Number of states with a given angular momentum for identical fermions and bosons

Y. M. Zhao[1,2,3] and A. Arima[4]
[1]Cyclotron Center, Institute of Physical Chemical Research (RIKEN), Hirosawa 2-1,Wako-shi, Saitama 351-0198, Japan
[2]Department of Physics, Southeast University, Nanjing 210018, China
[3]Department of Physics, Saitama University, Saitama-shi, Saitama 338, Japan
[4]The House of Councilors, 2-1-1 Nagatacho, Chiyodaku, Tokyo 100-8962, Japan
(Received 24 April 2003; published 16 October 2003)

In this paper we study the number of angular momentum I states for both fermions and bosons. Suppose that n is the particle number, $\mathcal{M}=I_{max}-I$, and that J refers to the angular momentum of a single-particle orbit for fermions or the spin L carried by bosons. We prove that the number of states is independent of J if $\mathcal{M}\leqslant(2J-n+1)$ for fermions and $\mathcal{M}\leqslant 2J$ for bosons, and that the number of I states is independent of both n and J if $\mathcal{M}\leqslant\min(n,2J+1-n)$ for fermions and $\mathcal{M}\leqslant\min(n,2J)$ for bosons. We also present in this paper empirical formulas for the number of I states of three and four identical fermions or bosons.

DOI: PACS number(s): 21.60.Cs, 21.60.Fw, 21.45.+v, 02.90.+p

I. INTRODUCTION

The enumeration of the number of total angular momentum I states is a very common practice in nuclear structure theory. One usually obtains this number by subtracting the combinatorial number of the states with a total angular momentum projection $M=I+1$ from that with $M=I$. The combinatorial numbers of different M's seem to be irregular, and such an enumeration procedure would be prohibitively tedious without a computer for a very large single-j shell. The numbers of states of a few nucleons in a single-j (j is a half integer) shell are usually tabulated in textbooks, for example, Ref. [1], for sake of convenience.

Another well-known solution was given by Racha [2] in terms of the seniority scheme, where one has to introduce (usually by computer choice) additional quantum numbers. More than one decade ago, a third route was studied by Katriel *et al.* [3] and Sunko and co-workers [4], who constructed generating functions of the number of states for fermions in a single-j shell or bosons with spin L.

These works are very interesting and important. On the other hand, the number of states therein are not algebraic. There was only one algebraic formula available for $I=0$ states with four fermions [5]. It would be very interesting to have more general algebraic formulas, if possible. In another context, it was noted [6] that the angular momentum 0 ground state probability of four fermions in a single-j shell in the presence of random interactions has a synchronous staggering with an increase of the number of $I=0$ states by one when j increases. It has not been clarified yet whether there is a deep relation between these two quantities or this synchronism is just a coincidence. This "coincidence" also motivated the present study of algebraic formulas of number of states.

In this paper the number of states is denoted as $D(n,I)_j$, $D(n,I)_l$, and $D(n,I)_L$ for n fermions in a single-j shell, n fermions in a single-l shell, and n bosons with spin L, respectively, where j is a half integer, and l and L are integers. For convenience we use $D(n,I)_J$ to refer to all these three cases, where J can be j, l, or L. Suppose that $\mathcal{M}=I_{max}-I$, where I_{max} is the maximum of I. For convenience we also denote the number of states as $\mathcal{D}(n,\mathcal{M})_J$, where the variable is $\mathcal{M}$ instead of I. In Sec. II, we shall show that the $\mathcal{D}(n,\mathcal{M})_J$ is independent of J if $\mathcal{M}\leqslant(2J-n+1)$ for fermions or $\mathcal{M}\leqslant 2L$ for bosons, and that the $\mathcal{D}(n,\mathcal{M})_J$ is independent of both J and n if $\mathcal{M}\leqslant\min(n,2J+1-n)$ for fermions and $\mathcal{M}\leqslant\min(n,2L)$ for bosons. In Sec. III, we shall construct empirical formulas for all I states of three and four identical particles. Although these formulas are obtained empirically, we have confirmed them by computer for $j\leqslant 999/2$, $l\leqslant 500$, or $L\leqslant 500$, and for many cases with much larger J's which were taken randomly. One therefore can use them "safely" in practice. A summary is given in Sec. IV. In the Appendix we shall present a few formulas for the cases of five particles.

II. A REGULARITY OF $D(n,I)_J$ FOR LARGE I CASES

In this section, we shall first show the following regularity, that is, for fermions with $\mathcal{M}\leqslant 2J-n+1$ [i.e., $I\geqslant(n-2)J-\frac{1}{2}(n-1)(n-2)$] and for bosons with $\mathcal{M}\leqslant 2J$ [i.e., $I\geqslant(n-2)J$], $\mathcal{D}(n,\mathcal{M})_J$ is independent of J. For an example, $\mathcal{D}(5,\mathcal{M})_J$=1, 0, 1, 1, 2, 2, 3, 3, 5, 5, 7, 7, 10, 10, 13, 14, 17, 18, 22, 23, 28, 29, 34, 36, 42, 44, 50, 53, 60, 63,…, for $\mathcal{M}=0,1,2,\ldots,28\ldots$, if $\mathcal{M}\leqslant 2J-4$ for fermions and $\mathcal{M}\leqslant 2L$ for bosons.

Below we prove this observation. Let $\mathcal{P}(n,\mathcal{M})$ be the number of partitions of $\mathcal{M}=i_1+i_2+\cdots+i_n$, with $0\leqslant i_1\leqslant i_2\leqslant\cdots\leqslant i_n$. Suppose that the angular momentum projection of the kth particle is $m_k (k=1,2,\ldots,n)$. Let us denote m_k of the $I=I_{\rm max}$ state as m_k^0. The m_k^0 is $J-k+1$ for fermions and $J=L$ for bosons.

Let $\mathcal{P}(n,0)=\mathcal{D}(n,0)_J=1$. It is easily noted that $\mathcal{D}(n,\mathcal{M})_J=\mathcal{P}(n,\mathcal{M})-\mathcal{P}(n,\mathcal{M}-1)$ if all $m_k=m_k^0-i_k\geqslant -J$. One should be aware that in the m scheme, a state with a given value of $M=I_{max}-\mathcal{M}$ is obtained from the state with $I=I_{max}$ by subtracting summands i_k of $\mathcal{M}$ from the corresponding m_k^0 [7], with the largest summand i_n subtracted from the smallest m_n value. Thus the condition $m_k\geqslant -J$ here means that $\min(m_k)=m_n\geqslant -J$, i.e., $i_1=i_2=\cdots=i_{n-1}=0$ and $i_n\leqslant m_n^0-(-J)$. The maximum of $\mathcal{M}$ which satisfies the condition of all $m_k \geqslant -J$ is therefore $2J-n+1$ for fermions and $2J$ for bosons. Namely, when $\mathcal{M} \leqslant 2J-n+1$ [or $I \geqslant (n-2)J - \frac{1}{2}(n-1)(n-2)$] for fermions and $\mathcal{M} \leqslant 2J$ [or $I \geqslant (n-2)J$] for bosons, $\mathcal{D}(n,\mathcal{M})_J$ are independent of J. We denote this maximum of $\mathcal{M}$ of n particles as $\mathcal{M}^n$. One sees that not all partitions with $\mathcal{M} > \mathcal{M}^n$ lead to physical states.

As a consequence of the above result, we shall be able to present in Sec. III *unified* formulas for $n=3$ and 4 under the condition that $\mathcal{M} \leqslant \mathcal{M}^n$, i.e., $I \geqslant J-1$ $(I \geqslant 2J-3)$ for three (four) fermions and $I \geqslant J$ $(I \geqslant 2J)$ for three (four) bosons.

Now we discuss the range in which $\mathcal{D}(n,\mathcal{M})_J$ is independent of both J and n. Here an additional condition should be satisfied: $\mathcal{M} \leqslant n$. In another sentence, when $\mathcal{M} \leqslant \min(n,\mathcal{M}^n)$, i.e., $\mathcal{M} \leqslant \min(n,2J-n+1)$ for fermions and $\mathcal{M} \leqslant \min(n,2J)$ for bosons, $\mathcal{D}(n,\mathcal{M})_J$ is the same for all n. For example, $\mathcal{D}(5,\mathcal{M}=5)_{j=31/2}(=2)$ is equal to $\mathcal{D}(5,\mathcal{M}=5)_{j=21/2}$ and $\mathcal{D}(10,\mathcal{M}=5)_{L=30}$. The universal $\mathcal{D}(n,\mathcal{M})_J$ series is 1, 0, 1, 1, 2, 2, 4, 4, 7, 8, 12, 14, 21, 24, 34, 41, 55, 66, 88, 105, 137, 165, 210, 253, 320, for $\mathcal{M}=0,1,\ldots,24$, respectively.

III. EMPIRICAL FORMULAS OF $D(n,I)_J$ FOR $n=3$ AND 4

For $I \leqslant J$, we empirically obtain

$$D(3,I)_j = \left[\frac{2I+3}{6}\right],$$

$$D(3,I)_l = \left[\frac{I}{3}\right] + \frac{1}{2}(1-(-)^{I+l}),$$

$$D(3,I)_L = \left[\frac{I}{3}\right] + \frac{1}{2}(1+(-)^{I+L}), \quad (1)$$

where [] means to take the largest integer not exceeding the value inside.

For fermions with $I \geqslant J-1$ or bosons with $n=3$ and $I \geqslant J$, the $D(3,I)_J$'s can be empirically given in a unified form:

$$D(3,I)_J = \left[\frac{I_{max}-I}{6}\right] + \delta_I, \quad (2)$$

where

$$\delta_I = \begin{cases} 0 & \text{if } (I_{max}-I)\,\%\,6 = 1 \\ 1 & \text{otherwise.} \end{cases}$$

In this paper $a\%b$ is the smallest integer congruent to a (mod b), where a is a non-negative integer and b is a natural number, i.e., $a\%b = a - b[a/b]$; for example, $7\%3=1$ and $27\%10=7$.

According to Eq. (1), $D(3,\frac{1}{2})_j=0, D(3,0)_l=1$ (or 0) if l is odd (or even), and $D(3,0)_L=1$ (or 0) if l is even (or odd). It is noted that there are overlaps of I's between the range described by Eq. (1) and that by Eq. (2), and that one may use either of them to obtain these $D(3,I)_J$.

The cases of four particles are more complicated. However, a regular staggering of $D(4,I)_J$ can be easily noticed if $I \leqslant 2J$. We define $\eta_I^J = D(4,I)_{J+1} - D(4,I)_J$, and $\Delta(I) = D(4,I)_J - D(4,I+3)_J$ (even I). It is found that η_I^J **(**$\Delta(I)$**)** changes periodically as $J(I)$ changes by three (four) for $I \leqslant 2J$. Table I shows that the $(\eta_I^J, \eta_I^{J+1}, \eta_I^{J+2})$ of four particles is the partition of $(I/2+1)$ with the conditions that $\eta_I^J \geqslant \eta_I^{J+1} \geqslant \eta_I^{J+2}$ and $(\eta_I^J - \eta_I^{J+1}) \leqslant 1, (\eta_I^{J+1} - \eta_I^{J+2}) \leqslant 1$. Here I is an even number. Although the origin of this regularity is not known, one may make use of this to construct the formulas of $D(4,I)_J$ for $I \leqslant 2J$. The cases of $I \geqslant 2J$ will be addressed later in this section.

TABLE I. For the cases with $n=4$, η_I^J, η_I^{J+1}, and η_I^{J+2} of angular momenta I and $I+3$ (I is even) states change periodically at an interval $\Delta_J=3$ when $I \leqslant 2J$. One thus easily constructs formulas for $D(n,I)_J$'s for states with $I \leqslant 2J$.

I	$I+3$	η_I^J	η_I^{J+1}	η_I^{J+2}
0	3	1	0	0
2	5	1	1	0
4	7	1	1	1
6	9	2	1	1
8	11	2	2	1
10	13	2	2	2
12	15	3	2	2
14	17	3	3	2
16	19	3	3	3
⋮	⋮	⋮	⋮	⋮

For $I \leqslant 2J$ with I being even, we empirically obtain

$$D(4,I)_J = \left[\frac{\mathcal{L}-I/2}{3}\right] \times \left(\frac{I}{2}+1\right) + C(I)m - \delta + d(I), \quad (3)$$

where $m = (\mathcal{L}-I/2)\%3$. For fermions in a single-j shell, the coefficients in Eq. (3) are given by

$$\mathcal{L} = j - \frac{1}{2}, \quad C(I) = \left[\frac{I}{6}\right] + 1,$$

$$\delta = \begin{cases} \delta_{m2} & \text{if } I\,\%\,6 = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$d(I) = 3K(K-1) + K + (1+K)K + \delta_{\mathcal{K},4} + \delta_{\mathcal{K},5},$$

where $K = [(I+4)/12]$, and $\mathcal{K} = ((I+4)\%12)/2$. For fermions in a single-l shell,

$$\mathcal{L} = l, \quad C(I) = \left[\frac{I+4}{6}\right],$$

$$\delta = \begin{cases} \delta_{m2} & \text{if } (I+4)\,\%\,6 = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$d(I) = 3K(K-1) + K + (1+K)K + \delta_{K,4} + \delta_{K,5},$$

where $K = [(I+2)/12]$, and $\mathcal{K} = ((I+2)\%12)/2$. For bosons with spin L,

$$\mathcal{L}=L, \quad C(I)_J=\left[\frac{I+4}{6}\right],$$

$$\delta=\begin{cases}\delta_{m2} & \text{if } (I+4)\,\%\,6=0\\ 0, & \text{otherwise},\end{cases}$$

$$d(I)=3K(K-1)+K+(1+\mathcal{K})K+\delta_{\mathcal{K},4}+\delta_{\mathcal{K},5},$$

where $K=[(I+8)/12]$ and $\mathcal{K}=\big((I+8)\%12\big)/2$.

For $I\leqslant 2J$ with I being odd, we introduce $I=I_0+3$, and obtain

$$D(4,I)_j=D_j(4,I_0)-\left[\frac{I}{4}\right]-1,$$

$$D(4,I)_l=D_l(4,I_0)-\left[\frac{I+2}{4}\right],$$

$$D(4,I)_L=D_L(4,I_0)-\left[\frac{I}{4}\right]-1. \quad (4)$$

One easily sees that the $D(4,1)_J$'s are always zero.

For $I\geqslant 2J-3$, we define

$$\text{for even } I: \quad I=I_{max}-2m,$$

$$\text{for odd } I: \quad I=I_{max}-3-2m.$$

We let $K=[m/6]$, $\mathcal{K}=m\%6$, and obtain

$$D(4,I)_J=3K(K+1)-K+(K+1)(\mathcal{K}+1)+\delta_{\mathcal{K}0}-1 \quad (5)$$

for fermions with $I\geqslant 2J-\frac{1}{2}(n-1)(n-2)=2J-3$ and for bosons with $I\geqslant 2L=I_{max}-2L$.

It is noted that for fermions $D(4,I)_J$ of $I=(2J-3,\ 2J-2,\ 2J-1$, and $2J)$ can be obtained either by Eqs. (3) and (4) or by Eq. (5). The formulas of $D(4,I)_J$ present an even-odd staggering of the number of states: the number of states with even number of I is not smaller and mostly larger than those of their odd-I neighbors. A similarity between the formulas for four fermions (in both half-integer j orbit and integer l orbit) and bosons is also easily noticed.

The situation of $n=5$ is much more complex, and we are unable to construct simple and unified formulas. In the Appendix we list a few formulas for the cases with $I\sim I_{min}$.

IV. SUMMARY AND DISCUSSION

To summarize, we have shown in this paper that the number of states $\mathcal{D}(n,\mathcal{M})_J$ is independent of J if $\mathcal{M}\leqslant 2J-n+1$ for fermions in a single-j or l shell, and $\mathcal{M}\leqslant 2L$ for bosons with spin L. The $\mathcal{D}(n,\mathcal{M})_J$ is independent of both n and J when $\mathcal{M}\leqslant\min(n,2J+1-n)$ for fermions and $\mathcal{M}\leqslant\min(n,2J)$ for bosons. These facts have eluded from observation in the long history of enumerating the $D(n,I)_J$.

We have also found that there are simple structures in the number of states of three and four identical particles, which enabled us to construct empirical formulas for $n=3$ and 4. For $n=5$ we presented formulas for a few lowest I states. We have confirmed the validity of these empirical formulas for $j\leqslant 999/2$ and $l(L)\leqslant 500$, which are large enough for practical use.

ACKNOWLEDGMENTS

The authors would like to thank Dr. J. N. Ginocchio for discussions on this work and Dr. W. Bentz for his reading of the manuscript. This work was supported in part by the Japan Society for the Promotion of Science (Contract No. P01021).

APPENDIX: NUMBER OF LOW *I* STATES FOR FIVE PARTICLES

First, we come to the case with five fermions in a single-j shell. We define $A(j,j_0,d)=[(j-j_0)/d]$ and $B(j,j_0,d)=(j-j_0)\%d$, and obtain

$$D\left(5,\frac{1}{2}\right)_j=6A^2\left(j,\frac{9}{2},12\right)+3A\left(j,\frac{9}{2},12\right)+\left[A\left(j,\frac{9}{2},12\right)+1\right]\left[B\left(j,\frac{9}{2},12\right)+1\right]+\delta_{B(j,9/2,12)}, \quad (A1)$$

where

$$\delta_{B(j,9/2,12)}=\begin{cases}-B\left(j,\frac{9}{2},12\right) & \text{if } B\left(j,\frac{9}{2},12\right)\leqslant 2\\ -2 & \text{if } 3\leqslant B\left(j,\frac{9}{2},12\right)\leqslant 4\\ -3 & \text{otherwise;}\end{cases}$$

$$D\left(5,\frac{3}{2}\right)_j=3A^2\left(j,\frac{11}{2},6\right)+4A\left(j,\frac{11}{2},6\right)+\left[A\left(j,\frac{11}{2},6\right)+1\right]\left[B\left(j,\frac{11}{2},6\right)+1\right]+\delta_{B(j,11/2,6)}+1, \quad (A2)$$

where

$$\delta_{B(j,11/2,6)}=\begin{cases}1 & \text{if } B\left(j,\frac{11}{2},6\right)\geqslant 4\\ 0 & \text{otherwise;}\end{cases}$$

$$D\left(5,\frac{5}{2}\right)_j=2A^2\left(j,\frac{9}{2},4\right)+3A\left(j,\frac{9}{2},4\right)+\left[A\left(j,\frac{9}{2},4\right)+1\right]\times\left[B\left(j,\frac{9}{2},4\right)+1\right]+\delta_{B(j,9/2,4)}+1, \quad (A3)$$

where

$$\delta_{B(j,9/2,4)}=\begin{cases}1 & \text{if } B\left(j,\frac{9}{2},4\right)\geqslant 2\\ 0 & \text{otherwise;}\end{cases}$$

and

$$D\left(5,\frac{7}{2}\right)_j = 6A^2\left(j,\frac{3}{2},6\right) + \left[2A\left(j,\frac{3}{2},6\right)+1\right] \times\left[B\left(j,\frac{3}{2},6\right)+1\right] + \delta_{B(j,3/2,6)} - 1, \quad (A4)$$

where

$$\delta_{B(j,3/2,6)} = \begin{cases} -1 & \text{if } 1 \leqslant B\left(j,\frac{3}{2},6\right) \leqslant 3 \\ 0 & \text{otherwise} \end{cases}.$$

An interesting behavior is that there exists an approximate relation for five fermions in a single-j shell: $D(5,I)_j \sim (I+\frac{1}{2})D(5,\frac{1}{2})_j$ when $I<j$.

Next, we come to five fermions in a single-l shell. For $I=0$, we define

$$k = \begin{cases} (l-2)/2 & \text{if } l\,\%\,2 = 0 \\ (l-11)/2 & \text{if } l\,\%\,2 = 1, \end{cases} \quad K = [k/6], \quad \mathcal{K} = k\,\%\,6,$$

and obtain

$$D(5,0)_l = 3K(K+1) - K + (\mathcal{K}+1)(K+1) + \delta_{\mathcal{K}0} - 1 \quad (A5)$$

for $I=1$, we define

$$k = \begin{cases} (l-1)/2 & \text{if } l\,\%\,2 = 1 \\ (l-4)/2 & \text{if } l\,\%\,2 = 0, \end{cases} \quad K = [k/2], \quad \mathcal{K} = k\,\%\,2,$$

and obtain

$$D(5,1)_l = K(K+1) + \mathcal{K}(K+1). \quad (A6)$$

We finally come to the case of five bosons with spin L. For $I=0$, we define

$$k = \begin{cases} L/2 & \text{if } L\,\%\,2 = 0 \\ (L-9)/2 & \text{if } L\,\%\,2 = 1, \end{cases} \quad K = [k/6], \quad \mathcal{K} = k\,\%\,6,$$

and obtain

$$D(5,0)_L = 3K(K+1) - K + (\mathcal{K}+1)(K+1) + \delta_{\mathcal{K}0} - 1. \quad (A7)$$

For $I=1$, we define

$$k = \begin{cases} L & \text{if } L\,\%\,2 = 1 \\ (L-3) & \text{if } L\,\%\,2 = 0, \end{cases} \quad K = [k/4],$$

$$\mathcal{K} = [(k\,\%\,4) - 1]/2,$$

and obtain

$$D(5,1)_L = (K+1)(K+\mathcal{K}+1). \quad (A8)$$

The formulas for larger I's with $n=5$ are more complicated and are not addressed in this paper.

---

[1] For example, P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, Berlin, 1980), p. 53.
[2] G. Racha, Phys. Rev. **63**, 367 (1943); A. de-Shalit and I. Talmi, *Nuclear Shell Model Theory* (Academic, New York, 1963).
[3] J. Katriel, R. Pauncz, and J. J. C. Mulder, Int. J. Quantum Chem. **23**, 1855 (1983); J. Katriel and A. Novoselsky, J. Phys. A **A22**, 1245 (1989).
[4] D. K. Sunko and D. Svrtan, Phys. Rev. C **31**, 1929 (1985); D. K. Sunko, *ibid.* **33**, 1811 (1986); **35**, 1936 (1987).
[5] J. N. Ginocchio and W. C. Haxton, in *Symmetries in Science VI*, edited by B. Gruber and M. Ramek (Plenum, New York, 1993), p. 263.
[6] Y. M. Zhao, A. Arima, and N. Yoshinaga, Phys. Rev. C **66**, 064322 (2002); D. Mulhall, A. Volya, and V. Zelevinsky, Nucl. Phys. **A682**, 229c (2001).
[7] For example, R. D. Lawson, *Theory of Nuclear Shell Model* (Clarendon, Oxford, 1980), p. 8–20.