Structure function of a damped harmonic oscillator

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Following the Caldeira-Leggett approach to describe dissipative quantum systems the structure function for a harmonic oscillator with Ohmic dissipation is evaluated by an analytic continuation from Euclidean to real time. The analytic properties of the Fourier transform of the structure function with respect to the energy transfer (the "characteristic function") are studied and utilized. In the one-parameter model of Ohmic dissipation we show explicitly that the broadening of excited states increases with the state number without violating sum rules. Analytic and numerical results suggest that this is a phenomenologically relevant, consistent model to include the coupling of a single (sub)nuclear particle to unobserved and complex degrees of freedom.

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I. INTRODUCTION

The structure functions measured in deep inelastic lepton scattering provide important information about the nature and momentum distribution of the constituents of the target. This is particularly true in the relativistic domain where the pointlike building blocks of matter, the quarks and gluons, have been discovered and studied by inclusive scattering of multi-GeV electrons, muons, or neutrinos. In a nonrelativistic description the (longitudinal) structure function for one scalar particle is given by

$$S(q,\nu)=\sum_n \int \delta(\nu-(E_n-E_0))|\langle n|\exp(iq\cdot\hat{x})|0\rangle|^2, \tag{1}$$

where the summation is over all discrete and continuum states which are excited by the probe. Here q,ν denote momentum and energy transfer and E_n the excitation energies of the target. Very often the confinement of quarks is described by using rising potentials which lead to a purely discrete spectrum. The simplest version is given by a harmonic oscillator (ho) potential which has a structure function [1]

$$S^{\mathrm{ho}}(q,\nu)=\sum_{n=0}^{\infty}\frac{\delta(\nu-n\omega_0)}{n!}\left(\frac{q^2b_0^2}{2}\right)^n\exp\left(-\frac{q^2b_0^2}{2}\right). \tag{2}$$

The oscillator frequency and length are denoted by ω_0 and $b_0=1/\sqrt{m\omega_0}$, where m is the mass of the struck particle. For simplicity, all many-body and recoil effects have been neglected and the elastic line $(n=0)$ is included in the sum over excited states. However, the observed structure functions are smooth due to hadronization and/or final state interactions. A number of recent theoretical studies have accounted for that by simply smearing out the δ functions in Eq. (2) by a Breit-Wigner distribution with a constant width or averaged over nearby ν bins [2]. It is obvious that this is not only *ad hoc* but also may violate general properties of the structure function, e.g., the fact that it has to vanish below the first (positive) excitation energy.

It is the purpose of this paper to demonstrate that a consistent quantum-mechanical framework exists which allows to treat couplings to unobserved degrees of freedom in a simple manner. Several methods have been used in the past to achieve that, for example, for Coulomb excitation of particle-unstable states [3]. For quasielastic scattering of electrons from nuclei, Horikawa *et al.* [4] first have included multinucleon channels by employing an optical potential without violating the nonenergy-weighted sum rule (NEWSR)

$$\int_0^\infty d\nu S(q,\nu)=1. \tag{3}$$

However, there exists a simpler treatment based on the description of dissipative quantum systems within the path integral formalism [5]. This originates in the celebrated work of Feynman and Vernon [6] and Caldeira and Leggett [7] who have modeled the coupling of the system to an environment of $N(\to\infty)$ harmonic oscillators,

$$H=\frac{p^2}{2m}+V(x)+\sum_{n=1}^{N}\left(\frac{p_n^2}{2m_n}+\frac{1}{2}m_n\omega_n^2x_n^2\right)-x\sum_{n=1}^{N}c_nx_n+x^2\sum_{n=1}^{N}\frac{c_n^2}{2m_n\omega_n^2} \tag{4}$$

with a bilinear coupling. The limit $N\to\infty$ of the number of environmental oscillators is essential in preventing the bounded motion of all particles to come back to the initial state after some time: the so-called Poincaré recurrence time then tends to infinity [8] and irreversibility becomes possible—still in a unitary quantum-mechanical framework. The infinite number of degrees of freedom also allows for strong damping even if each environmental oscillator couples only weakly to the system. This mechanism leads to

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a broadening (and a shift) of the δ functions in the structure function of the confined system without violating the sum rules.

In this approach the path integral description of the system offers particular advantages since the bath oscillators can be integrated out exactly. This gives rise to a retarded two-time action for the single particle which does not have a Hamiltonian counterpart anymore similar as in the time-honored polaron problem [9]. In particular, no Schrödinger equation for the single-particle motion is available. However, if this particle moves in a harmonic potential

$$V(x)=\tfrac{1}{2}m\omega_0^2x^2 \tag{5}$$

then the remaining path integral can also be done exactly. In Sec. II we will employ this formalism and the explicit results for the damped harmonic oscillator to obtain the Fourier transform of the structure function with respect to the energy transfer. As thermal physics lives in Euclidean times one needs an analytic continuation to real times which is performed in Sec. III. Numerical results are presented in Sec. IV while the conclusions are given in the final section. Some technical details are collected in the two appendixes.

II. CHARACTERISTIC FUNCTION AND CORRELATION FUNCTIONS OF DISSIPATIVE SYSTEMS

We will calculate the structure function from its Fourier transform,

$$S(q,\nu)=:\frac{1}{2\pi}\int_{-\infty}^{+\infty}dt e^{i\nu t}\Phi_q(t), \tag{6}$$

the "characteristic function" [10]. For the pure harmonic oscillator one has

$$\Phi_q^{\rm ho}(t)=\exp\left[-\frac{1}{2}q^2b_0^2(1-e^{-i\omega_0 t})\right], \tag{7}$$

which after expansion and integration leads to the result given in Eq. (2). For the damped harmonic oscillator the characteristic function can be related to the particular correlation function in *Euclidean* time τ,

$$T(q,\tau)=\langle 0|\mathcal{T}(e^{-iq\cdot\hat{x}(\tau)}e^{iq\cdot\hat{x}(0)})|0\rangle \tag{8}$$

by an analytic continuation. From the spectral representations

$$\left.\begin{matrix}\Phi_q(t)\\ T(q,\tau)\end{matrix}\right\}=\sum_n |\langle n|\exp(iq\cdot\hat{x})|0\rangle|^2\times\left\{\begin{matrix}\exp[-i(E_n-E_0)t]\\ \exp[-(E_n-E_0)|\tau|],\end{matrix}\right. \tag{9}$$

one sees that both expressions coincide for positive Euclidean time and the correct analytic continuation is therefore obtained by considering $T(q,\tau>0)$ and replacing $\tau\to it$.

In the path integral approach we have

$$T(q,\tau)=\lim_{\beta\to\infty}\frac{\int\mathcal{D}x\exp\{-iq\cdot[x(\tau)-x(0)]\}\exp\{-\mathcal{A}[x]\}}{\int\mathcal{D}x\exp\{-\mathcal{A}[x]\}}. \tag{10}$$

Here $\mathcal{A}[x]$ is the effective action of the particle after the oscillators of the environment have been integrated out and the limit $\beta\to\infty$ of the final Euclidean time projects out the ground state of the system [13]. In this limit the boundary conditions for the path integrals in Eq. (10) do not matter; therefore we may set $x(-\beta/2)=x(\beta/2)=x$ and integrate over x, i.e., perform the thermodynamical trace. This allows us to directly take over results from dissipative quantum systems where similar correlation functions (e.g., for the position operator) have been evaluated at finite temperature, i.e., finite β [5]. Since this is quite standard now we can be brief and immediately use results from the nice review by Ingold [14], in particular from Chap. 4.3 with the driving force $F(\sigma)=iq[\delta(\sigma-\tau)-\delta(\sigma)]$. In the limit $\beta\to\infty$ the sum over Matsubara frequencies turns into an integral so that the final result for the Euclidean correlation function reads

$$T(q,\tau)=\exp\left[-\frac{q^2}{2m}\frac{1}{\pi}\int_{-\infty}^{+\infty}dE\frac{1-\cos(E\tau)}{E^2+|E|\gamma(|E|)+\omega_0^2}\right]. \tag{11}$$

Here

$$\gamma(E)=\frac{2}{\pi m}\int_0^\infty d\omega\frac{J(\omega)}{\omega}\frac{E}{E^2+\omega^2} \tag{12}$$

is the damping kernel which is produced by the coupling of the system to the heat bath. One does not have to specify all masses, frequencies, and coupling constants in Eq. (4) but only the spectral density $J(\omega)$ of the environment oscillators. The simplest assumption is *Ohmic dissipation*

$$J_{\rm Ohm}=m\gamma\omega\Rightarrow\gamma_{\rm Ohm}(E)=\gamma. \tag{13}$$

Although some observables (e.g., the ground state energy) need high-frequency cutoffs [15] this form can be used with impunity for the structure function where only energy differences matter. For simplicity it will also be employed in the following. We then obtain

$$T(q,\tau)=\exp\{-2q^2[x_V(0)-x_V(\tau)]\} \tag{14}$$

with

$$x_V(\tau)=\frac{1}{2m\pi}\int_0^\infty dE\frac{\cos(E\tau)}{E^2+\gamma E+\omega_0^2}\quad(\tau\ \text{real}). \tag{15}$$

The analytic continuation of the above function which coincides with $x_V(\tau)$ for positive Euclidean time

$$\xi_V(\tau)=x_V(\tau)\ (\tau\geqslant 0) \tag{16}$$

may be called the *Vineyard function*, since many years ago Vineyard [17] derived an identical form of the structure function for inclusive scattering of slow neutrons from quantum liquids. Its nice feature is the clean separation between the squared momentum transfer and the variable t which is conjugate to the energy transfer. Such a form was also obtained in a (zeroth-order) variational calculation of relativistic deep inelastic scattering from a scalar particle where the broadening of the elastic line was due to multiple meson production [11]. We thus have found a simple description of inclusive scattering with one additional parameter γ which accounts for the coupling of the particle to additional degrees of freedom [18]. These could be the continuum and/or many-nucleon emission in the case of quasielastic scattering from nuclei or the production of colorless hadrons in the case of scattering from quarks.

Note that $T(q,\tau=0)=1$ so that the sum rule (3) is fulfilled. We will also see that the resulting structure function has the correct support, i.e., no unphysical excitations occur. This is because the Caldeira-Leggett model is based on a consistent many-body Hamiltonian and the environmental degrees of freedom have been integrated out without approximation. In contrast, other descriptions of the damped harmonic oscillator (dho) [20], friction [21] or time asymmetry [22] in general require a modification of usual quantum mechanics.

III. ANALYTIC CONTINUATION

The main task left over is to perform the analytic continuation to get the characteristic function $\Phi_q(t)$ from the correlation function $T(q,\tau)$. This requires the analytic continuation of the Vineyard function $\xi_V(\tau\to it)$.

It should be emphasized that $x_V(\tau)$ and $\xi_V(\tau)$ are *different* functions which only coincide for $\tau\geqslant 0$. In particular, although from Eq. (15) $x_V(\tau)$ is even in τ we will see that the Vineyard function has a logarithmic cut on the negative real τ axis. In the following we will always indicate the range of validity of the representation for the Vineyard function in parenthesis as was done in Eq. (16). It is obvious that this form cannot be used for analytic continuation to Minkowski time (where scattering occurs) because the cosine function would blow up. As shown in Appendix A we may, however, distort the integration contour as in Ref. [23] and obtain for $\tau>0$,

$$\xi_V(\tau)=\frac{\gamma}{2m\pi}\int_0^\infty dE\frac{E}{(E^2-\omega_0^2)^2+\gamma^2E^2}e^{-E\tau}$$
$$=:\int_0^\infty dE\,\rho(E)e^{-E\tau}\quad(\mathrm{Re}\tau\geqslant 0). \tag{17}$$

This now allows an analytic continuation $\tau\to it$ to obtain the characteristic function

$$\Phi_q^{\mathrm{dho}}(t)=\exp\{-2q^2[\xi_V(0)-\xi_V(it)]\}. \tag{18}$$

To see the effects of the damping we insert Eq. (17) into Eqs. (18) and (6), expand the exponential, and perform the t integration. This gives

$$S^{\mathrm{dho}}(q,\nu)=e^{-q^2b^2/2}\delta(\nu)+\sum_{n=1}^{\infty}\frac{(2q^2)^n}{n!}e^{-q^2b^2/2}\int_0^\infty dE_1\cdots dE_n\times\delta\left(\nu-\sum_{k=1}^{n}E_k\right)\rho(E_1)\cdots\rho(E_n). \tag{19}$$

The first term is the elastic line with the square of the typical Gaussian form factor for the harmonic oscillator. However, the oscillator length is renormalized by the interaction with the environment:

$$b_0^2\to b^2=4\xi_V(0)=b_0^2\frac{\omega_0}{\Omega}\frac{2}{\pi}\arctan\left(\frac{2\Omega}{\gamma}\right)\leqslant b_0^2 \tag{20}$$

with

$$\Omega=\sqrt{\omega_0^2-\frac{\gamma^2}{4}}. \tag{21}$$

(We are only considering the underdamped case $\gamma<2\omega_0$.) Compared with Eq. (2) all excited states are now broadened; in particular, the $(n=1)$ term just maps the weight function

$$2q^2e^{-q^2b^2/2}\Theta(\nu)\rho(\nu)=\frac{q^2b_0^2}{2}e^{-q^2b^2/2}\frac{2\omega_0}{\nu+\omega_0}\frac{\Theta(\nu)}{2\pi}\times\frac{\Gamma(\nu)}{(\nu-\omega_0)^2+\Gamma^2(\nu)/4}. \tag{22}$$

Apart from the additional factor $2\omega_0/(\nu+\omega_0)$ the line shape is just a one-sided Breit-Wigner distribution with the energy-dependent width

$$\Gamma(E)=\frac{2E\gamma}{E+\omega_0}. \tag{23}$$

Note that this distribution vanishes at threshold and is identically zero for unphysical negative energy transfers. The same holds for all other terms in expansion (19) due to the δ function and the fact that all $E_k\geqslant 0$. Thus the structure function of the damped harmonic oscillator has the correct support. Unfortunately, it is not possible to evaluate the higher-order terms analytically. Only in the *narrow-width approximation*

$$\Gamma(E)\approx\Gamma(\omega_0)=\gamma \tag{24}$$

a simple result is found when additionally the prefactor is also evaluated at $\nu=\omega_0$ and the E integration extended to $-\infty$:

$$\xi_V(it)\approx\frac{b_0^2}{4}\exp\left(-i\omega_0t-\frac{\gamma}{2}|t|\right). \tag{25}$$

Note that an *exact* expression of this form together with a real correction $r(t)$ is derived in Appendix A. One then sees from Eqs. (18) and (6) that the δ function in the nth excited state contribution to the structure function (2) of the harmonic oscillator turns into a Breit-Wigner function with a width

$$\Gamma_n \approx n\gamma. \tag{26}$$

This result is well known from the density of states of the damped harmonic oscillator [24] and has also been discussed for the width of multiphonon giant resonances [25]. In the present context, however, it should be stressed that it is only approximate and leads to a small, but nonvanishing structure function for $\nu<0$ [26]. This is due to the wrong analytic behavior of the Vineyard function in approximation (25) where $|t|=\sqrt{t^2}$ also produces a cut for $\mathrm{Im}t<0$. In contrast, the exact expression has only a logarithmic cut in the upper-half t plane in accord with general properties of the characteristic function. This can be best seen in the explicit expression of the Vineyard function with Ohmic damping in terms of the standard exponential integral which is derived in Appendix A. From Eq. (A7) it takes the form

$$\xi_V(\tau) = \text{regular function} - \frac{1}{2m\pi\Omega}\sinh(\Omega\tau) \times \sin\left(\frac{\gamma}{2}\tau\right)\ln(\omega_0\tau), \quad |\arg\tau|<\pi \tag{27}$$

for arbitrary complex τ away from the cut. Evaluating Eq. (6) for $\nu<0$ by closing the integration contour in the lower-half t plane, one therefore encounters no singularities and the structure function vanishes identically. The logarithmic cut of the Vineyard function also shows up in the low-t expansion [see Eqs. (A18) and (A19)]

$$\xi_V(it) = \xi_V(0) - \frac{it}{4m} + \frac{\gamma}{4m\pi}t^2\ln(i\omega_0 t) - \frac{t^2}{4m\pi\Omega}\left[\left(\Omega^2 - \frac{\gamma^2}{4}\right)\arctan\left(\frac{2\Omega}{\gamma}\right) + \gamma\Omega\left(\frac{3}{2} - \gamma_E\right)\right] + O(t^3), \tag{28}$$

which has important consequences: first, we see that also the energy-weighted sum rule (EWSR)

$$\int_0^\infty d\nu\, \nu S(q,\nu) = i\,\Phi_q'(0) = \frac{q^2}{2m} \tag{29}$$

is conserved [27] whereas higher-energy moments of the structure function diverge. This high-energy tail reflects, of course, the insufficient suppression of high frequencies in the simple model of Ohmic dissipation which may need modification for phenomenological applications in nuclear and quark physics. Second, it is even possible to determine the tail for large energy transfer analytically: as shown in Appendix B one obtains

$$S^{\mathrm{dho}}(q,\nu) \overset{\nu\to\infty}{\longrightarrow} \frac{\gamma}{m\pi}\frac{q^2}{\nu^3} + \frac{3}{2}\frac{\gamma}{m^2\pi}\frac{q^4}{\nu^4} + O\left(\frac{\ln\nu}{\nu^5}\right). \tag{30}$$

Compared with Eq. (19) one sees that the suppression of the low-lying states by the square of the elastic form factor has disappeared and the asymptotic form (30) does not depend anymore on ω_0 or the oscillator parameter b—a property which roughly resembles the conjectured "quark-hadron duality" [2].

Another consequence of Eq. (28) is that logarithmic corrections to y scaling will persist even for large momentum transfer. This is because interaction times $t\sim 1/q$ are probed in that limit [28] and therefore $q^2t^2\ln(i\omega_0 t)$ remains unbounded for $q\to\infty$.

IV. NUMERICAL RESULTS

Sticking to pure Ohmic dissipation we next try to evaluate the structure function quantitatively. Only for $q\to 0$ (i.e., photoabsorption) expansion (19) into a sum over excited states is useful. For arbitrary momentum transfer the numerical problem is much harder as one has to calculate the *inelastic* structure function

$$S^{\mathrm{dho}}_{\mathrm{inelastic}}(q,\nu) = \frac{1}{2\pi}\exp[-2q^2\xi_V(0)] \times \int_{-\infty}^{+\infty} dt e^{i\nu t}\{\exp[2q^2\xi_V(it)]-1\} \tag{31}$$

after subtraction of the elastic line (a δ function) as a Fourier transform over an infinite interval. One may alleviate the numerical problem slightly by expressing the inelastic structure function as a sine transform over the imaginary part of the characteristic function as demonstrated in Ref. [12],

$$S^{\mathrm{dho}}_{\mathrm{inelastic}}(q,\nu) = e^{-(1/2)q^2b^2}\frac{2}{\pi}\int_0^\infty dt\,\sin(\nu t)(-\mathrm{Im}) \times\{\exp[2q^2\xi_V(it)]\}. \tag{32}$$

This holds since $\xi_V(it)$ vanishes as $1/t^2$ in a sector of the complex t plane which includes the lower half plane (see Appendix A). Thus one may write a Cauchy integral representation for the (inelastic) characteristic function and express its real part in terms of the imaginary part. We have used the adaptive integration routine D01ASF from the NAG library together with the explicit representations (A8) and (A13) of the Vineyard function to perform the numerical evaluation of Eq. (32).

Figures 1 and 2 show the results of the calculation for several momentum transfers and damping parameters. It is seen that the excitation of individual levels gradually moves into the broad structure of the quasielastic peak as the momentum transfer increases. The value $\gamma/\omega_0=0.2$ corresponds roughly to the one used in Ref. [29] where a parametrization of the response function was fitted to photoabsorption and electron scattering data in ^{12}C. A peak position of 22.7 MeV

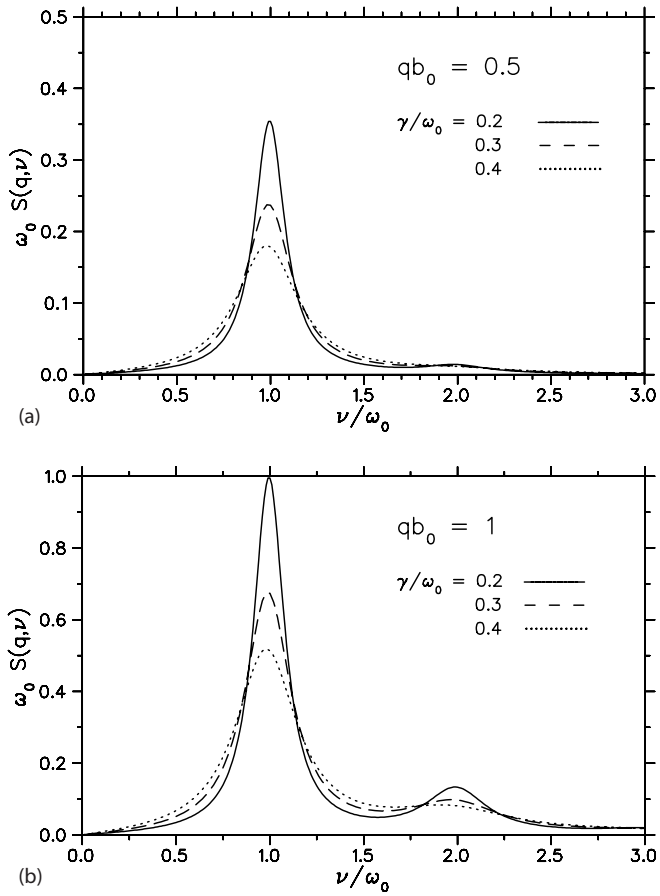

FIG. 1. The structure function of a damped harmonic oscillator as a function of the energy transfer for momentum transfers $qb_0 = 0.5$ (top) and $qb_0 = 1$ (bottom). Note the different scales in both plots. The solid curves are for a value of the Ohmic damping parameter $\gamma/\omega_0 = 0.2$, the dashed one for $\gamma/\omega_0 = 0.3$, and the dotted one for $\gamma/\omega_0 = 0.4$. The undamped oscillator length and frequency are denoted by b_0 and ω_0, respectively.

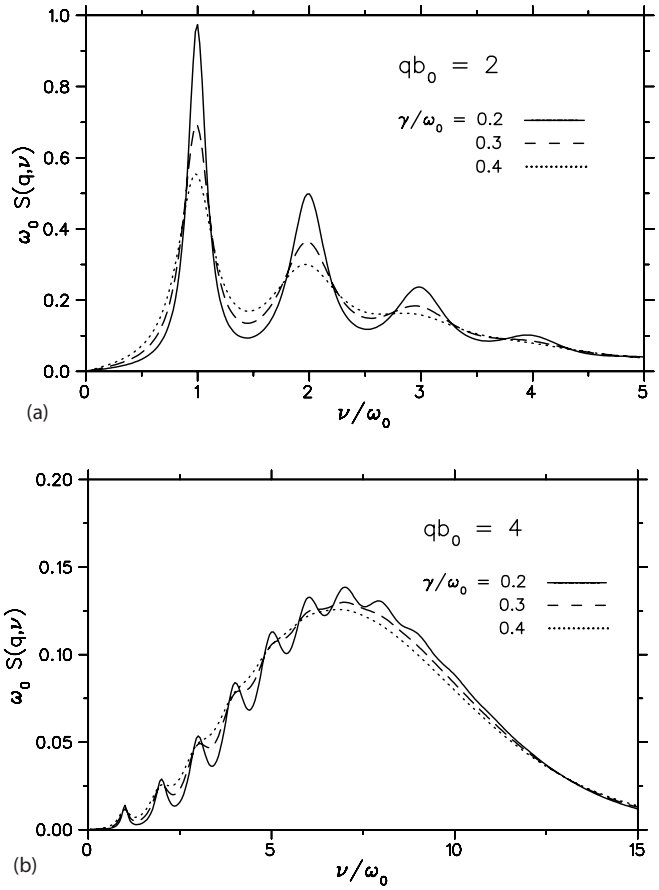

FIG. 2. Same as in Fig. 1 but for $qb_0 = 2$ (top) and $qb_0 = 4$ (bottom).

and a full width at half maximum of $(2\sqrt{\ln 2} \times 2.6 = 4.3)$ MeV was obtained for the giant resonance. Although accounting for the coupling of this excitation to many-body states and the continuum this value of γ/ω_0 gives too much structure from individual levels at higher q compared with typical experimental cross section. This may indicate that a modification of the assumed Ohmic damping is needed for a description of medium-energy inclusive scattering data from nuclei.

That the sum rules are well preserved can be seen in Tables I and II where the relative size of different contributions is presented for $\gamma/\omega_0 = 0.2$ (other values of the damping parameter yield similar results). The numerical part was obtained by integrating the structure functions shown in Figs. 1 and 2 with weight $\nu^n, n = 0,1$ from $\nu = 0$ to $\nu = \nu_{max}$ in steps of $\Delta\nu$ by means of a Simpson rule. The NEWSR, of course, also gets a contribution $\exp(-q^2b^2/2)$ from the elastic line. Also listed are the asymptotic contributions obtained from Eq. (30) by integrating from ν_{max} to ∞. It is seen that the asymptotic contributions substantially improve the convergence to the correct values except for a few cases at high momentum where ν_{max} is not large enough.

Due to its weighting the EWSR is, of course, not as well fulfilled as the NEWSR, but the agreement is very satisfactory and sufficient to demonstrate the consistency of *all* parts of the calculation: elastic line, inelastic excitations, and asymptotic behavior of the structure function.

Although the NAG routine D01ASF does an excellent job in evaluating the oscillating integral over the characteristic function one may ask whether it is possible to introduce additional damping by deforming the integration contour in Eq. (31) in the upper-half t plane. Indeed, running along both sides of the cut would eliminate the oscillating exponential factor altogether.

However, this is not possible since as shown in Appendix A there are *Stokes* lines in the upper-half t plane which separate the powerlike decrease of the Vineyard function $\xi_V(it)$ from an exponential increase which would overwhelm the exponential damping from the factor $\exp(i\nu t)$. The optimal damping which is achievable without contributions from the arcs at infinity is a rotation by an angle

$$\varphi_1 = \arctan\left(\frac{\gamma}{2\Omega}\right) \tag{33}$$

TABLE I. Test of the non-energy-weighted sum rule (3) for different momentum transfers q and upper limits ν_{max} of the numerical integration performed in steps of $\Delta\nu$. The parameter for Ohmic damping is taken as $\gamma/\omega_0=0.2$. The columns labeled "elast." and "num." denote the contribution of the elastic line and the result from numerical integration up to $\nu=\nu_{max}$, respectively. The columns "asy." list the leading-order (LO) and next-to-leading (NLO) asymptotic contributions from ν_{max} to infinity. In places with no entry the previous value applies.

qb_0	ν_{max}/ω_0	$\Delta\nu/\omega_0$	elast.	num.	asy. (LO)	asy. (NLO)	Total
0.5	3.0	0.01	0.88904	0.10966	0.00088	0.00007	0.99965
	4.0			0.11037	0.00050	0.00003	0.99994
	5.0			0.11061	0.00031	0.00002	0.99998
1.0	3.0		0.62471	0.36250	0.00354	0.00118	0.99193
	5.0			0.37334	0.00127	0.00025	0.99957
	7.0			0.37448	0.00065	0.00009	0.99993
2.0	5.0	0.02	0.15230	0.80314	0.00509	0.00407	0.96460
	8.0			0.84239	0.00200	0.00099	0.99768
	10.0			0.84532	0.00127	0.00051	0.99940
4.0	15.0	0.05	0.00054	0.97227	0.00226	0.00241	0.97749
	20.0			0.99455	0.00127	0.00102	0.99738
	25.0			0.99744	0.00082	0.00052	0.99932

which is depicted in Fig. 3. We have evaluated

$$S^{\rm dho}_{\rm inelastic}(q,\nu)=e^{-(1/2)q^2b^2}\frac{1}{\pi}{\rm Re}\left(e^{i\varphi}\int_0^\infty dx\,\exp[i\nu x e^{i\varphi}]\times\{\exp[2q^2\,\xi_V(\tau=ixe^{i\varphi})]-1\}\right) \tag{34}$$

by standard Gaussian integration after mapping the infinite interval to a finite one. As expected the damping factor $\exp(-\nu x\sin\varphi)$ is most beneficial for large energy transfer whereas a greater number of integration points is needed to avoid a negative numerical result for the structure function at small ν. Thus the contour rotation method is complementary to an explicit summation over excited lines as given in Eq. (19).

V. SUMMARY

We have shown that the Caldeira-Leggett model of the damped harmonic oscillator also provides a consistent model for inclusive processes where it accounts for the coupling of a single particle to more complicated states and additional degrees of freedom. With only one additional (damping) parameter in the simplest version with purely Ohmic dissipation a highly nontrivial structure function has been obtained which conserves the sum rules. Its characteristic function has been given in closed analytic form with a clear separation between momentum transfer and "time," the variable conjugate to the energy transfer in the process. This allowed us to concentrate on the study of a function of a single variable—the Vineyard function—and its analytic properties which determine the dynamics. As an extension to the many-body case is straightforward further applications in nuclear or hadronic physics seem to be possible.

TABLE II. Same as in Table I but for the energy-weighted sum rule (29) divided by $q^2/(2m)$. Note that no elastic contribution exists in this case. The Ohmic parameter is again $\gamma/\omega_0=0.2$.

qb_0	ν_{max}/ω_0	$\Delta\nu/\omega_0$	num.	asy. (LO)	asy. (NLO)	Total
0.5	3.0	0.01	0.9448	0.0424	0.0027	0.9899
	4.0		0.9638	0.0318	0.0015	0.9971
	5.0		0.9723	0.0255	0.0009	0.9987
1.0	3.0		0.8882	0.0425	0.0106	0.9413
	5.0		0.9653	0.0255	0.0038	0.9946
	7.0		0.9785	0.0182	0.0020	0.9987
2.0	5.0	0.02	0.8512	0.0255	0.0153	0.8920
	8.0		0.9668	0.0159	0.0060	0.9887
	10.0		0.9797	0.0127	0.0038	0.9962
4.0	15.0	0.05	0.9356	0.0085	0.0068	0.9509
	20.0		0.9819	0.0064	0.0038	0.9921
	25.0		0.9898	0.0051	0.0024	0.9973

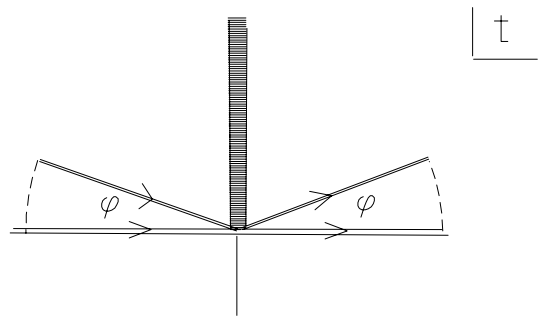

FIG. 3. Integration contours in the complex t plane for the numerical evaluation of the structure from the characteristic function: original path along the real axis (solid line) and path rotated by an angle φ into the upper half plane (double line). The arcs at infinity (dashed curves) do not give a contribution as long as $\varphi<\varphi_1$ where φ_1 is defined in Eq. (33). The logarithmic cut along the positive imaginary axis is shown as hatched strip.

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APPENDIX A: THE VINEYARD FUNCTION FOR THE HARMONIC OSCILLATOR WITH OHMIC DAMPING

Here we derive and collect a few properties of the Vineyard function defined by Eqs. (16) and (15) in Euclidean time $\tau\geqslant 0$,

$$\xi_V(\tau)=\frac{1}{2m\pi}\int_0^\infty dE\frac{\cos(E\tau)}{E^2+\gamma E+\omega_0^2}. \tag{A1}$$

One may decompose the cosine function into exponentials and deform the integration path such that it runs along the imaginary axes. There is no contribution from poles of the integrand (which are all in the left-hand E plane) and one therefore obtains (first for $\tau\geqslant 0$)

$$\xi_V(\tau)=\frac{\gamma}{2m\pi}\int_0^\infty dE\frac{E}{(E^2-\omega_0^2)^2+\gamma^2E^2}e^{-E\tau}\quad(\mathrm{Re}\tau\geqslant 0). \tag{A2}$$

This is suitable for analytic continuation $\tau\to it$ since the integral also converges for $\mathrm{Re}\tau\geqslant 0$. Using partial fractions one gets

$$\begin{aligned}\xi_V(\tau)&=\frac{1}{8m\pi i\Omega}\sum_{r,s=\pm 1}rs\int_0^\infty dE\frac{1}{E+E_{r,s}}e^{-E\tau}\\&=\frac{1}{8m\pi i\Omega}\sum_{r,s=\pm 1}rse^{E_{r,s}\tau}E_1(E_{r,s}\tau),\quad\tau\geqslant 0\end{aligned} \tag{A3}$$

with

$$E_{r,s}=r\Omega+is\frac{\gamma}{2},\quad r,s=\pm 1. \tag{A4}$$

Here

$$E_1(z)=-\gamma_E-\ln z-\sum_{n=1}^\infty\frac{(-z)^n}{nn!}=:-\gamma_E-\ln z+\mathrm{Ein}(z) \tag{A5}$$

is the standard exponential integral and $\gamma_E=0.577\,215\,66\ldots$ Euler's number. A careful evaluation of the arguments of the logarithm for $\tau\geqslant 0$ then gives

$$\begin{aligned}\xi_V(\tau)=\frac{1}{8m\pi\Omega}\sum_{r,s=\pm 1}e^{E_{r,s}\tau}\Bigg\{&\arctan\left(\frac{2\Omega}{\gamma}\right)-r\frac{\pi}{2}\\&+irs[\gamma_E+\ln(\omega_0\tau)-\mathrm{Ein}(E_{r,s}\tau)]\Bigg\},\end{aligned} \tag{A6}$$

which defines the Vineyard function for *arbitrary* complex τ with $|\arg\tau|<\pi$. Since $\mathrm{Ein}(z)$ is an entire function (Ref. [30], Chap. 5.1, footnote 3) one sees that

$$\begin{aligned}\xi_V(\tau)=&\,\text{regular function}\\&-\frac{1}{2m\pi\Omega}\sinh(\Omega\tau)\sin\left(\frac{\gamma}{2}\tau\right)\ln(\omega_0\tau),\end{aligned} \tag{A7}$$

which allows to determine the discontinuity across the cut.

For purely imaginary arguments $\tau=it,t$ real, Eq. (A6) can be written as

$$\xi_V(it)=\frac{1}{4m\Omega}\exp\left(-i\Omega t-\frac{\gamma}{2}|t|\right)-r(t), \tag{A8}$$

$$r(t)=\frac{1}{4m\pi\Omega}\mathrm{Im}[e^{z_\gamma(t)}E_1(z_\gamma(t))-(\gamma\to-\gamma)],$$

$$z_\gamma(t)\equiv\left(i\Omega+\frac{\gamma}{2}\right)|t|. \tag{A9}$$

The first term corresponds to the narrow-width approximation (25) whereas the last term (which is real, even in t and vanishing for $\gamma=0$) corrects for its deficiencies.

Another representation of the remainder function $r(t)$ is obtained by using the identity

$$\begin{aligned}&\frac{E}{(E^2-\omega_0^2)^2+\gamma^2E^2}\\&\quad=\frac{1}{4\Omega}\left[\frac{1}{(E-\Omega)^2+\gamma^2/4}-\frac{1}{(E+\Omega)^2+\gamma^2/4}\right],\end{aligned} \tag{A10}$$

which shows that the weight function $\rho(E)$ can also be considered as a (anti)symmetrized Breit-Wigner distribution around $E=\Omega$. Inserting this into Eq. (A2) one obtains for $\tau=it, t$ real,

$$\xi_V(it)=\frac{\gamma}{8m\pi\Omega}\left\{\int_{-\infty}^{+\infty}dE\,\frac{\exp(-iEt)}{(E-\Omega)^2+\gamma^2/4}-\int_{-\infty}^{0}dE\,\frac{\exp(-iEt)}{(E-\Omega)^2+\gamma^2/4}-\int_{0}^{\infty}dE\frac{\exp(-iEt)}{(E+\Omega)^2+\gamma^2/4}\right\}$$
$$=\frac{\gamma}{8m\pi\Omega}\left\{\frac{2\pi}{\gamma}\exp\left(-i\Omega t-\frac{\gamma}{2}|t|\right)-2\int_{0}^{\infty}dE\,\frac{\cos(Et)}{(E+\Omega)^2+\gamma^2/4}\right\}. \tag{A11}$$

Hence

$$r(t)=\frac{\gamma}{4m\pi\Omega}\int_0^\infty dE\,\frac{\cos(Et)}{(E+\Omega)^2+\gamma^2/4},\quad t\ \text{real}. \tag{A12}$$

Again one may distort the integration path such that it runs along the imaginary axis. Decomposing the cosine function into exponentials and realizing that the integrand in Eq. (A12) has only poles at $E=-\Omega\pm i\gamma/2$, one then obtains a representation

$$r(t)=\frac{\gamma}{2m\pi}\int_0^\infty dE\frac{E}{(E^2-\omega_0^2)^2+4\Omega^2E^2}e^{-E|t|},\quad t\ \text{real} \tag{A13}$$

which is very well suited for numerical evaluation. Indeed, we have checked the routine which calculates the Vineyard function $\xi_V(it)$ based on the exponential integral (ei) representation (A9) by a direct Gaussian integration of Eq. (A13) and found a relative deviation

$$\left|\frac{\xi_V^{\text{ei}}(it)-\xi_V^{\text{num}}(it)}{\xi_V^{\text{num}}(it)}\right|<2\times10^{-6} \tag{A14}$$

for all real t and $\gamma/\omega_0\leqslant0.4$. In this comparison the complex exponential integral $E_1(z)$ was calculated by the rational approximations with $n=10$ terms given in Ref. [31]: Table 64.4 was used for $|z|<9$ and Table 64.5 for $|z|\geqslant9$ and checked against values listed in Table 5.6 of Ref. [30]. Figure 4 shows $\xi_V(\tau)$ [relative to the undamped case $b_0^2\exp(-\omega_0\tau)/4$] and Fig. 5 the remainder function $r(t)$ for the chosen values of the damping parameter γ.

Note that one has a simple result for the *imaginary* part (which is odd in t),

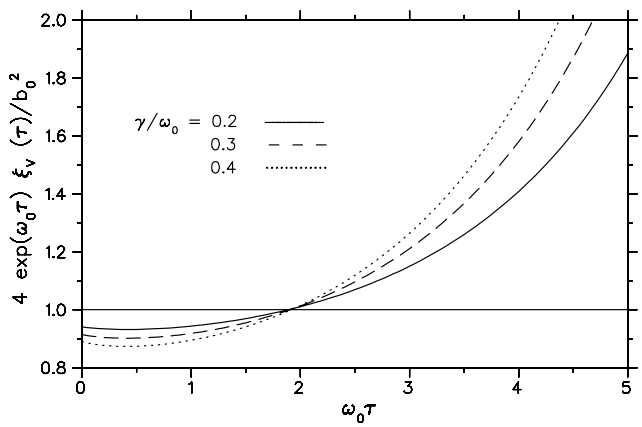

FIG. 4. The Vineyard function $\xi_V(\tau)$ normalized to the undamped case as a function of the Euclidean time τ. The solid curve is for a value of the Ohmic damping parameter $\gamma/\omega_0=0.2$, the dashed one for $\gamma/\omega_0=0.3$, and the dotted one for $\gamma/\omega_0=0.4$.

$$\text{Im}\,\xi_V(it)=-\frac{1}{4m\Omega}\sin(\Omega t)e^{-\gamma|t|/2}, \tag{A15}$$

but that the real part is more involved. This can also be seen if we use the standard representation of the step function

$$\Theta(E)=-\frac{1}{2\pi i}\int_{-\infty}^{+\infty}ds\,\frac{\exp(-isE)}{s+i\epsilon} \tag{A16}$$

to extend the integration range in Eq. (A2) to $-\infty$. Using identity (A10) simple manipulations then give

$$\xi_V(it)=\frac{1}{4m\pi\Omega}\int_{-\infty}^{+\infty}ds\,\frac{\sin(\Omega s)}{s-t+i\epsilon}e^{-\gamma|s|/2},\quad t\ \text{real} \tag{A17}$$

which shows that the real part is determined by a principal-value integral whereas the imaginary part is given by Eq. (A15).

Finally we consider the behavior of the Vineyard function for small and large τ. From Eq. (A6) one immediately obtains

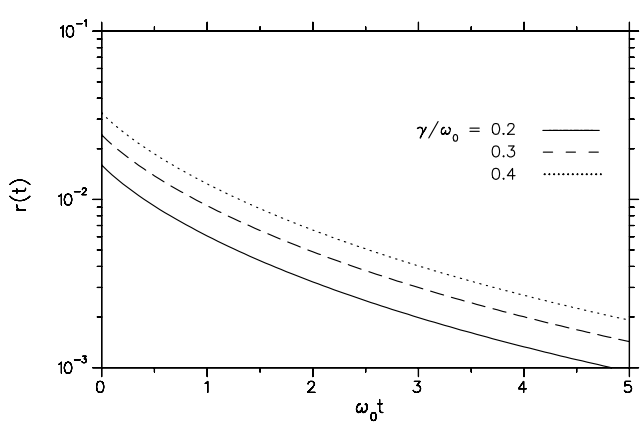

FIG. 5. The correction term $r(t)$ to the narrow-width approximation (A8) for the Vineyard function $\xi_V(it)$ as a function of the time t. Note the logarithmic scale for $r(t)$.

$$\xi_V(\tau)=\sum_{k=0}^{\infty} a_k\tau^k+\sum_{k=1}^{\infty} b_{2k}\tau^{2k}\ln(\omega_0\tau) \tag{A18}$$

with

$$a_0=\frac{1}{2m\pi\Omega}\arctan\left(\frac{2\Omega}{\gamma}\right),\quad a_1=-\frac{1}{4m},$$

$$a_2=\frac{1}{4m\pi\Omega}\left[\gamma\Omega\left(\frac{3}{2}-\gamma_E\right)+\left(\Omega^2-\frac{\gamma^2}{4}\right)\arctan\left(\frac{2\Omega}{\gamma}\right)\right],$$

$$a_3=-\frac{\gamma}{48m}\left(\Omega^2-\frac{3\gamma^2}{2}\right), \tag{A19}$$

$$\vdots$$

$$b_2=-\frac{\gamma}{4m\pi},\quad b_4=-\frac{\gamma}{24m\pi}\left(\Omega^2-\frac{\gamma^2}{4}\right),\ \ldots\ .$$

Another possibility is to use the differential equation

$$\xi_V^{\mathrm{IV}}(\tau)-(2\omega_0^2-\gamma^2)\xi_V''(\tau)+\omega_0^4\xi_V(\tau)=\frac{\gamma}{2m\pi}\frac{1}{\tau^2} \tag{A20}$$

with the appropriate boundary conditions, which follows directly from Eq. (A2).

The asymptotic behavior for arbitrary τ is more involved. Equation (A2) may be used in the right-hand τ plane: one simply has to expand $\rho(E)$ for small E and integrate term by term to find

$$\xi_V(\tau)\overset{\tau\to\infty}{\longrightarrow}\frac{\gamma}{2m\pi\omega_0^4}\frac{1}{\tau^2}+\frac{6\gamma}{m\pi\omega_0^8}\left(\Omega^2-\frac{\gamma^2}{4}\right)\frac{1}{\tau^4}+\cdots$$

$$\mathrm{Re}\,\tau\geqslant 0. \tag{A21}$$

This is consistent with the result from the differential equation (A20) for large τ if one considers the derivatives as corrections. To find the asymptotic behavior in the left-hand τ plane we may use the explicit representation (A6) and reintroduce the exponential integral as this function has a simple asymptotic behavior (see, e.g., Eq. (5.1.51) in Ref. [30]),

$$E_1(z)\sim\frac{e^{-z}}{z}\left[1-\frac{1!}{z}+\frac{2!}{z^2}-\cdots\right],\quad |\mathrm{arg}z|<3\pi/2. \tag{A22}$$

However, care is needed when replacing $\mathrm{Ein}(z)$ by the exponential integral since the correct addition theorem of the logarithm with complex arguments,

$$\ln(ab)=\ln(a)+\ln(b)+2\pi i[\Theta(-\mathrm{Im}\,a)\Theta(-\mathrm{Im}\,b)$$
$$\times\Theta(\mathrm{Im}(ab))-\Theta(\mathrm{Im}\,a)\Theta(\mathrm{Im}\,b)\Theta(-\mathrm{Im}(ab))], \tag{A23}$$

has to be used for $\ln(E_{r,s}\tau)$. Consequently,

$$\xi_V(\tau)=-\frac{1}{8m\pi\Omega}\sum_{r,s=\pm1} re^{E_{r,s}\tau}[2\pi\Theta(s\mathrm{Im}\,\tau)$$
$$\times\Theta(-s\mathrm{Im}(E_{r,s}\tau))+is\,E_1(E_{r,s}\tau)],$$
$$|\mathrm{arg}\,\tau|<\pi. \tag{A24}$$

Although Eq. (A24) is less suited to display the analytic structure of the Vineyard function it allows to find the Stokes lines for the asymptotic behavior. These are the rays in the complex τ plane which divide the powerlike decrease of Eq. (A21) from an exponential increase. Indeed, due to Eq. (A22) the last term in the square brackets of Eq. (A24) gives rise to the powerlike decrease of $\xi_V(\tau)$ but this will be overwhelmed by the exponential increase of the first term if

$$\Theta(\mathrm{Re}(E_{r,s}\tau))\Theta(s\mathrm{Im}\,\tau)\Theta(-s\mathrm{Im}(E_{r,s}\tau))\neq 0 \tag{A25}$$

for any $r,s=\pm1$. Writing

$$E_{1,1}=\Omega+i\frac{\gamma}{2}=\omega_0e^{i\varphi_1},\quad \varphi_1=\arctan\left(\frac{\gamma}{2\Omega}\right),\quad 0\leqslant\varphi_1\leqslant\frac{\pi}{2} \tag{A26}$$

a straightforward analysis of condition (A25) shows that an exponential increase only occurs for

$$|\mathrm{arg}\,\tau|>\frac{\pi}{2}+\varphi_1, \tag{A27}$$

i.e., inside a sector around the cut with opening angle $\pi/2-\varphi_1=\arctan(2\Omega/\gamma)$. This means that Eq. (A21) holds for the wider range $|\mathrm{arg}\,\tau|<\pi/2+\varphi_1$.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF THE STRUCTURE FUNCTION

Here we derive the asymptotic behavior of the structure function when the energy transfer ν becomes very large. This is done by standard asymptotic analysis: for example, one may apply Eq. (30) in Ref. [32] to our Eq. (32). Then one obtains

$$S(q,\nu)\overset{\nu\to\infty}{\longrightarrow}\frac{2}{\pi}\left[-\frac{1}{\nu}\mathrm{Im}\,\Phi_q(0)+\frac{1}{\nu^3}\mathrm{Im}\,\Phi_q''(0)\right.$$
$$\left.-\frac{1}{\nu^5}\mathrm{Im}\,\Phi_q^{\mathrm{IV}}(0)+\cdots\right]$$
$$=\frac{\gamma}{m\pi}\frac{q^2}{\nu^3}+\cdots, \tag{B1}$$

since $\mathrm{Im}\,\Phi_q(0)=0$. Here we have used the representation of the characteristic function in terms of the Vineyard function $\xi_V(it)$ and the low-t expansion of the latter. Note that the leading contribution comes from the logarithmic term in Eq. (28) which produces an imaginary part for $\xi_V''(0)$.

However, higher-order terms cannot be calculated by means of Eq. (B1) since $\mathrm{Im}\,\Phi_q^{\mathrm{IV}}(0)$ does not exist. This shows that the next-to-leading asymptotic term is *not* falling off like $1/\nu^5$. To determine this term we use the exponential representation (6), well-known results for the Fourier transform of generalized functions and the low-t behavior of the Vineyard function. Writing

$$2q^2[\xi_V(it)-\xi_V(0)]=:f(t)+g(t)\ln(i\omega_0 t)$$
$$=f(t)+g(t)\left[\ln(\omega_0|t|)+i\frac{\pi}{2}\mathrm{sgn}\,t\right] \tag{B2}$$

with

$$f(t)\overset{t\to 0}{\to}-i\frac{q^2}{2m}t+O(t^2),\qquad g(t)\overset{t\to 0}{\to}\frac{\gamma}{\pi}\frac{q^2}{2m}t^2+O(t^4) \tag{B3}$$

one simply gets by expanding the exponential

$$\Phi_q(t)\overset{t\to 0}{\to}1+f(t)+g(t)\left[\ln(\omega_0|t|)+i\frac{\pi}{2}\mathrm{sgn}\,t\right]$$
$$+\frac{1}{2}f^2(t)+f(t)g(t)\left[\ln(\omega_0|t|)+i\frac{\pi}{2}\mathrm{sgn}\,t\right]$$
$$+\frac{1}{2}g^2(t)\left[\ln(\omega_0|t|)+i\frac{\pi}{2}\mathrm{sgn}\,t\right]^2+\cdots. \tag{B4}$$

As the Fourier transform of powers of t gives derivatives of δ functions we see that all regular terms do not contribute to the asymptotic behavior for large ν. The contribution of the nonanalytic terms can be taken from Table 1 of Ref. [32] [setting $y=-\nu/(2\pi)$],

$$\int_{-\infty}^{+\infty}dt\,t^n\mathrm{sgn}\,t\,e^{i\nu t}=2\frac{n!}{(-i\nu)^{n+1}}, \tag{B5}$$

$$\int_{-\infty}^{+\infty}dt\,t^n\ln|t|e^{i\nu t}=i\pi\frac{n!}{(-i\nu)^{n+1}}\mathrm{sgn}\,\nu. \tag{B6}$$

One then realizes that the leading contribution in the asymptotic expansion arises from the last term in the first line of Eq. (B4),

$$S(q,\nu)\overset{\nu\to\infty}{\to}\frac{1}{2\pi}\frac{\gamma q^2}{2m\pi}\int_{-\infty}^{+\infty}dt\,t^2\left[\ln(\omega_0|t|)+i\frac{\pi}{2}\mathrm{sgn}\,t\right]e^{i\nu t}$$
$$=\frac{\gamma}{m\pi}\frac{q^2}{\nu^3}, \tag{B7}$$

in agreement with Eq. (B1). The subleading contribution stems from the last term in the second line,

$$\Delta S(q,\nu)\overset{\nu\to\infty}{\to}\frac{1}{2\pi}\left(-i\frac{q^2}{2m}\right)\frac{\gamma q^2}{2m\pi}$$
$$\times\int_{-\infty}^{+\infty}dt\,t^3\left[\ln(\omega_0|t|)+i\frac{\pi}{2}\mathrm{sgn}\,t\right]e^{i\nu t}$$
$$=\frac{3\gamma}{2m^2\pi}\frac{q^4}{\nu^4}, \tag{B8}$$

whereas one can show that the last line gives a contribution of the order of $(\ln\nu)/\nu^5$. The asymptotic expansion makes sense if the subleading term is much smaller than the leading one which requires $\nu\gg q^2/(2m)$, i.e., excitation energies much larger than the maximum of the quasielastic peak.

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