Integral formula for calculating rigorously the full Glauber series of the elastic scattering between two cluster nuclei

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In a previous work, the full Glauber series of the elastic scattering between two cluster nuclei has been evaluated in an approximate way. The present paper introduces a more elaborate technique to calculate such series rigorously. The deuteron-carbon elastic angular distribution is calculated by the two approaches and the results showed that the inaccuracy of the previous approach is significant at large angles. Impressive fit is observed to the experimental data when the phase variation is included in the present analysis.

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I. INTRODUCTION

During the past three decades, hadron-nucleus and nucleus-nucleus high-energy elastic collisions have been extensively analyzed by the multiple scattering theory of Glauber and his co-worker [1,2]. The attractive point of applying this model is that even with simple uncorrelated wave functions and effective nucleon-nucleon (*NN*) scattering amplitudes, one can obtain a microscopic description for these reactions. As a matter of fact, the theory has shown great success in describing the elastic scattering of hadrons from various target nuclei. The data are excellently reproduced especially when the higher orders of the multiple scattering series are taken into account [3–5]. For nucleus-nucleus collisions, a definite answer about the success of this theory is still lacking. The reason is that the actual calculation of the scattering amplitude becomes very tedious and difficult for projectile and target mass numbers greater than or equal to 4. In such cases, the full Glauber multiple scattering series contains numerous terms so that its complete summation is impractical. Moreover, the higher-order multiple scatterings involve multdimensional integrals, which are cumbersome to evaluate even for simple Gaussian forms of the nuclear densities and *NN* scattering amplitudes. In the early stages of the analyses made, the theory has been applied by proposing different approximate methods in which the series is truncated [6–9]. The results showed that such incomplete calculations are clearly less well founded, especially at higher angle cross sections. Eventually, the question of the significance of the higher-order corrections in improving the predictive power of the Glauber model cannot be assessed unless one retains the full multiple scattering series in the calculations.

Later, with the aid of permutation group, Yin, Tan, and Chen [10] have succeeded in classifying the multiple scattering terms into sets; each set contains the terms that have equal contribution to the scattering amplitude. As a result, the terms of the same contribution are represented by one typical term, referred to as an orbit, and the number of these equally contributing terms is referred to as the length of this orbit. Furthermore, using Gaussian forms for the *NN* scattering amplitude and the nuclear density, they have transferred the multdimensional integrals corresponding to the typical terms (orbits) into simple recursion formulas. Their results obtained for α-α collision showed that the evaluation of the full series brings the Glauber model predictions closer to the experimental data over the available range of momentum transfer [4,10].

The preliminary applications of this method have been restricted to study the elastic scattering between two very light nuclei (mass numbers less than or equal to 4), because the number of generators of the permutation group grows rapidly for heavier systems [11,12]. As an attempt to extend the application of this method, Huang [13] has proposed a technique, in which the cluster structures are assumed for the colliding nuclei. The idea is that the multiple scattering occurring by the collision between clusters and the subcollision between nucleons in such clusters will consider fewer number of particles so that the method developed by Yin, Tan, and Chen can be used. However, this attempt does not account properly for the possible sub-collisions between these clusters [14]. Actually, exact classification of the multiple scattering terms obtained by this technique is lengthy and time consuming. We have to classify first the multiple scattering terms representing the collisions between clusters into orbits, then the subcollision terms between the nucleons of the clusters involved in each orbit. In fact, each cluster-cluster orbit contains a different kind and number of subcollisions, and therefore their classification will depend on which clusters are colliding. To avoid this trouble, El-Gogary and co-workers [15,16] have approximated the multiple scattering picture corresponding to this technique by treating the cluster-cluster orbits of the same order as if they have equal contribution to the scattering amplitude, and this contribution has the value of the single scattering orbit result raised to a power equal to the order of each orbit. Using this approximation, the angular distributions of the reactions α-^{12}C, α-^{40}Ca, ^{12}C-^{12}C, ^{16}O-^{12}C, and ^{16}O-^{16}O have been calculated and compared with the experimental data and the comparison has shown that a clear difference at large angles still persists.

In the present work, the analysis [16] is improved by accounting for the multiple scattering series between two cluster nuclei accurately. The multiple scattering terms of such series have been classified into the exact set of orbits and the corresponding multidimensional integrals have been calculated analytically using single Gaussian forms for the nuclear densities and the *NN* scattering amplitudes. The results obtained by the developed formulas are used to account for the inaccuracy of the previous approximation and the reliability of the Glauber model in explaining the nucleus-nucleus scattering data. In Sec. II the optical phase-shift function resulting from the scattering between two cluster nuclei is derived rigorously. Section III contains a comparison between the results of the present formula and the previous one [16], by taking the D-^{12}C collision as an example. The method of integration is shown in Appendix A, while the orbits and lengths needed in the calculations are exhibited in Appendix B.

II. NUCLEUS-NUCLEUS PHASE SHIFT FUNCTION UNDER CLUSTER STRUCTURE

In Glauber theory the nucleus-nucleus elastic scattering amplitude $F_{AB}(q)$ is specified by the optical phase-shift function $\chi_{AB}(\vec{b})$ as [9]

$$F_{AB}(q) = ik\, H(q)\int_0^\infty J_0(qb)\{1-\exp[i\chi_{AB}(\vec{b})]\}b\, db \quad (1)$$

$$= ik\int_0^\infty J_0(qb)\{1-\exp[i\bar{\chi}_{AB}(\vec{b})]\}b\, db, \quad (2)$$

where q is the momentum transferred from the projectile nucleus A to the target nucleus B, k is the incident momentum of the projectile nucleus, and $\vec{b}$ is the impact parameter vector. $H(q)$ is the correction factor arising from the effect of the center-of-mass correlations [6]. $\bar{\chi}_{AB}(\vec{b})$ stands for the phase-shift function containing such a correction consistently [9] and it is related to the uncorrelated one, $\chi_{AB}(\vec{b})$, by

$$\exp[i\bar{\chi}_{AB}(\vec{b})] = \int_0^\infty J_0(qb)H(q)q\, dq\int_0^\infty J_0(qb')\times\exp[i\chi_{AB}(\vec{b}')]b'\, db'. \quad (3)$$

From the multiple scattering picture of Glauber, $\chi_{AB}(\vec{b})$ can be related to the elemental *NN* phase shifts $\chi_{ij}(\vec{b})$ as

$$\chi_{AB}(\vec{b},\{\vec{s}_i\},\{\vec{s}_j'\}) = \sum_{i=1}^{A}\sum_{j=1}^{B}\chi_{ij}(\vec{b}+\vec{s}_i-\vec{s}_j')$$

and is given by

$$\exp[i\chi_{AB}(\vec{b})] = \langle\Psi_A(\{\vec{r}_i\})\Psi_B(\{\vec{r}_j'\})|\exp[i\chi_{AB}(\vec{b},\{\vec{s}_i\},\{\vec{s}_j'\})]|\Psi_A\Psi_B\rangle, \quad (4)$$

where, $\Psi_A(\{\vec{r}_i\})$ $[\Psi_B(\{\vec{r}_j'\})]$ is the projectile [target] wave function that depends on the position vectors $\{\vec{r}_i\}$ $[\{\vec{r}_j'\}]$ of the projectile [target] nucleons whose projections on the impact parameter plane are $\{\vec{s}_i\}[\{\vec{s}_j'\}]$. With the definition of the *NN* profile function, $\Gamma(\vec{b})=1-\exp[i\chi(\vec{b})]$, the phase-shift operator is given by

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_i\},\{\vec{s}_j'\})] = \prod_{i=1}^{A}\prod_{j=1}^{B}[1-\Gamma_{ij}(\vec{b}+\vec{s}_i-\vec{s}_j')]. \quad (5)$$

As seen in Eq. (5) the multiple scattering between nucleons will contain too many terms for $A,B\geqslant 4$. For more tractable calculations, the projectile and target nuclei are assumed to have cluster structure with M_N nucleons in each cluster, leading to M_A clusters in nucleus A and M_B clusters in nucleus B (note that M_N is a common divisor for A and B). Under this treatment, Eq. (5) is reexpressed as [13]

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}_{j\delta}'\})] = \prod_{i=1}^{M_A}\prod_{j=1}^{M_B}[1-\Gamma_{ij}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}_{j\delta}'\})] \quad (6)$$

with

$$\Gamma_{ij}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}_{j\delta}'\}) = 1-\prod_{\alpha=1}^{M_N}\prod_{\delta=1}^{M_N}[1-\Gamma_{i\alpha,j\delta}(\vec{b}+\vec{s}_{i\alpha}-\vec{s}_{j\delta}')], \quad (7)$$

where Γ_{ij} represents the profile function of scattering between the ith cluster in nucleus A and jth cluster in B and $\Gamma_{i\alpha,j\delta}$ is the scattering between the αth nucleon of the ith cluster in A and δth nucleon of jth cluster in B. One can simplify further this problem by applying the permutation group method of Yin, Tan, and Chen [10]. In this method, the multiple scattering terms are classified into sets of terms, each set contains the terms of equal contribution to the scattering amplitude. All terms in each set are represented by one typical term, referred to as an “orbit,” and the number of terms in this set is referred to as the “length” of that orbit. Now, having classified the terms of the multiple scattering between clusters into n_1 orbits, Eq. (6) can be rewritten as

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}_{j\delta}'\})] = 1+\sum_{\upsilon=1}^{n_1}T_1(\upsilon)\prod_{i=1}^{M_A}\prod_{j=1}^{M_B}[-\Gamma_{ij}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}_{j\delta}'\})]^{\Delta_{ij}(\upsilon)}. \quad (8)$$

In this equation, each cluster-cluster orbit υ is represented by an $(M_A\times M_B)$-dimensional matrix $\Delta(\upsilon)$, with element $\Delta_{i,j}(\upsilon)$ either equal to 1 or 0, and $T_1(\upsilon)$ is the number of repetitions (length) of the corresponding orbit. The matrix $\Delta(\upsilon)$ corresponds uniquely to a typical term expressing the multicluster collision and its element $\Delta_{i,j}(\upsilon)$ is equal to “1” or “0” according to whether Γ_{ij} is considered or not in the typical term (orbit). Of course, the number of elements having value 1, in a matrix, is the order of scattering of the collision term expressed by this matrix. Applying Eq. (7) into Eq. (8), it can be written in terms of the *NN* collisions as

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}'_{j\delta}\})] = 1+\sum_{v=1}^{n_1} T_1(v)\prod_{i=1}^{M_A}\prod_{j=1}^{M_B}\left\{\sum_{\eta_{ij}=1}^{m_1}\prod_{\alpha=1}^{M_N}\prod_{\delta=1}^{M_N}[-\Gamma_{i\alpha,j\delta} \times(\vec{b}+\vec{s}_{i\alpha}-\vec{s}'_{j\delta})]^{\Delta_{i\alpha,j\delta}(\eta_{ij})}\right\}^{\Delta_{ij}(v)}. \qquad (9)$$

Here, $m_1=(2^{M_N^2}-1)$ is the number of NN collisions between two clusters. The matrix whose elements $\Delta_{i\alpha,j\delta}(\eta_{ij})$ is an $(M_A\times M_B)$- dimensional matrix, where each ith row and jth column element in this matrix is also a matrix of dimension $M_N\times M_N$. Expanding the product signs on i and j in Eq. (9), we can get the NN collision terms corresponding to each cluster-cluster orbit by

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}'_{j\delta}\})] = 1+\sum_{v=1}^{n_1} T_1(v)\left[\prod_{i=1}^{M_A}\prod_{j=1}^{M_B}\left(\sum_{\eta_{ij}=1}^{m_1}\right)^{\Delta_{ij}(v)}\right] \times\left[\prod_{i=1}^{M_A}\prod_{j=1}^{M_B}\prod_{\alpha=1}^{M_N}\prod_{\delta=1}^{M_N}[-\Gamma_{i\alpha,j\delta}(\vec{b}+\vec{s}_{i\alpha}-\vec{s}'_{j\delta})]^{\Delta_{i\alpha,j\delta}(\eta_{ij})}\right], \qquad (10)$$

provided that, if $\Delta_{i,j}(v)=0$ the corresponding summation sign $\Sigma_{\eta_{ij}}^{m_1}$ is dropped out and all the elements $\Delta_{i\alpha,j\delta}(\eta_{ij})$ of the absent index η_{ij} are omitted. Clearly, Eq. (10) shows that the subcollisions proceeded by a cluster-cluster orbit v depend on which elements $\Delta_{ij}(v)$ equal to 1. Thus, the present form accounts for the multiple scattering between two cluster nuclei more comprehensively than the previous formula in Ref. [16]. Now, after classifying the subcollision terms corresponding to each cluster-cluster orbit, Eq. (10) takes the form

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}'_{j\delta}\})] = 1+\sum_{v=1}^{n_1} T_1(v)\left\{\sum_{\mu=1}^{n_2(v)} T_2(\mu,v)\prod_{i=1}^{M_A}\prod_{j=1}^{M_B}\prod_{\alpha=1}^{M_N}\prod_{\delta=1}^{M_N}[-\Gamma_{i\alpha,j\delta} \times(\vec{b}+\vec{s}_{i\alpha}-\vec{s}'_{j\delta})]^{\Delta_{i\alpha,j\delta}(\mu,v)}\right\}, \qquad (11)$$

where μ is the serial index to number the subcollision orbits representing the collision between clusters with an orbit v and $T_2(\mu,v)$ is the length of these orbits. $\Gamma_{i\alpha,j\delta}$ is related to the NN scattering amplitude $f_{i\alpha,j\delta}$ by

$$\Gamma_{i\alpha,j\delta}(\vec{b})=\frac{1}{2\pi i k_N}\int d^2\vec{q}e^{-(i\vec{q}\cdot\vec{b})}f_{i\alpha,j\delta}(\vec{q}), \qquad (12)$$

where k_N is the wave number of the incident nucleon.

Assuming, for simplicity, that all the NN amplitudes are equal (which is approximately true at high energy) and neglecting further the spin effects, $f_{i\alpha,j\delta}$ can be parametrized by [9]

$$f_{i\alpha,j\delta}(\vec{q})=\frac{k_N\sigma}{4\pi}(i+\rho)e^{-\bar{\beta}q^2/2}, \qquad (13)$$

where σ is the total NN cross section, and ρ is the ratio of the real part to the imaginary part of the forward NN scattering amplitude. Here, $\bar{\beta}$ is taken to be complex: $\bar{\beta}=\beta^2+i\gamma^2$, the real part β^2 is typically the slope parameter of the NN differential cross section while the imaginary part γ^2 is a free parameter introducing a phase variation of the NN scattering amplitude. Inserting Eq. (13) into Eq. (12), we obtain

$$\Gamma_{i\alpha,j\delta}(\vec{b}+\vec{s}_{i\alpha}-\vec{s}'_{j\delta})=g\,\exp[-(\vec{b}+\vec{s}_{i\alpha}-\vec{s}'_{j\delta})^2/2\bar{\beta}] \qquad (14)$$

with

$$g=\frac{\sigma}{4\pi\bar{\beta}}(1-i\rho).$$

Substituting Eq. (14) into Eq. (11) gives the result

$$\exp[i\chi_{AB}(\vec{b},\{\vec{s}_{i\alpha}\},\{\vec{s}'_{j\delta}\})] = 1+\sum_{v=1}^{n_1} T_1(v)\left\{\sum_{\mu=1}^{n_2(v)} T_2(\mu,v)(-g)^{V_2(\mu,v)}\exp\left[-\sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\sum_{\alpha=1}^{M_N}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,v)(\vec{b}+\vec{s}_{i\alpha}-\vec{s}'_{j\delta})^2/2\bar{\beta}\right]\right\} \qquad (15)$$

with

$$V_2(\mu,v)=\sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\sum_{\alpha=1}^{M_N}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,v).$$

Let us consider the wave function of the system to have the form

$$|\Psi_A\Psi_B|^2=\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N}\rho_A(\vec{r}_{i\alpha})\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N}\rho_B(\vec{r}'_{j\delta}), \qquad (16)$$

where ρ_A and ρ_B are the normalized single particle density functions and they are chosen to be of the single Gaussian type

$$\rho_i(r)=\frac{\alpha_i^3}{\pi^{3/2}}e^{-\alpha_i^2 r^2}, \quad i=A,B.$$

Adopting wave function (16) and the phase-shift function (15), we can perform the integration of Eq. (4) analytically and get

$$\exp[i\chi_{AB}(b)] = 1 + c_A c_B \sum_{\upsilon=1}^{n_1} T_1(\upsilon)\left\{\sum_{\mu=1}^{n_2(\upsilon)} T_2(\mu,\upsilon)(-g)^{V_2(\mu,\upsilon)} \times R(\mu,\upsilon)\exp[-W(\mu,\upsilon)b^2]\right\} \quad (17)$$

with

$$C_r = \left[\frac{\alpha_r^2}{\pi}\right]^{M_r M_N}, \quad r = A, B,$$

$$R(\mu,\upsilon) = \left[\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N}[4\pi\bar{\beta}^2 w_{i\alpha}(\mu,\upsilon)]\right]\left[\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N}\left(\frac{\pi}{a_{j\delta,j\delta}(j,\delta)}\right)\right]$$

and

$$W(\mu,\upsilon) = \sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\left[\alpha_B^2 - \frac{c_{j\delta}^2(j,\delta)}{4a_{j\delta,j\delta}(j,\delta)}\right].$$

The details of the integration process and the definition of $w_{i\alpha}$ and a 's and c's coefficients are given in Appendix A. Now, the single Gaussian model chosen for the nuclear density leads to an exact result for the center-of-mass correlation function $H(q)$ given by [6]

$$H(q) = \exp\left[\frac{q^2}{4}\left(\frac{1}{A\alpha_A^2} + \frac{1}{B\alpha_B^2}\right)\right]. \quad (18)$$

Incorporating Eqs. (17) and (18) into Eq. (3), the correlated phase-shift function $\bar{\chi}_{AB}(b)$ can be obtained as

$$\exp[i\bar{\chi}_{AB}(b)] = 1 + c_A c_B \sum_{\upsilon=1}^{n_1} T_1(\upsilon)\left\{\sum_{\mu=1}^{n_2(\upsilon)} \mathrm{T}_2(\mu,\upsilon)(-g)^{V_2(\mu,\upsilon)} \times \bar{R}(\mu,\upsilon)\exp[-\bar{W}(\mu,\upsilon)b^2]\right\}, \quad (19)$$

where

$$\bar{W} = \left[\frac{1}{W} - \left(\frac{1}{A\alpha_A^2} + \frac{1}{B\alpha_B^2}\right)\right]^{-1}$$

and

$$\bar{R} = \frac{R\bar{W}}{W}.$$

With the result of Eq. (19), the integration in Eq. (2) gives the scattering amplitude by

$$F_{AB}(q) = 1 + c_A c_B \sum_{\upsilon=1}^{n_1} T_1(\upsilon)\left\{\sum_{\mu=1}^{n_2(\upsilon)} T_2(\mu,\upsilon)(-g)^{V_2(\mu,\upsilon)} \times \frac{\bar{R}(\mu,\upsilon)}{2\bar{W}(\mu,\upsilon)}\exp\left[-\frac{q^2}{4\bar{W}(\mu,\upsilon)}\right]\right\}. \quad (20)$$

The angular distribution of the elastic scattering is then determined by

$$\frac{d\sigma(q)}{d\Omega} = |F_{AB}(q)|^2. \quad (21)$$

TABLE I. Parameters of the nucleon-nucleon amplitude.

E/A (MeV/nucleon)	σ (fm^2)	ρ	$\bar{\beta}$ (fm^2)	Ref.
47	11.951	0.97	0.464-i1.25	[17,18]
62.5	9.15	1.17	0.375-i0.89	[19]
78	6.79	1.32	0.325-i0.51	[18,19]
85	5.99	1.00	0.238-i0.39	[17]
212.5	3.28	0.93	1.240-i0.98	[17,20]
325	2.86	0.53	0.061-i1.05	[21]

III. APPLICATION TO D-^{12}C SCATTERING

In Sec. II, using simple Gaussian forms for the nuclear densities and the NN scattering amplitudes, the nuclear phase-shift function representing the full Glauber multiple scattering series between two cluster nuclei has been derived analytically. The present analysis is an exact treatment to the approximation introduced in Ref. [16]. To test the significance of such a correction in an application the exact and the approximate phase-shift formulas are applied to calculate the angular distribution of the D-^{12}C elastic scattering. As a matter of significance of this comparison, the effect of the Coulomb field is neglected in these calculations. The lab energy of the deuteron is taken to be 94, 125, 156, 170, 425, and 650 MeV, where the corresponding experimental data are available [22–25]. The cluster structure specific to the D-^{12}C system is assumed as M_A=1, M_B=6, and M_N=2. The orbits, lengths, and Δ-matrices required in the present formula for this structure are exhibited in the Tables III–IX, in Appendix B. The formula in Ref. [16] requires only Tables III and IV representing the cluster-cluster collisions, and the sub-collisions corresponding to the single scattering between these clusters, respectively. The subcollisions corresponding to the higher orders are approximated in terms of the single scattering ones. The input parameters are those associated with the NN scattering amplitude and the nuclear densities. For the parameters of the NN scattering amplitude, the values used corresponding to above energies are listed in Table I.

In this table, the usual NN parameters σ, ρ, and β^2 (the real part of "$\bar{\beta}$") are obtained by averaging the values of the neutron and proton parameters available in the references stated. The values of ρ, and β^2, which are not available, have been determined either by interpolating between the available ones or by requiring the unitarity condition [18] that

$$\beta^2 = [(1+\rho^2)/16\pi]\sigma.$$

The phase-variation parameter γ^2 (the imaginary part of "$\bar{\beta}$") is obtained by comparing the calculated and the ex-

TABLE II. Nuclear rms radii [9].

Nucleus	P	D	^{12}C
$\langle r^2\rangle^{1/2}$ (fm)	0.81	2.17	2.453

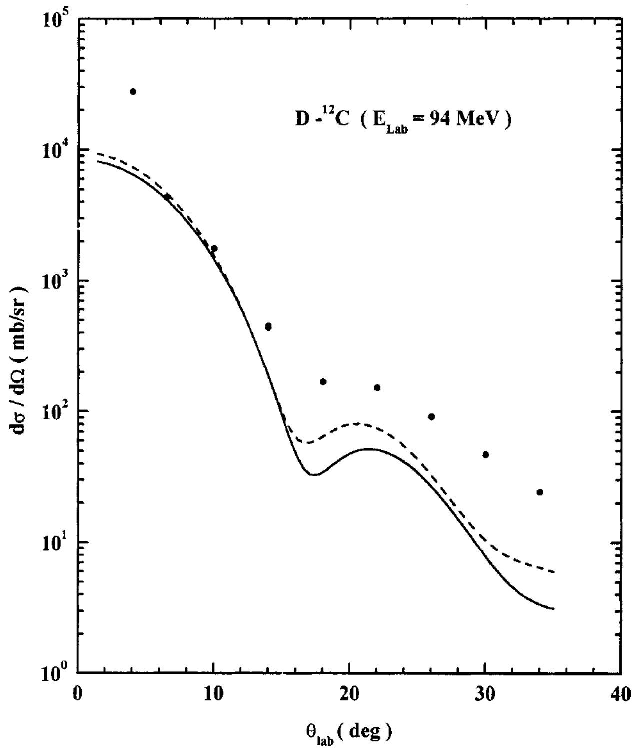

FIG. 1. Full Glauber series analysis for angular distribution of the D-^{12}C elastic scattering at 94 MeV. The curves compare the results obtained by the nuclear phase-shift formula of this work (the solid curve) and the corresponding formula of Ref. [16] (the dashed curve). The dashed curve is an approximation to the solid curve calculations. The curves display the predictions obtained by taking the phase-variation parameter $\gamma^2=0$. The dots are the experimental data [22].

perimental D-^{12}C angular distributions. The density parameter is obtained from [9]

$$\alpha_i^2 = \left(\frac{3}{2}\right)\frac{(1-1/i)}{[\langle r_i^2\rangle - \langle r_p^2\rangle]}, \qquad i = A, B$$

where $\langle r_i^2\rangle$ and $\langle r_p^2\rangle$ are the mean square radii of the colliding nuclei and the proton, respectively, and the measured values of their square roots are given in Table II.

At the beginning, the accuracy of the present analysis has been checked by reproducing the exact Glauber results of two applications, calculated independently using the same parametrization for the inputs needed: First, the cross sections of 1980, 2570, and 4200 MeV α-α scattering are calculated by taking the cluster structure (M_A=2, M_B=2, and M_N=2) and once again by considering (M_A=1, M_B=1, and M_N=4). The results obtained from the two structures are found identical and in agreement with the exact Glauber series calculations given in Ref. [4]. This ensures that the present approach accounts properly for the multiple scattering between the clusters considered in the nuclear system, the fact that the previous analysis does not verify. Second, using this analysis the calculation of the D-^{12}C total cross section at 4200 MeV yields the exact Glauber result reported in Ref. [9] given by $\sigma_{\rm tot}$=620.46 mb. Now, the D-^{12}C angular

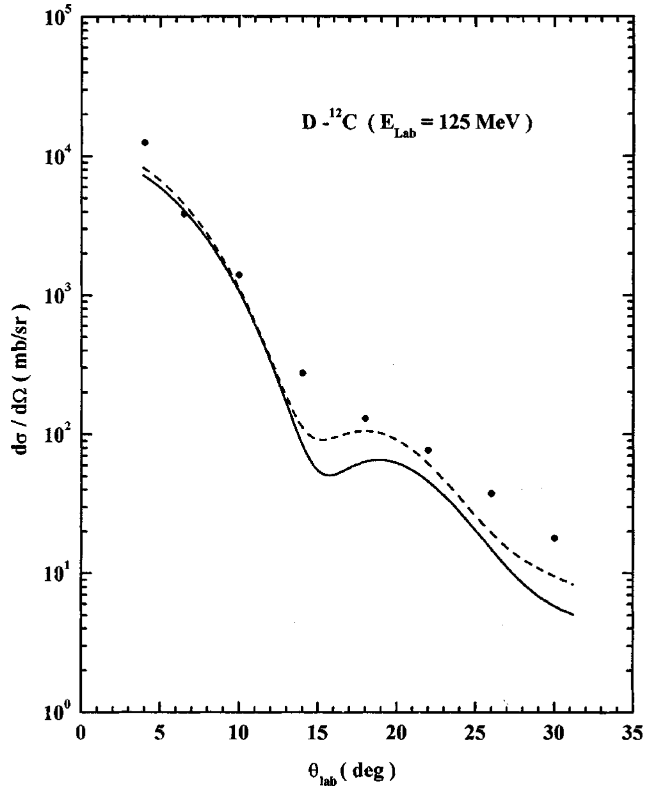

FIG. 2. Full Glauber series analysis for angular distribution of the D-^{12}C elastic scattering at 125 MeV. The curves are labeled as in Fig. 1. The dots are the experimental data [22].

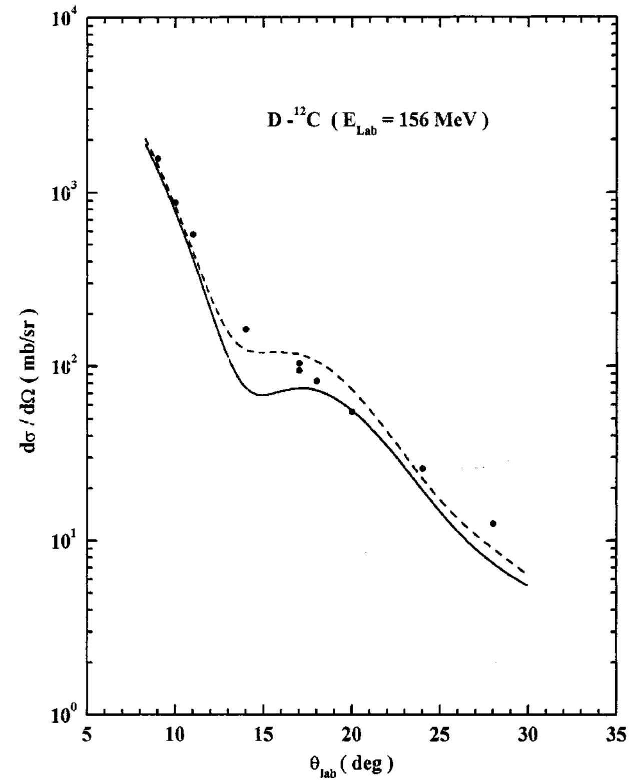

FIG. 3. Full Glauber series analysis for angular distribution of the D-^{12}C elastic scattering at 156 MeV. The curves are labeled as in Fig. 1. The dots are the experimental data [22].

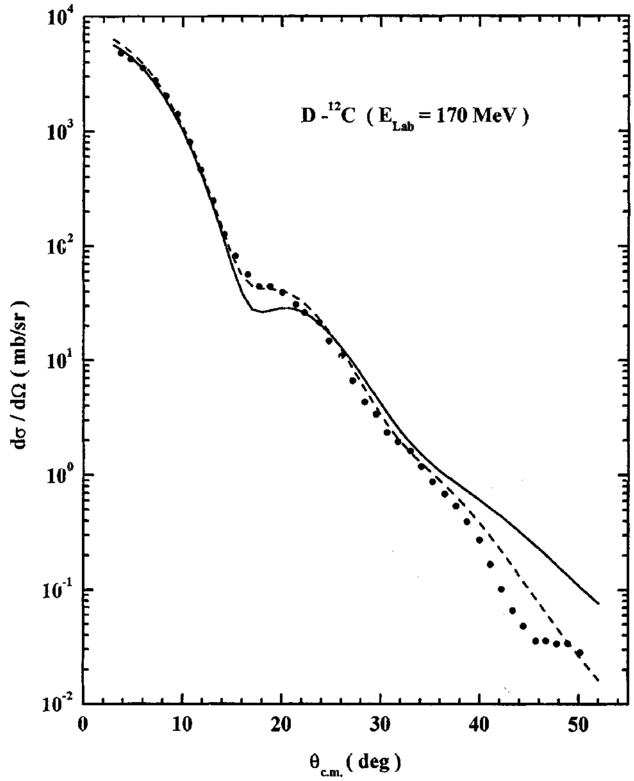

FIG. 4. Full Glauber series analysis for angular distribution of the D-^{12}C elastic scattering at 170 MeV. The curves are labeled as in Fig. 1. The dots are the experimental data [23].

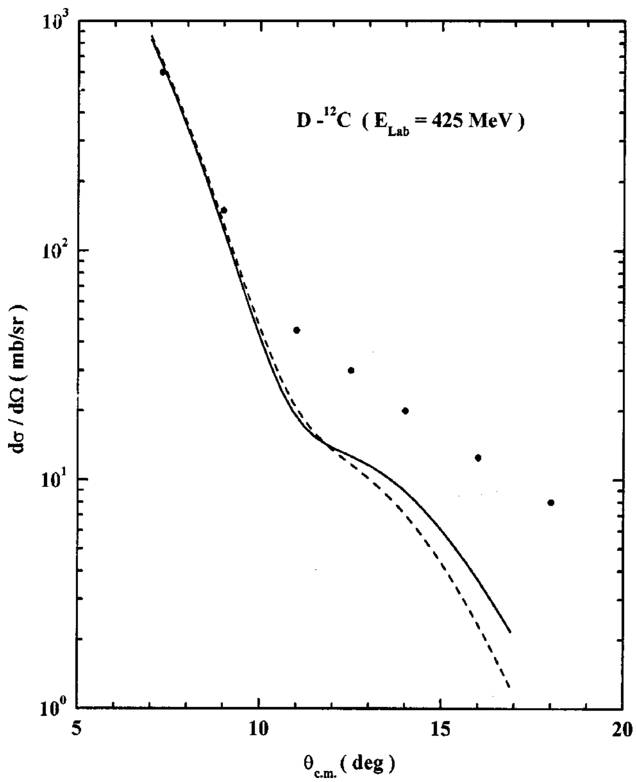

FIG. 5. Full Glauber series analysis for angular distribution of the D-^{12}C elastic scattering at 425 MeV. The curves are labeled as in Fig. 1. The dots are the experimental data [24].

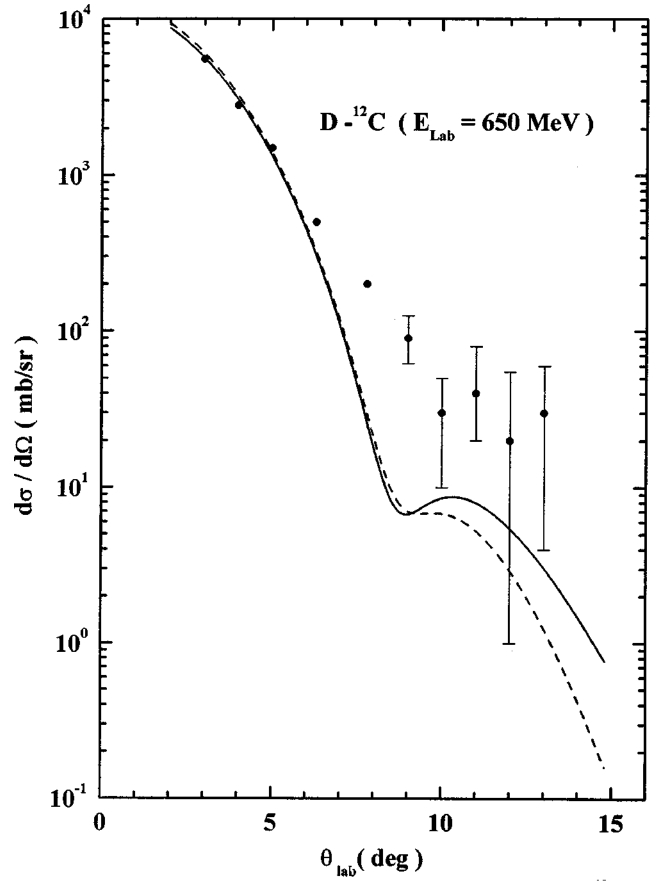

FIG. 6. Full Glauber series analysis for angular distribution of the D-^{12}C elastic scattering at 650 MeV. The curves are labeled as in Fig. 1. The dots are the experimental data [25].

distributions resulting from the present approach (solid curve) and previous approximation (dashed curve) are compared with the experimental data (dots) in Figs. 1–6. The curves are calculated by setting the phase-variation parameter γ^2 equal to zero. Figures 1–6 show that the two approaches yield similar results at small angles and give significantly different ones at large momentum transfers. As mentioned in Sec. I, the inaccuracy of the previous approach relative to the present one is neglecting the actual subcollisions of the higher-order multiple scattering between clusters and approximating their contributions as powers of the single scattering ones. So the significance of such a difference would be at large angles where the contributions of higher orders are dominant. Of course, the heavier the system applied, the bigger the error resulting from this approximation. Concerning their agreement with the experimental data, all calculations as shown do not agree with the measurements at large angles. Also, it is clear that the results obtained with the previous approximation (dashed curves) are closer to the large angle data at energies 94, 125, 156, and 170 MeV than the exact calculation ones (solid curves). This trend has been justified previously in Refs. [5,21] as a result of the approximation made in the calculation of the multiple scattering series. On the other hand, the reverse situation obtained here for energies 425 and 650 MeV reflects that the energies having this trend are not sufficiently high for the Glauber theory.

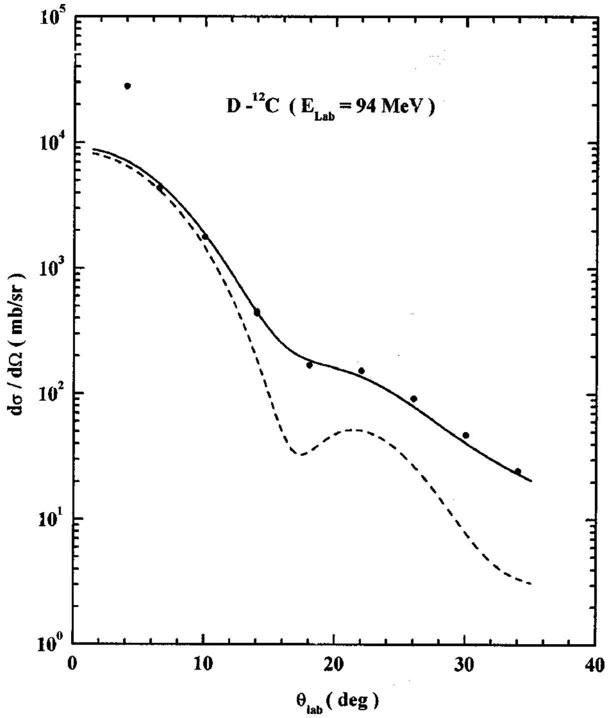

FIG. 7. Effect of phase varaition on the angular distribution of 94 MeV D-^{12}C elastic scattering. All curves are obtained from the nuclear phase-shift formula introduced in this work. The solid curve shows $\gamma^2 \neq 0$ calculated result with the value given in Table I. The dashed curve shows $\gamma^2 = 0$ calculated result. The dots are the experimental data [22].

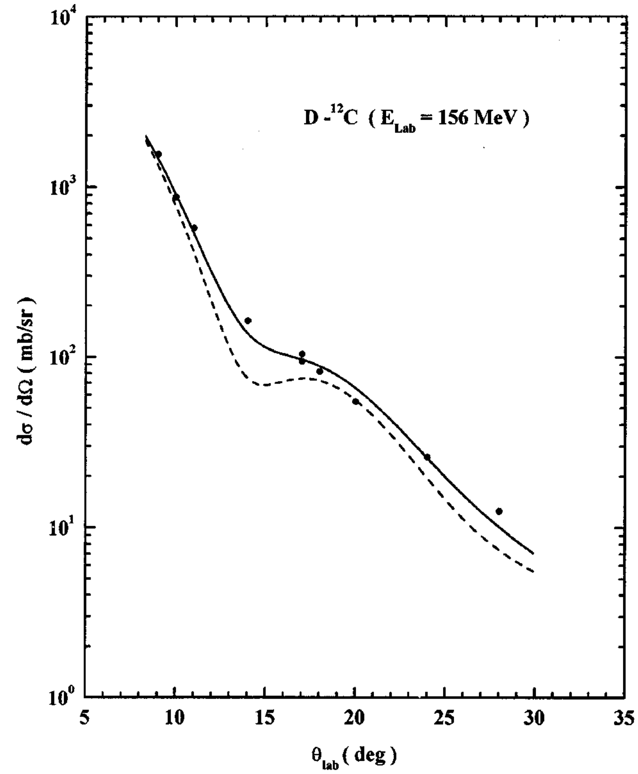

FIG. 9. Effect of phase variation on the angular distribution of 156 MeV D-^{12}C elastic scattering. The curves are labeled as in Fig. 7. The dots are the experimental data [22].

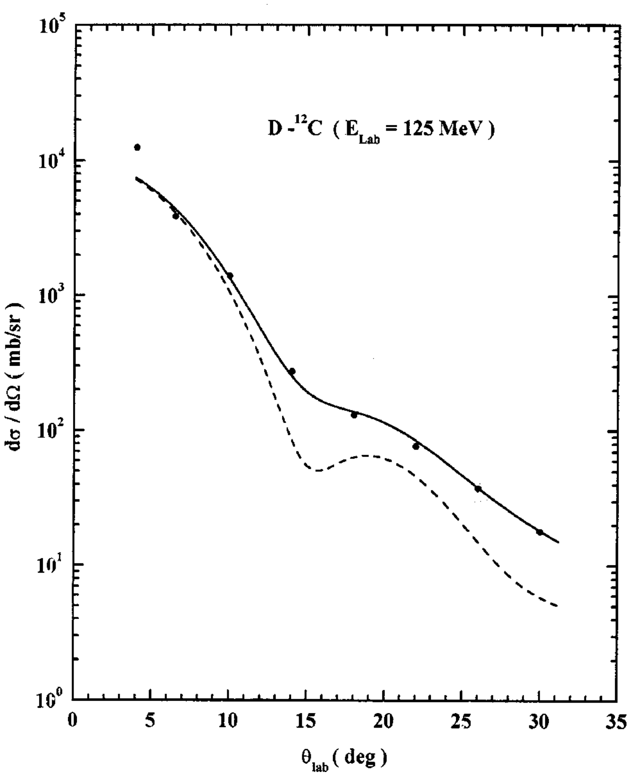

FIG. 8. Effect of phase variation on the angular distribution of 125 MeV D-^{12}C elastic scattering. The curves are labeled as in Fig. 7. The dots are the experimental data [22].

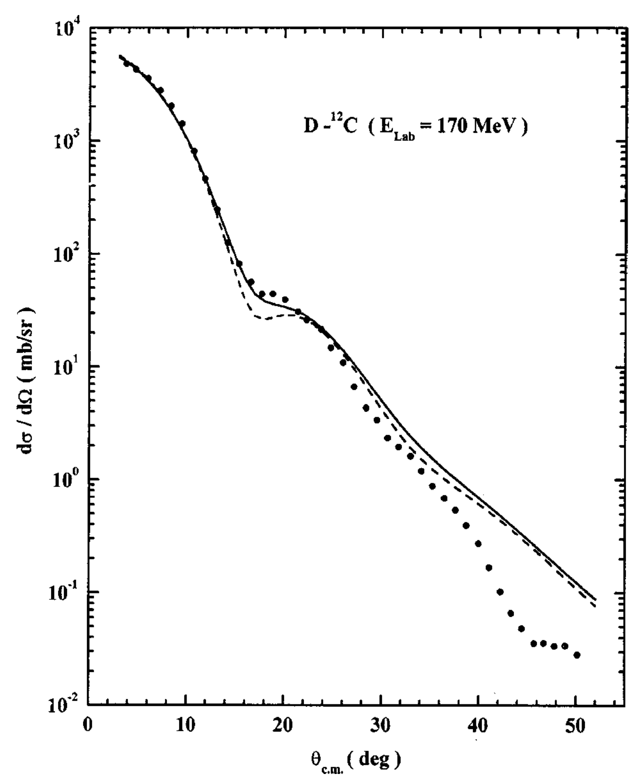

FIG. 10. Effect of phase variation on the angular distribution of 170 MeV D-^{12}C elastic scattering. The curves are labeled as in Fig. 7. The dots are the experimental data [23].

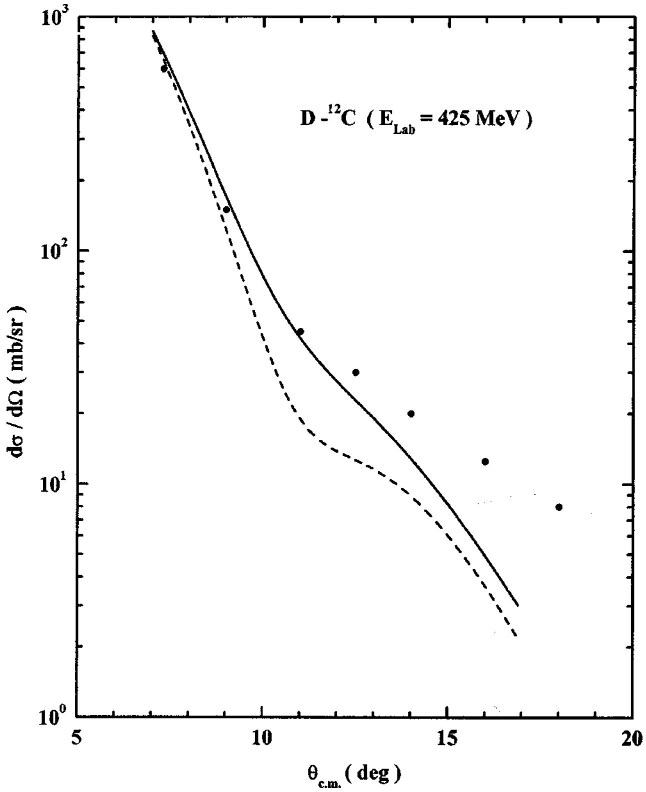

FIG. 11. Effect of phase variation on the angular distribution of 425 MeV D-^{12}C elastic scattering. The curves are labeled as in Fig. 7. The dots are the experimental data [24].

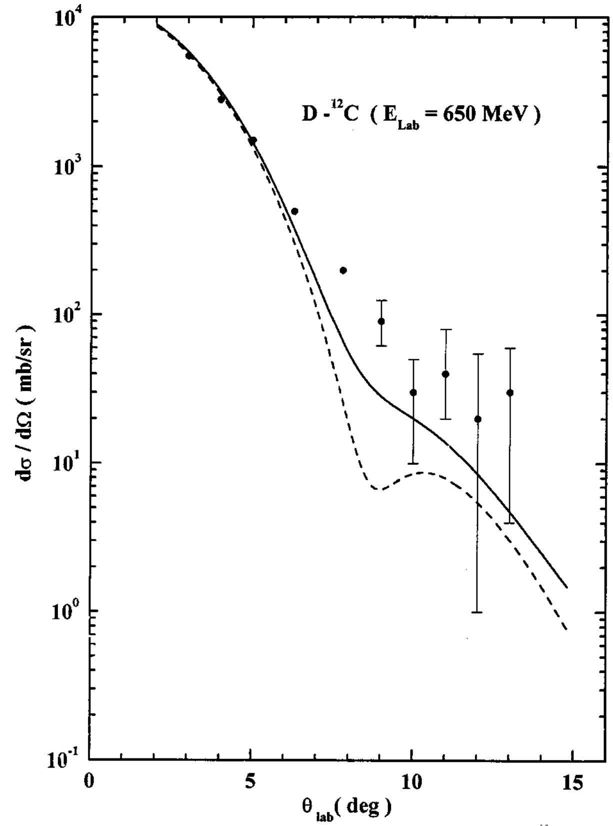

FIG. 12. Effect of phase varaition on the angular distribution of 650 MeV D-^{12}C elastic scattering. The curves are labeled as in Fig. 7. The dots are the experimental data [15].

In general, the full Glauber series calculation performed with the present analysis does not reproduce well the experimental data of the D-^{12}C angular distributions. However, better agreement may be obtained if one improves the accuracy of the values used for the *NN* amplitude parameters and the unrealistic Gaussian form used for the nuclear density.

As a matter of fact, the exact Glauber theory results of hadron-nucleus and α-α collisions have shown very close agreement with the experimental data when the phase variation is invoked in their calculations [4,15]. The authors in these attempts have treated the effect of this phase as an overall q-dependent phase factor multiplied by the *NN* scattering amplitude. Taking a nonzero value for γ^2 will introduce such a factor in the present analysis. As in their calculations, no restriction has been imposed on the values of the phase-variation parameter except the fitting with the scattering data. Just γ^2 is varied as a free parameter and the best fit of the $\gamma^2 \neq 0$ calculated results with the data is achieved at the values given in Table I. The effect of such a phase in the exact Glauber calculations of the D-^{12}C angular distribution is presented in Figs. 7–12. We can see from these figures that

TABLE III. Orbits, lengths, and Δ matrices representing the terms of the multiple scattering between M_A=1 and M_B=6 clusters. The elements $\Delta_{1j}(\nu)$ in the third column are given in the same order as the Γ's obtained by the cluster-cluster collision term $\prod_{j=1}^{6}[\Gamma_{1j}]^{\Delta_{1j}}$. Total number of collisions terms is 63.

ν	$T_1(\nu)$	$\Delta_{1j}(\nu)$
1	6	100000
2	15	110000
3	20	111000
4	15	111100
5	6	111110
6	1	111111

TABLE IV. Orbits, lengths, and Δ matrices representing the subcollisions between the constituents of the cluster-cluster scattering expressed by the term Γ_{11}. The elements $\Delta_{1\alpha,1\delta}$ in this table are given in the same order as the Γ's obtained by the subcollision term $\prod_{\alpha=1}^{2}\prod_{\delta=1}^{2}[\Gamma_{1\alpha,1\delta}]^{\Delta_{1\alpha,1\delta}}$. Total number of subcollisions terms is 15.

μ	$T_2(\mu,1)$	$\Delta_{1\alpha,1\delta}(\mu,1)$
1	4	1
2	2	5
3	2	6
4	2	7
5	4	11
6	1	15

TABLE V. Orbits, lengths, and Δ matrices representing the subcollisions between the constituents of the cluster-cluster scattering expressed by the term $\prod_{j=1}^{2}[\Gamma_{1j}]$. The elements $\Delta_{1\alpha,j\delta}$ in this table are given in the same order as the Γ's obtained by the subcollision term, $\prod_{j=1}^{2}[\prod_{\alpha=1}^{2}\prod_{\delta=1}^{2}(\Gamma_{1\alpha,j\delta})^{\Delta_{1\alpha,j\delta}}]$. Total number of subcollisions terms is 225.

μ	$T_2(\mu,2)$	$\Delta_{1\alpha,j\delta}(\mu,2)$	
1	8	1	1
2	8	1	3
3	8	1	6
4	16	1	5
5	24	3	6
6	2	6	6
7	24	1	11
8	8	6	7
9	4	5	5
10	24	5	7
11	6	6	10
12	8	6	11
13	24	1	15
14	24	7	11
15	12	6	15
16	4	5	15
17	12	7	15
18	8	11	15
19	1	15	15

consideration of the phase improves remarkably the agreement of the Glauber model predictions with the data at energies 94, 125, and 156 MeV and relatively at 170, 425, and 650 MeV. Furthermore, this phase has its strongest effect at large angles where higher orders of interference are dominant. The physical origin of the phase variation has not yet been settled, despite its impressive role in fitting various experimental data. Also, the values obtained in Table I show that the energy dependence of γ^2 is varying. It should be mentioned here that the phase-shift analyses could impose some constraints on the phase-variation parameter of the *NN* scattering amplitude generated from a particular parametrization for the *NN* potential. The analysis in Ref. [26], for example, provides values for γ^2 at energies 210 and 325 MeV/nucleon considered here. The values are obtained from the parametrization of the spin- and isospin-averaged *NN* amplitude, generated from the phase shifts, with a series of Gaussians. Particularly, the spin independent part of such a parametrization has yielded for γ^2 the value -0.49 fm^2 at 210 MeV and the value -0.75 fm^2 at 325 MeV (see Tables III and IV of Ref. [26]). In this work the Gaussian form of the *NN* amplitude that neglects the spin effects has yielded numerically higher values -0.98 and -1.05 fm^2 at the corresponding energies (see Table I). The difference occurs between the predictions of the two approaches arising from the reliability of their dependencies, namely, the inputs needed and the approximation made [20]. Unfortunately, the errors associated with these dependencies are not known in order to

TABLE VI. Orbits, lengths, and Δ matrices representing the subcollisions between the constituents of the cluster-cluster scattering expressed by the term $\prod_{j=1}^{3}[\Gamma_{1j}]$. The elements $\Delta_{1\alpha,j\delta}$ in this table are given in the same order as the Γ's obtained by the subcollision term $\prod_{j=1}^{3}[\prod_{\alpha=1}^{2}\prod_{\delta=1}^{2}(\Gamma_{1\alpha,j\delta})^{\Delta_{1\alpha,j\delta}}]$. Total number of subcollisions terms is 3375.

μ	$T_2(\mu,3)$	$\Delta_{1\alpha,j\delta}(\mu,3)$		
1	16	1	1	1
2	48	1	1	3
3	24	1	1	6
4	48	1	1	5
5	96	1	1	7
6	48	1	3	5
7	72	1	3	7
8	12	1	6	6
9	96	1	5	6
10	60	3	6	6
11	48	1	5	5
12	288	3	5	6
13	120	3	6	7
14	2	6	6	6
15	60	1	6	11
16	12	6	6	7
17	144	1	1	15
18	240	3	6	11
19	30	6	6	10
20	8	5	5	5
21	144	5	5	7
22	180	5	7	7
23	20	7	7	7
24	12	6	6	11
25	120	5	6	11
26	60	6	7	11
27	96	5	5	11
28	360	5	7	11
29	120	7	7	11
30	30	6	6	15
31	120	5	6	15
32	120	6	7	15
33	12	5	5	15
34	120	5	7	15
35	90	7	7	15
36	40	6	11	15
37	60	5	11	15
38	120	7	11	15
39	30	6	15	15
40	6	5	15	15
41	30	7	15	15
42	12	11	15	15
43	1	15	15	15

TABLE VII. Orbits, lengths, and Δ matrices representing the subcollisions between the constituents of the cluster-cluster scattering expressed by the term $\prod_{j=1}^{4}[\Gamma_{1j}]$. The elements $\Delta_{1\alpha,j\delta}$ in this table are given in the same order as the Γ's obtained by the subcollision term $\prod_{j=1}^{4}[\prod_{\alpha=1}^{2}\prod_{\delta=1}^{2}(\Gamma_{1\alpha,j\delta})^{\Delta_{1\alpha,j\delta}}]$. Total number of subcollisions terms is 50 625.

μ	$T_2(\mu,4)$	$\Delta_{1\alpha,j\delta}(\mu,4)$				μ	$T_2(\mu,4)$	$\Delta_{1\alpha,j\delta}(\mu,4)$			
1	32	1	1	1	1	42	70	7	7	7	7
2	128	1	1	1	3	43	16	6	6	6	11
3	96	1	1	3	3	44	336	1	6	6	15
4	64	1	1	1	6	45	112	6	6	7	11
5	128	1	1	1	5	46	960	1	5	6	15
6	320	1	1	1	7	47	1680	1	6	7	15
7	384	1	1	3	5	48	336	6	7	7	11
8	640	1	1	3	7	49	320	1	5	5	15
9	48	1	1	6	6	50	2880	1	5	7	15
10	320	1	1	1	11	51	3360	1	7	7	15
11	288	1	1	6	7	52	560	7	7	7	11
12	192	1	1	5	5	53	56	6	6	6	15
13	1280	1	1	5	7	54	560	1	6	11	15
14	720	1	1	7	7	55	336	6	6	7	15
15	192	1	3	5	5	56	720	5	5	6	15
16	960	1	3	5	7	57	2240	5	6	7	15
17	480	1	3	7	7	58	840	6	7	7	15
18	16	1	6	6	6	59	32	5	5	5	15
19	288	1	1	6	11	60	720	5	5	7	15
20	112	1	6	6	7	61	1680	5	7	7	15
21	640	1	1	1	15	62	560	7	7	7	15
22	1440	1	1	7	11	63	112	6	6	11	15
23	336	1	6	7	7	64	560	1	11	11	15
24	128	1	5	5	5	65	560	6	7	11	15
25	1920	1	1	3	15	66	288	1	5	15	15
26	2880	1	5	7	7	67	1680	1	7	15	15
27	560	1	7	7	7	68	1120	7	7	11	15
28	2	6	6	6	6	69	140	6	6	15	15
29	112	1	6	6	11	70	336	1	11	15	15
30	16	6	6	6	7	71	560	6	7	15	15
31	720	1	1	6	15	72	24	5	5	15	15
32	672	1	6	7	11	73	336	5	7	15	15
33	56	6	6	7	7	74	420	7	7	15	15
34	640	1	1	5	15	75	112	6	11	15	15
35	2880	1	1	7	15	76	112	1	15	15	15
36	1680	1	7	7	11	77	336	7	11	15	15
37	112	6	7	7	7	78	56	6	15	15	15
38	16	5	5	5	5	79	8	5	15	15	15
39	640	1	3	5	15	80	56	7	15	15	15
40	2160	1	3	7	15	81	16	11	15	15	15
41	1120	3	7	7	11	82	1	15	15	15	15

assess the uncertainty in these predictions. Surely, the potential model approaches are more consistent and comprehensive in restricting the *NN* parameters, but the phase-variation parameter may change substantially with more elaborate potential models. However, Franco and Yin [4] have obtained a similar difference between the two findings at 643 and 1050 MeV and Ahmed and co-workers [27] have clarified that the values of γ^2 provided by the potential model calculation are not necessarily the same as the values obtained with a phenomenological factor describing at best the small angle data. The best determination in the present analysis would be obtained when one uses realistic nuclear density,

TABLE VIII. Orbits, lengths, and Δ matrices representing the subcollisions between the constituents of the cluster-cluster scattering expressed by the term $\prod_{j=1}^{5}[\Gamma_{1j}]$. The elements $\Delta_{1\alpha,j\delta}$ in this table are given in the same order as the Γ's obtained by the subcollision term, $\prod_{j=1}^{5}[\prod_{\alpha=1}^{2}\prod_{\delta=1}^{2}(\Gamma_{1\alpha,j\delta})^{\Delta_{1\alpha,j\delta}}]$. Total number of subcollisions terms is 759 375.

μ	$T_2(\mu,5)$	$\Delta_{1\alpha,j\delta}(\mu,5)$					μ	$T_2(\mu,5)$	$\Delta_{1\alpha,j\delta}(\mu,5)$				
1	64	1	1	1	1	1	71	720	15	6	6	6	1
2	320	1	1	1	1	3	72	180	11	6	6	6	7
3	640	1	1	1	3	3	73	4480	15	5	6	6	1
4	160	6	1	1	1	1	74	5040	15	7	6	6	1
5	320	5	1	1	1	1	75	720	11	7	6	6	7
6	960	7	1	1	1	1	76	5600	15	15	1	1	1
7	1280	5	1	1	1	3	77	2240	15	11	7	1	1
8	2400	7	1	1	1	3	78	1512	15	7	7	6	1
9	960	5	1	1	3	3	79	1680	11	7	7	6	7
10	1600	7	1	1	3	3	80	960	15	5	5	5	1
11	160	6	6	1	1	1	81	1680	15	5	5	7	1
12	960	11	1	1	1	1	82	4480	15	5	7	7	1
13	1120	6	7	1	1	1	83	2520	15	7	7	7	1
14	640	5	5	1	1	1	84	2520	11	7	7	7	7
15	4800	5	7	1	1	1	85	90	15	6	6	6	6
16	3360	7	7	1	1	1	86	1680	15	11	6	6	1
17	1920	5	5	1	1	3	87	720	15	6	6	6	7
18	9600	5	7	1	1	3	88	5600	15	15	6	1	1
19	5600	7	7	1	1	3	89	1008	15	11	6	7	1
20	80	6	6	6	1	1	90	2520	15	6	6	7	7
21	1120	11	6	1	1	1	91	3360	15	15	5	1	1
22	640	6	6	7	1	1	92	2240	15	15	7	1	1
23	2400	15	1	1	1	1	93	2520	15	11	7	7	1
24	6720	11	7	1	1	1	94	5040	15	6	7	7	7
25	2240	6	7	7	1	1	95	80	15	5	5	5	5
26	640	5	5	5	1	1	96	3360	15	5	5	5	7
27	9600	5	5	7	1	1	97	1680	15	5	5	7	7
28	1680	5	7	7	1	1	98	1680	15	5	7	7	7
29	4480	7	7	7	1	1	99	3150	15	7	7	7	7
30	640	5	5	5	1	3	100	240	15	11	6	6	6
31	7200	5	5	7	1	3	101	2520	15	15	6	6	1
32	1120	5	7	7	1	3	102	1680	15	11	6	6	7
33	2800	7	7	7	1	3	103	4480	15	15	5	6	1
34	20	6	6	6	6	1	104	1260	15	15	7	6	1
35	640	11	6	6	1	1	105	5040	15	11	7	6	7
36	180	6	6	6	7	1	106	1120	15	15	5	5	1
37	3360	5	5	6	6	1	107	1344	15	15	5	7	1
38	4480	11	6	7	1	1	108	2520	15	15	7	7	1
39	720	6	6	7	7	1	109	8400	15	11	7	7	7
40	3200	15	5	1	1	1	110	420	15	15	6	6	6
41	1680	15	7	1	1	1	111	2520	15	15	11	6	1
42	1344	11	7	7	1	1	112	2520	15	15	6	6	7
43	1680	6	7	7	7	1	113	2240	15	15	15	1	1
44	320	5	5	5	5	1	114	1008	15	15	11	7	1
45	9600	15	5	1	1	3	115	6300	15	15	6	7	7
46	3360	15	7	1	1	3	116	80	15	15	5	5	5
47	2240	11	7	7	1	3	117	2240	15	15	5	5	7

TABLE VIII. (*Continued.*)

μ	$T_2(\mu,5)$	$\Delta_{1\alpha,j\delta}(\mu,5)$					μ	$T_2(\mu,5)$	$\Delta_{1\alpha,j\delta}(\mu,5)$				
48	2520	6	7	7	7	3	118	7560	15	15	5	7	7
49	2	6	6	6	6	6	119	4200	15	15	7	7	7
50	180	11	6	6	6	1	120	504	15	15	11	6	6
51	20	6	6	6	6	7	121	1680	15	15	15	6	1
52	2240	15	6	6	1	1	122	2520	15	15	11	6	7
53	1440	11	6	6	7	1	123	640	15	15	15	5	1
54	90	6	6	6	7	7	124	5040	15	15	15	7	1
55	5600	15	11	1	1	1	125	5040	15	15	11	7	7
56	1344	15	6	7	1	1	126	420	15	15	15	6	6
57	5040	11	6	7	7	1	127	720	15	15	15	11	1
58	240	6	6	7	7	7	128	1680	15	15	15	6	7
59	2400	15	5	5	1	1	129	40	15	15	15	5	5
60	2240	15	5	7	1	1	130	720	15	15	15	5	7
61	3360	15	7	7	1	1	131	1260	15	15	15	7	7
62	1008	11	7	7	7	1	132	240	15	15	15	11	6
63	420	6	7	7	7	7	133	180	15	15	15	15	1
64	32	5	5	5	5	5	134	720	15	15	15	11	7
65	2400	15	5	5	1	3	135	90	15	15	15	15	6
66	1680	15	5	7	1	3	136	10	15	15	15	15	5
67	2240	15	7	7	1	3	137	90	15	15	15	15	7
68	6300	5	7	7	7	7	138	20	15	15	15	15	11
69	252	6	7	7	7	10	139	1	15	15	15	15	15
70	20	11	6	6	6	6							

precise nucleon-nucleon amplitudes, full multiple scattering series, and coupling of inelastic channels for the reaction concerned.

Finally, one can summarize the findings of the present study as follows.

(1) The calculation of the full Glauber series of the multiple scattering between two nuclei has been simplified by introducing an approach in which the colliding nuclei are decomposed into clusters and the scattering between these clusters is classified by the permutation group method of Yin, Tan, and Chen [10]. As an example, the cluster approach introduced in this work has reduced the straightforward calculation of $(2^{24}-1)$ multiple scattering terms of the D-^{12}C collision into merely 513 equivalent orbits (the typical terms and their repetitions). In such a way, the scattering amplitude is very easy to compile and at the same time is rigorous like the formula containing all terms in Ref. [21]. In comparison with an approximation [16] to the present approach, the results show that it is necessary to include all the terms of the Glauber multiple scattering series in order to obtain accurate values for the differential cross section, especially at large momentum transfer.

(2) In general, c1assifying the terms of the multiple scattering series by Yin's method without clustering the colliding nuclei is practically limited to mass numbers less than or equal to 4 [4,11–13]. The cluster approach makes this method more practical and efficient so it can be used now to extend the full Glauber series calculations to systems with greater mass numbers. In particular, proposing a cluster picture for the composite-composite scattering gives an advantage in the following technical points. First, the classification in this picture is performed in parts depending on the order of scattering between clusters, therefore the application of Yin's method will concern only the particles contained in the colliding clusters. Second, the method of Yin in this picture catches a larger number of terms having equal contribution to the scattering amplitude and consequently obtains a smaller number of orbits than the situation where no clustering is used.

(3) The $\gamma^2 \neq 0$ calculated results presented here have shown a strong signature that the effect of the phase variation must be considered in Glauber theory calculations.

APPENDIX A

In this appendix, the detailed derivation of the analytic formula of the nuclear phase-shift function given by Eq. (17) is presented. The formula is developed by performing the integration of Eq. (4) via Eq. (16) for the nuclear density and Eq. (15) for phase-shift operator. With these ingredients the integration over z coordinates is straightforward and Eq. (4) with the remaining integral becomes

TABLE IX. Orbits, lengths, and Δ matrices representing the subcollisions between the constituents of the cluster-cluster scattering expressed by the term $\prod_{j=1}^{6}[\Gamma_{1j}]$. The elements $\Delta_{1\alpha,j\delta}$ in this table are given in the same order as the Γ's obtained by the subcollision term $\prod_{j=1}^{6}[\prod_{\alpha=1}^{2}\prod_{\delta=1}^{2}(\Gamma_{1\alpha,j\delta})^{\Delta_{1\alpha,j\delta}}]$. Total number of subcollisions terms is 11 390 625.

μ	$T_2(\mu,6)$	$\Delta_{1\alpha,j\delta}(\mu,6)$						μ	$T_2(\mu,6)$	$\Delta_{1\alpha,j\delta}(\mu,6)$					
1	1280	1	1	1	3	3	3	110	403200	15	5	5	7	7	1
2	1920	1	1	1	1	3	3	111	80640	15	5	5	5	6	3
3	768	1	1	1	1	1	3	112	2688	15	5	5	5	5	1
4	128	1	1	1	1	1	1	113	7920	11	6	7	7	7	7
5	13440	7	1	1	1	3	3	114	110880	15	6	7	7	7	1
6	7680	5	1	1	1	3	3	115	302400	15	11	7	7	1	1
7	8064	7	1	1	1	1	3	116	201600	15	15	7	1	1	1
8	3840	5	1	1	1	1	3	117	26880	15	15	5	1	1	1
9	2688	7	1	1	1	1	1	118	3960	11	6	6	7	7	7
10	768	5	1	1	1	1	1	119	47520	15	6	6	1	7	7
11	384	6	1	1	1	1	1	120	100800	15	11	6	1	1	7
12	16800	7	7	1	1	3	3	121	40320	15	15	6	1	1	1
13	26880	5	7	1	1	3	3	122	1320	11	6	6	6	7	7
14	5760	5	5	1	1	3	3	123	11880	15	6	6	6	1	7
15	26880	7	7	1	1	1	3	124	14400	15	11	6	6	1	1
16	40320	5	7	1	1	1	3	125	264	11	6	6	6	6	7
17	7680	5	5	1	1	1	3	126	1320	15	6	6	6	6	1
18	13440	7	7	1	1	1	1	127	24	11	6	6	6	6	6
19	16128	5	7	1	1	1	1	128	16632	15	7	7	7	7	7
20	1920	5	5	1	1	1	1	129	138600	15	5	7	7	7	7
21	3840	6	7	1	1	1	1	130	252000	15	15	7	7	1	3
22	2688	11	1	1	1	1	1	131	120960	15	15	5	7	1	3
23	480	6	6	1	1	1	1	132	13440	15	5	5	5	5	7
24	40320	7	7	7	1	1	3	133	192	15	5	5	5	5	5
25	134400	5	7	7	1	1	3	134	27720	15	6	7	7	7	7
26	80640	5	5	7	1	1	3	135	221760	15	11	7	7	7	1
27	7680	5	5	5	1	1	3	136	378000	15	15	7	7	1	1
28	26880	7	7	7	1	1	1	137	161280	15	15	5	7	1	1
29	80640	5	7	7	1	1	1	138	13440	15	15	5	5	1	1
30	40320	5	5	7	1	1	1	139	15840	15	6	6	7	7	7
31	2560	5	5	5	1	1	1	140	110880	15	11	6	1	7	7
32	11520	6	7	7	1	1	1	141	151200	15	15	6	1	1	7
33	26880	11	7	1	1	1	1	142	40320	15	15	11	1	1	1
34	8064	15	1	1	1	1	1	143	5940	15	6	6	6	7	7
35	2880	6	6	1	1	1	7	144	31680	15	11	6	6	1	7
36	3840	11	6	1	1	1	1	145	25200	15	15	6	6	1	1
37	320	6	6	6	1	1	1	146	1320	15	6	6	6	6	7
38	15120	7	7	7	7	1	3	147	3960	15	11	6	6	6	1
39	100800	5	7	7	7	1	3	148	132	15	6	6	6	6	6
40	134400	15	7	1	1	3	3	149	55440	15	11	7	7	7	7
41	40320	15	5	1	3	1	3	150	277200	15	15	7	7	7	1
42	1920	5	5	5	5	1	3	151	302400	15	15	5	7	7	1
43	25200	6	7	7	7	1	3	152	80640	15	15	5	5	7	1
44	161280	11	7	7	1	1	3	153	3840	15	15	5	5	5	1
45	201600	15	7	1	1	1	3	154	36960	15	11	6	7	7	7
46	53760	15	5	3	1	1	1	155	166320	15	15	6	1	7	7
47	1920	5	5	5	5	1	1	156	151200	15	15	11	1	1	7
48	14400	6	7	7	7	1	1	157	26880	15	15	15	1	1	1
49	80640	11	7	7	1	1	1	158	15840	15	11	6	6	7	7
50	80640	15	7	1	1	1	1	159	55440	15	15	6	6	1	7

TABLE IX. (*Continued.*)

μ	$T_2(\mu,6)$			$\Delta_{1\alpha,j\delta}(\mu,6)$				μ	$T_2(\mu,6)$			$\Delta_{1\alpha,j\delta}(\mu,6)$			
51	13440	15	5	1	1	1	1	160	30240	15	15	11	6	1	1
52	5400	6	6	1	1	7	7	161	3960	15	11	6	6	6	7
53	23040	11	6	1	1	1	7	162	7920	15	15	6	6	6	1
54	13440	15	6	1	1	1	1	163	440	15	11	6	6	6	6
55	1200	6	6	6	1	1	7	164	34650	15	15	7	7	7	7
56	2880	11	6	6	1	1	1	165	110880	15	15	5	7	7	7
57	120	6	6	6	6	1	1	166	75600	15	15	5	5	7	7
58	11088	6	7	7	7	7	3	167	11520	15	15	5	5	5	7
59	151200	11	7	7	7	1	3	168	240	15	15	5	5	5	5
60	403200	15	7	7	1	1	3	169	55440	15	15	6	7	7	7
61	268800	15	5	7	1	1	3	170	166320	15	15	11	7	7	1
62	40320	5	5	5	5	7	1	171	100800	15	15	15	7	1	1
63	768	5	5	5	5	5	1	172	11520	15	15	15	5	1	1
64	7920	6	7	7	7	7	1	173	27720	15	15	6	6	7	7
65	100800	11	7	7	7	1	1	174	66528	15	15	11	6	1	7
66	241920	15	7	7	1	1	1	175	25200	15	15	15	6	1	1
67	134400	15	5	7	1	1	1	176	7920	15	15	6	6	6	7
68	13440	15	5	5	1	1	1	177	11088	15	15	11	6	6	1
69	3960	6	6	7	7	7	1	178	990	15	15	6	6	6	6
70	43200	11	6	7	7	1	1	179	55440	15	15	11	7	7	7
71	80640	15	6	7	1	1	1	180	110880	15	15	15	7	7	1
72	26880	15	11	1	1	1	1	181	43200	15	15	15	5	7	1
73	1320	6	6	6	1	7	7	182	2880	15	15	15	5	5	1
74	10800	11	6	6	1	1	7	183	33264	15	15	11	6	7	7
75	11520	15	6	6	1	1	1	184	55440	15	15	15	6	7	1
76	264	6	6	6	6	1	7	185	14400	15	15	15	11	1	1
77	1200	11	6	6	6	1	1	186	11088	15	15	11	6	6	7
78	24	6	6	6	6	6	1	187	11088	15	15	15	6	6	1
79	924	7	7	7	7	7	7	188	1584	15	15	11	6	6	6
80	33264	5	7	7	7	7	7	189	18480	15	15	15	7	7	7
81	189000	15	7	7	7	1	3	190	23760	15	15	15	5	7	7
82	268800	15	5	7	7	1	3	191	5400	15	15	15	5	5	7
83	100800	15	5	5	7	1	3	192	160	15	15	15	5	5	5
84	8064	15	5	5	5	1	3	193	27720	15	15	15	6	7	7
85	64	5	5	5	5	5	5	194	31680	15	15	15	11	7	1
86	1584	6	7	7	7	7	7	195	5400	15	15	15	15	1	1
87	55440	11	7	7	7	7	1	196	11088	15	15	15	6	6	7
88	302400	15	7	7	7	1	1	197	7920	15	15	15	11	6	1
89	403200	15	5	7	7	1	1	198	1848	15	15	15	6	6	6
90	134400	15	5	5	7	1	1	199	15840	15	15	15	11	7	7
91	8064	15	5	5	5	1	1	200	11880	15	15	15	15	7	1
92	990	6	6	7	7	7	7	201	1200	15	15	15	15	5	1
93	31680	11	6	1	7	7	7	202	7920	15	15	15	11	6	7
94	151200	15	6	1	1	7	7	203	3960	15	15	15	15	6	1
95	161280	15	11	1	1	1	7	204	1584	15	15	15	11	6	6
96	33600	15	15	1	1	1	1	205	2970	15	15	15	15	7	7
97	440	6	6	6	7	7	7	206	1320	15	15	15	15	5	7
98	11880	11	6	6	1	7	7	207	60	15	15	15	15	5	5
99	43200	15	6	6	1	1	7	208	3960	15	15	15	15	6	7
100	26880	15	11	6	1	1	1	209	1320	15	15	15	15	11	1
101	132	6	6	6	6	7	7	210	990	15	15	15	15	6	6
102	2640	11	6	6	6	1	7	211	1320	15	15	15	15	11	7
103	5400	15	6	6	6	1	1	212	264	15	15	15	15	15	1

TABLE IX. (*Continued.*)

μ	$T_2(\mu,6)$	$\Delta_{1\alpha,j\delta}(\mu,6)$						μ	$T_2(\mu,6)$	$\Delta_{1\alpha,j\delta}(\mu,6)$					
104	24	6	6	6	6	6	7	213	440	15	15	15	15	11	6
105	264	11	6	6	6	6	1	214	132	15	15	15	15	15	7
106	2	6	6	6	6	6	6	215	12	15	15	15	15	15	5
107	11088	11	7	7	7	7	7	216	132	15	15	15	15	15	6
108	166320	15	7	7	7	7	1	217	24	15	15	15	15	15	11
109	504000	15	5	7	7	7	1	218	1	15	15	15	15	15	15

$$\begin{aligned}\exp[i\chi_{AB}(b)] &= \langle \Psi_A(\{\vec{S}_{i\alpha}\})\Psi_B(\{\vec{S}'_{j\delta}\})|\exp[i\chi_{AB}(\vec{b},\{\vec{S}_{i\alpha}\},\{\vec{S}'_{j\delta}\})]|\Psi_A\Psi_B\rangle\\ &= 1 + c_A c_B \sum_{\upsilon=1}^{n_1} T_1(\upsilon)\Bigg\{\sum_{\mu=1}^{n_2(\upsilon)} T_2(\mu,\upsilon)(-g)^{V_2(\mu,\upsilon)} \int \left(\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N} d\vec{S}_{i\alpha}\right)\int\left(\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N} d\vec{S}'_{j\delta}\right)\\ &\quad\times\exp\left[-\sum_{i=1}^{M_A}\sum_{\alpha=1}^{M_N}\alpha_A^2 S_{i\alpha}^2 - \sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\alpha_B^2 S'^2_{j\delta} - \sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\sum_{\alpha=1}^{M_N}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)(\vec{b}+\vec{S}_{i\alpha}-\vec{S}'_{j\delta})^2/(2,\bar{\beta})\right]\Bigg\}\\ &= 1 + c_A c_B \sum_{\upsilon=1}^{n_1} T_1(\upsilon)\left\{\sum_{\mu=1}^{n_2(\upsilon)} T_2(\mu,\upsilon)(-g)^{V_2(\mu,\upsilon)} I_{\mu,\upsilon}(b)\right\}.\end{aligned} \tag{A1}$$

From $(\vec{b}+\vec{S}_{i\alpha}-\vec{S}'_{j\delta})^2=(b_x+x_{i\alpha}-x'_{j\delta})^2+(b_y+y_{i\alpha}-y'_{j\delta})^2$, and

$$d\tau = \left(\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N}\overrightarrow{dS_{i\alpha}}\right)\left(\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N}\overrightarrow{dS'_{j\delta}}\right) = \left(\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N} dx_{i\alpha}dy_{i\alpha}\right)\left(\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N} dx'_{j\delta}dy'_{j\delta}\right),$$

the integral $I_{\mu,\upsilon}(b)$ can be separated into two similar forms in x and y coordinates as

$$I_{\mu,\upsilon}(b) = I_{\mu,\upsilon}(b_x)I_{\mu,\upsilon}(b_y), \tag{A2}$$

where, the form of x is given by

$$\begin{aligned}I_{\mu,\upsilon}(b_x) &= \int\left(\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N} dx_{i\alpha}\right)\int\left(\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N} dx'_{j\delta}\right)\exp\Bigg[-\sum_{i=1}^{M_A}\sum_{\alpha=1}^{M_N}\alpha_A^2 x_{i\alpha}^2 - \sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\alpha_B^2 x'^2_{j\delta}\\ &\quad - \sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\sum_{\alpha=1}^{M_N}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)(b_x+x_{i\alpha}-x'_{j\delta})^2/(2\bar{\beta})\Bigg].\end{aligned} \tag{A3}$$

Let $u_{j\delta}=x'_{j\delta}-b_x$, then we get

$$\begin{aligned}I_{\mu,\upsilon}(b_x) &= \int\left(\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N} du_{j\delta}\right)\exp\left[-\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\alpha_B^2(u_{j\delta}+b_x)^2 - \sum_{i=1}^{M_A}\sum_{j=1}^{M_B}\sum_{\alpha=1}^{M_N}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)u_{j\delta}^2/(2\bar{\beta})\right]\\ &\quad\times\left\{\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N}\int dx_{i\alpha}\exp\left[-\left(\alpha_A^2+\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)/(2\bar{\beta})\right)x_{i\alpha}^2+\left(\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)u_{j\delta}/\bar{\beta}\right)x_{i\alpha}\right]\right\}.\end{aligned} \tag{A4}$$

Using the integral formula

$$\int_{-\infty}^{\infty}\exp(-p^2x^2 \pm qx)dx = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p}, \quad p>0,$$

the integration in the curly bracket of Eq. (A4) is performed and we obtain

$$I_{\mu,\upsilon}(b_x)\left[\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N}\{4\pi\bar{\beta}^2 w_{ia}(\mu,\upsilon)\}^{1/2}\right]\exp\left[-\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\alpha_B^2 b_x^2\right]X(\{M_B\},\{M_N\}), \tag{A5}$$

where,

$$w_{i\alpha}(\mu,\upsilon)=\left[2\bar{\beta}\left(2\bar{\beta}\alpha_A^2+\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)\right)\right]^{-1}$$

and

$$X(\{M_B\},\{M_N\})=\int\left(\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N}du_{j\delta}\right)\exp\left[-\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}a_{j\delta,j\delta}(M_B,M_N)u_{j\delta}^2+\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N-1}\sum_{\beta=\delta+1}^{M_N}a_{j\delta,j\beta}(M_B,M_N)u_{j\delta}u_{j\beta}\right.$$
$$\left.+\sum_{j=1}^{M_B-1}\sum_{l=j+1}^{M_B}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N}a_{j\delta,l\beta}(M_B,M_N)u_{j\delta}u_{l\beta}-b_x\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}c_{j\delta}(M_B,M_N)u_{j\delta}\right].$$

The sets $\{M_B\},\{M_N\}$ marked in $X(\{M_B\},\{M_N\})$ characterize the indices of the variables we shall integrate over and the coefficients a's and c's are given by

$$a_{j\delta,j\delta}(M_B,M_N)=\alpha_B^2-\sum_{i=1}^{M_A}\sum_{a=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)w_{i\alpha}(\mu,\upsilon)+\sum_{i=1}^{M_A}\sum_{\alpha=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)/2\bar{\beta},$$

$$a_{j\delta,j\beta}(M_B,M_N)=2\sum_{i=1}^{M_A}\sum_{\alpha=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)\Delta_{i\alpha,j\beta}(\mu,\upsilon)w_{i\alpha}(\mu,\upsilon),$$

$$a_{j\delta,l\beta}(M_B,M_N)=2\sum_{i=1}^{M_A}\sum_{\alpha=1}^{M_N}\Delta_{i\alpha,j\delta}(\mu,\upsilon)\Delta_{i\alpha,l\beta}(\mu,\upsilon)w_{i\alpha}(\mu,\upsilon),\qquad\text{(A6)}$$

$$c_{j\delta}(M_B,M_N)=2\alpha_B^2.$$

Now, $X(\{M_B\},\{M_N\})$ can be evaluated recursively by the following procedure: First, separate the integration over the variables of index M_B as

$$X(\{M_B\},\{M_N\})=\int\left(\prod_{\beta=1}^{M_N}du_{1\beta}\right)\left(\prod_{\beta=1}^{M_N}du_{2\beta}\right)\cdots\left(\prod_{\beta=1}^{M_N}du_{(M_B-1)\beta}\right)\exp\left[-\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N}a_{j\delta,j\delta}(M_B,M_N)u_{j\delta}^2\right.$$
$$+\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N-1}\sum_{\beta=\delta+1}^{M_N}a_{j\delta,j\beta}(M_B,M_N)u_{j\delta}u_{j\beta}+\sum_{j=1}^{M_B-2}\sum_{l=j+1}^{M_B-1}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N}a_{j\delta,l\beta}(M_B,M_N)u_{j\delta}u_{l\beta}$$
$$\left.-b_x\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N}c_{j\delta}(M_B,M_N)u_{j\delta}\right]\int\left(\prod_{\beta=1}^{M_N}du_{M_B\beta}\right)\exp\left[-\sum_{\delta=1}^{M_N}a_{M_B\delta,M_B\delta}(M_B,M_N)u_{M_B\delta}^2\right.$$
$$\left.+\sum_{\delta=1}^{M_N-1}\sum_{\beta=\delta+1}^{M_N}a_{M_B\delta,M_B\beta}(M_B,M_N)u_{M_B\delta}u_{M_B\beta}+\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N}a_{j\delta,M_B\beta}(M_B,M_N)u_{j\delta}u_{M_B\beta}-b_x\sum_{\delta=1}^{M_N}C_{M_B\delta}(M_B,M_N)u_{M_B\delta}\right],$$

$$X(\{M_B\},\{M_N\})=X(\{M_B-1\},\{M_N\})X(M_B,\{M_N\}).\qquad\text{(A7)}$$

$X(M_B,\{M_N\})$ can be expressed also as

$$X(M_B,\{M_N\})=\int\left(\prod_{\delta=1}^{M_N-1}du_{M_B\delta}\right)\exp\left[-\sum_{\delta=1}^{M_N-1}a_{M_B\delta,M_B\delta}(M_B,M_N)u_{M_B\delta}^2+\sum_{\delta=1}^{M_N-2}\sum_{\beta=\delta+1}^{M_N-1}a_{M_B\delta,M_B\beta}(M_B,M_N)u_{M_B\delta}u_{M_B\beta}\right.$$
$$\left.+\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N-1}a_{j\delta,M_B\beta}(M_B,M_N)u_{j\delta}u_{M_B\beta}-b_x\sum_{\delta=1}^{M_N-1}c_{M_B\delta}(M_B,M_N)u_{M_B\beta}\right]$$
$$\times\int du_{M_BM_N}\exp\left[-a_{M_BM_N,M_BM_N}(M_B,M_N)u_{M_BM_N}^2+\left(\sum_{\delta=1}^{M_N-1}a_{M_B\delta,M_BM_N}(M_B,M_N)u_{M_B\delta}\right.\right.$$
$$\left.\left.+\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N}a_{j\delta,M_BM_N}(M_B,M_N)u_{j\delta}-b_xc_{M_BM_N}(M_B,M_N)\right)u_{M_BM_N}\right].\qquad\text{(A8)}$$

Integrating over the variable $u_{M_BM_N}$in Eq. (A8), we get

$$X(M_B,\{M_N\}) = \left(\frac{\pi}{a_{M_BM_N,M_BM_N}(M_B,M_N)}\right)^{1/2}\exp\left(\frac{c^2_{M_BM_N}(M_B,M_N)}{4a_{M_BM_N,M_BM_N}(M_B,M_N)}b_x^2\right)X(M_B,\{M_N-1\}), \tag{A9}$$

where the integration over the remaining variables of index M_B takes the form

$$X(M_B,\{M_N-1\}) = \int\left(\prod_{\delta=1}^{M_N-1} du_{M_BM_\delta}\right)\exp\Bigg[-\sum_{\delta=1}^{M_N-1} a_{M_B\delta,M_B\delta}(M_B,M_N-1)u^2_{M_B\delta} + \sum_{\delta=1}^{M_N-2}\sum_{\beta=\delta+1}^{M_N-1} a_{M_B\delta,M_B\beta}(M_B,M_N-1)u_{M_B\delta}u_{M_B\beta}$$
$$+\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N-1} a_{j\delta,M_B\beta}(M_B,M_N-1)u_{j\delta}u_{M_B\beta} - b_x\sum_{\delta=1}^{M_N-1} c_{M_B\delta}(M_B,M_N-1)u_{M_B\delta}\Bigg]$$

with coefficients given by

$$a_{M_B\delta,M_B\delta}(M_B,M_N-1) = a_{M_B\delta,M_B\delta}(M_B,M_N) - [a^2_{M_B\delta,M_BM_N}(M_B,M_N)/4a_{M_BM_N,M_BM_N}(M_B,M_N)],$$

$$a_{M_B\delta,M_B\beta}(M_B,M_N-1) = a_{M_B\delta,M_B\beta}(M_B,M_N) + [a_{M_B\delta,M_BM_N}(M_B,M_N)a_{M_B\beta,M_BM_N}(M_B,M_N)/2a_{M_BM_N,M_BM_N}(M_B,M_N)],$$

$$a_{j\delta,M_B\beta}(M_B,M_N-1) = a_{j\delta,M_B\beta}(M_B,M_N) + [a_{j\delta,M_BM_N}(M_B,M_N)a_{M_B\beta,M_BM_N}(M_B,M_N)/2a_{M_BM_N,M_BM_N}(M_B,M_N)],$$

$$c_{M_B\delta}(M_B,M_N-1) = c_{M_B\delta}(M_B,M_N) + [c_{M_BM_N}(M_B,M_N)a_{M_B\delta,M_BM_N}(M_B,M_N)/2a_{M_BM_N,M_BM_N}(M_B,M_N)]. \tag{A10}$$

Using Eqs. (A6) and (A10), we can deduce recursively all the coefficients of the different $(M_B,\{M_N-1\})$ multiple integrals. The solution of $X(M_B,\{M_N\})$ can be obtained in terms of these coefficients as

$$X(M_B,1) = \int du_{M_B1}\exp\left[-a_{M_B1,M_B1}(M_B,1)u^2_{M_B1} + \left(\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N} a_{j\delta,M_B1}(M_B,1)u_{j\delta} - b_xc_{M_B1}(M_B,1)\right)u_{M_B1}\right]$$
$$= \left(\frac{\pi}{a_{M_B1,M_B1}(M_B,1)}\right)^{1/2}\exp\left(\frac{c^2_{M_B1}(M_B,1)}{4a_{M_B1,M_B1}(M_B,1)}b_x^2\right)\exp\Bigg[\Bigg\{\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N} a^2_{j\delta,M_B1}(M_B,1)u^2_{j\delta}$$
$$+2\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N-1}\sum_{\beta=\delta+1}^{M_N} a_{j\delta,M_B1}(M_B,1)a_{j\beta,M_B1}(M_B,1)u_{j\delta}u_{j\beta} + 2\sum_{j=1}^{M_B-2}\sum_{l=j+1}^{M_B-1}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N} a_{j\delta,M_B1}(M_B,1)a_{l\beta,M_B1}(M_B,1)u_{j\delta}u_{l\beta}$$
$$-2b_xc_{M_B1}(M_B,1)\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N} a_{j\delta,M_B1}(M_B,1)u_{j\delta}\Bigg\}\Bigg/4a_{M_B1,M_B1}(M_B,1)\Bigg].$$

Therefore, $X(M_B,\{M_N\})$ can be determined by

$$X(M_B,\{M_N\}) = \prod_{\delta=1}^{M_N} X(M_B,\delta). \tag{A11}$$

Substituting with the result of Eq. (A11) into Eq. (A7), it gives

$$X(\{M_B\},\{M_N\}) = \left[\prod_{\delta=1}^{M_N}\left(\frac{\pi}{a_{M_B\delta,M_B\delta}(M_B,\delta)}\right)^{1/2}\right]\exp\left[\sum_{\delta=1}^{M_N}\frac{c^2_{M_B\delta}(M_B,\delta)b_x^2}{4a_{M_B\delta,M_B\delta}(M_B,\delta)}\right]X(\{M_B-1\},\{M_N\}), \tag{A12}$$

where

$$X(\{M_B-1\},\{M_N\}) = \int\left(\prod_{\beta=1}^{M_N} du_{1\beta}\right)\left(\prod_{\beta=1}^{M_N} du_{2\beta}\right)\cdots\left(\prod_{\beta=1}^{M_N} du_{(M_B-1)\beta}\right)\exp\Bigg[-\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N} a_{j\delta,j\delta}(M_B-1,M_N)u^2_{j\delta}$$
$$+\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_{N-1}}\sum_{\beta=\delta+1}^{M_N} a_{j\delta,j\beta}(M_B-1,M_N)u_{j\delta}u_{j\beta} + \sum_{j=1}^{M_B-2}\sum_{l=j+1}^{M_B-1}\sum_{\delta=1}^{M_N}\sum_{\beta=1}^{M_N} a_{j\delta,l\beta}(M_B-1,M_N)u_{j\delta}u_{l\beta}$$
$$-b_x\sum_{j=1}^{M_B-1}\sum_{\delta=1}^{M_N} c_{j\delta}(M_B-1,M_N)u_{j\delta}\Bigg]$$

with new coefficients given by

$$a_{j\delta,j\delta}(M_B-1,M_N)=a_{j\delta,j\delta}(M_B,M_N)-\sum_{k=1}^{M_N}[a^2_{j\delta,M_Bk}(M_B,k)/4a_{M_Bk,M_Bk}(M_B,k)],$$

$$a_{j\delta,j\beta}(M_B-1,M_N)=a_{j\delta,j\beta}(M_B,M_N)+\sum_{k=1}^{M_N}[a_{j\delta,M_Bk}(M_B,k)a_{j\beta,M_Bk}(M_B,k)/2a_{M_Bk,M_Bk}(M_B,k)],$$

$$a_{j\delta,l\beta}(M_B-1,M_N)=a_{j\delta,l\beta}(M_B,M_N)+\sum_{k=1}^{M_N}[a_{j\delta,M_Bk}(M_B,k)a_{l\beta,M_Bk}(M_B,k)/2a_{M_Bk,M_Bk}(M_B,k)],$$

$$c_{j\delta}(M_B-1,M_N)=c_{j\delta}(M_B,M_N)+\sum_{k=1}^{M_N}[c_{M_Bk}(M_B,k)a_{j\delta,M_Bk}(M_B,k)/2a_{M_Bk,M_Bk}(M_B,k)]. \tag{A13}$$

Thus, repeating the above steps for M_B-1 variables and the lower ones we obtain

$$X(\{M_B\},\{M_N\})=\left[\prod_{l=1}^{M_B}\prod_{\delta=1}^{M_N}\left(\frac{\pi}{a_{j\delta,j\delta}(j,\delta)}\right)^{1/2}\right]\exp\left[\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\frac{c^2_{j\delta}(j,\delta)b_x^2}{4a_{j\delta,j\delta}(j,\delta)}\right]. \tag{A14}$$

Substituting Eq. (A14) into Eq. (A5), $I_{\mu,\upsilon}(b_x)$ and similarly $I_{\mu,\upsilon}(b_y)$ are obtained and then Eq. (A2) gives

$$\mathrm{I}_{\mu,\upsilon}(b)=\left[\prod_{i=1}^{M_A}\prod_{\alpha=1}^{M_N}\{4\pi\bar{\beta}^2w_{i\alpha}(\mu,\upsilon)\}\right]\left[\prod_{j=1}^{M_B}\prod_{\delta=1}^{M_N}\left(\frac{\pi}{a_{j\delta,j\delta}(j,\delta)}\right)\right]\times\exp\left[-\left(\sum_{j=1}^{M_B}\sum_{\delta=1}^{M_N}\alpha_B^2-\frac{c^2_{j\delta}(j,\delta)}{4a_{j\delta,j\delta}(j,\delta)}\right)b^2\right]. \tag{A15}$$

Thus, applying Eq. (A15) in Eq. (A1) the formula given by Eq. (17) is obtained.

APPENDIX B

The D-^{12}C scattering is assumed to have a cluster structure with M_A=1 cluster in nucleus A, M_B=6 clusters in nucleus B, and M_N=2 nucleons in each cluster. Using the permutation method introduced in Ref. [20], the multiple scattering terms representing the collisions between these clusters and those representing the subcollisions between their constituent nucleons have been classified into sets; each set contains the terms that have equal contribution to the scattering amplitude. All terms in each set are accounted for by one of them as a typical term (referred to as an orbit) and their number (called the length of this orbit). A set of (0, 1) matrices is used to express the typical terms (they are called Δ matrices for short). The orbits, lengths, and Δ matrices representing the full Glauber series of the D-^{12}C scattering are tabulated in this appendix. They are obtained by enumerating and investigating the possible cluster-cluster collisions and the possible subcollisions proceed between the nucleons involved in each cluster-cluster orbit. Trails are now undertaken by the author to translate the details of this procedure into a computer program. Table III contains the orbits, lengths, and Δ matrices representing the possible collisions between M_A=1 and M_B=6 clusters. ν in the first column represents the serial index of the orbit, and $T_1(\nu)$ in the second column represents the length of this orbit. $\Delta_{1j}(\nu)$ refers to the elements of the Δ matrices representing the cluster-cluster orbits and the six binary numbers in the third column are the values of the elements $\Delta_{1j}(\nu), j=1,2,\ldots,6$, respectively.

Tables IV–IX exhibit the orbits, lengths, and Δ matrices representing the subcollisions resulting from the cluster-cluster collisions given in Table III. In these tables, μ represents the serial index of the subcollision orbit due to the collision orbit ν. $T_2(\mu,\nu)$ represents the length of the subcollision orbit (μ,ν). The elements of the Δ matrices representing the subcollisions orbits are $\Delta_{i\alpha,j\delta}$. In the $\Delta_{1\alpha,j\delta}(\mu,\upsilon)$ columns of these, the corresponding binary numbers are abbreviated with code numbers referring to their structure

TABLE X. The sets of binary numbers represented by the code numbers given in the $\Delta_{1\alpha,j\delta}(\mu,\upsilon)$ columns of the Tables IV–IX.

Code number	Corresponding set of binary numbers			
1	1	0	0	0
2	0	1	0	0
3	0	0	1	0
4	0	0	0	1
5	1	0	1	0
6	1	1	0	0
7	1	0	0	1
8	0	1	1	0
9	0	1	0	1
10	0	0	1	1
11	1	1	1	0
12	1	0	1	1
13	1	1	0	1
14	0	1	1	1
15	1	1	1	1

(the sets of binary numbers represented by these codes are shown in Table X). The binary numbers corresponding to the elements $\Delta_{i\alpha,j\delta}(\mu,\upsilon)$ can be obtained from these codes as follows: The first code number represents the values of the elements $\Delta_{11,j1}$, $\Delta_{11,j2}$, $\Delta_{12,j1}$, $\Delta_{12,j2}$, $j=1$, respectively, as in shown Table IV; the second number represents the Δ's of $j=2$, as in shown Table V;…, the six code represents the Δ's of $j=6$, as shown in Table IX.

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