Relativistic instant-form approach to the structure of two-body composite systems: Nonzero spin

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The relativistic approach to electroweak properties of two-particle composite systems developed in A.F. Krutov and V.E. Troitsky, Phys. Rev. C **65**, 045501 (2002) is generalized here to the case of nonzero spin. In developed technique the parametrization of matrix elements of electroweak current operators in terms of form factors is a realization of the Wigner-Eckart theorem on the Poincaré group and form factors are reduced matrix elements. The ρ-meson charge form factor is calculated as an example.

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A new relativistic approach to electroweak properties of composite systems has been proposed in our recent paper [1]. The approach is based on the use of the instant form (IF) of relativistic Hamiltonian dynamics (RHD). The detailed description of RHD can be found in the review [2]. Some other references as well as some basic equations of RHD approach are given in Ref. [1].

Now our aim is to generalize the approach to composite systems of two particles of spin $\frac{1}{2}$ with nonzero values of total angular momentum, total orbital momentum, and total spin. The main problem is the construction of electromagnetic current operator satisfying standard conditions (see, e.g., Refs. [1,3]).

The basic point of our approach [1] to the construction of the electromagnetic current operator is the general method of relativistic invariant parametrization of local operator matrix elements proposed long ago in 1963 by Cheshkov and Shirokov [4]. This canonical parametrization of local operators matrix elements was generalized to the case of composite systems of free particles in Refs. [5,6]. This parametrization is a realization of the Wigner-Eckart theorem for the Poincaré group and so it enables one, for given matrix element of arbitrary tensor dimension, to separate the reduced matrix elements (form factors) that are invariant under the Poincaré group.

Physical approximations that we use in our approach are formulated in terms of reduced matrix elements, for example, the well known relativistic impulse approximation. In our method this approximation does not violate the standard conditions for the current.

In the present paper we propose a general formalism for the operators diagonal in the total angular momentum. The details of calculations can be found in Ref. [3].

Let us consider the operator $j_\mu = j_\mu(0)$ that describes a transition between two states of a composite two-constituent system. Let us neglect temporarily, for simplicity, the conditions of self-adjointness, conservation law, and parity conservation. The Wigner-Eckart decomposition of the matrix element has the form [4]

$$\langle \vec{p}_c, m_{Jc}|j_\mu|\vec{p}_c\,', m'_{Jc}\rangle = \langle m_{Jc}|D^{J_c}(p_c, p'_c)[F_1^c K'_\mu + F_2^c \Gamma_\mu(p'_c) + F_3^c R_\mu + F_4^c K_\mu]|m'_{Jc}\rangle, \tag{1}$$

$$F_i^c = \sum_{n=0}^{2J_c} f_{in}^c(Q^2)[i p_{c\mu}\Gamma^\mu(p'_c)]^n. \tag{2}$$

Here $K_\mu = (p_c - p'_c)_\mu = q_\mu$, $K'_\mu = (p_c + p'_c)_\mu$, $R_\mu = \epsilon_{\mu\nu\lambda\rho} p_c^\nu p_c'^\lambda \times \Gamma^\rho(p'_c)$; $(p_c - p'_c)^2 = -Q^2$, $p_c^2 = p'^2_c = M_c^2$, M_c, J_c are the mass and spin of the composite particle, m_{Jc} is spin projection, $\Gamma^\rho(p'_c)$ is the spin four-vector defined with the use of the Pauli-Lubansky vector [1], f_{in}^c are reduced matrix elements, $\epsilon_{\mu\nu\lambda\rho}$ is a completely antisymmetric pseudotensor in four-dimensional space-time with $\epsilon_{0123} = -1$.

In the framework of RHD, the form factors of composite systems f_{in}^c are to be expressed in terms of RHD wave functions and constituents form factors.

In RHD a state of two-particle interacting system is described by a vector in the direct product of two one-particle Hilbert spaces (see, e.g., Ref. [1]). So, the matrix element in RHD can be decomposed in the basis [1]

$$|\vec{P}, \sqrt{s}, J, l, S, m_J\rangle. \tag{3}$$

Here $P_\mu = (p_1 + p_2)_\mu$, $P_\mu^2 = s$, $\sqrt{s}$ is the invariant mass of the two-particle system, l is the orbital angular momentum in the center-of-mass frame (c.m.), S is the total spin in the c.m., J is the total angular momentum with the projection m_J.

$$\langle \vec{p}_c, m_J|j_\mu|\vec{p}_c\,', m'_J\rangle = \sum \int \frac{d\vec{P}\,d\vec{P}\,'}{N_{CG}N'_{CG}} d\sqrt{s}\,d\sqrt{s'} \times\langle \vec{p}_c, m_{Jc}|\vec{P}, \sqrt{s}, J, l, S, m_J\rangle \times\langle \vec{P}, \sqrt{s}, J, l, S, m_J|j_\mu|\vec{P}\,', \sqrt{s'}, J', l', S', m_{J'}\rangle \times\langle \vec{P}\,', \sqrt{s'}, J', l', S', m_{J'}|\vec{p}_c\,', m'_{Jc}\rangle. \tag{4}$$

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Here the sum is over variables $J,J',l,l',S,S',m_J,m_{J'}$, and $\langle\vec{P}',\sqrt{s'},J',l',S',m_{J'}|\vec{p}_c\,',m'_{Jc}\rangle$ is the wave function in the sense of IF RHD.

$$\langle\vec{P},\sqrt{s},J,l,S,m_J|\vec{p}_c,m_{J_c}\rangle=N_c\delta(\vec{P}-\vec{p}_c)\delta_{J_cJ}\delta_{m_{J_c}m_J}\varphi^{J_c}_{lS}(k). \tag{5}$$

Here $k=\sqrt{\lambda(s,M_1^2,M_2^2)}/(2\sqrt{s})$, M_1,M_2 are masses of constituents, $\lambda(a,b,c)=a^2+b^2+c^2-2(ab+bc+ac)$.

The RHD wave function of constituents relative motion with fixed total angular momentum is defined as

$$\varphi^{J_c}_{lS}(k(s))=\sqrt{\sqrt{s}(1-\eta^2/s^2)}\,u_{lS}(k)k, \tag{6}$$

and is normalized by the condition

$$\sum_{lS}\int u^2_{lS}(k)k^2dk=1. \tag{7}$$

Here $\eta=M_1^2-M_2^2, u_{lS}(k)$ is a model wave function.

The main difficulty arising in this case is the following. In expression (1) we were dealing with the parametrization of local operator matrix elements in the case when the transformations of the state vectors and of the operators were defined by one and the same representation of the quantum mechanical Poincaré group.

A different situation takes place in the case of the matrix element in the right-hand side (rhs), of Eq. (4). The operator describes the system of two interacting particles and transforms following the representation with Lorentz boosts generators depending on the interaction [1]. The state vectors physically describe the system of two free particles and present the basis of a representation with interaction-independent generators. So, the Wigner-Eckart decomposition cannot be applied directly to the matrix element in the integrand in the rhs of Eq. (4). This is caused by the fact that it is impossible to construct four-vectors describing the matrix element transformation properties under the action of Lorentz boosts from the variables entering the state vectors [contrary to the case of, e.g., Eq. (1)]. In fact, the possibility of matrix element representation in form (1) is based on the following fact. Let us act by Lorentz transformation on operator $\hat{U}^{-1}(\Lambda)j^\mu\hat{U}(\Lambda)=\tilde{j}^\mu$. We obtain the following chain of equalities:

$$\begin{aligned}\langle\vec{p}_c,m_{Jc}|\tilde{j}^\mu|\vec{p}_c\,',m'_{Jc}\rangle&=\langle\vec{p}_c,m_{Jc}|\hat{U}^{-1}(\Lambda)j^\mu\hat{U}(\Lambda)|\vec{p}_c\,',m'_{Jc}\rangle\\&=\sum_{\tilde{m}_{Jc},\tilde{m}'_{Jc}}\langle m_{Jc}|[D^{J_c}(R_\Lambda)]^{-1}|\tilde{m}_{Jc}\rangle\\&\quad\times\langle\Lambda\vec{p}_c,\tilde{m}_{Jc}|j^\mu|\Lambda\vec{p}_c\,',\tilde{m}'_{Jc}\rangle\\&\quad\times\langle\tilde{m}'_{Jc}|D^{J_c}(R_\Lambda)|m'_{Jc}\rangle.\end{aligned} \tag{8}$$

Here $D^{J_c}(R_\Lambda)$ is rotation matrix realizing the angular momentum transformation under the action of Lorentz transformations. Equalities (8) show that the transformation properties of the current as a four-vector can be described using the four-vectors of the initial and the final states. This means that the canonical parametrization [4] is the realization of the Wigner-Eckart theorem on the Poincaré group.

In the case of the current matrix element in the rhs of Eq. (4) relations (8) are not valid and direct application of the Wigner-Eckart theorem is impossible.

However, it can be shown that for the matrix element in Eq. (4) considered as a generalized function [that is considered as an object having sense only under integrals and sums in Eq. (4)], the equality (8) is valid in the weak sense.

Let us consider the matrix element in question as a regular Lorentz covariant generalized function (see, e.g., Ref. [7]). Using Eq. (5), let us rewrite Eq. (4) in the following form:

$$\begin{aligned}&\langle\vec{p}_c,m_{Jc}|j_\mu|\vec{p}_c\,',m'_{Jc}\rangle\\&=\sum_{l,l',S,S'}\int\mathcal{N}d\sqrt{s}d\sqrt{s'}\varphi^{Jc}_{lS}(s)\varphi^{Jc}_{l'S'}(s')\\&\quad\times\langle\vec{p}_c,\sqrt{s},J_c,l,S,m_{Jc}|j_\mu|\vec{p}_c\,',\sqrt{s'},J_c,l',S',m'_{Jc}\rangle.\end{aligned} \tag{9}$$

Here it is taken into account that the current operator j_μ is diagonal in total angular momentum of the composite system, $\mathcal{N}=N_cN'_c/N_{CG}N'_{CG}$.

Let us make use of the fact that the set of states (3) is complete:

$$\hat{I}=\sum\int\frac{d\vec{P}}{N_{CG}}d\sqrt{s}|\vec{P},\sqrt{s},J,l,S,m_J\rangle\langle\vec{P},\sqrt{s},J,l,S,m_J|. \tag{10}$$

Here the sum is over the discrete variables of basis (3).

Under the integral the matrix element of the transformed current satisfies the following equalities [Eqs. (5) and (10) are taken into account]:

$$\begin{aligned}&\sum\int\mathcal{N}d\sqrt{s}d\sqrt{s'}\varphi^{Jc}_{lS}(s)\varphi^{Jc}_{l'S'}(s')\langle\vec{p}_c,\sqrt{s},J_c,l,S,m_{Jc}|\hat{U}^{-1}(\Lambda)j_\mu\hat{U}(\Lambda)|\vec{p}_c\,',\sqrt{s'},J_c,l',S',m'_{Jc}\rangle\\&=\sum\int\mathcal{N}d\sqrt{s}d\sqrt{s'}\varphi^{Jc}_{lS}(s)\varphi^{Jc}_{l'S'}(s')\sum_{\tilde{m}_{Jc},\tilde{m}'_{Jc}}\langle m_{Jc}|[D^{J_c}(R_\Lambda)]^{-1}|\tilde{m}_{Jc}\rangle\\&\quad\times\langle\Lambda\vec{p}_c,\sqrt{s},J_c,l,S,\tilde{m}_{Jc}|j_\mu|\Lambda\vec{p}_c\,',\sqrt{s'},J_c,l',S',\tilde{m}'_{Jc}\rangle\langle\tilde{m}'_{Jc}|D^{J_c}(R_\Lambda)|m'_{Jc}\rangle.\end{aligned} \tag{11}$$

It is easy to see that under the integral the current matrix element satisfies the equalities analogous to Eq. (8), so now it is possible to use the parametrization under the integral, that is to use the Wigner-Eckart theorem in the weak sense. The rhs of Eq. (9) can be written as a functional on the space of test functions of the form [see Eq. (6), too)] $\psi^{ll'SS'}(s,s') = u_{lS}(k(s))u_{l'S'}(k(s'))$, and Eq. (9) can be rewritten as a functional in $\mathbf{R}^2$ with variables (s,s'):

$$\langle\vec{p}_c, m_{Jc}|j_\mu(0)|\vec{p}_c\,', m'_{Jc}\rangle = \sum_{l,l',S,S'}\int d\mu(s,s')\mathcal{N}\psi^{ll'SS'}(s,s') \times\langle\vec{p}_c,\sqrt{s},J_c,l,S,m_{Jc}|j_\mu|\vec{p}_c\,',\sqrt{s'},J_c,l',S',m'_{Jc}\rangle. \tag{12}$$

Here the measure is chosen with the account of the relativistic density of states, subject to normalization (6) and (7):

$$d\mu(s,s') = 16\theta(s-[M_1+M_2]^2)\,\theta(s'-[M_1+M_2]^2) \times\sqrt{\sqrt{s}(1-\eta^2/s^2)\sqrt{s'}(1-\eta^2/s'^2)} \times d\mu(s)d\mu(s'). \tag{13}$$

Here $d\mu(s) = (1/4)k d\sqrt{s}$.

The sums over discrete invariant variables can be transformed into integrals by introducing the adequate δ functions. The obtained expressions are functionals in $\mathbf{R}^6$.

The functional in the rhs of Eq. (12), defines a Lorentz covariant generalized function, generated by the current operator matrix element.

Taking into account Eq. (11), we decompose the matrix element in the rhs of Eq. (12) into the set of linearly independent scalars entering the rhs of Eq. (1):

$$\mathcal{N}\langle\vec{p}_c,\sqrt{s},J_c,l,S,m_{Jc}|j_\mu|\vec{p}_c\,',\sqrt{s'},J_c,l',S',m'_{Jc}\rangle = \langle m_{Jc}|D^{J_c}(p_c,p'_c)\sum_{n=0}^{2J_c}[ip_{c\mu}\Gamma^\mu(p'_c)]^n \times\mathcal{A}_{n\mu}^{ll'SS'}(s,Q^2,s')|m'_{Jc}\rangle. \tag{14}$$

Here $\mathcal{A}_{n\mu}^{ll'SS'}(s,Q^2,s')$ is a Lorentz covariant generalized function.

Making use of Eq. (14) and comparing the rhs of Eq. (1) with Eq. (12), we obtain

$$\sum_{l,l',S,S'}\int d\mu(s,s')\psi^{ll'SS'}(s,s') \times\langle m_{Jc}|\mathcal{A}_{n\mu}^{ll'SS'}(s,Q^2,s')|m'_{Jc}\rangle = \langle m_{Jc}|[f_{1n}^c K'_\mu + f_{2n}^c\Gamma_\mu(p'_c) + f_{3n}^c R_\mu + f_{4n}^c K_\mu]|m'_{Jc}\rangle. \tag{15}$$

All the form factors in the rhs of Eq. (15) are nonzero if the generalized function $\mathcal{A}$ contains parts that are diagonal ($\mathcal{A}_1$) and nondiagonal ($\mathcal{A}_2$) in m_{Jc}, m'_{Jc}. For the diagonal part we have, from Eq. (15),

$$\sum_{l,l',S,S'}\int d\mu(s,s')\psi^{ll'SS'}(s,s')\langle m_{Jc}|\mathcal{A}_{1n\mu}^{ll'SS'}(s,Q^2,s')|m_{Jc}\rangle = \langle m_{Jc}|[f_{1n}^c[\psi]K'_\mu + f_{4n}^c[\psi]K_\mu]|m_{Jc}\rangle. \tag{16}$$

The notation $f_{in}^c[\psi]$ in the rhs emphasizes the fact that form factors of composite systems are functionals on the wave functions of the intrinsic motion, and so, on the test functions.

Let equality (16) be valid for any test function $\psi^{ll'SS'}(s,s')$. When the test functions (the intrinsic motion wave functions) are changed, the vectors in the rhs are not changed because according to the essence of the parametrization (1) they do not depend on the model for the particle intrinsic structure. So, when the test functions are varied the vector of the rhs of Eq. (16) remains in the hyperplane defined by the vectors K_μ, K'_μ.

When test functions are varied arbitrarily, the vector in lhs of Eq. (16) can take, in general, an arbitrary direction. So, the requirement of the validity of Eq. (16) in the whole space of our test functions is that the lhs generalized function has the form

$$\mathcal{A}_{1n\mu}^{ll'SS'}(s,Q^2,s') = K'_\mu G_{1n}^{ll'SS'}(s,Q^2,s') + K_\mu G_{4n}^{ll'SS'}(s,Q^2,s'). \tag{17}$$

Here $G_{in}^{ll'SS'}(s,Q^2,s')$, $i=1,4$ are Lorentz invariant generalized functions. Substituting Eq. (17) in Eq. (16) and taking into account Eqs. (6) and (13), we obtain the following integral representations:

$$f_{in}^c(Q^2) = \sum_{l,l',S,S'}\int d\sqrt{s}\,d\sqrt{s'}\,\varphi_{lS}^{Jc}(s)\varphi_{l'S'}^{Jc}(s')G_{in}^{ll'SS'}(s,Q^2,s') \tag{18}$$

for $i=1,4$. In the case of the matrix element in Eq. (15) nondiagonal in m_{Jc}, m'_{Jc} we can proceed in an analogous way and obtain an analogous integral representations for $f_{in}^c(Q^2)$, $i=2,3$.

So, the matrix element in the rhs of Eq. (12) considered as Lorentz covariant generalized function can be written as the following decomposition of the type of Wigner-Eckart decomposition:

$$\langle\vec{p}_c,\sqrt{s},J_c,l,S,m_{Jc}|j_\mu|\vec{p}_c\,',\sqrt{s'},J_c,l',S',m'_{Jc}\rangle = \frac{1}{\mathcal{N}}\langle m_{Jc}|D^{J_c}(p_c,p'_c)[\mathcal{F}_1K'_\mu + \mathcal{F}_2\Gamma^\mu(p'_c) + \mathcal{F}_3R_\mu + \mathcal{F}_4K_\mu]|m'_{Jc}\rangle. \tag{19}$$

$$\mathcal{F}_i = \sum_{n=0}^{2J_c} G_{in}^{ll'SS'}(s,Q^2,s')[ip_{c\mu}\Gamma^\mu(p'_c)]^n. \tag{20}$$

In Eqs. (19) and (20) the form factors $G_{in}^{ll'SS'}(s,Q^2,s')$ contain all the information about the physics of the transition described by operator j_μ. They are connected with the composite particle form factors (1) and (2) through Eq. (18). In particular, physical approximations are formulated in our approach in terms of form factors $G_{in}^{ll'SS'}(s,Q^2,s')$ (see Ref. [1] for details). The matrix element transformation properties are given by the four-vectors in the rhs of Eq. (19).

It is worth emphasizing that it is necessary to consider the composite system form factors as the functionals generated by the Lorentz invariant generalized functions $G_{in}^{ll'SS'}(s,Q^2,s')$.

Now let us impose the conditions of self-adjointness, conservation law, and parity conservation on the matrix elements in Eqs. (1) and (19). The rhs of equalities (1) and (19) contain the same four-vectors and the same sets of Lorentz scalars (2) and (20), so, to take into account the additional conditions, it is necessary to redefine these four-vectors and functions $G_{in}^{ll'SS'}(s,Q^2,s')$. For example, the conservation law gives $F_4^c=0$ and $\mathcal{F}_4=0$.

Let us write parametrizations (19) and (20) for the particular case of composite particle electromagnetic current with quantum numbers $J=J'=S=S'=1$, which is realized, for example, in the case of deuteron. Separating the quadrupole form factor and using Eqs. (19) and (20), we obtain the following form:

$$\langle \vec{p}_c,\sqrt{s},J_c,l,S,m_{Jc}|j_\mu|\vec{p}_c{}',\sqrt{s'},J_c,l',S',m'_{Jc}\rangle = \frac{1}{\mathcal{N}}\langle m_{Jc}|D^1(p_c,p_c')\left[\tilde{\mathcal{F}}_1K'_\mu+\frac{i}{M_c}\tilde{\mathcal{F}}_3R_\mu\right]|m'_{Jc}\rangle. \quad (21)$$

$$\tilde{\mathcal{F}}_1=\tilde{G}_{10}^{ll'}(s,Q^2,s')+\tilde{G}_{12}^{ll'}(s,Q^2,s')\left\{[ip_{c\nu}\Gamma^\nu(p_c')]^2-\frac{1}{3}\mathrm{Sp}[ip_{c\nu}\Gamma^\nu(p_c')]^2\right\}\frac{2}{\mathrm{Sp}[p_{c\nu}\Gamma^\nu(p_c')]^2},$$

$$\tilde{\mathcal{F}}_3=\tilde{G}_{30}^{ll'}(s,Q^2,s'). \quad (22)$$

We have taken into account that equation $\tilde{G}_{21}^{ll'}(s,Q^2,s')=0$ is valid in weak sense. Parametrizations (1) and (2) take forms (21) and (22) with $\tilde{G}_{in}^{ll'}(s,Q^2,s')\to\tilde{f}_{in}^c(Q^2)$. It is easy to see that for the redefined form factors equality (18) remains valid. Form factors $\tilde{f}_{in}^c(Q^2)$ are connected with charge, quadrupole, and magnetic Sachs form factors: $G_C(Q^2)=\tilde{f}_{10}^c(Q^2)$, $G_Q(Q^2)=(2M_c^2/Q^2)\tilde{f}_{12}^c(Q^2)$, $G_M(Q^2)=-M_c\tilde{f}_{30}^c(Q^2)$. The modified impulse approximation (MIA) can be formulated in terms of form factors $\tilde{G}_{iq}^{ll'}(s,Q^2,s')$. The physical meaning of this approximation is considered in detail in Ref. [1,3].

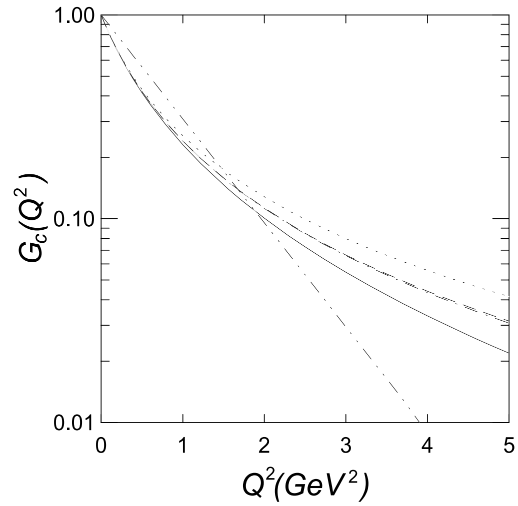

FIG. 1. The results of the calculations of the ρ-meson charge form factor with different model wave functions [1,3]. The solid line represents the relativistic calculation with the wave function of harmonic oscillator, the dashed line—with the power-law wave function for $n=3$; dash-dot-line—with the wave function with linear confinement; dotted line—with the power-law wave function for $n=2$; dot-dot-dash-line—the nonrelativistic calculation with the wave function of harmonic oscillator. The wave functions parameters are obtained from the fitting of ρ meson MSR. The sum of quark anomalous magnetic moments is taken as $\kappa_u+\kappa_{\bar{d}}=0.09$ in natural units. The quark mass is $M=0.25$ GeV.

The results of calculations for the ρ-meson charge form factor in MIA ($l=l'=0$) are represented in Fig. 1.

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[1] A.F. Krutov and V.E. Troitsky, Phys. Rev. C **65**, 045501 (2002).
[2] B.D. Keister and W. Polyzou, Adv. Nucl. Phys. **20**, 225 (1991).
[3] A.F. Krutov and V.E. Troitsky, hep-ph/0210046.
[4] A.A. Cheshkov and Yu.M. Shirokov, Zh. Eksp. Teor. Fiz. **44**, 1982 (1963).
[5] V.E. Troitsky and Yu. M Shirokov, Teor. Mat. Fiz. **1**, 213 (1969).
[6] V.P. Kozhevnikov, V.E. Troitsky, S.V. Trubnikov, and Yu.M. Shirokov, Teor. Mat. Fiz. **10**, 47 (1972).
[7] N. N. Bogoliubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *General Principles of Quantum Field Theory* (Kluwer Academic, Dodrecht, 1990).
[8] D. Melikhov and S. Simula, Phys. Rev. D **65**, 094043 (2002); S. Simula, Phys. Rev. C **66**, 035201 (2002).