Faddeev calculations for the $A=5,6$ $\Lambda\Lambda$ hypernuclei

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Faddeev calculations are reported for ${}^{5}_{\Lambda\Lambda}$H, ${}^{5}_{\Lambda\Lambda}$He, and ${}^{6}_{\Lambda\Lambda}$He in terms of two Λ hyperons plus ^{3}H, ^{3}He, and ^{4}He nuclear clusters, respectively, using $\Lambda\Lambda$ central potentials considered in past non-Faddeev calculations of ${}^{6}_{\Lambda\Lambda}$He. The convergence with respect to the partial-wave expansion is studied and comparison is made with some of these $\Lambda\Lambda$ hypernuclear calculations. The $\Lambda\Lambda$-ΞN mixing effect is briefly discussed.

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I. INTRODUCTION

The recent report of a ${}^{6}_{\Lambda\Lambda}$He uniquely identified event in nuclear emulsion [1], with $\Lambda\Lambda$ binding-energy value $B_{\Lambda\Lambda}=7.25\pm0.19^{+0.18}_{-0.11}$ MeV, has triggered renewed interest in the physics of double-Λ hypernuclei, particularly for light species. The previous report of $B_{\Lambda\Lambda}=10.8\pm0.6$ MeV [2] which has been considered dubious by the hypernuclear community implied a fairly strong $\Lambda\Lambda$ interaction potential, considerably stronger than the ΛN interaction potential deduced from studying single-Λ hypernuclei and at odds with one-boson-exchange models [3]. In contrast, the new event is compatible with a fairly weak $\Lambda\Lambda$ interaction, with scattering length $a_{\Lambda\Lambda}\sim-0.8$ fm [4,5], considerably smaller in magnitude than $a_{\Lambda N}\sim-2$ fm for the ΛN interaction [6]. With such a weak $\Lambda\Lambda$ interaction, it becomes interesting to explore the onset of $\Lambda\Lambda$ binding in nuclei. Our earlier Faddeev calculations [4,5] of the $A=5$ $\Lambda\Lambda$ hypernuclei ${}^{5}_{\Lambda\Lambda}$H and ${}^{5}_{\Lambda\Lambda}$He suggested that these species are comfortably particle stable for weak $\Lambda\Lambda$ interaction potentials, and this has been recently confirmed by the variational calculation of Ref. [7]. However, for $A=4$ the situation is unclear, with conflicting calculational conclusions [8,9] for the $\Lambda\Lambda pn$ four-body ${}^{4}_{\Lambda\Lambda}$H hypernucleus which has been recently conjectured to exist in the experimental report of Ref. [10]. A three-body $\Lambda\Lambda N$ bound state is ruled out on general grounds [11].

In our earlier work [4,5], the $A=5,6$ $\Lambda\Lambda$ hypernuclei ${}^{5}_{\Lambda\Lambda}$H, ${}^{5}_{\Lambda\Lambda}$He, and ${}^{6}_{\Lambda\Lambda}$He were considered as three-body systems $\Lambda\Lambda C$, where the (assumed inert) nuclear cluster C stands for ^{3}H, ^{3}He, or ^{4}He respectively. The Faddeev equations were then solved for model $\Lambda\Lambda$ interactions under an s-wave approximation (to be specified below) using ΛC interaction potentials fitted to the observed ΛC binding energies. It was argued, by comparing to earlier non-Faddeev calculations for ${}^{6}_{\Lambda\Lambda}$He, that the use of this approximation incurred an error of roughly 0.2 MeV. In the present work, we extend our earlier Faddeev calculations relaxing the s-wave approximation and testing the convergence of these calculations with respect to the partial wave expansion. To this end, we have followed the formulation and numerical solution method outlined and tested by Bernabéu *et al.* [12] for Faddeev equations in configuration space. The calculations here reported do confirm our earlier estimates. Since ${}^{6}_{\Lambda\Lambda}$He serves in most applications as the primary normalizing datum for extracting phenomenologically the $\Lambda\Lambda$ interaction, it is desirable to improve as much as possible the calculational aspects of the ${}^{6}_{\Lambda\Lambda}$He binding-energy evaluation in order to gain confidence in such extraction. We therefore compare our Faddeev calculations to other non-Faddeev calculations of the $A=5,6$ $\Lambda\Lambda$ hypernuclei. Finally, we comment on the order of magnitude expected for dynamical effects due to $\Lambda\Lambda$-ΞN mixing.

II. METHODOLOGY

A. Faddeev equations

The bound states of the three-body systems considered in this work are obtained by solving the differential Faddeev equations [13]

$$\{H_0+V_\alpha(u_\alpha)-E\}\Psi_\alpha(\mathbf{u}_\alpha,\mathbf{v}_\alpha)=-V_\alpha(u_\alpha)\sum_{\beta\neq\alpha}\Psi_\beta(\mathbf{u}_\beta,\mathbf{v}_\beta), \tag{1}$$

where V_α is a short-range pair interaction in the channel α, $H_0=-\Delta_{\mathbf{u}_\alpha}-\Delta_{\mathbf{v}_\alpha}$ is the internal kinetic energy operator, E is the total energy and the wave function of the three-body system is given as a sum $\Psi=\Sigma_{\alpha=1}^{3}\Psi_\alpha$ over the three Faddeev components, corresponding to the two-body rearrangement channels. The Faddeev components are functions of spin-isospin variables and of the relative Jacobi coordinate vectors, here given in terms of the particle coordinates $\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3$ by

$$\mathbf{u}_\alpha=\left(\frac{2m_\beta m_\gamma}{m_\beta+m_\gamma}\right)^{1/2}(\mathbf{r}_\beta-\mathbf{r}_\gamma),$$

$$\mathbf{v}_\alpha=\left(\frac{2m_\alpha(m_\beta+m_\gamma)}{M}\right)^{1/2}\left(\frac{m_\beta\mathbf{r}_\beta+m_\gamma\mathbf{r}_\gamma}{m_\beta+m_\gamma}-\mathbf{r}_\alpha\right), \tag{2}$$

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where (α,β,γ) is a cyclic permutation of the subscripts 1, 2, and 3 and where M is the total mass. The Jacobi coordinate vectors for different α's are linearly related by an orthogonal transformation

$$\begin{pmatrix}\mathbf{u}_\alpha\\ \mathbf{v}_\alpha\end{pmatrix}=\begin{pmatrix} C_{\alpha\beta} & S_{\alpha\beta}\\ -S_{\alpha\beta} & C_{\alpha\beta}\end{pmatrix}\begin{pmatrix}\mathbf{u}_\beta\\ \mathbf{v}_\beta\end{pmatrix},\quad C_{\alpha\beta}^2+S_{\alpha\beta}^2=1, \tag{3}$$

where

$$C_{\alpha\beta}=\delta_{\alpha\beta}-(1-\delta_{\alpha\beta})\sqrt{\frac{m_\alpha m_\beta}{(M-m_\alpha)(M-m_\beta)}},$$

$$S_{\alpha\beta}=(-)^{\beta-\alpha}\mathrm{sgn}(\beta-\alpha)\sqrt{1-C_{\alpha\beta}^2}. \tag{4}$$

The partial-wave analysis of Eq. (1), by separating the angular variables (see, for instance, Ref. [14]), leads to a system of integrodifferential equations, which in the polar coordinates $\rho^2=u_\alpha^2+v_\alpha^2$, $\tan\theta=v_\alpha/u_\alpha$, has the form

$$\left\{-\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+V_\alpha^{\lambda l}(\rho,\theta)+\frac{l(l+1)}{\rho^2\cos^2\theta}+\frac{\lambda(\lambda+1)}{\rho^2\sin^2\theta}-E\right\}\Psi_\alpha^{\lambda l}(\rho,\theta)$$
$$=-\frac{1}{2}V_\alpha^{\lambda l}(\rho,\theta)\sum_{\beta\neq\alpha}(h_{\lambda l,\lambda' l'}^{L\alpha\beta}\Psi_\beta^{\lambda' l'})(\rho,\theta). \tag{5}$$

Here, $\mathbf{L}$ is the total orbital angular momentum of the system, $\mathbf{L}=\boldsymbol{\lambda}+\boldsymbol{l}$, where $\boldsymbol{l}$ is the relative orbital angular momentum of the pair α ($\alpha=1,2,3$) and $\boldsymbol{\lambda}$ is the orbital angular momentum of the spectator particle relative to the center of mass of the pair α. Note that the hyper-radius ρ is independent of the channel label α. In a bispherical basis, the integral operator has the form

$$(h_{\lambda l,\lambda' l'}^{L\alpha\beta}\Psi_\beta^{\lambda' l'})(\rho,\theta)=\int_{-1}^{+1}dt\frac{\sin\theta\cos\theta}{\sin\theta'\cos\theta'}h_{\lambda l,\lambda' l'}^{L\alpha\beta}(\theta,\theta')\times\Psi_\beta^{\lambda' l'}(\rho,\theta'), \tag{6}$$

where

$$\cos^2\theta'(t,\theta)=C_{\alpha\beta}^2\cos^2\theta+2tC_{\alpha\beta}S_{\alpha\beta}\cos\theta\sin\theta+S_{\alpha\beta}^2\sin^2\theta. \tag{7}$$

An explicit representation of the operator $h_{\lambda l,\lambda' l'}^{L\alpha\beta}(\theta,\theta')$ is given in the Appendix. In Eq. (5), the potential $V_\alpha^{\lambda l}(\rho,\theta)$ has the form

$$V_\alpha^{\lambda l}=\langle\lambda l|V_\alpha|\lambda l\rangle \tag{8}$$

in terms of its matrix elements in the bispherical basis of eigenfunctions of the total angular momentum operator. The standard substitution $\Psi_\alpha=\rho^{-1/2}U_\alpha$ eliminates the first radial derivative, reducing Eq. (5) to the form

$$\left\{-\frac{\partial^2}{\partial\rho^2}-\frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}+V_\alpha^{\lambda l}(\rho,\theta)+\frac{l(l+1)}{\rho^2\cos^2\theta}+\frac{\lambda(\lambda+1)}{\rho^2\sin^2\theta}-\frac{1}{4\rho^2}-E\right\}U_\alpha^{\lambda l}(\rho,\theta)$$
$$=-\frac{1}{2}V_\alpha^{\lambda l}(\rho,\theta)\sum_{\beta\neq\alpha}(h_{\lambda l,\lambda' l'}^{L\alpha\beta}U_\beta^{\lambda' l'})(\rho,\theta). \tag{9}$$

To solve the eigenvalue problem in the region $\rho\in[0,\infty)$, $\theta\in[0,\pi/2]$, Eq. (9) must be supplemented by the boundary conditions

$$U_\alpha(0,\theta)=U_\alpha(\infty,\theta)=0,$$

$$U_\alpha(\rho,0)=U_\alpha(\rho,\pi/2)=0. \tag{10}$$

B. Systems with two identical particles

For a three-body system generically of the form $\Lambda\Lambda C$ (C=core), for example, Λ hyperons in ${}^{6}_{\Lambda\Lambda}$He ($\Lambda\Lambda\alpha$), the coupled set of Faddeev equations simplifies as follows:

$$(H_0+V_{\Lambda\Lambda}-E)\Psi_{C\text{-}(\Lambda\Lambda)}=-V_{\Lambda\Lambda}(1-P_{12})\Psi_{\Lambda\text{-}(\Lambda C)}, \tag{11}$$

$$(H_0+V_{\Lambda C}-E)\Psi_{\Lambda\text{-}(\Lambda C)}=-V_{\Lambda C}(\Psi_{C\text{-}(\Lambda\Lambda)}-P_{12}\Psi_{\Lambda\text{-}(\Lambda C)}),$$

where P_{12} is a permutation operator for the Λ hyperons. The total wave function is then given by

$$\Psi=\Psi_{C\text{-}(\Lambda\Lambda)}+(1-P_{12})\Psi_{\Lambda\text{-}(\Lambda C)}. \tag{12}$$

The total orbital angular momentum may be represented in two forms

$$\mathbf{L}=\boldsymbol{l}_{\Lambda\Lambda}+\boldsymbol{\lambda}_{C\text{-}(\Lambda\Lambda)}=\boldsymbol{l}_{\Lambda C}+\boldsymbol{\lambda}_{\Lambda\text{-}(\Lambda C)}. \tag{13}$$

Other three-body systems studied in the present work are the isodoublet ${}^{5}_{\Lambda\Lambda}$H, ${}^{5}_{\Lambda\Lambda}$He charge-symmetric hypernuclei, here considered as $\Lambda\Lambda{}^3$H and $\Lambda\Lambda{}^3$He, respectively. For the ground state ($\frac{1}{2}^+$) of these systems, after separation of spin variables, the Faddeev equations assume the form

$$(H_0+V_{\Lambda\Lambda}-E)\Psi_{C\text{-}(\Lambda\Lambda)}=-V_{\Lambda\Lambda}A(1+P_{12})\boldsymbol{\Psi}_{\Lambda\text{-}(\Lambda C)}, \tag{14}$$

$$(H_0+V_{\Lambda C}-E)\boldsymbol{\Psi}_{\Lambda\text{-}(\Lambda C)}=-V_{\Lambda C}(A^T\Psi_{C\text{-}(\Lambda\Lambda)}+BP_{12}\boldsymbol{\Psi}_{\Lambda\text{-}(\Lambda C)}),$$

where the exchange operator P_{12} acts on coordinates only, $V_{\Lambda\Lambda}=v_{\Lambda\Lambda}^s$ is the singlet $\Lambda\Lambda$ potential, $V_{\Lambda C}=\mathrm{diag}\{v_{\Lambda C}^s,v_{\Lambda C}^t\}$ is a diagonal 2×2 matrix with $v_{\Lambda C}^s$ and $v_{\Lambda C}^t$ the singlet and triplet ΛC interaction potentials, respectively, and

$$A=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right), \quad B=\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$\Psi_{\Lambda\text{-}(\Lambda C)}=\begin{pmatrix} \Psi^{s}_{\Lambda\text{-}(\Lambda C)} \\ \Psi^{t}_{\Lambda\text{-}(\Lambda C)} \end{pmatrix}. \tag{15}$$

Note that the squares of elements of A correspond then to a $(2J+1)$ average over the $J=0,1$ (singlet and triplet, respectively) ${}^{4}_{\Lambda}$H and ${}^{4}_{\Lambda}$He states. Neglecting the spin dependence of $V_{\Lambda C}$, i.e., using $V_{\Lambda C}=v_{\Lambda C}I$, where I is the 2×2 unit matrix in spin space, it is possible to reduce Eqs. (14) to the spinless Eqs. (11) where $\Psi_{\Lambda\text{-}(\Lambda C)}$ in Eqs. (11) stands for $A_1\Psi^{s}_{\Lambda\text{-}(\Lambda C)}+A_2\Psi^{t}_{\Lambda\text{-}(\Lambda C)}$. This procedure, for $v_{\Lambda C}=\frac{1}{4}v^{s}_{\Lambda C}+\frac{3}{4}v^{t}_{\Lambda C}$, will be compared below with the variational spin-averaged calculation of Ref. [7].

C. Potentials

The $\Lambda\Lambda$ interaction potentials in the ${}^{1}S_0$ channel which are used as input to the Faddeev equations are of a three-range Gaussian form

$$V_{\Lambda\Lambda}=\sum_{i=1}^{3} v^{(i)}\exp(-r^2/\beta_i^2), \tag{16}$$

following the work of Hiyama *et al.* [15] where a phase-equivalent $\Lambda\Lambda$ potential of this soft-core form was fitted to the Nijmegen model D (ND) hard-core interaction [16] assuming the same hard core for the NN and $\Lambda\Lambda$ potentials in the ${}^{1}S_0$ channel. For other interactions, notably the Nijmegen soft-core model NSC97 [3], we have renormalized the strength of the medium-range attractive component ($i=2$) of this potential fitting as closely as possible the scattering length and the effective range. The appropriate range and strength parameters are listed in Tables 1 and 2 of Ref. [5]. For the NSC97e interaction, Myint *et al.* [7] have used a different parametrization which is denoted by *e* and is listed in Table 3 of their paper. Finally, some older works used purely attractive one-range Gaussian forms or two-range Gaussian forms with inner repulsion and outer attraction (this kind of potential is also called 'Isle').

For the $\Lambda\alpha$ interaction potential, we have followed the Isle potential due to Ref. [17] which was shown to provide good agreement with the measured mesonic weak decay rate of ${}^{5}_{\Lambda}$He. The resulting Λ-hyperon density distribution has been shown very recently [18] to closely resemble that due to a microscopic calculation of ${}^{5}_{\Lambda}$He using YN interactions which simulate those of model NSC97 [6]. The parameters of this potential are listed in Table 3 of Ref. [5]. Myint *et al.* [7] recently have used different parameters for this *s*-wave interaction potential and also kept an option for using a weaker *p*-wave potential. These various $\Lambda\alpha$ potentials are shown in Fig. 1. Similar Isle potentials were constructed for the Λ-^{3}H and Λ-^{3}He singlet and triplet interactions by fitting to the observed binding energies for the $0^+,1^+$ ground-state doublet in ${}^{4}_{\Lambda}$H-${}^{4}_{\Lambda}$He, respectively [5,7]. Some of the older works used purely attractive ΛC potentials which are no longer considered realistic ones.

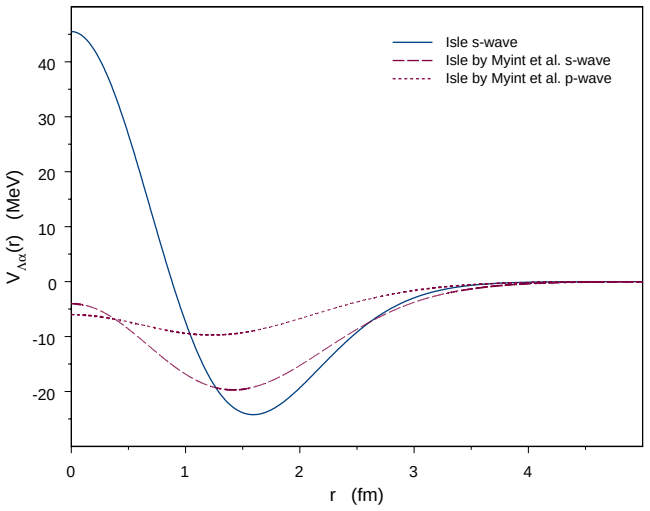

FIG. 1. $\Lambda\alpha$ Isle potentials: solid curve from Ref. [17] and other ones from Ref. [7].

III. RESULTS AND DISCUSSION

In this section, we report binding-energy results of solving the coupled Faddeev equations for $\Lambda\Lambda C$ systems with $A=5,6$, using a sufficiently large cutoff value $l_{\max}=6$ for the angular momenta of the partial two-body systems. Our results are compared to those of several non-Faddeev calculations and a brief discussion is offered.

A. ${}^{6}_{\Lambda\Lambda}$He

In Table I, we show results of our Faddeev calculations for ${}^{6}_{\Lambda\Lambda}$He using purely attractive Gaussian $\Lambda\alpha$ and $\Lambda\Lambda$ potentials taken from Ref. [19] and which act in all the allowed partial waves. In particular, the effective-range parameters of

TABLE I. $B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\text{He})$ calculated for the purely attractive Gaussian $\Lambda\alpha$ and $\Lambda\Lambda$ interaction potentials used by Ikeda *et al.* [19].

Reference	$l_{\max}$	$l_{\Lambda\alpha}$	$l_{\Lambda\Lambda}$	$B_{\Lambda\Lambda}$ (MeV)
FGS	0	0	0	11.15
	1	0,1	0	11.19
	2	0,1,2	0,2	11.21
	4	0,1,2,3,4	0,2,4	11.21
	6	0,1,2,3,4,5,6	0,2,4,6	11.21
[19]	[a]		0	10.3
	[a]		0,2,4	10.8
[20]	[a]		0,2,4	11.207
[21]	0	0	0	11.2

[a]Since $l_{\Lambda\alpha}$ is not assigned specific values, no definite value holds for $l_{\max}=\max(l_{\Lambda\alpha},l_{\Lambda\Lambda})$.

the $\Lambda\Lambda$ interaction potential are $a_{\Lambda\Lambda}=-1.76$ fm, $r_{\Lambda\Lambda}=2.11$ fm, indicating a fairly strong $\Lambda\Lambda$ interaction aimed at reproducing the older value for the binding energy $B_{\Lambda\Lambda}=10.8\pm0.6$ MeV [2]. For these central interactions, the Pauli spin is conserved and $S=0$ generally holds for the 0^+ ground state. Hence, $L=0$ and $\lambda_{\alpha\text{-}(\Lambda\Lambda)}=l_{\Lambda\Lambda}$, with $l_{\Lambda\Lambda}$ running over even values in order to respect the Pauli principle. Similarly, $\lambda_{\Lambda\text{-}(\Lambda\alpha)}=l_{\Lambda\alpha}$. The calculated $B_{\Lambda\Lambda}$ values are listed in order of increasing $l_{\max}$, where $l_{\max}=\max(l_{\Lambda\alpha},l_{\Lambda\Lambda})$, and are seen to increase monotonically with $l_{\max}$. Convergence is reached already for $l_{\max}=2$, merely 0.06 MeV higher than the $B_{\Lambda\Lambda}$ value corresponding to the $l_{\max}=0$ *s*-wave approximation. Our Faddeev calculations (marked FGS) are compared in the table to the Ikeda *et al.* [19] Schrödinger-equation calculations which were restricted to the α-$(\Lambda\Lambda)$ rearrangement channel, disregarding the Λ-$(\Lambda\alpha)$ rearrangement channel. Therefore, calculations of this latter type offer neither a way to sort out a range of values for $l_{\Lambda\alpha}$ nor a meaning for *s*-wave approximation; indeed, improving over what would have been perceived as an *s*-wave approximation ($l_{\Lambda\Lambda}=0$) amounts in Ref. [19] to 0.5 MeV, about eight times the corresponding improvement for our Faddeev calculation. More importantly, our Faddeev calculations demonstrate that the Ikeda *et al.* calculation misses our converged value of $B_{\Lambda\Lambda}$ by about 0.4 MeV (which is a sizable miss in this three-body trade).

A comparison is also offered in Table I with the variational calculation by Portilho and Coon [20], extrapolated from $l_{\max}=4$. The agreement with our exact Faddeev calculation is remarkable. This variational calculation uses a large basis of harmonic oscillator wavefunctions, with a variable spring constant, in the α-$(\Lambda\Lambda)$ rearrangement channel. It works well because these wave functions within a given dimension may be cast into a similar basis of the same dimensionality in the Λ-$(\Lambda\alpha)$ rearrangement channel, a property which is exclusively specific to harmonic oscillator wave functions. Finally, we quote in the table the result of the pioneering Dalitz and Rajasekaran [21] variational calculation, using a ${}^{6}_{\Lambda\Lambda}$He ansatz wave function

$$\Psi=F(r_{\Lambda_1\alpha})F(r_{\Lambda_2\alpha})G(r_{\Lambda_1\Lambda_2}), \tag{17}$$

which accounts through its variational parameters for short-range correlations as well as for obvious long-range asymptotic requirements. In this outstanding calculation, Dalitz and Rajasekaran used the same $\Lambda\Lambda$ interaction as used in the subsequent calculations listed in Table I, but their $\Lambda\alpha$ interaction was slightly different, although it was constrained by fitting to the observed ${}^{5}_{\Lambda}$He binding energy. For this reason, we hesitate to claim that their 40 years old variational calculation matches today's Faddeev techniques, yet even if they missed the exact result by only 0.1 MeV (or less) it is a tribute to the essential physics requirements imposed on the parametrization of the functions F and G of which the wave function Ψ in Eq. (17) consists of.

In Table II, we show $B_{\Lambda\Lambda}$ values calculated by us (marked FGS), this time using a $\Lambda\Lambda$ Isle potential together with a purely attractive $\Lambda\alpha$ interaction potential, both taken from the work of Khan and Das [22]. These potentials, again, act in all the allowed partial waves. Our *s*-wave approximation, here too, works very well (to order of 0.04 MeV), in contrast with the hyperspherical harmonics (HH) calculation by Khan and Das which falls short of 1 MeV in its first round. Nevertheless, the HH calculation is a systematical one, treating *correctly* all the three-body degrees of freedom in the limit of going to infinitely large values of the hyperangular momentum K which serves as a measure of the size of space in which the coupled three-body equations are solved. For $l_{\max}=6$ where comparison is possible, the agreement between the Faddeev calculation and the HH calculation is excellent. However, the HH calculation may suggest that higher values of $l_{\max}$ beyond $l_{\max}=6$ are needed in our Faddeev calculation in order to reach convergence, whereas this Faddeev calculation appears already converged at $l_{\max}=6$.

TABLE II. $B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\text{He})$ calculated for the purely attractive Gaussian $\Lambda\alpha$ and the Isle $\Lambda\Lambda$ interaction potentials used in model B of Khan and Das [22].

Reference	$l_{\max}$	$l_{\Lambda\alpha}$	$l_{\Lambda\Lambda}$	$B_{\Lambda\Lambda}$ (MeV)
FGS	0	0	0	10.732
	6	0,1,2,3,4,5,6	0,2,4,6	10.770
[22]	[a]		0	9.707
	[a]		0,2,4	10.687
	[a]		0,2,4,6	10.767
	[a]		0,2,4,6,8,10	10.816

[a]Since $l_{\Lambda\alpha}$ is not assigned specific values, no definite value holds for $l_{\max}=\max(l_{\Lambda\alpha},l_{\Lambda\Lambda})$.

We now switch to a more realistic *s*-wave $\Lambda\alpha$ Isle interaction, with a repulsive core as shown by the solid curve in Fig. 1. The $\Lambda\Lambda$ interaction which is of the generic form Eq. (16) also has a repulsive core. These *s*-wave interactions were recently used within an *s*-wave Faddeev calculation by Filikhin and Gal [4,5]. Here, we report on direct solutions of the coupled Faddeev equations for three specific $\Lambda\Lambda$ interaction potentials, NSC97b, NSC97e, and ND, with parameters specified in Tables 1 and 2 of Ref. [5]. The $\Lambda\alpha$ and $\Lambda\Lambda$ potentials act in all the allowed partial waves. The resulting $B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\text{He})$ values are shown in Table III for $l_{\max}=0,1,\ldots,6$. It is seen that the deviation of $B_{\Lambda\Lambda}$ for a given value of $l_{\max}$ from its *s*-wave approximation ($l_{\max}=0$) grows monotonically with $l_{\max}$, reaching about 0.2 MeV for $l_{\max}=6$. While it is still reasonably small, of the order of magnitude of the experimental error [1] derived from the observed emulsion tracks for ${}^{6}_{\Lambda\Lambda}$He, this deviation is several times bigger than for the calculations summarized in the previous tables which shared purely attractive $\Lambda\alpha$ potentials in common. This is due to the contribution of $l_{\Lambda\alpha}=1$ getting enhanced for $\Lambda\alpha$ interactions of an Isle form, compared to such contribution for a purely attractive potential. However, since it is unrealistic to use the same *s*-wave $\Lambda\alpha$ interaction for all $\Lambda\alpha$ partial waves, one should expect the above devia-

TABLE III. $B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\mathrm{He})$ (in MeV) calculated for the Isle $\Lambda\alpha$ potential and for several simulations of Nijmegen models for the $\Lambda\Lambda$ interaction used by Filikhin and Gal [4,5]. These *s*-wave $\Lambda\alpha$ and $\Lambda\Lambda$ potentials act in all the allowed partial waves.

Reference	$l_{\max}$	$l_{\Lambda\alpha}$	$l_{\Lambda\Lambda}$	$B_{\Lambda\Lambda}$(NSC97b)	$B_{\Lambda\Lambda}$(NSC97e)	$B_{\Lambda\Lambda}$(ND)
FGS	0	0	0	6.491	6.710	8.947
	1	0,1	0	6.593	6.793	8.982
	2	0,1,2	0,2	6.653	6.853	9.061
	3	0,1,2,3	0,2	6.676	6.879	9.123
	4	0,1,2,3,4	0,2,4	6.690	6.894	9.148
	5	0,1,2,3,4,5	0,2,4	6.695	6.900	9.167
	6	0,1,2,3,4,5,6	0,2,4,6	6.698	6.903	9.176
[5]	0	0	0	6.60	6.82	9.10
[7]	0	0	0		6.70	
	>0	>0	>0		6.90	

[a] $l_{\max}=\max(l_{\Lambda\alpha},l_{\Lambda\Lambda})$.

tion to be smaller once a more realistic (weaker) *p*-wave potential component is introduced. For example, using for $l_{\Lambda\alpha}=1$, the *p*-wave Isle potential due to Myint *et al.* [7] which is shown by the short-dash curve in Fig. 1, we get within the NSC97e calculation $B_{\Lambda\Lambda}=6.74$ MeV for $l_{\max}=1$ ($l_{\Lambda\Lambda}=0,l_{\Lambda\alpha}=0,1$) compared to 6.79 MeV (cf. Table III) when the standard *s*-wave Isle potential (solid curve in Fig. 1) is used for $l_{\Lambda\alpha}=1$ as well as for $l_{\Lambda\alpha}=0$.

Also shown in Table III are the results of the Filikhin and Gal [5] *s*-wave calculations which exceed by 0.11–0.15 MeV the corresponding $l_{\max}=0$ present (FGS) results. This discrepancy is due to the slow and nonmonotonic convergence in the cluster-reduction method used in Refs. [4,5], particularly for interactions, such as here, consisting of a repulsive core plus an attractive tail. Finally, we compare our Faddeev calculation with the recent variational calculation by Myint *et al.* [7] for the $\Lambda\Lambda$ interaction potential NSC97e. Their method is based on using Gaussian wave function expansion and allowing for all the rearrangement channels in the trial three-body wave function. The agreement between our calculation and their calculation is excellent both for the *s*-wave approximation ($l_{\max}=0$) as well as for the full calculation, assuming that our Faddeev calculation is very close to convergence for $l_{\max}=6$.

B. ${}^{6}_{\Lambda\Lambda}$He and ${}^{5}_{\Lambda\Lambda}$H-${}^{5}_{\Lambda\Lambda}$He

In Table IV, we show our Faddeev calculation results (FGS) for ${}^{6}_{\Lambda\Lambda}$He and also for the $A=5$ charge symmetric ${}^{5}_{\Lambda\Lambda}$H-${}^{5}_{\Lambda\Lambda}$He hypernuclei, using *s*-wave $\Lambda\alpha$ and $\Lambda\Lambda$ potentials due to Myint *et al.* [7]. These *s*-wave interactions are used in *all* partial waves. The results for ${}^{6}_{\Lambda\Lambda}$He are very similar in character to those of the previous table, indicating again that the *s*-wave approximation ($l_{\max}=0$) holds to order of 0.2 MeV. For ${}^{5}_{\Lambda\Lambda}$H-${}^{5}_{\Lambda\Lambda}$He, we have a similar pattern of results, where the *s*-wave approximation holds to about 0.13 MeV. Here, in order to provide direct comparison with the variational results of Myint *et al.*, we have solved the spin-averaged form of the Faddeev equations as described here in Sec. II A, using the *spin-averaged* Λ-^{3}H and Λ-^{3}He interactions listed in their paper [7]. The results of Myint *et al.* are given in the last row of the table and are in very good agreement with our *s*-wave approximation results. It appears that by limiting the variational calculation to $l_{\Lambda\alpha}=0$, as might be understood from the discussion at the beginning of their Sec. 3, the whole evaluation turned out to be limited to the *s*-wave approximation. We have also pursued the *full* spin-dependent calculation for the $A=5$ $\Lambda\Lambda$ hypernuclei and found out that it yields 0.08 MeV higher binding than for the spin-averaged results shown in Table IV.

TABLE IV. $B_{\Lambda\Lambda}$ (in MeV) calculated for the $A=5,6$ $\Lambda\Lambda$ hypernuclei using the *s*-wave Isle ΛC potentials and two $\Lambda\Lambda$ potentials *e* and *e*1 due to Myint *et al.* [7]. These *s*-wave potentials act in all the allowed partial waves.

Reference	$l_{\max}$	${}^{6}_{\Lambda\Lambda}$He (*e*)	${}^{5}_{\Lambda\Lambda}$He (*e*)	${}^{5}_{\Lambda\Lambda}$H (*e*)	${}^{6}_{\Lambda\Lambda}$He (*e*1)	${}^{5}_{\Lambda\Lambda}$He (*e*1)	${}^{5}_{\Lambda\Lambda}$H (*e*1)
FGS	0	6.880	3.527	3.002	7.254	3.810	3.261
	1	6.987	3.608	3.061	7.341	3.882	3.311
	2	7.045	3.640	3.088	7.405	3.918	3.343
	3	7.078	3.657	3.103	7.443	3.939	3.360
	4	7.095	3.665	3.111	7.463	3.451	3.370
	5	7.103	3.669	3.114	7.473	3.954	3.374
	6	7.107	3.671	3.115	7.477	3.956	3.376
[7]	0	6.88	3.51	2.99	7.25	3.80	3.26

[a] $l_{\max}=\max(l_{\Lambda\alpha},l_{\Lambda\Lambda})$.

TABLE V. Low-energy parameters (in fm) and $B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\mathrm{He})$ values (in MeV) calculated for several interaction models depending on whether or not or how the $\Lambda\Lambda$-ΞN coupling is incorporated.

Reference	V(model)	$a_{\Lambda\Lambda}$	$r_{\Lambda\Lambda}$	$B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\mathrm{He})$
FGS	$V_{\Lambda\Lambda}(e)$	−0.27	19.07	6.664
	$V^{\mathrm{eff}}_{\Lambda\Lambda}(e)$	−0.50	8.51	7.107
	$V_{\Lambda\Lambda}(e1)$	−0.43	10.40	6.964
	$V^{\mathrm{eff}}_{\Lambda\Lambda}(e1)$	−0.73	5.59	7.477
[23]	$V_{\Lambda\Lambda}$(ND)			9.508
	$V^{\mathrm{eff}}_{\Lambda\Lambda}$(ND)	−1.91	3.36	10.007
	V(ND)			9.738
[25]	$V_{\Lambda\Lambda}$(ND)			9.4
	$V^{\mathrm{eff}}_{\Lambda\Lambda}$(ND)	−1.91	3.36	
	V(ND)			9.8

C. $\Lambda\Lambda$-ΞN coupling effects in ${}^{6}_{\Lambda\Lambda}$He

The input $\Lambda\Lambda$ interaction potentials to the Faddeev calculations summarized in Tables III and IV are *effective* single-channel simulations $V^{\mathrm{eff}}_{\Lambda\Lambda}$ of the Nijmegen meson-exchange models ND and NSC97 (except for $e1$ which is a slight variation on NSC97e). $V^{\mathrm{eff}}_{\Lambda\Lambda}$ represents the combined effect of the $\Lambda\Lambda$ and ΞN channels, including the coupling between these channels, and it is more attractive than the $\Lambda\Lambda$ single-channel potential $V_{\Lambda\Lambda}$ which does not include the effect of the $\Lambda\Lambda$-ΞN coupling. In Table V, we give the low-energy scattering parameters and the ${}^{6}_{\Lambda\Lambda}\mathrm{He}(0^{+})$ binding-energy values calculated by us (marked FGS) for potentials e and $e1$ of Myint *et al.* [7], using their parametrization of the corresponding potentials $V_{\Lambda\Lambda}$ and $V^{\mathrm{eff}}_{\Lambda\Lambda}$. It is seen that the inclusion of a coupling potential, motivated by the NSC97e free-space interaction, increases the calculated binding energy by 0.48 ± 0.04 MeV. We note that the simulation of the $\Lambda\Lambda$-ΞN coupling potential $V_{\Lambda\Lambda\text{-}\Xi N}$ in Ref. [7] consists exclusively of an attractive component, unlike the common practice for the $\Lambda\Lambda$ diagonal potentials, and this is likely to inflate the effect calculated for the $\Lambda\Lambda$-ΞN coupling. The $\Lambda\Lambda-\Xi N$ coupling potentials due to model NSC97 involve a subtle pattern of cancellations between pseudoscalar (K), vector (K^{*}), and scalar (κ) meson-exchange contributions, the net result being considerably weaker than assumed by the parametrization in Tables 2 and 3 of Ref. [7]. Carr *et al.* [23], for stronger potentials (diagonal as well as off-diagonal, each consisting of a short-range repulsive component plus a longer-range attractive component motivated by model ND), found that by including the $\Lambda\Lambda$-ΞN coupling one adds 0.50 MeV as shown in the table too. A consequence of their methodology is that for the considerably weaker NSC97 interactions, the total effect of including the $\Lambda\Lambda$-ΞN coupling would amount to much less, as argued recently by Afnan and Gibson [24].

An important consideration in the discussion of the $\Lambda\Lambda$-ΞN coupling effect is the extent to which this coupling is Pauli suppressed. For example, in ${}^{6}_{\Lambda\Lambda}\mathrm{He}$ transitions creating a fifth nucleon in the $1s$ shell are Pauli forbidden. This means that $V^{\mathrm{eff}}_{\Lambda\Lambda}$ is less attractive in ${}^{6}_{\Lambda\Lambda}\mathrm{He}$ than in free space where it was derived. Myint *et al.* [7] estimated the corresponding suppression for potentials e and $e1$ to be *at least* as large as 0.43 MeV, almost saturating the *total* 0.5 MeV $\Lambda\Lambda$-ΞN coupling effect calculated for these same potentials. This casts doubts on the validity of their method to estimate the Pauli suppression effect. We note that the contribution due to what they perceive as the Pauli blocked transition is inversely proportional to the mass difference ΔM between the initial ${}^{6}_{\Lambda\Lambda}\mathrm{He}(0^{+})$ ground state and the intermediate ${}^{6}_{\Xi}\mathrm{He}$ forbidden state. This mass difference is estimated by them to be $\Delta M\sim32$ MeV, ignoring the substantial binding that a fifth nucleon in the $1s$ shell would have acquired in the field of the ^{4}He core. Estimating this extra binding to be of the order of 25 MeV, the mass difference would reduce to merely $\Delta M\sim7$ MeV, resulting in an unacceptably large Pauli suppression effect of the order of 2 MeV which exceeds substantially the total 0.5 MeV $\Lambda\Lambda$-ΞN coupling effect. The strong dependence on ΔM makes the whole approach questionable. Similar objections hold for the estimates given in Ref. [7] for medium effects in the $A=5$ isodoublet ${}^{5}_{\Lambda\Lambda}\mathrm{H}$-${}^{5}_{\Lambda\Lambda}\mathrm{He}$. For a realistic account of the Pauli suppression effect and other medium effects, the explicit introduction of the ΞN channel is unavoidable, as applied by Carr *et al.* [23] and by Yamada and Nakamoto [25] who used a properly defined Pauli-suppression projection operator within a genuine coupled-channel calculation. Carr *et al.* [23] calculated the suppression effect to be 0.27 MeV out of a total of 0.50 MeV, as shown in Table V, for the considerably stronger ND interactions. The table also shows similar results from Ref. [25]. The Pauli-suppressed coupled-channel potential is denoted by **V**(ND).

IV. SUMMARY

In this work, we have studied light $\Lambda\Lambda$ hypernuclei ($A=5,6$) which may be described in terms of $\Lambda\Lambda C$ (C=cluster) systems and treated by solving the three-body Faddeev equations. Our calculations confirm the estimates made by Filikhin and Gal [4,5] that the s-wave approximation ($l_{\max}=0$) works fairly well and that the contribution of higher partial waves is small (<0.2 MeV) if ordered according to increasing $l_{\max}$. This is not necessarily the case for the other, non-Faddeev methods chosen for comparison in which the partial-wave ordering of successive approximations is defined only via $l_{\Lambda\Lambda}$, irrespective of $l_{\Lambda\alpha}$. A direct comparison between these two classes of calculation becomes fully meaningful only in the limit $\max(l,\lambda)\rightarrow\infty$.

For ${}^{6}_{\Lambda\Lambda}\mathrm{He}$, we have also studied the model dependence on the partial-wave composition of the $\Lambda\alpha$ interaction potential, in particular, when weakening this interaction in odd states relative to the even-state strength. This model dependence introduces as big uncertainty into the binding-energy calculation as incurred by sticking to the s-wave approximation. A proper microscopic construction of the $\Lambda\alpha$ interaction potential leads necessarily to a nonlocal potential (e.g., Ref. [15]) and is beyond the scope and aim of the present work.

For the $A=5$ $\Lambda\Lambda$ hypernuclei, we have tested the accuracy of averaging over the spins of the ΛC subsystems which was found to miss by somewhat less than 0.1 MeV the

binding energy due to the full, spin-dependent calculation.

Finally, we commented on the size expected for the $\Lambda\Lambda$-ΞN mixing effect in these light $\Lambda\Lambda$ hypernuclei. For models such as NSC97e which are close to describing well the $\Lambda\Lambda$ interaction as deduced from $B_{\Lambda\Lambda}({}^{6}_{\Lambda\Lambda}\mathrm{He})$, we have argued that the $\Lambda\Lambda$-ΞN coupling effect should not exceed 0.2 MeV in ${}^{6}_{\Lambda\Lambda}$He, and a similar order of magnitude is expected for this and other medium effects in the $A=5$ $\Lambda\Lambda$ hypernuclei. For comparison with the better studied $S=-1$ sector, we mention the 0^{+}-1^{+} binding-energy difference in ${}^{4}_{\Lambda}$He, calculated recently by Akaishi *et al.* [26] using a simulation of model NSC97e with and without including the powerful $\Lambda N-\Sigma N$ coupling which arises primarily from one-pion exchange. Compared to the 0.57 MeV effect of the ΛN-ΣN coupling which these authors calculated, we anticipate a considerably smaller effect for the light $\Lambda\Lambda$ hypernuclei considered in the present work, due to strange-meson exchange which underlies the $\Lambda\Lambda$-ΞN coupling in $\Lambda\Lambda$ hypernuclei. Further work on $\Lambda\Lambda$-ΞN mixing is necessary in order to tell whether or not the above argument is supported by a solid calculation.

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APPENDIX

The expression for the integral operator $h^{L\alpha\beta}_{\lambda l,\lambda' l'}$ is well known [14]. For particles of unequal masses this function has the form

$$h^{L\alpha\beta}_{\lambda l,\lambda' l'}(\theta,\theta')=(-)^{L+l'}(2\lambda'+1)(2l'+1)[(2\lambda')!(2l')!(2\lambda+1)(2l+1)]^{1/2}$$
$$\times\sum_{\substack{\lambda_1+\lambda_2=\lambda'\\ l_1+l_2=l'}}\frac{\sin^{\lambda_1+l_1}\theta\cos^{\lambda_2+l_2}\theta}{\sin^{\lambda'}\theta'\cos^{l'}\theta'}\frac{C^{\lambda_1+l_2}_{\alpha\beta}S^{\lambda_2+l_1}_{\alpha\beta}}{[(2\lambda_1)!(2\lambda_2)!(2l_1)!(2l_2)!]^{1/2}}\sum_{\lambda''l''}(2\lambda''+1)(2l''+1)\begin{pmatrix}\lambda_1 & l_1 & \lambda''\\ 0 & 0 & 0\end{pmatrix}$$
$$\times\begin{pmatrix}\lambda_2 & l_2 & l''\\ 0 & 0 & 0\end{pmatrix}\sum_{k=0}(-)^k(2k+1)P_k(t)\begin{pmatrix}k & \lambda'' & \lambda\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}k & l'' & l\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}l & \lambda & L\\ \lambda'' & l'' & k\end{Bmatrix}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda'\\ l_1 & l_2 & l'\\ \lambda'' & l'' & L\end{Bmatrix}, \quad \text{(A1)}$$

in terms of Legendre polynomials and $3j$, $6j$, and $9j$ symbols. The index k runs in Eq. (A1) from zero to $(\lambda'+l'+\lambda+l)/2$. The $C_{\alpha\beta}$, $S_{\alpha\beta}$, and $\cos^2\theta'$ are defined in the main text. For zero total orbital angular momentum $L=0$ $(\lambda=l,\lambda'=l')$, all the summations in the expression above may be carried out to obtain a simpler expression of the form

$$h^{\alpha\beta}_{ll'}(\theta,\theta')=(-)^{l+l'}\sqrt{(2l+1)(2l'+1)}P_l(t)P_{l'}(t'), \quad \text{(A2)}$$

where

$$t'=\frac{-\cos(2\theta)+(C^2_{\alpha\beta}-S^2_{\alpha\beta})\cos(2\theta')}{2C_{\alpha\beta}S_{\alpha\beta}\sin(2\theta')} \quad \text{(A3)}$$

is the cosine of the angle between the vectors $\mathbf{u}'_\alpha$ and $\mathbf{v}'_\alpha$.

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