Shapes of the proton

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A model proton wave function, constructed using Poincaré invariance and constrained by recent electromagnetic form factor data, is used to study the shape of the proton. Spin-dependent quark densities are defined as matrix elements of density operators in proton states of definite polarization, and shown to have an infinite variety of nonspherical shapes. For high momentum quarks with spin parallel to that of the proton, the shape resembles that of a peanut, but for quarks with antiparallel spin the shape is that of a bagel.

The recent data [1,2] showing that the ratio of the proton's electric and magnetic form factor G_E/G_M falls with increasing momentum transfer Q^2 for $1 < Q^2 < 6$ GeV^2 are very interesting. This behavior, or the equivalent statement that the ratio of Pauli to Dirac form factors, $QF_2(Q^2)/F_1(Q^2)$, is approximately constant, indicates that quarks in the proton carry considerable orbital angular momentum [3–6].

This paper is concerned with the relation between the shape of the proton and the quark oribtal angular momentum. Several subtleties and difficulties enter in determining the shape. As a particle of spin-$\frac{1}{2}$, the proton can have no quadrupole moment, according to the Wigner-Eckhart theorem. Thus any nonspherical manifestation must reside in fluctuating components of the quantum wave function. Another difficulty occurs in using experiments in which momentum is transferred to the proton. The proton's final state carries different total momentum than that of its initial states, and the effects of boosts cause the initial and final state wave functions to differ. For example, one often thinks of a particle in relativistic motion having a pancake shape because of the effects of Lorentz contraction. Such an effect is caused by external influence, and is therefore not a manifestation of the intrinsic shape of the proton. Another possible problem occurs through the use of light cone coordinates, which involves a separation of coordinates into longitudinal and transverse, that complicates a simple interpretation. Our technique is to use a model of the proton wave function [7,5,8], constructed with Poincaré invariance and parameters constrained by data, to compute specific matrix elements using only the proton's rest frame wave function and ordinary coordinates.

The basic experimental observables concerning us are the form factors

$$F_1(Q^2) = \frac{1}{2P^+}\langle N,\uparrow|J^+|N,\uparrow\rangle,$$

$$Q\kappa F_2(Q^2) = \frac{-2M_N}{2P^+}\langle N,\uparrow|J^+|N,\downarrow\rangle, \tag{1}$$

with F_1 obtained from a non-spin-flip matrix element and F_2 obtained from a spin-flip term. In evaluating the right-hand side of Eq. (1) we take J^+ to be that of free quarks [9], γ^+ times the quark charge. We use a frame in which $q^\nu = (p' - p)^\nu, Q^2 = -q^2, q^+ = 0, \mathbf{q} = Q\mathbf{e}_x = -2\mathbf{p}$.

Our three-quark wave function is constructed using symmetries [10,11]. It is antisymmetric, expressed in terms of relative momentum variables, an eigenstate of the spin operator defined by the Pauli-Lubanski vector, is rotationally invariant, and reduces to nonrelativistic SU(6) wave function in the limit that the quark mass goes to infinity. The wave function is given by

$$\Psi_s(p_i) = \Phi(M_0^2)u(p_1)u(p_2)u(p_3)\psi_s(p_1,p_2,p_3),$$

$$p_i = \mathbf{p}_i s_i, \tau_i, \tag{2}$$

where ψ_s is a spin-isospin color amplitude factor [11], the $\mathbf{p}_i$ are expressed in terms of relative coordinates (with $-\mathbf{p}_{3\perp} = \mathbf{K}$), and the $u(p_i)$ are Dirac spinors. The spin-independent wave function Φ is a function of the mass-squared operator for a noninteracting system [12]:

$$M_0 = (\mathbf{K}^2 + M_3^2)^{1/2} + (\mathbf{K}^2 + m^2)^{1/2}, \quad M_3 \equiv 2(\mathbf{k}^2 + m^2)^{1/2}, \tag{3}$$

where $\mathbf{k}$ is the relative momentum between quarks labeled 1 and 2 and $-\mathbf{K}$ is the momentum of the third quark [13]. We use the Schlumpf [12] form

$$\Phi(M_0^2) = \frac{N}{(M_0^2 + \beta^2)^\gamma}, \quad \beta = 0.607 \text{ GeV}, \quad \gamma = 3.5,$$

$$m = 0.267 \text{ GeV}. \tag{4}$$

The value of γ is chosen so that $Q^4 G_M(Q^2)$ is approximately constant for $Q^2 > 4$ GeV^2. The values of β and m are determined by the charge radius and magnetic moment of the proton. Slightly different values of β, γ, and m are used when the effects of the pion cloud are incorporated [8]. Using these newer values would cause very small differences here.

The calculation of electromagnetic form factors is completely defined once this wave function and operator J^+ are specified. Although the evaluation was presented long ago [7] and explained recently [5], it is worthwhile to briefly explain how the constant nature of the ratio QF_2/F_1 emerges from the relativistic nature of the calculation. The wave function Eq. (2) is completely antisymmetric, so we may take J^+ to act only on the third quark which absorbs the

virtual photon's momentum. The average charge of the third quark in the mixed-symmetric component of Eq. (2) vanishes, so only the mixed-antisymmetric component in which the first two quarks have a vanishing total angular momentum enters here. Then the spin of the proton s is governed by that of the third quark. The relevant Dirac spinor is

$$u(p_3 = \mathbf{K}, s) = \frac{1}{\sqrt{E(K)+m}} \begin{pmatrix} [E(K)+m]|s\rangle \\ \boldsymbol{\sigma} \cdot \mathbf{K}|s\rangle \end{pmatrix}, \qquad (5)$$

with $E(K) = (K^2 + m^2)^{1/2}$. The total angular momentum of the proton is denoted by s and the lower component contains a term $\boldsymbol{\sigma} \cdot \mathbf{K}$ that allows the quark to have a spin opposite to that of the proton's total angular momentum. The vector $\mathbf{K}$ reveals the presence of the quark orbital angular momentum: the struck quark may carry a spin that is opposite to that of the proton. Consequently nucleon helicity is not conserved [3,4,14].

Suppose Q is very large compared to β and the third quark changes its momentum from $\mathbf{K}$ to $\mathbf{K}'$. Then the form factor depends on the matrix element of $\bar{u}(K',s')\gamma^+ u(K,s)$. If we keep only the largest terms, those proportional to Q, we find

$$\bar{u}(K',s')\gamma^+ u(K,s) \sim \langle s'|Q + \frac{K^+}{M_0}Qi\sigma_y|s\rangle. \qquad (6)$$

The spin-flip term proportional to $i\sigma_y$ arises from the lower component of the spinor of Eq. (5), and this term has the same large factor Q as the nonspin-flip term, implying via Eq. (1) that QF_2 and F_1 have the same dependence on Q, and thus a constant ratio.

The essential feature is that construction of an eigenstate of spin mandates the use of Dirac spinors. The lower components of these spinors carry the quark orbital angular momentum responsible for the constant nature of QF_2/F_1.

Our aim is to interpret these features of the wave function in terms of the shapes of the proton. The technique of defining spin-dependent density operators is introduced by presenting two simple examples. Consider a nonrelativistic nuclear wave function consisting of a proton outside a 0^+ inert core. The proton is bound by a combination of a central and spin-orbit potential and the single particle wave function is an eigenfunction of total angular momentum. Consider the case $(l, js) = (1, 1/2s)$, with one unit of orbital angular momentum. Then the proton wave function is a two-component Pauli spinor: $\langle \mathbf{r}_p | \psi_{1,1/2s} \rangle = R(r_p)\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}_p |s\rangle$, where $\mathbf{r}_p$ is the proton's position and $|s\rangle$ is a Pauli spinor. The charge density of this system is the expectation value of the density operator $\delta(\mathbf{r} - \mathbf{r}_p)$. Then $\rho(r) = \langle \psi_{1,1/2s}|\delta(\mathbf{r} - \mathbf{r}_p)|\psi_{1,1/2s}\rangle = R^2(r)$, and is spherically symmetric. However, consider instead the case that we require the proton at a position $\mathbf{r}$ to have a spin in a direction defined by a unit vector $\mathbf{n}$. Then we define a spin-dependent density:

$$\rho(\mathbf{r}, \mathbf{n}) = \langle \psi_{1,1/2s}|\delta(\mathbf{r} - \mathbf{r}_p)\frac{(1+\boldsymbol{\sigma} \cdot \mathbf{n})}{2}|\psi_{1,1/2s}\rangle$$

$$= \frac{R^2(r)}{2}\langle s|1 + 2\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}\mathbf{n} \cdot \hat{\mathbf{r}} - \boldsymbol{\sigma} \cdot \mathbf{n}|s\rangle. \qquad (7)$$

An interesting special case is to take $\mathbf{n}$ parallel or antiparallel to the direction of the proton angular momentum $\mathbf{s}$. The direction of this vector defines an axis (the "z axis"), and the direction of vectors can be represented in terms of this axis: $\hat{\mathbf{s}} \cdot \hat{\mathbf{r}} = \cos\theta$. With this notation $\rho(\mathbf{r}, \mathbf{n} = \hat{\mathbf{s}}) = R^2(r)\cos^2\theta$, $\rho(\mathbf{r}, \mathbf{n} = -\hat{\mathbf{s}}) = R^2(r)\sin^2\theta$ and the nonspherical shape is exhibited clearly. Note that the average of these two cases would give a spherical shape (as would an average over the direction of the total angular momentum), but the ability to define a spin-dependent density allows the presence of the orbital angular momentum to be revealed in the shape of the computed density.

Another useful example is that of the Dirac electron wave function of the hydrogen atom. This wave function is a four-component spinor given by $|\psi_e\rangle$ with

$$\langle \mathbf{r}_e | \psi_e \rangle = Nr_e^{\gamma}\exp(-m_e\alpha r_e)\begin{pmatrix} 1|s\rangle \\ i\alpha/2\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}_e|s\rangle \end{pmatrix}, \qquad (8)$$

where α is the fine structure constant and $\gamma = \sqrt{1-\alpha^2}$. We compute the expectation value of the spin-dependent density operator, in terms of Dirac matrices: $\delta(\mathbf{r} - \mathbf{r}_e)(1 + \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{n}\gamma_5)/2$, so that with $\mathbf{n} = \hat{\mathbf{s}}$, $\rho(\mathbf{r}, \hat{\mathbf{n}} = \hat{\mathbf{s}}) = r_e^{-2\gamma}\exp(-2m_e\alpha r_e)[1+\alpha^2/4\cos^2\theta]$. We see that the hydrogen atom is deformed. The angular dependence is $\sim 1 + 10^{-5}\cos^2\theta$ so that the shape is almost spherical. But the principle is clear: relativity as manifest by lower components of a Dirac wave function implies a deformed shape, if the matrix element is computed in a state of fixed total angular momentum. For $\mathbf{n} = -\hat{\mathbf{s}}$, $\rho(\mathbf{r}, \hat{\mathbf{n}} = -\hat{\mathbf{s}}) = \alpha^2\sin^2\theta/4$.

With these examples in hand we turn to the proton. Its wave function is specified in momentum space, so we define a momentum-space, spin-dependent, charge-density operator:

$$\hat{\rho}(\mathbf{K}, \mathbf{n}) = \int \frac{d^3r}{(2\pi)^3} e^{i\mathbf{K}\cdot\mathbf{r}} \bar{\psi}(\mathbf{r})\frac{\hat{Q}}{e}(\gamma^0 + \boldsymbol{\gamma} \cdot \mathbf{n}\gamma_5)\psi(\mathbf{0}), \quad (9)$$

where $\hat{Q}/e$ is the quark charge operator in units of the proton charge. The quark field operators are evaluated at equal time. In QCD, one would need to add a factor between the field operators to ensure gauge invariance. With the present model, any gluons are contained within constituent quarks, so the factor becomes unity. For our model, it is convenient to use first-quantized notation so that

$$\hat{\rho}(\mathbf{K}, \mathbf{n}) \equiv \sum_{i=1,3}(Q_i/e)\delta(\mathbf{K} - \mathbf{K}_i)[1 + (\gamma^0\boldsymbol{\gamma} \cdot \mathbf{n}\gamma_5)_i]/2. \qquad (10)$$
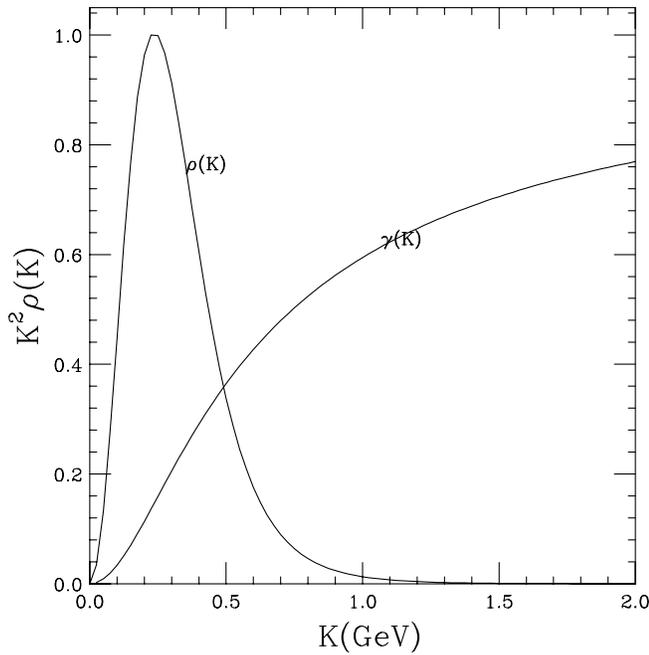
FIG. 1. Momentum probability $K^2\rho(K)$ in dimensionless units, arbitrarily normalized.

We may compute probabilities for a quark to have a momentum $\mathbf{K} \equiv (K, \theta, \phi)$ and spin direction $\mathbf{n}$, for a spin-polarized proton polarized in the $\hat{\mathbf{s}}$ direction. We find

$$\rho(\mathbf{K},\mathbf{n}) = \langle \Psi_s | \hat{\rho}(\mathbf{K},\hat{\mathbf{n}}) | \Psi_s \rangle$$

$$= \rho(K)\tfrac{1}{2}[1 + \mathbf{n}\cdot\hat{\mathbf{s}} + \gamma(K)(1 - \mathbf{n}\cdot\hat{\mathbf{s}} + 2\hat{\mathbf{K}}\cdot\mathbf{n}\hat{\mathbf{K}}\cdot\hat{\mathbf{s}})] \tag{11}$$

with

$$\rho(K) \equiv \int d^3k\,\Phi^2(k,K)[E(K)+m], \quad \gamma(K) \equiv \frac{E(K)-m}{E(K)+m}. \tag{12}$$

Some special cases of Eq. (11) are interesting. Suppose the quark spin is parallel to the proton spin, $\mathbf{n}=\hat{\mathbf{s}}$, then $\hat{\rho}(K,\mathbf{n}=\hat{\mathbf{s}}) = \rho(K)[1+\gamma(K)\cos^2\theta]$. For values of $K \ll m$ the shape is nearly spherical, but for $K \gg m$ the $\cos^2\theta$ term becomes prominent. But if the quark spin is antiparallel to the proton spin, $\mathbf{n}=-\hat{\mathbf{s}}$, we find: $\hat{\rho}(K,\mathbf{n}=-\hat{\mathbf{s}}) = \rho(K)\gamma(K)\sin^2\theta$, and the shape is that of a torus. We may also take the quark spin perpendicular to the proton spin $\mathbf{n}\cdot\mathbf{s}=0$, so that $\rho(\mathbf{K},\mathbf{n}\cdot\mathbf{s}=0) = \rho(K)[1+\gamma(K)]/2 + \gamma(K)\sin\theta\cos\theta(\cos\phi n_x + \sin\phi n_y)$, to display the dependence on the azimuthal angle. In each case, the nonspherical nature arises from the term proportional to $\gamma(K)$ caused by the lower components of the Dirac spinor (5).

We turn to the numerical evaluation and display of these shapes, see Fig. 1. The shape for a given value of K is determined by the ratio $\gamma(K)$ which reaches a value of 0.6 for $K=1$ GeV/c, implying considerable nonsphericity and providing a sharp contrast between the proton and the hydrogen
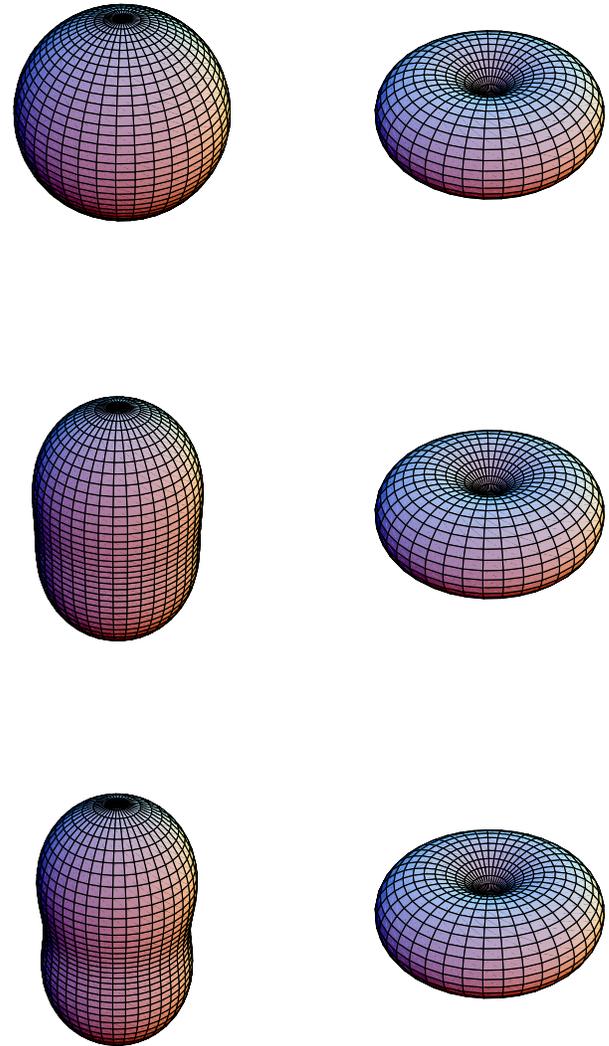
FIG. 2. (Color online) Shapes of the proton. $\mathbf{s}$ is in the vertical direction $\hat{\mathbf{z}}$. Left column quark spin parallel to nucleon spin $\mathbf{n}=\hat{\mathbf{s}}$. Right column: quark spin antiparallel to nucleon spin $\mathbf{n}=-\mathbf{s}$. The value of K increases from 0 to 1 to 4 GeV/c.

atom. The probability that a given value of K is reached is determined by the function $K^2\rho(K)$, displayed in Fig. 1. The most likely value of K corresponds to $\gamma(K)=0.16$.

The shapes for the cases of quark spin parallel and antiparallel to the polarization direction of the proton $\mathbf{s}$ are displayed in Fig. 2. As the value of K increases from 0 to 4 GeV/c the shape varies from that of a sphere to that of a peanut, if $\mathbf{n}\|\mathbf{s}$. The torus or bagel shape is obtained if $-\mathbf{n}\|\mathbf{s}$. Taking $\mathbf{n}\perp\mathbf{s}$ leads to some very unusual shapes shown in Fig. 3.

It is worthwhile to look at the shape of the proton in coordinate space. One can determine the probability for the separation of a quark from the center of mass to be $\mathbf{r}$ because $\mathbf{r}$ is canonically conjugate to $\mathbf{K}$. We define a coordinate-space, spin-dependent, charge-density operator that acts on the quarks in the proton rest frame:

$$\hat{\rho}(\mathbf{r},\mathbf{n}) \equiv \sum_{i=1,3}(Q_i/e)\delta(\mathbf{r}-\mathbf{r}_i)[1+(\gamma^0\boldsymbol{\gamma}\cdot\mathbf{n}\gamma_5)_i]/2 \tag{13}$$
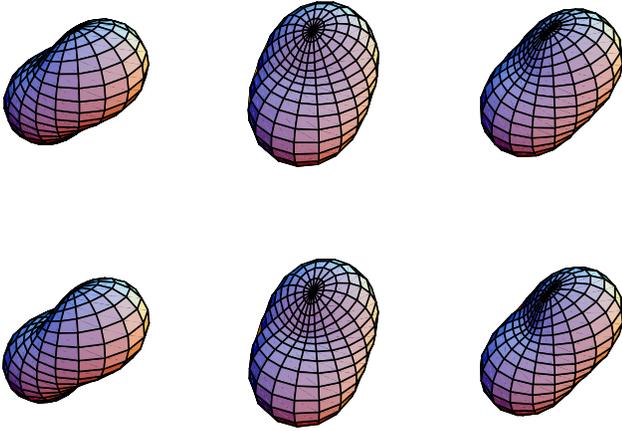
FIG. 3. (Color online) $\mathbf{n} \cdot \mathbf{s} = 0$. Left column, $\mathbf{n} = \hat{\mathbf{x}}$ (out of page), central: $\mathbf{n} = \hat{\mathbf{y}}$, right $\mathbf{n} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. The momentum K increases from 1 to 4 GeV/c.

FIG. 4. Coordinate space densities.

so that we may compute probabilities for a quark to have a position $\mathbf{r} \equiv (r, \theta, \phi)$ and spin direction $\mathbf{n}$. We find

$$\rho(\mathbf{r}, \mathbf{n}) = \langle \Psi_s | \hat{\rho}(\mathbf{r}, \mathbf{n}) | \Psi_s \rangle$$
$$= \int d^3 k \chi^\dagger(k, \mathbf{r}) \frac{1}{2} (1 + \gamma^0 \boldsymbol{\gamma} \cdot \mathbf{n} \gamma_5) \chi(k, \mathbf{r}), \quad (14)$$

with

$$\chi(k, \mathbf{r}) = \begin{pmatrix} F_k(r) | s \rangle \\ -i \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} G_k(r) | s \rangle \end{pmatrix}, \quad (15)$$

and $F_k(r) = \int d^3 K \Phi(k, K)[E(K) + M]^{1/2} e^{i\mathbf{K} \cdot \mathbf{r}}, G_k(r) = \partial/\partial r \int d^3 K \Phi(k, K) e^{i\mathbf{K} \cdot \mathbf{r}}/[E(K) + M]^{1/2}$. We find

$$\rho(\mathbf{r}, \mathbf{n}) = \rho_U(r) \frac{1}{2} (1 + \mathbf{n} \cdot \hat{\mathbf{s}}) + \rho_L(r) \frac{1}{2} (1 + 2\hat{\mathbf{r}} \cdot \hat{\mathbf{s}} \mathbf{n} \cdot \hat{\mathbf{r}} - \mathbf{n} \cdot \hat{\mathbf{s}}), \quad (16)$$

where $\rho_U(r) \equiv \int d^3 k F_k^2(r), \rho_L(r) \equiv \int d^3 k G_k^2(r)$. The pattern is similar to that in momentum space, with $\rho(\mathbf{r}, \mathbf{n} = \hat{\mathbf{s}}) = \rho_U(r) + \rho_L(r) \cos^2 \theta$, $\rho(\mathbf{r}, \mathbf{n} = -\hat{\mathbf{s}}) = \rho_L(r) \sin^2 \theta$, and $\rho(\mathbf{r}, \mathbf{n} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}) = \frac{1}{2} \rho_U(r) + \frac{1}{2} \rho_L(r) [1 + 2/\sqrt{2} \cos \theta \sin \theta (\cos \phi + \sin \phi)]$.

The ratio ρ_L/ρ_U, which determines the size of the relativistic effects, can be much larger (Fig. 4) than the factor $\gamma(K)$ (Fig. 1) controlling the momentum-space shapes, so that extreme deviations from a spherical shape are possible. The most likely value of ρ_L/ρ_U is about 0.25, but there is no limit. The case with $\rho_L/\rho_U = 3$ is shown in Fig. 5. A pretzel form is obtained if $\mathbf{n}$ is out of the page.

The shape of the proton may be defined in terms of matrix elements of spin-dependent density operators Eqs. (10) and (13) taken for protons in any fixed polarization state. Relativity mandates the use of Dirac spinors to describe the quarks. These components, embodied in Eq. (6), lead to a constant ratio of QF_2/F_1 in accord with observation, and also to shapes that depend strongly on the relative orientation

of the quark spin with respect to that of the proton total angular momentum, Eqs. (11) and (16) (Figs. 2, 3, and 5).

We next consider experiments aimed at measuring the matrix element $\hat{\rho}(\mathbf{K}, \mathbf{n})$, Eq. (9), for real nucleons $|N\rangle$. Observe that $\int d^3 K \hat{\rho}(\mathbf{K}, \mathbf{n})$ is a local operator. Its matrix element is a linear combination of the charge, integrals of spin-dependent structure functions Δq, and g_A that can be determined from previous measurements. We find

$$\int d^3 K \langle N | \hat{\rho}(\mathbf{K}, \mathbf{n} = \pm \hat{\mathbf{s}}) | N \rangle$$
$$= \frac{1}{2} \langle N | \bar{\psi}(0) \frac{\hat{Q}}{e} (\gamma^0 \pm \gamma^3 \gamma^5) \psi(0) | N \rangle$$
$$= \frac{1}{2} [1 \pm \frac{1}{6} (\Delta u + \Delta d + \Delta s) + \frac{1}{2} g_A] = 0.5 \pm 0.34, \quad (17)$$

in which numerical values of Δq are taken from Ref. [15]. The model we use gives 0.5 ± 0.37 for the above quantity,
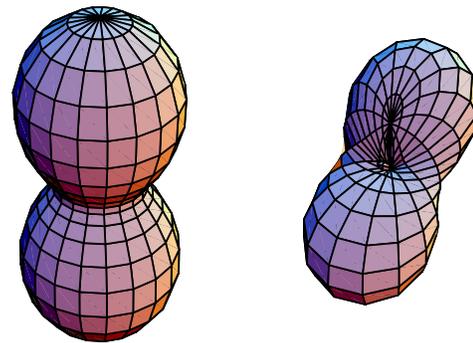
FIG. 5. (Color online) Shape of the proton coordinate space. Left, $\mathbf{n} = \mathbf{s}$; right, $\mathbf{n}$ points out of the page.

indicating that the results shown here may not be unrealistic.

The task of determining $\rho(\mathbf{K},\mathbf{n})$ as a function of $\mathbf{K}$ remains open. However, the specific relativistic effects of orbital angular momentum and the related spin-flip effects responsible for the nonspherical shapes can be expected to influence many measurable quantities. These include: spin-dependent structure functions g_1, g_2, the $N\Delta$ electromagnetic transition form factor, and spin-dependent cross sections in high momentum transfer scattering of polarized protons.

[1] M.K. Jones *et al.*, Jefferson Lab Hal A Collaboration, Phys. Rev. Lett. **84**, 1398 (2000).

[2] O. Gayou *et al.*, Phys. Rev. Lett. **88**, 092301 (2002).

[3] P. Jain, B. Pire, and J.P. Ralston, Phys. Rep. **271**, 67 (1996); T. Gousset, B. Pire, and J.P. Ralston, Phys. Rev. D **53**, 1202 (1996).

[4] V.M. Braun, A. Lenz, N. Mahnke, and E. Stein, Phys. Rev. D **65**, 074011 (2002).

[5] G.A. Miller and M.R. Frank, Phys. Rev. C **65**, 065205 (2002).

[6] A.V. Belitsky, X-d. Ji, and F. Yuan, arXiv:hep-ph/0212351.

[7] M.R. Frank, B.K. Jennings, and G.A. Miller, Phys. Rev. C **54**, 920 (1996).

[8] G.A. Miller, Phys. Rev. C **66**, 032201 (2002).

[9] When the pion cloud [16] is included [8], J^+ also includes the current of free pions, which are important at low values of Q^2.

[10] V.B. Berestetskii and M.V. Terent'ev, Sov. J. Nucl. Phys. **25**, 347 (1977).

[11] P.L. Chung and F. Coester, Phys. Rev. D **44**, 229 (1991); F. Coester, Prog. Part. Nucl. Phys. **29**, 1 (1992).

[12] F. Schlumpf and U. Zurich, Ph.D. thesis (unpublished), hep-ph/9211255.

[13] Calculations of the form factors are simplified by using light front dynamics to include the effects of the boost. For our model the difference between the light front and instant forms is contained in a coordinate transformation, which may or may not be made depending on the application. Here $\mathbf{q}=0$, so we use the instant form.

[14] The ratio of QF_2/F_1 is a ratio of a helicity-nonconserving to a helicity-conserving matrix element, if the reference frame of Eq. (1) is used.

[15] A. W. Thomas and W. Weise, *The Structure Of The Nucleon* (Wiley, Berlin, 2001).

[16] S. Théberge, A.W. Thomas, and G.A. Miller, Phys. Rev. D **22**, 2838 (1980); **23**, 2106(E) (1981); A.W. Thomas, S. Théberge, and G.A. Miller, *ibid.* **24**, 216 (1981); S. Théberge, G.A. Miller, and A.W. Thomas, Can. J. Phys. **60**, 59 (1982); G.A. Miller, A.W. Thomas, and S. Théberge, Phys. Lett. **91B**, 192 (1980).