

Dynamical and invariant supersymmetry in the fermion pairing problem

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We argue that fermion-boson mapping techniques represent a natural tool for studying many-body supersymmetry in fermionic systems with pairing. In particular, using the generalized Dyson mapping of a many-level fermion superalgebra with the $SU(2)$ type of pairing we investigate two kinds of supersymmetry connecting excitations in the systems with even and odd particle numbers: dynamical supersymmetry, which ensures a unified classification of states for both even and odd populations, and invariant supersymmetry with actual degeneracies of states within the same supermultiplet. Dynamical supersymmetries based on the dynamical algebra of the type $U(K/2\Omega)$ (where K and 2Ω denote the number of fermion-pair and single-fermion states, respectively) naturally arise in the bosonized description of the system. Conditions for invariant supersymmetry are determined in a restricted case of bilinear supercharge operators.

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I. INTRODUCTION

Supersymmetry (SUSY) is commonly known as a hypothetical algebraic scheme in quantum field theory that unifies internal and space-time symmetries and predicts elementary particles of matter occurring in boson-fermion doublets [1]. As shown by Witten [2], the same scheme can be applied in nonrelativistic quantum mechanics, yielding a generalized dynamical algebra (superalgebra) which leads to an analogous doublet structure (except for the unique ground state) in the spectra of relevant quantum systems. In the simplest case, the SUSY Hamiltonian $H = a^\dagger a + b^\dagger b$ is just the anticommutator of supercharges $Q = ba^\dagger$ and $Q^\dagger = ab^\dagger$ that change bosonic excitations (b^\dagger) into fermionic ones (a^\dagger) and vice versa, so that the excited states $|n_a, n_b\rangle$ (where n_a and n_b , respectively, are numbers of fermionic and bosonic quanta) exhibit the characteristic twofold degeneracy $H|0, n_b\rangle = H|1, n_b - 1\rangle$, coupling bosonlike and fermionlike excitations.

Applications of SUSY quantum mechanics soon followed. Analytically solvable and isospectral sets of potentials were constructed on the basis of SUSY (see Refs. [3,4] and references therein) and some of these potentials were found relevant for experimental spectroscopic data of certain atoms and ions. Thus a (approximate) manifestation of phenomenological supersymmetry was established in atomic physics [5,6]. Methods based on the SUSY formalism were also developed in random matrix theory and applied to systems that exhibit signatures of quantum chaos [7].

In nuclear physics, the concept of supersymmetry found a natural application (see, e.g., Refs. [8–12]) in the framework of the interacting boson-fermion model (IBFM) [13]. The embedding of the IBFM dynamical algebra of the type $U_B(K) \otimes U_F(2\Omega)$ [formed by K^2 bosonic generators $b_i^\dagger b_j$ and by $(2\Omega)^2$ fermionic generators $a_i^\dagger a_l$, where $i, j = 1, \dots, K$ and $k, l = 1, \dots, 2\Omega$ enumerate single-boson and -fermion states,

respectively] into the dynamical superalgebra $U(K/2\Omega)$ (with the mixed generators $b_i^\dagger a_l$ and $a_k^\dagger b_j$ added) makes it possible to simultaneously describe low-lying spectra of doublets [8–10,12] or quartets [11] of neighboring even and odd nuclei. (Note that in this paper direct products are used also for algebras, although rigorously we should speak about a direct sum of generators associated with the corresponding product of groups.)

It should be stressed that nuclear supersymmetry, introduced by Iachello [8] already in 1980, was historically the first application of the SUSY ideas in nonrelativistic physics. Their most detailed verification in nature up to date—in the recent experimental work by Metz *et al.* [14]—opened up further questions in the SUSY many-body physics. Unfortunately, discussions of supersymmetry on the phenomenological nuclear-structure level, and its relation to the notion of SUSY quantum mechanics, have not always clarified the distinction between them, nor focused on the microscopic basis of nuclear SUSY. It is therefore not totally surprising to find that somewhat negative opinions such as the following one by 't Hooft [15] have been voiced: “At first sight, the fact that supersymmetric patterns were discovered in nuclear physics has little to do with the question of supersymmetry among elementary particles, but it may indicate that, as the spectrum of particles is getting more and more complex, some supersymmetric patterns might easily arise, even if there is no ‘fundamental’ reason for their existence.”

Of particular interest in phenomenological SUSY studies are so-called *dynamical supersymmetries*, i.e., situations when the many-body Hamiltonian H can be written solely in terms of the Casimir invariants $C(\mathcal{A}_i)$ of (super)algebras in a particular chain $\mathcal{D} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$ decomposing the dynamical superalgebra \mathcal{D} . Nuclear SUSY focuses on various dynamical supersymmetries associated with $\mathcal{D} \equiv U(K/2\Omega)$ and on the transitions between them. Experimentally relevant dynamical SUSY chains always rely on an immediate decomposition

$$U(K/2\Omega) \supset U_B(K) \otimes U_F(2\Omega) \supset \dots \quad (1)$$

of the dynamical superalgebra into a product of the corresponding bosonic and fermionic algebras. For instance,

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the dynamical supersymmetry described in Refs. [11,14] is based on the decomposition $U_\nu(6/12) \otimes U_\pi(6/4) \supset U_{B\nu}(6) \otimes U_{B\pi}(6) \otimes U_{F\nu}(12) \otimes U_{F\pi}(4) \supset \dots$, where ν and π , respectively, stand for neutron and proton realizations of the above (super)algebras. This implies that the achieved SUSY description does not really go beyond an application of the same IBFM Hamiltonian (with fixed interaction strengths) to a quartet of neighboring even-even, even-odd, odd-even, and odd-odd nuclei. Indeed, since the dynamical-superalgebra irreps with more than one fermion in a given nucleus are highly excited, and thus do not mix with the low-energy spectrum, the SUSY Casimir invariants can be skipped as they only yield the same additive constants to level energies in all four nuclei considered [11,14]. The remaining Casimir invariants (of the embedded algebras) are precisely those that appear in the standard IBFM description.

As recently demonstrated in Ref. [16], any collective superalgebra of fermion-pair and single-fermion operators, linked to a *microscopic* perspective of the IBFM, can be naturally embedded into the phenomenological-type boson-fermionic dynamical superalgebra $U(K/2\Omega)$. Here K and 2Ω now denote the number of pair- and single-fermion operators, respectively, contained in the fermionic collective superalgebra. Moreover, the same analysis leads to the conclusion that all microscopically relevant dynamical supersymmetries must *separately* conserve the numbers of bosons and fermions, so that they are always of form (1) and should then only be referred to as dynamical boson-fermion symmetries. These symmetries and the transitions between them are extensions of the IBFM description, rather than what would primarily be considered a genuine realization of supersymmetry in nuclei (see also the discussion in Ref. [12] and the elaboration below).

An interesting, though so far mostly hypothetical possibility was, nevertheless, considered by Jolos and von Brentano [17]. It is based on the requirement that supercharges Q_i , generating the odd sector of a boson-fermion superalgebra $\mathcal{I} \subset U(K/2\Omega)$, together with generators of the even sector, commute with the Hamiltonian, and thus \mathcal{I} forms a superalgebra of *invariant supersymmetry* of the system. This represents principally the same kind of symmetry as the one introduced by the SUSY quantum mechanics, although the form of the Hamiltonian can be more general than simply the anticommutator of supercharges discussed above. It should be stressed that the bosons considered in this work are introduced on a purely phenomenological level.

The distinction between the above two SUSY schemes is similar to the difference between invariant and dynamical symmetries in standard quantum mechanics: in the former case, all generators of the given symmetry algebra commute with the Hamiltonian, while in the latter this is satisfied only for Casimir operators of a certain chain of subalgebras (the last algebra in the chain thus represents invariant symmetry). Dynamical supersymmetry, as presented in nuclear data, implies that all (low-energy) states in the neighboring even and odd systems are described by a single energy formula and completely classified by SUSY quantum numbers, but these states are not necessarily degenerate. (One can also deal with

nuclei along transitions between two or more dynamical supersymmetries [12]; then the integrability is lost but the simultaneous description of even and odd neighbors with generally nondegenerate states remains valid.) The invariant supersymmetry, on the other hand, does not fully classify the states, but results in actual degeneracy (at least on a relative scale) of a certain subset of states in the even-odd doublet, which has not been observed yet. Needless to say, that in field theory, as well, the SUSY scenario naturally assumes breaking of the supersymmetry between elementary bosons and fermions, as large mass differences for particles within the same supermultiplet must be accommodated.

In this work, we extend the approach of Ref. [16] to study microscopic conditions for supersymmetric schemes in fermionic many-body systems. Our present discussion aims to clarify that while dynamical boson-fermion symmetries and transitions between them seem to be firmly established, and in the final instance based on the nuclear interaction in situations where some collective pairs are favored (together with the utility of boson-fermion mappings, see Sec. II), the direct analog in many-body systems of SUSY quantum mechanics (invariant supersymmetry) still requires clarification and may indeed be difficult to realize. This is explicitly demonstrated in Sec. III. Again the precise nature of the interaction on the fermionic level will of course turn out to be crucial for the appearance of invariant SUSY.

II. BOSONIZED FORMS OF THE PAIRING SYSTEM

A. Pairing Hamiltonian

In this section we present a bosonized form of a many-level fermionic system with the $SU(2)$ type of pairing, a standard interaction to capture the essence of superconductivity in solids and nuclei [18]. We will deal with a set of single-particle states, enumerated by $k=1, \dots, \Omega$, and the respective time-reversal conjugate states, denoted by $\bar{k}=1, \dots, \bar{\Omega}$, occupied by an arbitrary number $N_F \in [0, 2\Omega]$ of fermions. The fermion creation and annihilation operators corresponding to individual states are $a_k^\dagger, a_{\bar{k}}^\dagger$ and $a_k, a_{\bar{k}}$. The Hamiltonian

$$\begin{aligned} H_F &= \sum_k E_k (a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}) - \sum_{k,l} G_{kl} a_k^\dagger a_{\bar{l}}^\dagger a_l a_{\bar{k}} \\ &= \sum_k E_k N_k - \sum_{k,l} G_{kl} B_k^\dagger B_l \end{aligned} \quad (2)$$

contains as free parameters single-particle energies E_k (states k and \bar{k} are degenerate due to the time-reversal invariance) and strengths of the pairing interaction $G_{kl} = G_{\bar{l}k}^*$. We can anticipate that the pairing interaction acts only between fermions in a certain subset $\mathcal{P} \equiv \{k_1, k_2, \dots, k_\Xi\}$ (with all k_i 's mutually different, $\Xi \leq \Omega$) of states—most likely states in some interval around the Fermi energy—so that $G_{kl} \neq 0$ only for $k, l \in \mathcal{P}$.

The operators

$$B_k^\dagger = a_{\bar{k}}^\dagger a_k^\dagger \quad (3)$$

and $B_k \equiv (B_k^\dagger)^\dagger = a_k a_{\bar{k}}$ create and annihilate, respectively, a

pair of fermions (*bifermion*) on the shell k (thus there are Ω such pairs), and together with the k -shell fermion number operator $N_k = a_k^\dagger a_k + a_{\bar{k}}^\dagger a_{\bar{k}}$ form the SU(2) algebra associated with each shell; $[B_k^\dagger, B_k] = N_k - 1$. Thus the Hamiltonian (2) leads to the dynamical algebra $\otimes_k \text{SU}(2)_k$.

For special choices of E_k and G_{kl} , the Hamiltonian can be rewritten via a smaller number ($K < \Omega$) of pairs,

$$C_\mu^\dagger = \sum_k \chi_{\mu k} a_{\bar{k}}^\dagger a_k^\dagger \quad (4)$$

and $C_\mu \equiv (C_\mu^\dagger)^\dagger$ ($\mu = 1, \dots, K$), with the coefficients $\chi_{\mu k}$ satisfying the normalization constraint $\sum_k \chi_{\mu k} \chi_{\nu k}^* = \delta_{\mu\nu}$. The condition for the Hamiltonian reads as $H_F = H_F^0$, where

$$\begin{aligned} H_F^0 &= \text{const} + \sum_{\mu, \nu} U_{\mu\nu} [C_\mu^\dagger, C_\nu] + \sum_{\mu, \nu} V_{\mu\nu} C_\mu^\dagger C_\nu \\ &= \text{const}' + \sum_k \left(\sum_{\mu, \nu} U_{\mu\nu} \chi_{\mu k} \chi_{\nu k}^* \right) N_k \\ &\quad + \sum_{k, l} \left(\sum_{\mu, \nu} V_{\mu\nu} \chi_{\mu k} \chi_{\nu l}^* \right) B_k^\dagger B_l \end{aligned} \quad (5)$$

($U_{\mu\nu} = U_{\nu\mu}^*$, $V_{\mu\nu} = V_{\nu\mu}^*$ are arbitrary interaction constants). Since creation and annihilation operators of *any* set of fermion pairs, together with their commutators, close under commutation, the dynamical algebra generating the Hamiltonian in Eq. (5) can be identified with the algebra associated with pairs (4) instead of those in Eq. (3). We call the C bifermions in Eq. (4) *collective*. Note that they couple fermions into pairs in entangled single-particle states, in contrast to the noncollective B bifermions (3) whose wave functions are separable (except, of course, unavoidable entanglement due to the antisymmetrization). Note that Eq. (4) represents the most general ansatz for the collective pairs that conserves the form of the Hamiltonian (2).

The possibility to rewrite the pairing Hamiltonian (2) via a collective algebra with a small number of C bifermions is not very exceptional. In general, it requires to satisfy a set of equations for unknown coefficients $\chi_{\mu k}$ and for interaction parameters $U_{\mu\nu}, V_{\mu\nu}$, cf. Eqs. (2) and (5). Dimensionality considerations suggest that some solutions should typically exist if $K/\Omega \geq \sqrt{1 + 1/\Omega + (\Xi/\Omega)^2} - 1$, which for $\Xi \ll \Omega$ indeed leads to $K \ll \Omega$. Let us stress here the importance of the requirement that also single-particle terms in H_F , not only the interactions, are expressed via the collective algebra, see Eq. (5). For instance, the constancy of interaction strengths in Eq. (2), $G_{kl} = G$, would not imply that the simple SU(2) collective algebra generated by the single bifermion $C^\dagger = \sum_{k \in \mathcal{P}} a_k^\dagger a_{\bar{k}}^\dagger / \sqrt{\Xi}$ is the dynamical algebra of the problem, unless all the single-particle states subject to interactions are degenerate, i.e., $E_k = E$ for $k \in \mathcal{P}$:

$$H_F^0 = \sum_{k \in \mathcal{P}} E_k N_k + E N_{\mathcal{P}} - G \Xi C^\dagger C, \quad (6)$$

where $N_{\mathcal{P}} = \sum_{k \in \mathcal{P}} N_k$.

Another example of the use of a collective algebra concerns a system of fermions on K single- j shells, j

$= j_1, j_2, \dots, j_K$, with a Hamiltonian expressed just via interactions of $J=0$ pairs on individual shells, $S_\mu^\dagger = (a_{j_\mu}^\dagger a_{j_\mu}^\dagger)^{(0)}$:

$$H_F^0 = \sum_\mu E_\mu N_\mu - \sum_{\mu, \nu} V_{\mu\nu} S_\mu^\dagger S_\nu \quad (7)$$

(E_μ are the j -shell energies, N_μ the corresponding occupation-number operators, and $V_{\mu\nu}$ the interaction strengths). Again, the Hamiltonian (7) is of the general form (2), with the conjugate states k and \bar{k} corresponding to opposite projections $+m_\mu$ and $-m_\mu$ of the angular momentum for the same level μ .

B. Mapping via noncollective pairs

The dynamical algebra of the fermion pairing problem can be translated into boson language with available boson mapping techniques [19], mostly explored for even fermion systems. If both even and odd particle numbers are to be described simultaneously, the algebra of fermion pairs must be extended to take into account also the odd degrees of freedom. This can most naturally be done by adding the single-fermion creation and annihilation operators to the given set of bifermions. The resulting collection of operators then forms a superalgebra with the algebra of pair operators as a subalgebra.

As shown in Refs. [20–22], partial bosonization of this extended superalgebra can be achieved by a generalized Dyson mapping which, in the case of the superalgebra based on the above noncollective pairs (3), leads to

$$B_k^\dagger \mapsto b_k^\dagger (1 - n_k - \mathcal{N}_k), \quad (8)$$

$$B_k \mapsto b_k, \quad (9)$$

$$N_k \mapsto 2n_k + \mathcal{N}_k \quad (10)$$

for the even sector, and

$$a_k^\dagger \mapsto b_k^\dagger \alpha_{\bar{k}} + \alpha_k^\dagger \frac{1 - \mathcal{N}_k - n_k}{1 - \mathcal{N}_k} - \alpha_k^\dagger \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}} \frac{1 - \mathcal{N}_k - n_k}{(2 - \mathcal{N}_k)(1 - \mathcal{N}_k)}, \quad (11)$$

$$a_k \mapsto \alpha_k + \alpha_{\bar{k}}^\dagger b_k \frac{1}{1 - \mathcal{N}_k} + \alpha_k^\dagger \alpha_{\bar{k}}^\dagger \alpha_k b_k \frac{1}{(2 - \mathcal{N}_k)(1 - \mathcal{N}_k)} \quad (12)$$

for the odd sector [22,16]. In the right-hand side images, b_k^\dagger (b_k) creates (annihilates) a *boson* of the k th type and $n_k = b_k^\dagger b_k$, while α_k^\dagger (α_k) create (annihilate) *ideal fermions* in the states k (similarly for \bar{k}), and $\mathcal{N}_k = \alpha_k^\dagger \alpha_k + \alpha_{\bar{k}}^\dagger \alpha_{\bar{k}}$. These bosons and ideal fermions are kinematically independent, i.e., all boson operators commute with all fermion operators. The images of a_k^\dagger and $a_{\bar{k}}$ can be deduced from Eqs. (11) and (12), respectively, after the $k \leftrightarrow \bar{k}$ exchange of indices in the fermionic operators and inverting signs of the first term in Eq. (11) and of the second and third terms in Eq. (12).

The term “ideal fermion” is used to distinguish the fermion-type particles resulting from the mapping (where

they appear as necessary ingredients of any superalgebraic extension of boson mapping techniques) from the real fermions in the original formulation of the problem. The Bogoliubov-Valatin-like structure of Eqs. (11) and (12) suggests a physical interpretation of ideal fermions as generalized quasiparticles [23]. Anticipating discussions that follow below, we already point out here that possible divergences associated with the denominators in these formulas simply do not appear in the physical subspace. Furthermore the loss of symmetry between creation and annihilation operators under Hermitian conjugation (a typical feature of the Dyson type of mappings) is accounted for in the calculation of observables within the general Dyson framework, see Refs. [19,16].

The mapped Hamiltonian reads as follows:

$$H_B = \sum_k E_k(2n_k + \mathcal{N}_k) - \sum_{k,l} G_{kl} b_k^\dagger b_l (1 + \delta_{kl} - n_k - \mathcal{N}_k). \quad (13)$$

It is clear that this Hamiltonian conserves both the total number of bosons, $N_B = \sum_k n_k$, and of ideal fermions, $\mathcal{N}_F = \sum_k \mathcal{N}_k$, so that by considering a fixed total number of *real* fermions, $N_F = \sum_k N_k$, we also fix the sum $2N_B + \mathcal{N}_F$ to the given value N_F ; see Eq. (10).

The Dyson mapping of states can be trivially deduced from Eqs. (8)–(12). Four real-fermion basis states in each k subspace yield the ideal boson-fermion images as follows: $|0\rangle \mapsto |0\rangle$, $a_k^\dagger|0\rangle \mapsto \alpha_k^\dagger|0\rangle$, $a_k|0\rangle \mapsto \alpha_k|0\rangle$, $B_k^\dagger|0\rangle \mapsto b_k^\dagger|0\rangle$; we denote real- and ideal-space vectors by angular and circled bras/kets, respectively, as in case of real and ideal vacua $|0\rangle$ and $|0\rangle$. All the states in the ideal Hilbert space that are *not* linear combinations of the images just given are spurious. In particular, the components containing in any k shell more than one ideal particle (whether bosons or fermions, or both) are nonphysical. We see, therefore, that the present model of pairing allows us to write the projector to the physical subspace in the following *explicit* form:

$$P_{ph} = \prod_k (P_{n_k=0} P_{\mathcal{N}_k=0} + P_{n_k=0} P_{\mathcal{N}_k=1} + P_{n_k=1} P_{\mathcal{N}_k=0}), \quad (14)$$

where P 's on the right-hand side represent projectors onto the ideal subspaces with the given number of ideal particles (bosons or fermions) of the k th type.

Because of the nonunitary character of the Dyson mapping, the boson-fermionic Hamiltonian in Eq. (13) (as well as images of other physical observables) is non-Hermitian with respect to the standard boson-fermion Fock space inner product. However, it is well known that Dyson mappings lead to so-called quasi-Hermitian structures which are consistent with standard quantum mechanics, and in particular guarantees real eigenvalues for observables (see Refs. [24,16,25]). Although explicit Hermitization has been achieved in some particular cases [25], it seems that this procedure will generally introduce higher order many-body interaction terms into the Hamiltonian. Fortunately, in the present case, it can be easily checked that the Hamiltonian in Eq. (13) is already Hermitian *within* the physical subspace, i.e., it satisfies $H_B P_{ph} = H'_B P_{ph}$, where

$$\begin{aligned} H'_B &= \sum_k E_k(2n_k + \mathcal{N}_k) - \sum_{k,l} G_{kl} b_k^\dagger b_l \\ &= \sum_k E_k \mathcal{N}_k + \sum_k (2E_k - G_{kk}) n_k - \sum_{k \neq l} G_{kl} b_k^\dagger b_l \end{aligned} \quad (15)$$

is manifestly Hermitian. We see that the Hamiltonian (15) contains no interaction terms—it is just a combination of bosonic and fermionic mean fields—although both boson-boson and boson-fermion interactions were present in the original mapped Hamiltonian (13). This implies that the interactions in Eq. (13) do not affect physical states.

The Hamiltonian (15) has the same form as the one in Eq. (3) of Ref. [26]. Note that while in their treatment of pairing the authors of the cited work introduce bosonlike particles, so-called cooperons, by modifying fundamental anticommutation relations of real fermions, our bosons result from the mapping of a conventional multifermionic superalgebra. In contrast to cooperons, the bosons in our case obey ordinary commutation relations, but for the limitations concerning the physical subspace they are in fact *hard-core bosons*, i.e., bosons with occupation numbers restricted to 0 and 1.

In spite of the single-particle form of the Hamiltonian (15), its diagonalization in terms of some new bosons $d_k^\dagger = \sum_k \beta_{lk} b_k^\dagger$ and identification of the ground state for even systems with a condensate

$$|\text{cond}\rangle \propto \left(\sum_k \eta_k b_k^\dagger \right)^{N_B} |0\rangle \quad (16)$$

(where $N_B = N_F/2$ and $\eta_k = \beta_{l_0 k}$, with l_0 denoting the d boson with minimum energy) are not physically allowed operations. Indeed, such a procedure does not keep under control physicality of the transformed states; the condensate state has a finite spurious admixture as it contains terms with more than one boson on a given k shell (cf. Ref. [27]). The overlap of state (16) with the physical subspace of the ideal space is given by the following expression:

$$\frac{|\langle \text{cond} | P_{ph} | \text{cond} \rangle|^2}{(\langle \text{cond} | \text{cond} \rangle)} = \frac{\left| \sum_{k_1 \neq k_2 \neq \dots \neq k_{N_B}} \eta_{k_1} \eta_{k_2} \dots \eta_{k_{N_B}} \right|^2}{\left(\sum_k |\eta_k|^2 \right)^{N_B} N_B!}, \quad (17)$$

which is less than unity for $1 < N_B \leq N_F/2$ and any non-trivial set of η 's. At the same time, $P_{ph} |\text{cond}\rangle$ is generally *not* an eigenstate of H'_B , nor of H_B .

On the other hand, properties of the ground state of an even system can be estimated from the bosonic equivalent of the BCS approximation [18], using the image of the BCS state $|\text{BCS}\rangle = \prod_k (u_k + v_k B_k^\dagger) |0\rangle$, namely,

$$|\text{BCS}\rangle = \prod_k (u_k + v_k b_k^\dagger) |0\rangle = P_{ph} \exp \sum_k \left(\ln u_k + \frac{v_k}{u_k} b_k^\dagger \right) |0\rangle \quad (18)$$

with the normalization condition $u_k^2 + v_k^2 = 1$ for each k . The trial wave function (18) yields the following energy functional:

$$\begin{aligned}
(\text{BCS}|H'_B - 2\lambda N_B|\text{BCS}) &= \sum_k (2E_k - G_{kk} - 2\lambda)v_k^2 \\
&\quad - \sum_{k \neq l} G_{kl}u_k v_l u_l v_l \quad (19)
\end{aligned}$$

(with λ denoting a Lagrange multiplier), the same expression as in the standard BCS approximation.

We see that the bosonic BCS state (18) is just an ordinary Glauber coherent state projected onto the physical subspace (while the original BCS state is a generalized coherent state of the fermion dynamical group [28]). If the exponential on the right-hand side in Eq. (18) is projected onto a fixed number of particles (instead of the projection onto the physical space) one would get exactly the condensate state (16) with $\eta_k = v_k/u_k$, but—as discussed above—this would not be a physically justified procedure. Instead, the use of P_{ph} leads to the cutoff of spurious components, which also modifies the normalization factor in front of $\exp \sum_k \eta_k b_k^\dagger |0\rangle$ in Eq. (18) with respect to the standard bosonic coherent state.

C. Mapping via collective pairs

If the pairing Hamiltonian can be expressed in form (5), one can perform the fermion-boson mapping via the corresponding collective pairs. This is particularly simple in both the SU(2) special cases discussed above, see Eqs. (6) and (7): The C bifermion from Eq. (6) yields a c boson, $C^\dagger \mapsto c^\dagger [1 - (n + \mathcal{N})/\Xi]$, while the S bifermions from Eq. (7) map onto s bosons as follows: $S_\mu^\dagger \mapsto s_\mu^\dagger [1 - (n_\mu + \mathcal{N}_\mu)/(j_\mu + \frac{1}{2})]$. Note that the bifermion annihilation operators are in both cases mapped trivially just onto the corresponding boson annihilation operators, and the fermion number operators from the collective algebras onto analogous combinations as in Eq. (10). For instance, $N_\mu \mapsto 2n_\mu + \mathcal{N}_\mu$, where n_μ and \mathcal{N}_μ stand for the boson- and ideal-fermion number operators, respectively, of the μ th level in Eq. (7). It is clear that in this case the physical space is not restricted to the s -boson occupation numbers $n_\mu = 0$ and 1, but contains states with n_μ up to $j_\mu + \frac{1}{2}$. The single-fermion operators, which in both of the discussed cases supplement the bifermion algebra to yield the corresponding collective superalgebra, are also mapped analogously as in the preceding section, cf. expressions in Refs. [22,16].

The collective algebra of bifermions (4) is not generally of the SU(2) type because different collective pairs C_μ^\dagger and C_ν^\dagger , $\mu \neq \nu$, may contain components with the same k . Nevertheless, using the formalism described in Refs. [21,16], the Dyson bosonic images can be easily constructed:

$$\begin{aligned}
C_\mu^\dagger \mapsto c_\mu^\dagger - \sum_\nu c_\nu^\dagger \left(\sum_k \chi_{\mu k}^* \chi_{\nu k} \mathcal{N}_k \right. \\
\left. + \sum_{\omega, \pi} \sum_k \chi_{\mu k} \chi_{\nu k}^* \chi_{\omega k}^* \chi_{\pi k} c_\omega^\dagger c_\pi \right), \quad (20)
\end{aligned}$$

$$C_\mu \mapsto c_\mu, \quad (21)$$

$$\begin{aligned}
[C_\mu^\dagger, C_\nu] \mapsto -\delta_{\mu\nu} + \sum_k \chi_{\mu k} \chi_{\nu k}^* \mathcal{N}_k \\
+ 2 \sum_{\omega, \pi} \sum_k \chi_{\mu k} \chi_{\nu k}^* \chi_{\omega k}^* \chi_{\pi k} c_\omega^\dagger c_\pi. \quad (22)
\end{aligned}$$

Now c_μ^\dagger and c_μ create and annihilate, respectively, the boson corresponding to the μ th collective bifermion. Note, however, that single-particle creation and annihilation operators generating the odd sector of the general collective superalgebra are more difficult to determine than in the SU(2)-based special cases [16].

It should be stressed that although the collective C bifermions are just linear combinations of the noncollective ones, $C_\mu^\dagger = \sum_k \chi_{\mu k} B_k^\dagger$, the respective collective and noncollective bosons are *not* connected by the same linear relation. Note that the two kinds of boson operators can be viewed to act formally in two different Hilbert spaces, so that their comparison requires to introduce an operator T which transforms the physical space of the collective mapping onto the physical space of noncollective mapping. Equation (21) and linearity of all mapping procedures trivially yield $T c_\mu T^{-1} \equiv \tilde{c}_\mu = \sum_k \chi_{\mu k}^* b_k$; it means that the linear relation between C and B bifermions is preserved for the respective boson annihilation operators. For the creation operators, however, the situation is much more complicated, as the application of T on the appropriate linear superposition of mapping (20) results in a difficult self-consistent relation containing combinations of operators $\tilde{c}_\mu^\dagger \equiv T c_\mu^\dagger T^{-1}$, b_k^\dagger , b_l , and also $\tilde{\mathcal{N}}_k \equiv T \mathcal{N}_k T^{-1}$. This relation cannot be generally satisfied by any ansatz of the type $\tilde{c}_\mu^\dagger = \sum_k \chi'_{\mu k} b_k^\dagger$, where $\chi'_{\mu k}$ would represent some unknown coefficients, and thus the correspondence between both kinds of mapping is not linear.

III. SUPERSYMMETRY OF THE PAIRING HAMILTONIAN

A. Possible SUSY schemes

As explicitly demonstrated by the Hamiltonian in Eq. (15), the total number of particles is a conserved quantity in the system of bosons and ideal fermions obtained by mapping (8)–(12). The same holds true also for any collective mapping, see Eqs. (20)–(22) (neither projection to the physical space nor the Hermitization can spoil this property [16]). Therefore, the dynamics of the mapped system does not rule out the use of the $U(K/2\Omega)$ superalgebra as the spectrum generating (dynamical) superalgebra. In fact, the superalgebra of real fermions and bifermions, either collective or noncollective, is mapped *into* (but generally not onto) this superalgebra.

For the noncollective mapping, for instance, $K = \Omega$ and the even sector of $U(\Omega/2\Omega)$ is formed by generators $b_k^\dagger b_l$, $\alpha_k^\dagger \alpha_l$, $\alpha_k^\dagger \alpha_l^\dagger$, and their Hermitian conjugates (all together $5\Omega^2$ operators), while the odd sector is generated by $b_k^\dagger \alpha_l$, $b_k^\dagger \alpha_l^\dagger$, and the conjugates (all together $4\Omega^2$ operators). We see, however, that the odd sector is not used in the construction of the Hamiltonian, so that $U_B(\Omega) \otimes U_F(2\Omega)$ can equally well be chosen as the dynamical algebra. This means that the decomposition in Eq. (1) is applicable, which preselects possible dynamical symmetries of the system and, consequently, also

the relevant transitional regimes. Realizing that all these symmetries are given by standard decompositions of the $U_B(K) \otimes U_F(2\Omega)$ algebra into chains of embedded subalgebras, we skip here their explicit discussion. In general, conditions for any specific dynamical symmetry, given by a chain $\mathcal{D} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \dots$ of decomposition of the boson-fermion dynamical algebra \mathcal{D} , are obtained from the set of constraints $[H_B, C(\mathcal{A}_i)] = 0$ required to hold in the physical space. This naturally yields specific (for each symmetry) sets of equations for parameters E_k and G_{kl} of the Hamiltonians (15) and (2).

Although—as we just saw—the odd sector of the dynamical superalgebra $U(K/2\Omega)$ can be ruled out, it is also clear that the SUSY quantum number $\aleph = N_B + \mathcal{N}_F$ still classifies all eigenstates of the mapped many-body Hamiltonian. Conservation of the total particle number is a common feature of the generalized Dyson mapping. Let us stress that this simple conclusion was not at all clear in the early days of nuclear supersymmetry, especially in view of the fact that the real fermion number maps onto the number of ideal fermions plus *twice* the number of bosons, $N_F \mapsto \mathcal{N}_F + 2N_B$, which excludes the particle-number conservation law for real fermions from the explanation of the conservation of \aleph . Indeed, as shown in Ref. [16], the conservation law for \aleph (as well as the separate conservation of N_B and \mathcal{N}_F) follows from a consistent choice of the real-fermion dynamical algebra, which must represent the interaction as well as single-particle terms of the fermion Hamiltonian. Therefore, it is only the possibility of fully *algebraic formulation* of the fermion pairing problem that automatically creates the $U(K/2\Omega)$ -based boson-fermion description, with all potential dynamical-supersymmetry chains and transitional classes included.

Potential *invariant* supersymmetries of the pairing Hamiltonian (2) also depend on the energies E_k and interaction strengths G_{kl} . On the fermion level, all such supersymmetries are described in the framework of the most general superalgebra of multifermion operators,

$$A_{k'_1 \dots k'_n, \bar{k}'_1 \dots \bar{k}'_n}^{k_1 \dots k_n, \bar{k}_1 \dots \bar{k}_n} = a_{k_1}^\dagger \dots a_{k_n}^\dagger a_{\bar{k}_1}^\dagger \dots a_{\bar{k}_n}^\dagger a_{k'_1} \dots a_{k'_n} a_{\bar{k}'_1} \dots a_{\bar{k}'_n}, \quad (23)$$

where a given element belongs to the even or odd sector, respectively, according to whether the difference $\Delta N_F = (n + \bar{n}) - (n' + \bar{n}')$ is even or odd. Operators of the even sector transform states with even and odd fermion numbers separately, while operators of the odd sector interconnect even and odd populations. The invariant SUSY takes place if the Hamiltonian (2) commutes with any of the operators (23) belonging to the odd sector, which implies the existence of a degenerate supermultiplet of states differing by a given odd number of ΔN_F fermions.

In the Hilbert space of bosons and ideal fermions created by mapping (8)–(12), similarly, one can consider the superalgebra of operators

$$B_{k'_1 \dots k'_n, \bar{k}'_1 \dots \bar{k}'_n}^{k_1 \dots k_n, \bar{k}_1 \dots \bar{k}_n} = \mathcal{A}_{k'_1 \dots k'_n, \bar{k}'_1 \dots \bar{k}'_n}^{k_1 \dots k_n, \bar{k}_1 \dots \bar{k}_n} b_{l_1}^\dagger \dots b_{l_m}^\dagger b_{l'_1} \dots b_{l'_m}, \quad (24)$$

where $\mathcal{A}_{k'_1 \dots k'_n, \bar{k}'_1 \dots \bar{k}'_n}^{k_1 \dots k_n, \bar{k}_1 \dots \bar{k}_n}$ is defined in the same way as $A_{k'_1 \dots k'_n, \bar{k}'_1 \dots \bar{k}'_n}^{k_1 \dots k_n, \bar{k}_1 \dots \bar{k}_n}$ in Eq. (23), but with the real-fermion operators substituted by the corresponding ideal-fermion operators. As in the previous case, even and odd sectors of the superalgebra (24) are distinguished according to the difference $\Delta \mathcal{N}_F$ between the numbers of creation and annihilation operators in $A_{k'_1 \dots k'_n, \bar{k}'_1 \dots \bar{k}'_n}^{k_1 \dots k_n, \bar{k}_1 \dots \bar{k}_n}$.

Although the mapping in Sec. II B was performed only for a small subset of the above general fermionic superalgebra, the images of operators (23) in the physical space can be constructed as the corresponding products of the images in Eqs. (11) and (12) [20]. Superalgebra (23) is thus mapped to series of operators (24), i.e., onto the given superalgebra acting in the space of bosons and ideal fermions. Trivially, even and odd sectors of superalgebra (23) are not mixed by the mapping—the image of an element with a given value of ΔN_F has $\Delta \mathcal{N}_F + 2\Delta N_B = \Delta N_F$, where $\Delta N_B = m - m'$, see Eq. (24). Taking $\Delta N_F = +1$, for example, we get an image with terms corresponding to $(\Delta \mathcal{N}_F, \Delta N_B) = (+1, 0), (-1, +1), (+3, -1), (-3, +2), \dots$

This expansion would hardly be tractable in its general form, but fortunately, it can be considerably simplified if states with more than one ideal fermion are physically irrelevant. Indeed, such states correspond to real-fermionic states with broken pairs which are supposed to have relatively high excitation energies. Under this restriction, the invariant-SUSY supercharges connect states in doublets with N_F and $N_F + 1$ fermions, and can be constructed just from the $(\Delta \mathcal{N}_F, \Delta N_B) = (+1, 0)$ and $(-1, +1)$ terms. The first term applies in case that N_F corresponding to the left system in the doublet is even, the second if N_F is odd. To make contact with the standard notion of a supercharge, we here only focus on the second term that changes a fermion into a boson (or vice versa). It reads as

$$Q^\dagger = \sum_{k,l} b_k^\dagger f_{kl} (b_i^\dagger b_j, \alpha_m^\dagger \alpha_n) (q_{kl} \alpha_l + \bar{q}_{kl} \alpha_l), \quad (25)$$

where f_{kl} are arbitrary functions of operators conserving separately the numbers of bosons and fermions, and q_{kl} and \bar{q}_{kl} are some coefficients.

In spite of the above restrictions, the commutator of supercharge (25) with the boson-fermion Hamiltonian still remains completely unknown without a further specification of f_{kl} . This makes the general theoretical determination of invariant supersymmetries in fermionic systems a rather non-trivial task, where no analytic insight seems available so far. Nevertheless, the mapping procedure, as outlined in Sec. II, provides a natural framework for the classification and analysis of various special cases. In the next paragraph, for example, we determine conditions for the invariant SUSY generated by supercharges (25) with $f_{kl} = 1$.

B. Invariant SUSY with bilinear supercharges

We will look for conditions that the mapped Hamiltonian from Sec. II B must satisfy to commute with the simplest bilinear supercharges of the form

$$Q^\dagger = \sum_{k,l} b_k^\dagger (q_{kl} \alpha_l + \bar{q}_{kl} \alpha_l^\dagger) \equiv \sum_{k,l} b_k^\dagger \hat{\alpha}_{kl}, \quad (26)$$

where b_k^\dagger and α_l (or α_l^\dagger) are noncollective bosons and ideal fermions, respectively, from Eqs. (8) and (12). If the supercharge is assumed to commute with the time reversal, the coefficients q_{kl} and \bar{q}_{kl} should be set equal (up to a phase factor), but this we do not generally require here.

The assumption $[H'_B, Q^\dagger]=0$ ($=[Q, H'_B]^\dagger$) leads to the following equation to be valid for any k and l :

$$(2E_k - E_l - G_{kk})q_{kl} - \sum_{m \neq k} G_{km}q_{ml} = 0, \quad (27)$$

and the same also with the change of q 's to \bar{q} 's. For a fixed value of l , the condition for the existence of non-trivial solutions q_{kl} and \bar{q}_{kl} reads as

$$\det \begin{pmatrix} G_{11} - 2E_1 + E_l & G_{12} & G_{13} & \dots \\ G_{21} & G_{22} - 2E_2 + E_l & G_{23} & \dots \\ G_{31} & G_{32} & G_{33} - 2E_3 + E_l & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} = 0. \quad (28)$$

This equation has rather clear physical interpretation: it says that diagonalization of the bosonic part of the single-particle Hamiltonian (15) gives one of the eigenvalues equal to the fermionic energy E_l . Mutual conversions of the corresponding boson and fermion thus do not change the total energy. One can apply Eq. (28) simultaneously to a certain subset of fermionic states, $l \in \mathcal{S}$, which is equivalent with taking $q_{kl} = \bar{q}_{kl} = 0$ for $l \notin \mathcal{S}$. The supersymmetry then concerns only a part of the spectrum.

An important special case of the above condition is obtained if one assumes supercharges with diagonal matrices of coefficients, $q_{kl} = q_k \delta_{kl}$ and $\bar{q}_{kl} = \bar{q}_k \delta_{kl}$. Equation (27) then leads to the following simple constraints: $G_{kl} = 0$ for $k \neq l \in \mathcal{S}$ and $E_k = G_{kk}$ for $k \in \mathcal{S}$, the coefficients q_k and \bar{q}_k being arbitrary for $k \in \mathcal{S}$ and zero otherwise. We thus have $H'_B = H'_{\text{SUSY}}$ with

$$H'_{\text{SUSY}} = \sum_{k \in \mathcal{S}} E_k (n_k + \mathcal{N}_k) + \sum_{k \notin \mathcal{S}} E_k (2n_k + \mathcal{N}_k) - \sum_{k,l \notin \mathcal{S}} G_{kl} b_k^\dagger b_l. \quad (29)$$

The interpretation of this solution is the same as above, with only the difference that the bosonic part of the Hamiltonian (15) is already supposed to be diagonalized *a priori*, allowing for the diagonal constraint on the supercharge coefficients. By inserting the above constraints into the original Hamiltonian (2), one can verify the existence of degenerated supermultiplets with $N_k = 1$ and 2 (for $k \in \mathcal{S}$) directly on the fermionic level.

The Hamiltonian (29) is trivially invariant under the set of transformations $b_k^\dagger \alpha_k$, $b_k^\dagger \alpha_k^\dagger$, $\alpha_k^\dagger \alpha_k$, $\alpha_k^\dagger \alpha_k^\dagger$, $\alpha_k^\dagger \alpha_k^\dagger$, $b_k^\dagger b_k$ (and their Hermitian conjugates) with $k \in \mathcal{S}$. These are generators of the

$\mathcal{I} = \otimes_{k \in \mathcal{S}} U_k(1/2)$ superalgebra, which thus forms the invariant-SUSY superalgebra of the problem. This is so in spite of the fact that the dynamical algebra of H'_{SUSY} in Eq. (29) is just an ordinary algebra $\mathcal{D} \subset U_B(\Omega) \otimes U_F(2\Omega)$ where seemingly no supersymmetry is involved. In view of the discussion above, the SUSY Hamiltonian obtained from the general solution of Eq. (28) in case of nondiagonal matrices of supercharge coefficients receives the same algebraic interpretation, but with the original bosons b_k^\dagger substituted by the new ones d_k^\dagger , resulting from the diagonalization of the bosonic Hamiltonian. Note that the above superalgebraic scheme represents a direct generalization of the minimal SUSY scheme [2], where the invariant-SUSY superalgebra $\text{sl}(1/1)$ is formed only by operators Q^\dagger , Q , and $H = \{Q^\dagger, Q\}$ [3,4]. In the present case, in particular, the relevant part of the Hamiltonian is not just a supercharge anticommutator (or a sum of such terms), but a linear combination of Casimir invariants of \mathcal{I} .

Let us turn, at last, to the crucial question of physicality of the above supersymmetric transformations. It was so far completely ignored in this section but it is indeed very relevant since the supercharge operator (26) can transform some of the physical states out of the physical space given by the projector in Eq. (14). In particular, the supercharge operator acting within the physical space may produce a non-physical state with $(n_k, \mathcal{N}_k) = (1, 1)$ or $(2, 0)$, this being not generally excluded—in case of nondiagonal matrices of supercharge coefficients—by the action of α_l or α_l^\dagger , associated with b_k^\dagger in Eq. (26). Consequently, some of the supersymmetric transformations that, as found above, constitute an invariant supersymmetry of the Hamiltonian H'_B may turn to be nonphysical.

To examine this question, let us introduce a physical supercharge operator

$$P_{ph} Q^\dagger P_{ph} = \tilde{Q}^\dagger P_{ph}, \quad (30)$$

where

$$\tilde{Q}^\dagger = \sum_{k \neq l} b_k^\dagger \hat{\alpha}_{kl} P_{n_k=0} P_{\mathcal{N}_k=0} + \sum_k b_k^\dagger \hat{\alpha}_{kk} P_{n_k=0}. \quad (31)$$

Similarly, we define

$$\tilde{H}_B = \sum_k E_k \mathcal{N}_k + \sum_k (2E_k - G_{kk}) n_k - \sum_{k \neq l} G_{kl} b_k^\dagger b_l P_{n_k=0} P_{\mathcal{N}_k=0} \quad (32)$$

that represents another form of the Hamiltonian (13). The condition for the commutation of the supercharge with the Hamiltonian in the physical space then reads as $[\tilde{H}_B, \tilde{Q}^\dagger] P_{ph} = 0$, which yields the following operator equality:

$$\begin{aligned}
& \sum_k (E_k - G_{kk}) b_k^\dagger \hat{\alpha}_{kk} + \sum_{k \neq l} (2E_k - E_l - G_{kk}) b_k^\dagger \hat{\alpha}_{kl} P_{n_k=0} P_{N_k=0} \\
& - \sum_{k \neq l} G_{kl} b_k^\dagger \hat{\alpha}_{ll} P_{n_k=0} P_{N_k=0} - \sum_{k \neq l} G_{kl} b_k^\dagger \hat{\alpha}_{lk} P_{n_l=0} P_{N_l=0} \\
& + \sum_{k \neq l \neq m} G_{kl} b_k^\dagger \hat{\alpha}_{lm} (P_{n_l=1} - P_{n_l=0}) P_{n_k=0} P_{N_l=0} P_{N_k=0} \\
& - \sum_{k \neq l \neq m} G_{kl} b_k^\dagger b_m^\dagger b_l \hat{\alpha}_{mk} P_{n_m=0} P_{N_m=0} = 0 \quad (33)
\end{aligned}$$

to be valid in the physical space. It is not difficult to see that Eq. (33) can only be satisfied by imposing the above SUSY conditions for supercharges with vanishing nondiagonal q_{kl} and \bar{q}_{kl} elements, see Eq. (29) and the text above. The Hamiltonian (29) is the *only* invariant-SUSY Hamiltonian with supermultiplet states contained entirely in the physical space that can be constructed through a bilinear supercharge in Eq. (26). The other solutions, see Eq. (28), are spurious, i.e., have no real counterpart on the level of the original fermion system.

Let us finally note that the search for invariant supersymmetries would be much more involved for the mapping performed via collective bifermions from Eq. (4). In this case, we again have to check if the expression $[\tilde{H}_B, \tilde{Q}^\dagger] P'_{ph}$ vanishes, where $\tilde{Q}^\dagger = \sum_{\mu,k} c_{\mu}^\dagger (q_{\mu k} \alpha_k + \bar{q}_{\mu k} \alpha_k)$ is a collective bilinear supercharge and P'_{ph} a projector onto the physical subspace obtained via the collective superalgebra. Unlike the noncollective case, boson-boson and boson-fermion interactions in the Hamiltonian \tilde{H}_B are generally relevant also in the physical subspace [16] and must be considered in the commutator. Moreover, the calculation is obscured by the fact that P'_{ph} has a more complicated form in the general case than for the noncollective superalgebra. The above-discussed relation between collective and noncollective bosons, see the end of Sec. II C, indicates that the use of collective supercharges has a similar effect as the generalization of Eq. (26) to form (25).

IV. CONCLUSIONS

We have pointed out the need to distinguish between two schemes in supersymmetric many-body physics (nuclear structure in particular). The first one relies on the definition of an appropriate dynamical superalgebra of the given system and leads to the well-established notion of dynamical supersymmetry. Phenomenological applications of the SUSY IBFM represent the best known example. The second scheme directly follows from SUSY quantum mechanics. It aims at the identification of a superalgebra of invariant supersymmetry, which—unlike dynamical supersymmetries—

leads to actual degeneracies of many-body states in neighboring even and odd fermionic systems.

In contrast to phenomenological studies based on the IBFM, our search for the SUSY patterns is derived from a purely fermionic level. Existing boson-fermion mapping techniques allow us to introduce the relevant boson and fermion degrees consistently, and facilitate our exploration of dynamical and invariant supersymmetries in the simple case of pairinglike Hamiltonians.

While the use of dynamical superalgebra $U(K/2\Omega)$ with various dynamical supersymmetries of form (1) and the corresponding transitional regions directly follows from the microscopic foundations of the IBFM, the problem of invariant supersymmetries seems more difficult to solve on the microscopic basis. Nevertheless, for Hamiltonians (2), and with supercharges restricted to the bilinear type (26), it turns out that invariant supersymmetry is ruled out for pairing other than the simplest diagonal form, see Eq. (29). This is not in contradiction with the result of Ref. [17] where the boson interaction does not follow from a mapped fermion interaction.

Although we have not exhausted all possibilities that may result in invariant supersymmetry for generalized pairing in a fermion system, going beyond the present analysis seems to imply a tough problem when pursued analytically.

From a microscopic point of view, considering and accounting for the physical subspace has been crucial for a complete analysis, as shown in detail in Secs. II B and III B. It is one of the main aims of this work to demonstrate on a relatively simple example the restrictive role that the physical subspace may play in transferring results of phenomenological SUSY to the microscopic framework.

Finally, as argued in greater detail in Ref. [29], and as is also clear from the present discussion, there is no reason to expect (as, e.g., in Ref. [30]) that transfer operators in the context of dynamical supersymmetry should be restricted to the simplest supercharges appearing in invariant supersymmetry situations. In fact, for dynamical symmetry the appropriate transfer operators are *mapped images* of single-fermion operators, as, e.g., in Eqs. (11) and (12). In our presentation these images may indeed be considered generalized supercharges, but in general they are not of the simple bilinear type.

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