\nu}_{JA,i,8} = -\epsilon_{3ij}\tau_j \frac{p_f^\nu - p_i^\nu}{t-m_\pi^2} g_{\pi N} F_\pi \bar{u}(p_f)\gamma_5 u(p_i), \tag{5.11a}$$

$$k_\mu \sum_{k=5}^{7} \mathcal{M}^{\mu\nu}_{JA,i,k} = \epsilon_{3ij} g_A \bar{u}(p_f)\left[\gamma^\nu\left(1+\frac{1}{6}\langle r^2\rangle_A t\right) - 2m_N(p_f-p_i)^\nu \frac{1}{6}\langle r^2\rangle_A\right]\gamma_5 \frac{\tau_j}{2}u(p_i). \tag{5.11b}$$

Adding the two terms and comparing with the result for the axial-vector current matrix element of Eqs. (4.11) and (4.12), we see that the chiral Ward identity of Eq. (5.10) is indeed satisfied.

3. Test of the Adler-Gilman relation

We are finally in the position to explicitly test the Adler-Gilman relation. Since this has been the key ingredient in many investigations of the connection between the PCAC hypothesis and threshold pion production, we will discuss the individual terms in detail.

Contracting the two s-channel diagrams of $\mathcal{M}^{\mu\nu}_{JA,i}$ with q_ν [see (1) and (5) of Fig. 9] we obtain

$$q_\nu(\mathcal{M}^{\mu\nu}_{JA,i,1} + \mathcal{M}^{\mu\nu}_{JA,i,5}) = i\left(\frac{F_\pi g_{\pi N}}{m_N}\frac{q^2}{q^2-m_\pi^2} - g_A\right)\bar{u}(p_f) \times\left(1-\not{q}\frac{2m_N}{s-m_N^2}\right)\gamma_5\frac{\tau_i}{2}\Gamma^\mu(k)u(p_i)$$
$$= \hat{m}i(\mathcal{M}^\mu_{JP,i,1} + \mathcal{M}^\mu_{JP,i,5}). \tag{5.12}$$

Note that Eq. (5.12) does *not* imply a one-to-one correspondence between diagrams (1) and diagrams (5) of Figs. 8 and 9, respectively. In a similar fashion we find for the u-channel diagrams

$$q_\nu(\mathcal{M}^{\mu\nu}_{JA,i,2} + \mathcal{M}^{\mu\nu}_{JA,i,6}) = i\left(\frac{F_\pi g_{\pi N}}{m_N}\frac{q^2}{q^2-m_\pi^2} - g_A\right)\bar{u}(p_f) \times\Gamma^\mu(k)\left(1-\not{q}\frac{2m_N}{u-m_N^2}\right)\gamma_5\frac{\tau_i}{2}u(p_i)$$
$$= \hat{m}i(\mathcal{M}^\mu_{JP,i,2} + \mathcal{M}^\mu_{JP,i,6}). \tag{5.13}$$

Let us now discuss (3) of Fig. 9:

$$q_\nu \mathcal{M}^{\mu\nu}_{JA,i,3} = \epsilon_{3ij}\tau_j \frac{F_\pi q^2}{q^2-m_\pi^2}\frac{(2q-k)^\mu}{t-m_\pi^2} g_{\pi N}\bar{u}(p_f)\gamma_5 u(p_i)$$
$$= \hat{m}i\mathcal{M}^\mu_{JP,i,3} + \epsilon_{3ij}\tau_j \frac{(2q-k)^\mu}{t-m_\pi^2} F_\pi g_{\pi N}\bar{u}(p_f)\gamma_5 u(p_i),$$
$$\equiv \hat{m}i\mathcal{M}^\mu_{JP,i,3} + \Delta\mathcal{M}^\mu_{i,3}, \tag{5.14}$$

where we introduced the "remainder" $\Delta\mathcal{M}^\mu_{i,3}$ for later purposes. In order to obtain Eq. (5.14) we made use of $q^2/(q^2-m_\pi^2) = 1 + m_\pi^2/(q^2-m_\pi^2)$. In a similar way we obtain for (4), (7), and (8) of Fig. 9:

$$q_\nu \mathcal{M}^{\mu\nu}_{JA,i,4} = \hat{m}i\mathcal{M}^\mu_{JP,i,4} + \Delta\mathcal{M}^\mu_{i,4}, \tag{5.15}$$

$$\Delta\mathcal{M}^\mu_{i,4} = \epsilon_{3ij}\frac{\tau_j}{2}F_\pi \bar{u}(p_f)\left\{\frac{g_{\pi N}}{m_N}\gamma^\mu + \frac{1}{6}\langle r^2\rangle_A \frac{g_A}{F_\pi}[(k-q)k\gamma^\mu - (k-q)^\mu \not{k}]\right\}\gamma_5 u(p_i), \tag{5.16}$$

$$q_\nu \mathcal{M}^{\mu\nu}_{JA,i,7} = \hat{m}i\mathcal{M}^\mu_{JP,i,7} + \Delta\mathcal{M}^\mu_{i,7}, \tag{5.17}$$

$$\Delta\mathcal{M}^\mu_{i,7} = -\epsilon_{3ij}\frac{\tau_j}{2}g_A\frac{1}{6}\langle r^2\rangle_A \bar{u}(p_f)[\not{q}(q-k)^\mu - \gamma^\mu(q-k)q]\gamma_5 u(p_i) - \epsilon_{3ij}\frac{\tau_j}{2}F_\pi \Delta_{\pi N}\frac{g_{\pi N}}{m_N}\bar{u}(p_f)\gamma^\mu\gamma_5 u(p_i), \tag{5.18}$$

$$q_\nu \mathcal{M}^{\mu\nu}_{JA,i,8} = \Delta\mathcal{M}^{\mu}_{i,8} = -\epsilon_{3ij}\tau_j g_{\pi N} F_\pi \frac{q^\mu}{t-m_\pi^2}\bar{u}(p_f)\gamma_5 u(p_i). \tag{5.19}$$

As a first observation, notice that by construction the sum of Eqs. (5.12)–(5.19) adds up to $\hat{m} i \mathcal{M}^{\mu}_{JP,i}$ plus the remainders. The sum of the latter is given by

$$\begin{aligned}\Delta\mathcal{M}^{\mu}_{i} &= \Delta\mathcal{M}^{\mu}_{i,3} + \Delta\mathcal{M}^{\mu}_{i,4} + \Delta\mathcal{M}^{\mu}_{i,7} + \Delta\mathcal{M}^{\mu}_{i,8}\\ &= \epsilon_{3ij}\tau_j \frac{(q-k)^\mu}{t-m_\pi^2} F_\pi g_{\pi N}\bar{u}(p_f)\gamma_5 u(p_i)\\ &\quad + \epsilon_{3ij}\frac{\tau_j}{2}\frac{F_\pi g_{\pi N}}{m_N}(1-\Delta_{\pi N})\bar{u}(p_f)\gamma^\mu\gamma_5 u(p_i)\\ &\quad + \epsilon_{3ij}\frac{\tau_j}{2}\frac{1}{6}\langle r^2\rangle_A g_A \bar{u}(p_f)[\gamma^\mu t - (k-q)^\mu(\not{k}-\not{q})]\gamma_5 u(p_i)\\ &= \epsilon_{3ij}\langle N(p_f)|A^\mu_j(0)|N(p_i)\rangle. \end{aligned} \tag{5.20}$$

We thus have explicitly verified the Adler-Gilman relation, Eq. (2.16), in the framework of the phenomenological approximation to $\mathcal{L}_{\text{eff}}$.

At this point it is appropriate to recollect that Eq. (2.16) is an *exact* relation among QCD Green functions. Of course, one would like to verify Eq. (2.16) in terms of an *ab initio* QCD calculation. On the other hand, a complete and systematic analysis of this chiral Ward identity in terms of effective degrees of freedom requires effective-field-theory techniques. At this stage one needs the most general effective Lagrangian which is chirally invariant under *local* $SU(2)_L \times SU(2)_R \times U(1)_V$ transformations provided the external fields are transformed accordingly [17,72]. This then allows one to deal with the chiral Ward identities in terms of an invariance property of the generating functional (see appendix A of Ref. [52] for a pedagogical illustration). Under these circumstances the chiral Ward identities of QCD (as well as their symmetry-breaking pattern) are encoded in the generating functional which is then given through the *effective* field theory. In the process of constructing the effective Lagrangian one necessarily also generates nonminimal terms such as, e.g., the b_{23} term in Eq. (3.14) which will be discussed in more detail below. As has been illustrated in Ref. [73] for the case of the pion electromagnetic vertex, such nonminimal terms are mandatory for reasons of consistency.

D. Comparison with previous calculations

1. Extrapolation from $q_\mu=0$ to $q_\mu=(m_\pi,0)$

The basic idea of traditional current-algebra or PCAC approaches consists of defining a function

$$\tilde{\mathcal{M}}^{\mu}_{i}(q) = -i\frac{\hat{m}}{m_\pi^2 F_\pi}(q^2-m_\pi^2)\mathcal{M}^{\mu}_{JP,i} \tag{5.21}$$

for arbitrary values of q,[14] with the property that the physical pion production matrix element [see Eq. (2.21)] is given by

$$\mathcal{M}^{\mu}_{i} = \tilde{\mathcal{M}}^{\mu}_{i}(q)|_{q^2=m_\pi^2}. \tag{5.22}$$

By applying the Adler-Gilman relation, Eq. (5.21) is reexpressed as

$$\tilde{\mathcal{M}}^{\mu}_{i}(q) = \frac{q^2-m_\pi^2}{m_\pi^2 F_\pi}(\epsilon_{3ij}\mathcal{M}^{\mu}_{A,j} - q_\nu \mathcal{M}^{\mu\nu}_{JA,i}), \tag{5.23}$$

and a constraint for $\tilde{\mathcal{M}}^{\mu}_{i}(q)$ is obtained by evaluating the right-hand side of Eq. (5.23) at $q_\mu=0$, which is traditionally referred to as the soft-pion limit. In the present work we prefer the terminology "soft-momentum limit" which avoids the notion "off-mass-shell pions." Rather, we consider the Green functions for finite quark masses (implying massive pions) at $q_\mu=0$, and the result is then translated into consistency conditions in terms of the invariant amplitudes parametrizing $\tilde{\mathcal{M}}^{\mu}_{i}(q)$.

From the first term of Eq. (5.23) one obtains the axial-vector current matrix element for a four-momentum transfer $k=p_f-p_i$. Out of the second term only the one-particle-reducible pole terms are candidates contributing to the soft-momentum limit [32].[15] Such $1/q$ singularities in $\mathcal{M}^{\mu\nu}_{JA,i}$ originate from pole diagrams where the vertex associated with the axial-current operator is attached to a nonterminating external nucleon line. In principle, these diagrams have to be evaluated using the most general renormalized, one-particle-irreducible half-off-shell electromagnetic and axial vertices in combination with the most general renormalized dressed propagator. However, expanding vertices and propagators around their on-shell values, all such off-shell effects become irrelevant in the soft-momentum limit. This statement requires that none of the off-shell vertices contain poles as $q\to 0$. In fact, such poles would have to be of a dynamical origin and are expected to be absent as long as the underlying dynamics does not contain massless particles.[16] Let us illustrate the above statement by use of a "generic" axial form factor contributing in the s-channel diagram,

$$\begin{aligned} G[q^2, m_N^2, (p_f+q)^2] &= G(q^2, m_N^2, m_N^2)\\ &\quad + (s-m_N^2)G'(q^2, m_N^2, m_N^2) + \cdots \end{aligned} \tag{5.24}$$

with analogous considerations for the electromagnetic vertex. Similarly, the renormalized dressed propagator can be written as

$$S(p_f+q) = S_F(p_f+q) + \text{regular terms}, \tag{5.25}$$

[14]Of course, four-momentum conservation $k+p_i=q+p_f$ is assumed.

[15]We take the soft-momentum limit by first setting $\vec{q}=0$ and then performing the limit $q_0\to 0$.

[16]The prototype of such a pole behavior is, of course, given by the induced pseudoscalar form factor in case of *massless pions*.

where $S_F(p)$ denotes the free propagator of a nucleon with mass m_N. Finally, noting[17]

$$\lim_{q\to 0}\frac{q_\nu}{\not p_f+\not q-m_N}=\lim_{q\to 0}\frac{q_\nu(\not p_f+\not q+m_N)}{2p_f q+q^2}=\frac{g_{\nu 0}(\not p_f+m_N)}{2E_f}, \tag{5.26a}$$

$$\lim_{q\to 0}\frac{q_\nu}{\not p_i-\not q-m_N}=-\frac{g_{\nu 0}(\not p_i+m_N)}{2E_i}, \tag{5.26b}$$

the soft-momentum limit of $q_\nu \mathcal{M}^{\mu\nu}_{JA,i}$ reads

$$\lim_{q\to 0} q_\nu \mathcal{M}^{\mu\nu}_{JA,i}=ig_A\bar u(p_f)\left[\frac{\gamma^0}{2E_f}\gamma_5\frac{\tau_i}{2}(\not p_f+m_N)\Gamma^\mu(p_f,p_i)-\Gamma^\mu(p_f,p_i)(\not p_i+m_N)\frac{\gamma_0}{2E_i}\gamma_5\frac{\tau_i}{2}\right]u(p_i)_{p_f-p_i=k}, \tag{5.27}$$

where $\Gamma^\mu(p_f,p_i)$ is given by

$$\Gamma^\mu(p_f,p_i)=\gamma^\mu\frac{F_1^s(k^2)+\tau_3F_1^v(k^2)}{2}+i\frac{\sigma^{\mu\nu}k_\nu}{2m_N}\frac{F_2^s(k^2)+\tau_3F_2^v(k^2)}{2},\quad k=p_f-p_i. \tag{5.28}$$

Here $F^{s/v}_{1/2}(k^2)$ refer to the isoscalar (isovector) Dirac and Pauli form factors of the nucleon. As already pointed out by Adler [32], the positive frequency projection operators $\not p_f+m_N$ and $\not p_i+m_N$ in the respective s- and u-channel contributions to Eq. (5.27) give rise to the fact that only the on-shell electromagnetic vertex enters into the soft-momentum result. Indeed, we have explicitly checked that inserting the on-shell equivalent parametrizations involving G_E and $G_{\mathcal{M}}$ or H_1 and H_2 (for a discussion see, e.g., Ref. [74]) generates the same soft-momentum limit. Moreover, Eq. (5.27) only contains the axial-vector coupling constant g_A but *not* the axial form factor.

Let us test the consistency of the procedure by contracting Eq. (5.27) with k_μ. According to Eq. (5.10) we have

$$\begin{aligned}k_\mu q_\nu \mathcal{M}^{\mu\nu}_{JA,i}&=\epsilon_{3ij}q_\nu\langle N(p_f)|A^\nu_j(0)|N(p_i)\rangle\\&=\epsilon_{3ij}[k_\nu-(p_f-p_i)]_\nu\langle N(p_f)|A^\nu_j(0)|N(p_i)\rangle,\end{aligned} \tag{5.29}$$

which clearly vanishes as $q\to 0$, i.e., for $k\to p_f-p_i$. On the other hand, from Eq. (5.27) we find that both the s- and u-channel contributions vanish separately which, of course, simply reflects the on-shell current conservation condition.

[17]A possible t-channel contribution remains finite, because

$$\lim_{q\to 0}\frac{q_\nu}{t-m_\pi^2+i\epsilon}=\lim_{q\to 0}\frac{q_\nu}{k^2-m_\pi^2+i\epsilon}=0.$$

Note that in the physical region $(p_f-p_i)^2\leqslant 0$ such that the denominator never vanishes.

The consistency relation can be summarized as

$$\begin{aligned}\lim_{q\to 0}\widetilde{\mathcal{M}}^\mu_i(q)=&-\frac{\epsilon_{3ij}}{F_\pi}\langle N(p_f)|A^\mu_j(0)|N(p_i)\rangle\Big|_{p_f-p_i=k}-i\frac{g_A}{F_\pi}\bar u(p_f)\\&\times\left[\left(1-\frac{m_N}{E_f}\gamma_0\right)\gamma_5\frac{\tau_i}{2}\Gamma^\mu(p_f,p_i)+\Gamma^\mu(p_f,p_i)\right.\\&\left.\times\left(1+\frac{m_N}{E_i}\gamma_0\right)\gamma_5\frac{\tau_i}{2}\right]u(p_i)\Big|_{pf-p_{i_x}=k}.\end{aligned} \tag{5.30}$$

Since the second part of Eq. (5.30) does *not* involve the axial form factor, the soft-momentum limit of Eq. (5.30) leaves no room for a cancellation of the axial form factor between the axial-vector current piece and the second contribution. In other words, the soft-momentum limit of $\widetilde{M}^\mu_i(q)$ unambiguously contains the axial form factor as well as the induced pseudoscalar form factor.

Although the specific form of the consistency relation depends on how the soft-momentum limit is taken, the above conclusion is not affected. First of all, the first part of Eq. (5.30) involving the axial-vector current matrix element is path independent, whereas the directional dependence of the second part is trivial. For example, if one wanted to take the soft-momentum limit using $q^\mu=|\vec q|(0,\hat q)$, instead of Eqs. (5.26) one would have to consider

$$\lim_{|\vec q|\to 0}=\frac{aq}{\not p_f+\not q-m_N}=\frac{\vec a\hat q}{2\vec p_f\hat q}(\not p_f+m_N), \tag{5.31a}$$

$$\lim_{|\vec q|\to 0}=\frac{aq}{\not p_i-\not q-m_N}=-\frac{\vec a\cdot\hat q}{2\vec p_i\cdot\hat q}(\not p_i+m_N), \tag{5.31b}$$

leading to analogous replacements in the second part of Eq. (5.30). Nevertheless, it would still only contain g_A because the soft-momentum limit of functions containing only invariants has no directional dependence.

Finally, as an explicit test we evaluate the soft-momentum limit of Eq. (5.21) in the framework of Eqs. (5.6),

$$\lim_{q\to 0}\widetilde{\mathcal{M}}^\mu_i(q)=\frac{1}{F_\pi}\lim_{q\to 0}\hat m i\mathcal{M}^\mu_{JP,i}, \tag{5.32}$$

and compare the result with the consistency relation of Eq. (5.30). Using

$$\frac{1}{q^2-m_\pi^2}\to-\frac{1}{m_\pi^2}, \tag{5.33a}$$

$$1-\frac{2m_N\not q}{s-m_N^2}\to 1-\frac{m_N}{E_f}\gamma_0, \tag{5.33b}$$

$$1-\frac{2m_N\not q}{u-m_N^2}\to 1+\frac{m_N}{E_i}\gamma_0, \tag{5.33c}$$

together with $2\hat m B=m_\pi^2$ and $(1-\Delta_{\pi N})g_{\pi N}/m_N=g_A/F_\pi$, we see that Eq. (5.6a) together with Eq. (5.6e) [Eq. (5.6b) together with Eq. (5.6f)] exactly generate the s-channel (u-channel) expression of Eq. (5.30), whereas the sum of Eqs. (5.6c), (5.6d), and (5.6g) yields

$$-\frac{1}{F_\pi}\epsilon_{3ij}\bar{u}(p_f)\left[\gamma^\mu g_A\left(1+\frac{1}{6}\langle r^2\rangle_A k^2\right)\right.$$
$$\left.+k^\mu\left(\frac{2g_{\pi N}F_\pi}{m_\pi^2-k^2}-\frac{m_N g_A}{3}\langle r^2\rangle_A\right)\right]\gamma_5\frac{\tau_j}{2}u(p_i), \quad (5.34)$$

which corresponds to the nucleon axial-vector current matrix element in Eq. (5.30). Thus, the calculation within the framework of $\mathcal{L}_{\rm eff}$ reproduces the constraint of Eq. (5.30).

On the other hand, we would like to emphasize that Eq. (5.30) does not imply a consistency condition for *every* pion production amplitude evaluated for off-shell pion momenta. This can easily be visualized by investigating Eqs. (5.3) in the limit $q\to 0$. We remind the reader that for $q^2\neq m_\pi^2$ the result does not correspond to an observable [75–77] but would, for example, be a building block of the reaction $\gamma N \to N\gamma\pi$ evaluated in the framework of $\mathcal{L}_{\rm eff}$. In fact, the (off-shell) soft-pion limit of $\mathcal{M}_i^\mu$ looks similar to $\tilde{\mathcal{M}}_i^\mu(0)$ with the difference that g_A/F_π in the pole terms of Eq. (5.30) is replaced by $g_{\pi N}/m_N$. The same is true for the $k^2=0$ limit of Eq. (5.3d) as compared with Eq. (5.34). This is an illustration for the fact that Eq. (5.30) does not yield a consistency relation for the soft-pion production amplitude for an *arbitrary* interpolating pion field.

At present, corrections to the soft-momentum result of Eq. (5.30) either have to be studied within specific models—thus obviously yielding model-dependent results—or can be addressed in the framework of ChPT. In the second context such corrections have systematically been analyzed at $O(p^3)$ in Refs. [35–37], where, essentially, a direct calculation of the pion production matrix element as in Sec. V A was performed. In particular, pion-loop corrections contributing at $q^2=m_\pi^2$ modify the soft-momentum result of Eq. (5.30) such that the threshold production amplitude $E_{0+}^{(-)}(k^2)$ obtains an additional term proportional to m_π^2/F_π^2 multiplied by a function $f(k^2/m_\pi^2)$ which vanishes at $k^2=0$. The subtle point about such corrections is that they invalidate the naive expectation that corrections to the soft-momentum result should be of order m_π or higher. The reason is that pion loops give rise to nonanalytic pieces [78], where the scale in the loop integrals is set by the pion mass originating from the propagators of internal lines. Since in ChPT the Green functions are evaluated at a fixed ratio $\hat{m}/p^2$, the function f counts as $O(p^0)$ in the momentum and quark mass expansion. The m_π^2 in front of the function f reflects the evolution from the soft-momentum limit $q^2=0$ to $q^2=m_\pi^2$.

An *explicit* test of the chiral Ward identity of Eq. (2.16) at $O(p^3)$ including the loop corrections is not yet available in the literature.

2. Comparison with Haberzettl

Recently, the question whether the axial radius of the nucleon can be obtained from threshold pion electroproduction data [26–31] has given rise to much controversy [41–45]. The discussion was triggered by a paper of Haberzettl [41], where it was argued that PCAC does not provide any additional constraints beyond the Goldberger-Treiman relation. Similar claims were made by Ohta in Ref. [57] some time ago.

In order to solve this seeming puzzle we need to have a closer look at the method used in Ref. [41]. Starting from the nucleon matrix element of the axial-vector current [see Eq. (4.11)] in combination with the constraint of Eq. (4.14), the axial-vector current was split into "weak" and "hadronic" parts, expressed in terms of G_A and $G_{\pi N}$, respectively. Such a splitting may be interpreted as resulting from the (formal) separation of the axial-vector current operator into a transversal part and a longitudinal one [79],

$$A_i^\mu(x)=\left[A_i^\mu(x)-\frac{\partial_\mu\partial_\nu}{\Box}A_i^\nu(x)\right]+\frac{\partial_\mu\partial_\nu}{\Box}A_i^\nu(x). \quad (5.35)$$

After this separation a formal expression for (the equivalent of) the Green function $\mathcal{M}_{JA,i}^{\mu\nu}$ of Eq. (2.14a) was constructed. This was done by inserting an external photon in all possible places in the diagram corresponding to the separation of the axial-vector current (see Figs. 1 and 3 of Ref. [41]). For the insertion *into* vertices the so-called gauge-derivative method of Ref. [80] was applied. For example, for the last diagram of Fig. 3 of Ref. [41] corresponding to diagram (4) of our Fig. 9 one needs the contact interaction of pion electroproduction as obtained from the insertion into the pion-nucleon vertex. For our case, this vertex is given by Eq. (4.10), and the application of the gauge-derivative method would simply produce the contact vertex

$$ie\epsilon_{3ij}\tau_j\frac{g_{\pi N}}{2m_N}\gamma^\mu\gamma_5. \quad (5.36)$$

Of course, in the present case, Eq. (5.36) is nothing else than what is generated by minimal substitution into the pseudovector pion nucleon interaction. However, this is *not* what chiral symmetry tells us. In order to see this we have to compare with the result for the $\gamma\pi NN$ vertex of Eq. (5.3d), namely,

$$ie\epsilon_{3ij}\tau_j\left\{\frac{g_{\pi N}}{2m_N}\gamma^\mu+\frac{g_A}{12F_\pi}\langle r^2\rangle_A[(k-q)k\!\!\!/\gamma^\mu-(k-q)^\mu k\!\!\!/]\right\}\gamma_5. \quad (5.37)$$

We conclude that the gauge derivative-method produces only part of the full interaction and is in conflict with the constraints of chiral symmetry. In the above case it does not generate the $\langle r^2\rangle_A$ term entering the charged-pion electroproduction amplitude.

E. The role of chiral symmetry

From the effective-field-theory point of view it is rather straightforward to understand how a quantity such as $\langle r^2\rangle_A$ enters different physical amplitudes. Due to spontaneous symmetry breaking, the chiral symmetry of QCD is realized nonlinearly on the effective degrees of freedom [16–19,60,81] [see Eqs. (3.4) and (3.10)]. In order to collect the chiral Ward identities in a generating functional one needs the most general locally invariant effective Lagrangian where the emphasis is on both *generality* and *local invariance*. In the present case we will have a closer look at the b_{23} term of Eq. (3.14) involving the quantity $f_{\mu\nu}^-$ of Eq. (3.12a):

$$
\begin{aligned}
f^{-}_{\mu\nu} &= u\{\partial_\mu(v_\nu - a_\nu) - \partial_\nu(v_\mu - a_\mu) - i[v_\mu - a_\mu, v_\nu - a_\nu]\}u^\dagger \\
&\quad - u^\dagger\{\partial_\mu(v_\nu + a_\nu) - \partial_\nu(v_\mu + a_\mu) - i[v_\mu + a_\mu, v_\nu + a_\nu]\}u \\
&= -2(\partial_\mu a_\nu - \partial_\nu a_\mu) + 2i([v_\mu, a_\nu] - [v_\nu, a_\mu]) \\
&\quad + \frac{i}{F}[\vec{\tau}\cdot\vec{\pi}, \partial_\mu v_\nu - \partial_\nu v_\mu] + \frac{1}{F}[\vec{\tau}\cdot\vec{\pi}, [v_\mu, v_\nu]] \\
&\quad + \frac{1}{F}[\vec{\tau}\cdot\vec{\pi}, [a_\mu, a_\nu]] + O(\pi^2),
\end{aligned}
\tag{5.38}
$$

where we expanded u in terms of the pion field. We first note that $f^{-}_{\mu\nu}$ involves field-strength tensors as opposed to pure covariant-derivative terms. Moreover, due to the nonlinear realization it contains a string of terms with an increasing number of pion fields. The lowest-order term involving one external axial-vector field,

$$-2(\partial_\mu a_\nu - \partial_\nu a_\mu),$$

gives rise to a contribution to the axial-vector current matrix element. It is responsible for the identification of the b_{23} term with the axial radius. On the other hand, there is no term with only one pion, i.e., no contribution to the πNN vertex of Eq. (4.10). In addition, there is also no contribution to the strong form factor $G_{\pi N}$ of Eq. (4.6). The term

$$\frac{i}{F}[\vec{\tau}\cdot\vec{\pi}, \partial_\mu v_\nu - \partial_\nu v_\mu]$$

contributes to the $\gamma\pi NN$ vertex. Thus, we clearly see how chiral symmetry relates for this particular term (a part of) the axial-vector current vertex with (a part of) the $\gamma\pi NN$ vertex. On the other hand, this relation is not generated by the gauge-derivative method.

VI. SUMMARY AND CONCLUSIONS

We have reinvestigated Adler's PCAC relation in the presence of an external electromagnetic field [32] within the framework of QCD coupled to external fields [17,18]. With a suitable choice for the interpolating pion field the QCD result is of the same form as Adler's pre-QCD version. We then discussed the Adler-Gilman relation [6] as a chiral Ward identity in terms of QCD Green functions and established the connection with the pion electroproduction amplitude. In order to explain the consequences of the Adler-Gilman relation, we made use of a tree-level approximation to the Green functions at $O(p^3)$ within relativistic baryon chiral perturbation theory. As a reference point we first performed a direct calculation of the pion-production transition current, $\mathcal{M}^\mu_i$, in terms of the effective degrees of freedom. We saw explicitly how the axial radius enters charged-pion electroproduction at $O(p^3)$. As an alternative we calculated the Green function $\mathcal{M}^\mu_{JP,i}$ involving the electromagnetic current and the pseudoscalar density and, using the LSZ reduction formalism, explicitly verified the connection with the pion electroproduction transition current determined previously. Again we saw that the axial radius enters this particular Green function. As a test of our result we verified a chiral Ward identity relating the divergence of $\mathcal{M}^\mu_{JP,i}$ to the matrix element of the pseudoscalar density. We then calculated the Green function $\mathcal{M}^{\mu\nu}_{JA,i}$ involving the electromagnetic and axial-vector currents, tested the constraints due to gauge invariance, and, finally, explicitly verified the Adler-Gilman relation for *arbitrary* values of q^2. Thus, all three possibilities of calculating pion electroproduction—direct calculation, determination in terms of the QCD Green function $\mathcal{M}^\mu_{JP,i}$, or application of the Adler-Gilman relation—generate the same result.

We then made contact with the traditional current-algebra or PCAC techniques by defining a generalization $\tilde{\mathcal{M}}^\mu_i$ of the physical pion electroproduction transition current in terms of the QCD Green function $\mathcal{M}^\mu_{JP,i}$ for arbitrary values of q. We considered the soft-momentum limit of $\tilde{\mathcal{M}}^\mu_i$, $q^\mu \to 0$, and showed that the usual "soft-pion" results are recovered if the pseudoscalar density is used as the pion interpolating field. We pointed out how the nonlinear realization of chiral symmetry leads to an interplay between various vertices in the most general theory and how approximations such as minimal substitution may fail to be compatible with the strictures of chiral symmetry and can lead to erroneous conclusions.

Clearly, chiral perturbation theory has become the standard method to systematically deal with *corrections* to the current-algebra results beyond the phenomenological approximation. The contribution of loop diagrams is expected to separately satisfy the constraints due to the Ward identities. In the case of pion photo- and electroproduction such corrections were determined in Refs. [35–38] leading to additional terms beyond the current-algebra results. Obviously, it would be nice to have a fully relativistic calculation within the infrared regularization [24] or the extended-on-mass-shell scheme [25] including an explicit test in terms of the Adler-Gilman relation.

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