Relativistic quantum mechanics: Particle production and cluster properties

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This paper constructs relativistic quantum mechanical models of particles satisfying cluster properties and the spectral condition which do not conserve particle number. The treatment of particle production is limited to systems with a bounded number of bare-particle degrees of freedom. This paper focuses on the realization of cluster properties in these theories.

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I. INTRODUCTION

The purpose of this paper is to formulate a class of quantum theories of interacting particles with the following properties. They are Poincaré invariant, they satisfy cluster properties, the four-momentum operator has a spectrum supported in the future-pointing light cone, and they allow particle production. These theories are applicable to problems in strong interaction physics where relativistic invariance is an important symmetry. Cluster properties can be used to systematically build many-particle models from few-body models that are constrained by experiment. Relativistic quantum theories with cluster properties are essential to the relevance of the few-body program at accelerators such as TJNAF.

The formulation of Poincaré invariant quantum theories satisfying cluster properties for systems of a fixed number of particles has been discussed in Refs. [1–5]. This paper discusses the modifications to the fixed-N construction, needed to extend that construction to treat a class of models with particle production. The theories discussed in this paper are limited to systems with a finite number of bare-particle degrees of freedom. A complete treatment of particle production, with no restrictions on the number of bare-particle degrees of freedom, is beyond the scope of this paper.

The physical properties mentioned in the first paragraph are the minimal physical requirements for a realistic description of a system of strongly interacting particles. The physical motivation for each of these requirements is discussed below.

Poincaré invariance is the requirement that the group of continuous Poincaré transformations is a symmetry of the theory. In 1939 Wigner [6] showed that this is equivalent to the existence of a unitary representation of the Poincaré group on the Hilbert space of the quantum theory.

Poincaré invariance is essential for a consistent interpretation of any reaction with strong binding or particle production. For reactions where the initial and final states have different inertial masses, momentum conservation cannot be simultaneously satisfied in the laboratory and center of momentum frames in a Galilean invariant quantum theory.

Cluster properties require that isolated subsystems have the same properties as the system. They relate interactions in subsystem Hamiltonians to interactions in the system Hamiltonian. Cluster properties provide the connection between the few- and many-body problem and the justification for experiments that are performed on isolated targets at modern accelerators.

The spectral condition is essential for the stability of matter. The mathematical requirement is that the eigenvalue spectrum of Hamiltonian is bounded from below.

Theories must be able to model reactions that change particle number. These reactions are observed in experiments at almost all modern accelerators.

While the above discussion makes a case that the physical constraints discussed above are essential requirements for any reasonable quantum theory of strongly interacting particles, it is surprisingly difficult to formulate mathematically well-defined theories that are consistent with all of these properties.

Even in quantum field theory, the problem of identifying the physical Hilbert space $\mathcal{H}$ and finding a set of ten self-adjoint operators on this space that satisfy the Poincaré commutation relations is an unsolved problem, except for the case of free quantum fields. These are the minimal requirements for realizing the Poincaré symmetry in a quantum theory.

Some of the difficulties in formulating theories that are consistent with these physical constraints are discussed below.

The Poincaré group provides an infinite number of independent paths to the future and each path involves the dynamics. If one starts with a given state and transforms it to a future time using different combinations of Poincaré transformations, consistency requires that the resulting states are identical. For example, time evolution can be expressed in terms of rotationless Lorentz transformations and spatial translations. Consistency of the quantum initial value problem requires that if there are interactions in the Hamiltonian then there must be interactions in the infinitesimal generators of rotationless Lorentz transformations and/or spatial translations. This is a consequence of the commutation relation

$$[P_i, K_j] = i\delta_{ij} H, \tag{1}$$

which relates the Hamiltonian H to the linear momentum generators $\vec{P}$ and generators of rotationless Lorentz transformations, $\vec{K}$. The Poincaré commutation relations impose nonlinear constraints on these interactions.

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Cluster properties impose independent nonlinear constraints on the interactions. To see this, note that in the three-body problem, cluster properties fix the two-body interactions in each of the Poincaré generators up to an overall three-body interaction. However, because interactions involving different pairs of particles appear in more than one generator, the operators obtained by adding the required two-body interactions to the noninteracting generators fail to satisfy the commutation relations without additional three-body interactions. For example, if the generators of rotationless Lorentz transformations, $\vec{K}$, have interactions between particles 1 and 2 and the Hamiltonian has interactions between particles 2 and 3, then the commutator $[H,\vec{K}]=i\vec{P}$ will have three-body interactions involving particles 1, 2, and 3 unless H and/or $\vec{K}$ have three-body interactions that are designed to cancel the three-body operators generated by the commutator.

While the spectral condition is not difficult to satisfy, negative energy states have historically appeared when classical relativistic field theories, like the Klein-Gordon-Schrödinger and Dirac equations, are treated as quantum mechanical equations. The negative energy eigenstates of the Hamiltonian disappear when these equations are properly treated as equations for quantum fields.

Particle production requires a more critical analysis of cluster properties. For theories with a fixed number N of particles there is an ordering on particle number, and cluster properties define the relationship between the interactions in the $K<N$-body Poincaré generators and the N-body Poincaré generators. This leads to important relations between the dynamics of the system and its proper subsystems. These relations provide the justification for both theory and experiment on few-body systems.

The problem with formulating a useful cluster condition in theories with particle production is the absence of a few-body problem that puts useful constraints on the many-body dynamics. Specifically, in theories with particle production, states with a few physical particles generally involve an infinite number of bare particles.

The difficulties with formulating quantum theories with an infinite number of degrees of freedom are well known [7–9]. These difficulties are distinct from the specific problems that arise from particle production. In this paper these problems are deliberately separated by restricting considerations to a class of theories with a finite number of bare-particle degrees of freedom. This is achieved using conservation laws that limit the number of bare-particle degrees of freedom. It is possible to formulate cluster properties in these theories without having to confront the specific problems that arise due to the infinite number of degrees of freedom.

The class of models considered in this paper is designed to complement models based on formal quantum field theory. Quantum mechanical models of interacting particles have the advantage that (for systems of strongly interacting particles) they are mathematically well defined and can, in principle, be solved using convergent algorithms. It is for this reason that quantum theories of particles are often used to model few-body reactions involving composite systems or scattering from composite targets. Some recent applications can be found in Refs. [10–13]. Extending these theories to Poincaré invariant theories with cluster properties that allow particle production provides a more robust class of models.

The following section discusses the assumptions that are used to limit the number of bare-particle degrees of freedom. The structure of the model Hilbert space is given in Sec. III. It differs from the Hilbert space for a system of N particles in how it factors into subsystem spaces. This factorization and some of its properties are discussed in Sec. IV. In Sec. V the cluster property is formulated in a manner that is consistent with the modified factorization into subsystems. The formulation of scattering theory for reactions that do not conserve particle number is discussed in Sec. VI. Modification of the C^* algebra of asymptotic constants, which is a central element of the construction of a dynamics satisfying cluster properties in Ref. [5], is discussed in Sec. VII. The unitary elements of this algebra preserve the scattering observables, and can be used to restore cluster properties. The modifications to the general construction in Ref. [5] to treat a variable number of particles are summarized in Sec. VIII. Rather than giving a systematic description of the general construction, as was done in Ref. [5], the essential elements of the general construction are illustrated in Secs. IX-XI using a nontrivial example.

II. MOTIVATION AND ASSUMPTIONS

General features of the class of theories studied in this paper are motivated by comparing theories of a fixed number of particles to theories that change particle number. The construction in this paper extends the general fixed-N construction in Ref. [5]. In all that follows, we use notation from Ref. [5].

Consider a relativistic theory of N-interacting particles following the construction of Ref. [5]. Relativistic invariance is realized by a dynamical unitary representation $U[\Lambda,Y]$ of inhomogeneous $SL(2,C)$ $[ISL(2,C)]$ on the N-particle Hilbert space $\mathcal{H}$. $ISL(2,C)$ is the covering group of the Poincaré group; it is used because the relevant representations are single valued and computations are easier using 2×2 matrices.

The $SL(2,C)$ matrix Λ is related to a finite Lorentz transformation Λ^{μ}_{ν} by

$$\Lambda^{\mu}_{\nu}=\frac{1}{2}\mathrm{Tr}(\sigma_{\mu}\Lambda\sigma_{\nu}\Lambda^{\dagger}), \tag{2}$$

and the 2×2 Hermitian matrix Y parametrizes a space-time translation y^{μ} by

$$Y=y^{\mu}\sigma_{\mu}, \quad y^{\mu}=\frac{1}{2}\mathrm{Tr}(Y\sigma_{\mu}). \tag{3}$$

The group product is

$$(\Lambda_2,Y_2)(\Lambda_1,Y_1)=(\Lambda_2\Lambda_1,\Lambda_2Y_1\Lambda_2^{\dagger}+Y_2). \tag{4}$$

The resulting $U[\Lambda,Y]$ satisfies cluster properties and the spectrum of the Hamiltonian H is bounded from below.

Assume that in this model some isolated subsystems can form bound states. Then cluster properties imply that the isolated bound subsystems have the same Poincaré transformation properties as elementary particles with the same mass and spin. With respect to their Poincaré transformation properties, there is no distinction between elementary and composite particles.

Treating the asymptotically stable subsystems as physical particles, the relativistic theory described in Ref. [5] can be interpreted as a theory of fixed number of bare particles with a variable number of physical particles. The physical particles in the above sense are needed to formulate scattering asymptotic conditions and cluster properties.

This can be compared to local quantum field theory, where physical particles, defined as discrete eigenstates of the mass and spin, also have a composite bare-particle content. An important distinction is that in local field theory the physical particles involve an infinite number of bare-particle degrees of freedom, while in the relativistic quantum mechanics case discussed above, the composite systems involve a fixed finite number of bare-particle degrees of freedom.

In this paper the fixed-N construction is generalized by replacing the N-constituent particles by a set of conserved additive quantum numbers. These quantum numbers have no physical interpretation; they are introduced to provide a mechanism to control the number of degrees of freedom. These quantum numbers are called charges and they are assumed to satisfy the following:

(a) There are K types of charges.

(b) Charges can have only *non-negative* integer values.

(c) Each bare particle of the model has a set of charges labeled by an n-tuple of integers $(n_1, \ldots, n_K)$ labeling the number of each of the K types of charges.

(d) The charge of a composite system is the sum of the charges of the constituents.

(e) Each bare particle of the theory has as least one nonzero charge.

(f) Interactions conserve all K types of charges.

The charge of a bare particle is *minimal* if it cannot be expressed as a sum of smaller charges corresponding to at least two bare particles.

The relativistic Lee model [2,14–16] provides a well-known example of a theory with this structure. The Lee model has three types of bare particles that can be suggestively called π, N, and Δ, with a vertex interaction $\pi + N \leftrightarrow \Delta$. There are two conserved charges (q_N, q_π) where the π has charge (0,1), the N has charge (1,0), and the Δ has charge (1,1). In this model the charges of the π and the N are minimal. The charge of the Δ is not minimal because the π-N system has the same charge as the Δ. In this model the Δ is called a *composite* bare particle. The vertex interaction conserves charge. Many isobar models also fall into this class.

Theories with an infinite number of degrees of freedom are obtained by dropping the assumptions (b) and (e). For example, if a neutral pion is assigned a charge zero and a neutron is assigned a charge equal to its baryon number, each fixed-charge subspace of the Hilbert space has subspaces with arbitrarily large numbers of pions and neutron-antineutron pairs. This paper only considers theories where conditions (b) and (e) are enforced. With these restrictions it is possible to define a meaningful ''few-charge'' problem.

III. HILBERT SPACE

The Hilbert space $\mathcal{H}_{\{N\}}$, corresponding to the value $\{N\} = (n_1, \ldots, n_K)$ of the conserved charges, is a direct sum of tensor products of bare-particle Hilbert spaces,

$$\mathcal{H}_{\{N\}} := \oplus_{i=1}^{n} \left(\otimes_{k=1}^{n_i} \mathcal{H}_{m_{ik} j_{ik}} \right), \tag{5}$$

where $\mathcal{H}_{mj}$ is the mass m, spin j irreducible representation space of $ISL(2,C)$. Each term of the direct sum has a *different bare-particle content*, but the same value of total charge, $\{N\}$.

In the Lee model example the Hilbert space,

$$\mathcal{H}_{\{1,1\}} = (\mathcal{H}_N \otimes \mathcal{H}_\pi) \oplus \mathcal{H}_\Delta \tag{6}$$

is the direct sum of the two-particle N-π space and the one-particle Δ space. Note that including a bare Δ particle in the model does not imply that the Δ will exist as a stable physical particle.

The irreducible representation spaces $\mathcal{H}_{mj}$ of $ISL(2,C)$ are spaces of square integrable functions of the eigenvalues of a maximal set of commuting self-adjoint functions of the single-particle generators. In general, this set includes the invariant mass and spin operators, and four additional functions [5] of the $ISL(2,C)$ generators, denoted by F^i. A typical choice of the operators F^i is the three components of the linear momentum and the $\hat{z}$ component of the canonical spin.

Denoting the eigenvalues of F^i, m, j^2 by f, m, and $j(j+1)$ gives basis vectors on each $\mathcal{H}_{mj}$ of the form

$$|f;m,j\rangle \tag{7}$$

with resolution of the identity and normalization given by

$$I = \int |f;m,j\rangle d\mu(f) \langle f;m,j|,$$

$$\langle f;m,j|f';m,j\rangle = \delta[f,f']. \tag{8}$$

In this expression $\int d\mu(f)$ denotes an integral over the continuous eigenvalues and a sum over the discrete eigenvalues of F^i. Likewise, $\delta[f,f']$ indicates a product of Dirac delta functions in the continuous variables and Kronecker delta functions in the discrete variables. This basis of the single-particle Hilbert space is called the f basis.

By assumption, $ISL(2,C)$ acts irreducibly on this space. In the f basis the action of $U[\Lambda,Y]$ is given by

$$U[\Lambda,Y]|f;m,j\rangle = \int |f';m,j\rangle d\mu(f') \mathcal{D}^{mj}_{f'f}[\Lambda,Y], \tag{9}$$

where

$$\mathcal{D}^{mj}_{f'f}[\Lambda,Y] := \langle f';m,j|U[\Lambda,Y]|f;m,j\rangle \tag{10}$$

is the mass m, spin j irreducible representation of $ISL(2,C)$ in the f basis. What is relevant for this paper is that the irreducible representations $\mathcal{D}^{mj}_{f'f}[\Lambda,Y]$ are known for each value of m and j. Explicit formulas for Poincaré $\mathcal{D}$ functions $\mathcal{D}^{mj}_{f'f}[\Lambda,Y]$, are given in Refs. [3,5,17].

The irreducible representation $U_{ik}[\Lambda,Y]$ of $ISL(2,C)$ on each of the subspaces $\mathcal{H}_{m_{ik}j_{ik}}$ can be used to construct a natural *noninteracting* representation $U_0[\Lambda,Y]$ on $\mathcal{H}_{\{N\}}$ given by

$$U_0[\Lambda,Y]=\sum_{i=1}^{n}(\otimes_{k=1}^{n_i}U_{ik}[\Lambda,Y]). \tag{11}$$

The Hilbert space $\mathcal{H}_{\{N\}}$ has two natural bases. The first is the tensor product of single bare-particle basis vectors. There are distinct basis functions corresponding to each orthogonal subspace in the direct sum (5).

The second is a basis that transforms irreducibly with respect to $U_0[\Lambda,Y]$. The irreducible basis is constructed, using the $ISL(2,C)$ Clebsch-Gordan coefficients [3–5,18–20], as a linear combination of the tensor product of irreducible representations. As in the case of the tensor product basis, there is a distinct orthogonal subspace corresponding to each term in the direct sum (5).

The two types of basis vectors on $\mathcal{H}_{\{\mathcal{N}\}}$ are denoted by

$$|\otimes f_i;j_i,m_i\rangle \tag{12}$$

and

$$|f,d;j,m\rangle, \tag{13}$$

respectively, where d denotes a set of invariant degeneracy quantum numbers. The m in Eq. (13) is the invariant mass of the system of noninteracting bare particles in the tensor product.

The second basis transforms irreducibly with respect to $U_0[\Lambda,Y]$,

$$U_0[\Lambda,Y]|f,d;m,j\rangle=\int|f',d;m,j\rangle d\mu(f')\mathcal{D}^{mj}_{f'f}[\Lambda,Y], \tag{14}$$

which has the same form as the transformation law for a single particle of mass m and spin j, while basis (12) transforms like

$$U_0[\Lambda,Y]|\otimes f_i;j_i,m_i\rangle=\int|\otimes f'_i;j_i,m_i\rangle \times\prod_l d\mu(f'_l)\mathcal{D}^{m_lj_l}_{f'_lf_l}[\Lambda,Y]. \tag{15}$$

IV. TENSOR PRODUCTS/FACTORIZATION

In nonrelativistic many-particle quantum mechanics the N-particle Hilbert space can be decomposed into a tensor product of Hilbert spaces with fewer particles. Cluster properties lead to an asymptotic factorization of the interacting representation, $U[\Lambda,Y]$, into a tensor product of subsystems $U_{a_i}[\Lambda,Y]$'s that act on each factor of the tensor product.

For models with conserved charges $\{N\}$ a similar, but slightly more complicated, relationship exists. To define this relationship, begin by labeling each charge. In this paper the charges are initially treated as distinguishable. Proper symmetry under exchange of identical particles can be restored after the Poincaré generators are constructed.

Let $\mathcal{H}_{\{\mathcal{N}\}}$ be the Hilbert space for a given set of charges. Partitions a of the labeled charges are identified with equivalence relations on the charges. The ith equivalence class is denoted by a_i, called the ith cluster of a. The set of all partitions of the charges is denoted by $\mathcal{P}_{\{\mathcal{N}\}}$.

To each partition a of the conserved charges, the Hilbert space $\mathcal{H}_{\{N\}}$ can be decomposed as an orthogonal direct sum of the form

$$\mathcal{H}_{\{N\}}=\mathcal{H}_a\oplus\mathcal{H}^a, \tag{16}$$

where

$$\mathcal{H}_a:=\otimes_{i=1}^{n_a}\mathcal{H}_{\{N_{a_i}\}} \tag{17}$$

is the tensor product of the subsystem Hilbert spaces associated with the charges in the ith cluster of a and $\mathcal{H}^a$ is the orthogonal complement of $\mathcal{H}_a$ in $\mathcal{H}_{\{N\}}$.

The residual space $\mathcal{H}^a$ appears because for each partition a the Hilbert space $\mathcal{H}_{\{\mathcal{N}\}}$ may have a subspace with bare particles having nonzero charges in *different* clusters of the partition a.

In the case of the $\{1,1\}$ sector of the Lee model [2,14–16] the factorization (16) has the form

$$\mathcal{H}_{\{1,1\}}=(\mathcal{H}_N\otimes\mathcal{H}_\pi)\oplus\mathcal{H}_\Delta, \tag{18}$$

where for $a=(N)(\pi)$, $\mathcal{H}_a=\mathcal{H}_N\otimes\mathcal{H}_\pi$ and $\mathcal{H}^a=\mathcal{H}_\Delta$. The Δ subspace is unimportant for understanding clustering into an asymptotically separated π and N.

The appearance and treatment of the residual space is the main technical difference between models of a fixed number of particles and models with production.

The partition of the conserved charges into disjoint equivalence classes has an obvious partial ordering given by $a\supseteq b$ if and only if conserved charge labels in the same equivalence class with respect to b are in the same equivalence class with respect to a. This means that the clusters of b are obtained by breaking up the clusters of a.

Given the partial ordering on $\mathcal{P}_{\{N\}}$, it is possible to define zeta and Möbius functions [22,23] for the partial ordering:

$$\zeta(a\supseteq b)=\begin{cases}1 & \text{for} \quad a\supseteq b\\ 0 & \text{otherwise}\end{cases} \tag{19}$$

and

$$\mu(a\supseteq b)=\zeta^{-1}(a\supseteq b)$$
$$=\begin{cases}(-)^{n_a}\prod_{i=1}^{n_a}(-)^{n_{b_i}}(n_{b_i}-1)! & \text{for} \quad a\supseteq b\\ 0 \quad \text{otherwise,}\end{cases} \tag{20}$$

where n_a is the number of clusters of the partition a and n_{b_i} is the number of clusters of b in the ith cluster of a.

It is a consequence of the definitions that the subspaces $\mathcal{H}_a$ and $\mathcal{H}^a$ satisfy the relations

$$b\supset a\Rightarrow\mathcal{H}_b\supset\mathcal{H}_a \text{ and } \mathcal{H}^a\supset\mathcal{H}^b. \tag{21}$$

This is equivalent to the observation that if a bare particle has nonzero charges in two different clusters of a partition a, then this is also true for any refinement of a. This means that for the purpose of studying cluster properties, the residual spaces $\mathcal{H}^a$ can be ignored when making refinements of partitions.

V. CLUSTER PROPERTIES

In order to formulate cluster properties, assume that it is possible to find the dynamics

$$U_{a_i}[\Lambda,Y] \tag{22}$$

associated with the conserved charges in the ith cluster of a.

The *interacting* representation $U[\Lambda,Y]$ of the system satisfies spacelike cluster properties if for each partition a of the conserved charges the following strong limits vanish:

$$\lim_{\min(y_i-y_j)^2\to\infty}(U[\Lambda,Y]-\otimes_{l=1}^{n_a}U_{a_l}[\Lambda,Y])$$
$$\otimes_{m=1}^{n_a}U_{a_m}[I,Y_m]\Pi_a=0, \tag{23}$$

where Π_a is the orthogonal projection onto the subspace $\mathcal{H}_a$ of $\mathcal{H}$. This projection is needed because $\otimes_{l=1}^{n_a}U_{a_l}[\Lambda,Y]$ and $\otimes_{m=1}^{n_a}U_{a_m}[I,Y_m]$ are only defined on $\mathcal{H}_a$. For successive limits the projections should be on the largest subspace that allows the charges to be asymptotically separated.

Equation (23) contains two conditions. First, it requires that when the interaction terms between particles with charges in different clusters of a are turned off, the projection of $U[\Lambda,Y]$ on $\mathcal{H}_a$ becomes a tensor product of subsystem representations. This property is referred to as the algebraic cluster property. This condition is nontrivial; when it fails, either the cluster limit does not exist, or interactions between particles in the *same* cluster of a vanish in the cluster limit [3].

The second condition is that the interaction between particles with charges in different clusters satisfy the short-range condition specified above. This can be reformulated as a ''Cook-like'' condition on the range of the residual interactions [21]. To see this, let $[\Lambda,Y]$ denote a fixed $ISL(2,C)$ transformation. To formulate the range condition, assume

$$U[\Lambda,Y]=e^{iG}, \tag{24}$$

where $G=G[\Lambda,Y]$ is a fixed linear combination of the generators $ISL(2,C)$ on $\mathcal{H}$.

The limiting form required by cluster properties when the clusters of a are asymptotically separated is

$$U_a[\Lambda,Y]=e^{i(\Sigma iG_{a_i})}=e^{iG_a}. \tag{25}$$

To formulate the cluster condition define the residual interaction by

$$V^a=G-G_a. \tag{26}$$

Consider

$$F(\alpha):=e^{i\alpha G}e^{-i\alpha G_a}, \tag{27}$$

where $e^{-i\alpha G_a}$ is extended to be the identity on $\mathcal{H}^a$. This satisfies the integral equation

$$F(\alpha)=I+i\int_0^\alpha F(\alpha')V^a(\alpha')d\alpha', \tag{28}$$

where

$$V^a(\alpha):=e^{i\alpha G_a}V^a e^{-i\alpha G_a}. \tag{29}$$

The cluster condition is equivalent to

$$\lim_{\min(y_i-y_j)^2\to\infty}\|[F(1)-I]\otimes_{l=1}^{n_a}U_{a_l}[I,Y_l]\Pi_a|\xi\rangle\|=0. \tag{30}$$

This limit is bounded by

$$\lim_{\min(y_i-y_j)^2\to\infty}\int_0^1\|V^a(\alpha')\otimes_{l=1}^{n_a}U_{a_l}[I,Y_l]\Pi_a|\xi\rangle\|d\alpha'. \tag{31}$$

The integrand is uniformly bounded in α' by

$$\|V^a\|\|\Pi_a|\xi\rangle\|<\infty \tag{32}$$

and each term in the integrand has the limit

$$\lim_{\min(y_i'-y_j')^2\to\infty}\|V^a\otimes_{l=1}^{n_a}U_{a_l}[I,Y_l']\Pi_a|\xi'\rangle\|=0, \tag{33}$$

where

$$Y_l'=\Lambda Y_l\Lambda^\dagger+Y \tag{34}$$

and

$$|\xi'\rangle=e^{-i\alpha G_a}|\xi\rangle. \tag{35}$$

In this expression Λ,Y is the $ISL(2,C)$ transformation defined by $U[\Lambda,Y]=e^{-i\alpha G}$.

It follows from the Lebesgue dominated convergence theorem [24] that the cluster limit vanishes, provided condition (33) holds for all $|\xi'\rangle$ and all asymptotic spacelike separations, $(y_i'-y_j')^2\to\infty$. This is the desired ''Cook-like'' con-

dition on the range of the intercluster interaction V^a. This is analogous to the cluster condition in nonrelativistic quantum models.

The difficult aspect of the cluster problem in relativistic quantum mechanics is to construct $U[\Lambda,Y]$ so it satisfies algebraic cluster properties. In this case algebraic cluster properties mean that

$$U[\Lambda,Y]\to U_a[\Lambda,Y] \tag{36}$$

when the interactions between particles in different clusters are turned off, where

$$U_a[\Lambda,Y]:=\begin{pmatrix} \otimes_{l=1}^{n_a} U_{a_l}[\Lambda,Y] & 0 \\ 0 & I \end{pmatrix} \tag{37}$$

on $\mathcal{H}=\mathcal{H}_a\oplus\mathcal{H}^a$. Extending $\otimes_{l=1}^{n_a} U_{a_l}[\Lambda,Y]$ to all of $\mathcal{H}$ by extending it as the identity on the subspace $\mathcal{H}^a$ is one of the modifications introduced because of factorization (16). The identity term is consistent with setting the corresponding generators to zero. This choice does not affect the cluster condition because the identity term is eliminated by the projector Π_a.

This formulation of cluster properties has the property that if the system is further sub-divided by $b\subset a$ then $\Pi_a\Pi_b=\Pi_b$. This is because all of the refinements of a are defined on $\mathcal{H}_a$. This property of the model Hilbert space ensures that system can continue to be subdivided until all that remains is a system of bare particles with minimal charges.

VI. SCATTERING

The formulation of scattering theory with particle production is identical to the two-Hilbert-space formulation used in Refs. [2,5] for a fixed number of particles.

To formulate the scattering theory assume that the dynamical representation $U[\Lambda,Y]$ of $ISL(2,C)$ on $\mathcal{H}$ is given. Assume that the representation $U[\Lambda,Y]$ has the following properties, which are consistent with the fixed N case.

(a) There are simultaneous eigenstates of M,j,F^i with positive discrete mass eigenvalues that transform irreducibly with respect to $U[\Lambda,Y]$.

(b) There are simultaneous eigenstates of M,j,F^i with positive eigenvalues in the absolutely continuous spectrum of M. These satisfy scattering asymptotic conditions.

(c) The bound and scattering eigenstates are complete on the model Hilbert space, with the incoming and outgoing wave scattering states each spanning the orthogonal complement of the subspace spanned by the bound states.

A bound state is a simultaneous eigenstate of F_i, M, and j^2,

$$|f;m,j\rangle \tag{38}$$

with discrete mass eigenvalue m. It transforms irreducibly under the action of the *dynamical* representation of $ISL(2,C)$

$$U[\Lambda,Y]|f;m,j\rangle=\int d\mu(f')|f';m,j\rangle\mathcal{D}^{mj}_{f'f}[\Lambda,Y]. \tag{39}$$

The function $\mathcal{D}^{mj}_{f'f}[\Lambda,Y]$ is the known mass-m and spin-j irreducible representation of $ISL(2,C)$.

Normalizable eigenstates of physical mass and spin can be expressed in the form

$$|\psi\rangle=\int |f;m,j\rangle d\mu(f)\chi(f) \tag{40}$$

for square integrable functions $\chi(f)$.

Each irreducible bound subspace defines a bound-state channel α. The channel Hilbert space $\mathcal{H}_\alpha$ is the space of square integrable functions $\chi(f)$ over the joint spectrum $\sigma(F)$ of the commuting operators F^i. Equation (40) can be interpreted as a mapping Φ_α from the bound channel Hilbert space $\mathcal{H}_\alpha$ to the physical Hilbert space $\mathcal{H}$,

$$|\psi\rangle=\Phi_\alpha|\chi\rangle. \tag{41}$$

This can be done for each bound channel. Note that, in general, $|\psi\rangle$ has components in all bare-particle sectors of $\mathcal{H}$.

In this notation, Eq. (39) can be expressed in the form

$$U[\Lambda,Y]\Phi_\alpha=\Phi_\alpha U_\alpha[\Lambda,Y], \tag{42}$$

where $U_\alpha[\Lambda,Y]$ is the irreducible unitary representation of $ISL(2,C)$ with kernel $\mathcal{D}^{mj}_{f'f}[\Lambda,Y]$.

Individual subsystem bound states are used to formulate the asymptotic condition for multiparticle scattering channels.

For a partition a of conserved charges the physical Hilbert space has the factorization $\mathcal{H}_{\{N\}}=\mathcal{H}_a\oplus\mathcal{H}^a$, where

$$\mathcal{H}_a=\otimes_{a_i}\mathcal{H}_{\{N_{a_i}\}}. \tag{43}$$

Assume that there is a subsystem dynamics,

$$U_{a_i}[\Lambda,Y]:\mathcal{H}_{\{N_{a_i}\}}\to\mathcal{H}_{\{N_{a_i}\}} \tag{44}$$

for the charges in the ith cluster of the partition a. There is a scattering channel α associated with the partition a if there is a bound channel for each of the subsystem $U_{a_i}[\Lambda,Y]$'s.

Following Eq. (42), for each bound subsystem there is an asymptotic Hilbert space $\mathcal{H}_{\alpha_i}$ and an injection operator Φ_{α_i},

$$\Phi_{\alpha_i}:\mathcal{H}_{\alpha_i}\to\mathcal{H}_{\{N_{a_i}\}} \tag{45}$$

with the property

$$U_{a_i}[\Lambda,Y]\Phi_{\alpha_i}=\Phi_{\alpha_i}U_{\alpha_i}[\Lambda,Y]. \tag{46}$$

Define the channel Hilbert space $\mathcal{H}_\alpha$ and the channel injection operator $\Phi_\alpha:\mathcal{H}_\alpha\to\mathcal{H}_a\subset\mathcal{H}$ by

$$\mathcal{H}_\alpha:=\otimes_{i=1}^{n_a}\mathcal{H}_{\alpha_i}, \tag{47}$$

$$\Phi_\alpha:=\otimes_{i=1}^{n_a}\Phi_{\alpha_i}. \tag{48}$$

Define

$$U_a[\Lambda,Y] \coloneqq \begin{pmatrix} \otimes_{i=1}^{n_a} U_{a_i}[\Lambda,Y] & 0 \\ 0 & I \end{pmatrix}, \tag{49}$$

where I acts on the residual subspace $\mathcal{H}^a$. Also define

$$U_\alpha[\Lambda,Y]:\mathcal{H}_\alpha \to \mathcal{H}_\alpha \tag{50}$$

by

$$U_\alpha[\Lambda,Y] \coloneqq \otimes_{i=1}^{n_a} U_{\alpha_i}[\Lambda,Y]. \tag{51}$$

With these definitions, relations (46) can be compactly expressed in the form

$$U_a[\Lambda,Y]\Phi_\alpha = \Phi_\alpha U_\alpha[\Lambda,Y]. \tag{52}$$

Following the case of a fixed number of particles, a scattering state is a solution

$$|\psi_\alpha^\pm(t)\rangle = U[I,T]|\psi_\alpha^\pm(0)\rangle \tag{53}$$

of the time-dependent Schrödinger equation satisfying the asymptotic condition

$$\lim_{t\to\pm\infty} \|\psi_\alpha^\pm(t)\rangle - U_a[I,T]\Phi_\alpha|\chi_\alpha\rangle\| = 0 \tag{54}$$

for all $|\chi_\alpha\rangle \in \mathcal{H}_\alpha$. Using the intertwining relations (52), this can be expressed as

$$\lim_{t\to\pm\infty} \|\psi_\alpha^\pm(0)\rangle - U^\dagger[I,T]\Phi_\alpha U_\alpha[I,T]|\chi_\alpha\rangle\| = 0. \tag{55}$$

As in the fixed number of particles case, this is automatically satisfied for the system bound states. When this limit exists, channel wave operators are defined by the strong limits

$$\Omega_\alpha^\pm = \lim_{t\to\pm\infty} U^\dagger[I,T]\Phi_\alpha U_\alpha[I,T]. \tag{56}$$

Cook's condition [21] for scattering provides a sufficient condition for the existence of the channel wave operators,

$$\int_c^{\pm\infty} \|V_\alpha U_\alpha[I,T]|\psi\rangle\| dt < \infty, \tag{57}$$

where

$$V_\alpha \coloneqq H\Phi_\alpha - \Phi_\alpha H_\alpha. \tag{58}$$

The scattering operator for scattering from channel β to channel α is the mapping from $\mathcal{H}_\alpha \to \mathcal{H}_\beta$ defined by

$$S_{\alpha\beta} \coloneqq \Omega_{\alpha+}^\dagger \Omega_{\beta-}. \tag{59}$$

This can be compactly expressed in the two-Hilbert-space formulation. The asymptotic Hilbert space is the direct sum of all of the channel subspaces, including all bound state channels,

$$\mathcal{H}_\mathcal{A} \coloneqq \oplus_{\alpha\in\mathcal{A}} \mathcal{H}_\alpha. \tag{60}$$

A two-Hilbert-space injection operator $\Phi_\mathcal{A}$ is a mapping from $\mathcal{H}_\mathcal{A}$ to $\mathcal{H}$ defined by

$$\Phi_\mathcal{A} \coloneqq \sum_{\alpha\in\mathcal{A}} \Phi_\alpha, \tag{61}$$

where each Φ_α acts on the subspace $\mathcal{H}_\alpha$ of $\mathcal{H}_\mathcal{A}$.

There is a natural unitary representation of $ISL(2,C)$ on $\mathcal{H}_\mathcal{A}$ defined by

$$U_\mathcal{A}[\Lambda,Y] \coloneqq \sum_{\alpha\in\mathcal{A}} U_\alpha[\Lambda,Y]. \tag{62}$$

The two-Hilbert-space wave operators can be expressed by the strong limits

$$\Omega_\pm(H,\Phi_\mathcal{A},H_\mathcal{A}) \coloneqq \lim_{t\to\pm\infty} U^\dagger[I,T]\Phi_\mathcal{A} U_\mathcal{A}[I,T], \tag{63}$$

where $T = t\sigma_0$.

In what follows, the dynamical model is assumed to have two-Hilbert-space wave operators that exist and are asymptotically complete in the sense that the S is unitary. With these assumptions and some restrictions on the interactions [5] the two-Hilbert-space wave operators are unitary operators from $\mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{A}$ satisfying the intertwining property:

$$U[\Lambda,Y]\Omega_\pm(H,\Phi_\mathcal{A},H_\mathcal{A}) = \Omega_\pm(H,\Phi_\mathcal{A},H_\mathcal{A})U_\mathcal{A}[\Lambda,Y]. \tag{64}$$

VII. SCATTERING EQUIVALENCES

For a fixed number of particles, a scattering equivalence is a unitary operator A satisfying

$$s - \lim_{t\to\pm\infty} (A-I)U_0[I,T] = 0 \tag{65}$$

for $T \coloneqq t\sigma_0$. The physical significance of asymptotic condition (65) is that the unitary transformation A transforms the Hamiltonian in a manner that leaves the spectrum and scattering observables unchanged, *without changing the representation of a free particle* used to formulate the asymptotic condition [5].

What this means is that if the relevant observables are S-matrix elements and spectral properties, and the dynamics is defined by adding interactions to a free-particle dynamics, there is a large class of interactions that give the same S-matrix elements and spectral properties.

Specifically, if $U'[\Lambda,Y] \coloneqq AU[\Lambda,Y]A^\dagger$, condition (65) is equivalent to

$$\Omega_\pm(H',\Phi'_\mathcal{A},H_\mathcal{A}) = A\Omega_\pm(H,\Phi_\mathcal{A},H_\mathcal{A}) \tag{66}$$

with $\Phi'_{\alpha_0} = \Phi_{\alpha_0}$ when α_0 is the N-body breakup channel.

Equation (66) ensures that all S-matrix elements are preserved,

$$S(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})=\Omega_+^\dagger(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})$$
$$=\Omega_+^\dagger(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})$$
$$=S(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}). \tag{67}$$

The distinction between minimally charged bare particles and ''composite'' bare particles is relevant for generalizing condition (65). The special property of minimally charged bare particles is that they exist as asymptotically separated stable bare particles.

The generalization of Eq. (65) is easy to formulate. First consider only those channels α_m where each asymptotic particle is a ''minimum-charge'' particle. Consider the class of unitary transformations A on $\mathcal{H}$ with the property that they preserve the full S matrix

$$H'=AHA^\dagger, \tag{68}$$

$$S=S', \tag{69}$$

where

$$S:=\Omega_+^\dagger(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}). \tag{70}$$

The equality of the scattering operators and asymptotic completeness imply

$$\Omega_+(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_+^\dagger(H',\Phi'_{,}H_{\mathcal{A}})$$
$$=\Omega_-(H',\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_-^\dagger(H',\Phi'_{,}H_{\mathcal{A}}). \tag{71}$$

A sufficient condition for this to be true is

$$\Omega_\pm(H',\Phi_{\mathcal{A}},H_{\mathcal{A}})=A\Omega_\pm(H,\Phi_{,}H_{\mathcal{A}}) \tag{72}$$

for *both* asymptotic conditions. It is a nontrivial condition that there is a single solution A to Eq. (72) for both asymptotic conditions. This is equivalent to the condition

$$s-\lim_{t\to\pm\infty}[A\Phi_{\mathcal{A}}-\Phi'_{\mathcal{A}}]U_{\mathcal{A}}[I,t]=0. \tag{73}$$

In general, given a unitary A, it is possible to define $\Phi'_{\mathcal{A}}:=A\Phi_{\mathcal{A}}$; however for minimal charge channels α_m, which have no asymptotic bound clusters, any reasonable model must also require

$$\Phi_{\alpha_m}=\Phi'_{\alpha_m}. \tag{74}$$

This requirement puts a nontrivial condition on A given by

$$s-\lim_{t\to\pm\infty}(A-I)\Phi_{\alpha_m}U_{\alpha_m}[I,T]=0, \tag{75}$$

which must hold for each minimal-charge channel α_m and both time limits.

For systems of a fixed number of particles, scattering equivalences are used to relate tensor product representations of $ISL(2,C)$, which are useful cluster limits of a satisfactory dynamical model, with representations where the mass commutes with the spin and a maximal set of functions of the noninteracting generators. Both representations are needed to combine interactions that appear in different asymptotic configurations. What makes this work is that the N-particle Hilbert space can be factored into a tensor product of subsystem spaces.

What is different for the models under consideration is that the tensor product of subsystem Hilbert spaces is not the whole Hilbert space. Specifically, for a partition a of charges the Hilbert space has the decomposition $\mathcal{H}=\mathcal{H}_a\oplus\mathcal{H}^a$ and Hilbert space for the tensor product of the subsystems defined by the partition a is $\mathcal{H}_a$. The construction in Ref. [5] naturally leads to a scattering equivalence A_a on $\mathcal{H}_a$. In order to treat cluster properties, A_a must be extended to all of $\mathcal{H}$. The obvious extension $\tilde{A}_a$ that preserves all of the required properties of A_a is

$$\tilde{A}_a:=\begin{pmatrix}A_a & 0\\ 0 & I^a\end{pmatrix}, \tag{76}$$

where $A_a:\mathcal{H}_a\to\mathcal{H}_a$ and I^a is the identity on $\mathcal{H}^a$. In addition, because $\mathcal{H}_{a\cap b}\subseteq\mathcal{H}_a$, the $\mathcal{H}^a$ is not relevant for cluster properties. In all that follows, the symbol A_a is used to denote both A_a and the extension $\tilde{A}_a$.

When the Hilbert space is extended to include composite bare particles, the range of Φ_{α_m}, corresponding to *minimal charge channels*, is orthogonal to all of the subspaces associated with composite bare particles. The condition on the scattering equivalences is that in addition to being unitary operators that preserve all scattering matrix elements, they satisfy condition (75) for each minimal-charge channel α_m.

The C^* algebra of asymptotic constants defined in Ref. [5] is replaced by the a new C^* algebra of operators Z subject to the asymptotic conditions

$$s-\lim_{t\to\pm\infty}Z\Phi_{\alpha_m}U_{\alpha_m}[I,T]=0, \tag{77}$$

$$s-\lim_{t\to\pm\infty}Z^\dagger\Phi_{\alpha_m}U_{\alpha_m}[I,T]=0 \tag{78}$$

for each minimal-charge channel α_m. This algebra is completed by including the identity. The importance of this algebra is that unitary elements of this algebra are scattering equivalences. Operations on the algebra provide a functional calculus for constructing new scattering equivalences that are functions of noncommuting scattering equivalences.

VIII. SUMMARY

In the previous sections the modifications of the fixed number of particles construction [5] necessary to treat particle production were discussed. The first new feature is that the Hilbert space does not factor into a tensor product of subsystems Hilbert spaces. Instead, for any decomposition into subsystems, there is a residual subspace $\mathcal{H}^a$ of Eq. (16) where the factorization is not compatible with the bare particle content on these subspaces. This led to modifications of the formulation of cluster properties (23), the asymptotic dynamics [Eqs. (37) and (49)], and the structure of the algebra of scattering equivalences [Eqs. (75)–(78)]. Charges were

introduced to replace particles and cluster properties were formulated with respect to partitions of charges. The partial ordering on charges had important consequences. The most important was that subsequent refinements of clusters never affected the residual component of the Hilbert space. Subsequent refinements only acted nontrivially on the tensor product subspace associated with the preceding refinement. While in the general case the addition of fictitious charges has no consequence, the requirement that all particles have positive charge limits the class of theories to theories with a bounded number of bare-particle degrees of freedom. The theories consistent with this requirement have structures like the Lee model and Isobar models. The value of this restriction is the existence of a meaningful few-body dynamics. Specifically, the dynamics of the N-charge system is determined by the dynamics of the $K<N$ charge systems up to N-charge interactions. While the ultimate goal is to remove the restriction to positive charges, the bounded charge theories exhibit all of required properties.

The modifications discussed are adequate to allow the general methods used in Ref. [5] to be extended to treat the class of models discussed in this paper. In the next three sections the modifications to the general construction are illustrated with an example in the three charge sector.

IX. THE TWO-CHARGE SYSTEM

The general construction of a dynamical representation of the Poincaré group satisfying cluster properties is inductive, starting with the simplest system. The construction is illustrated with a theory having two minimally charged bare particles and two additional composite-bare particles. In this example the induction starts with the two-charge sector. The first step is to specify the bare particles of the theory.

(a) There are two minimally charged bare particles, labeled N and π, with charges $(q_1,q_2)=(1,0)$ and (0,1), respectively.

(b) There are two composite bare particles labeled Δ and ρ with charges (1,1) and (0,2), respectively.

The charge conservation condition means that in addition to reactions that preserve particle number, the following basic reactions that change particle number are also possible

$$\pi+\pi\leftrightarrow\rho, \quad N+\pi\leftrightarrow\Delta. \tag{79}$$

The composite bare particles of this model do not necessarily correspond to stable physical particles. That will be true only if there are point eigenstates of the mass operator with the same charge as the bare particle. In a given model there could be zero, one, or several physical Δ or ρ particles. In addition, the dynamics could lead to new composite particles, such as a composite system of two or more N particles, which have different charges than any composite bare particle. These are analogous to bound states in the N-particle case.

The second step is to choose a representation to label the states of the bare particles of the theory. In this example the single-particle observables F_i are chosen as the three components of the linear momentum $\vec{p}$ and the helicity $h=\hat{p}\cdot\vec{j}$. Vectors in the bare particle Hilbert spaces for a bare-particle of mass m and spin j are represented by square integrable functions $\psi(\vec{p},\lambda)$ of the eigenvalues λ of the helicity $h:=\hat{p}\cdot\vec{j}$ and $\vec{p}$ of the linear momentum:

$$|\psi\rangle:=\sum_{\lambda=-j}^{j}\int|\vec{p},\lambda;m,j\rangle d^3p\,\psi(\vec{p},\lambda). \tag{80}$$

The bare-particle Hilbert spaces are denoted by $\mathcal{H}_N$, $\mathcal{H}_\pi$, $\mathcal{H}_\Delta$, and $\mathcal{H}_\rho$.

There are three two-charge problems corresponding to the total charges $(q_1,q_2)=(2,0),(1,1)$, and (0,2). The (2,0) problem corresponds to two N particles, and is an ordinary two-body problem that can be treated using the methods of Refs. [5,17]. The model Hilbert space for the (1,1) and (0,2) sectors each include a composite (total charge >1) bare particle.

The Hilbert spaces for the (1,1) and (0,2) sectors are

$$\mathcal{H}_{(1,1)}:=(\mathcal{H}_{m_N,j_N}\otimes\mathcal{H}_{m_\pi,j_\pi})\oplus\mathcal{H}_{m_\Delta,j_\Delta} \tag{81}$$

and

$$\mathcal{H}_{(0,2)}:=(\mathcal{H}_{m_{\pi_1},j_{\pi_1}}\otimes\mathcal{H}_{m_{\pi_2},j_{\pi_2}})\oplus\mathcal{H}_{m_\rho,j_\rho}. \tag{82}$$

The bare-particle spaces $\mathcal{H}_{m,j}$ are irreducible representation spaces [3,5] of $ISL(2,C)$. Noninteracting basis vectors on the charge-two Hilbert spaces $\mathcal{H}$ have the general form:

$$\begin{pmatrix}|\vec{p}_1,\lambda_1;m_1,j_1\rangle\otimes|\vec{p}_2,\lambda_2;m_2,j_2\rangle\\0\end{pmatrix} \tag{83}$$

in the two-particle sector $\mathcal{H}_a$ and

$$\begin{pmatrix}0\\|\vec{p}_1,\lambda_1;m_c,j_c\rangle\end{pmatrix} \tag{84}$$

in the one-particle sector $\mathcal{H}^a$.

There is a noninteracting unitary representation $U_0[\Lambda,Y]$ of $ISL(2,C)$ on $\mathcal{H}$ corresponding to Eq. (11), defined by

$$U_0[\Lambda,Y]:=\begin{pmatrix}U_1[\Lambda,Y]\otimes U_2[\Lambda,Y] & 0\\0 & U_c[\Lambda,Y]\end{pmatrix}, \tag{85}$$

where

$$U_i[\Lambda,Y]|\vec{p},\lambda;m_i,j_i\rangle \tag{86}$$

$$=\sum_{\lambda'=-j}^{j}|\vec{p}',\lambda';m_i,j_i\rangle\sqrt{\frac{\omega_i(p')}{\omega_i(p)}}D^j_{\lambda'\lambda}[R_H(\Lambda,p)]e^{ip'\cdot y} \tag{87}$$

with

$$\omega_i(p):=\sqrt{m_i^2+\vec{p}\cdot\vec{p}}, \tag{88}$$

$$p'^\nu:=\frac{1}{2}\mathrm{Tr}(\sigma_\nu\Lambda\sigma_\mu\Lambda^\dagger)p^\mu=\Lambda^\nu_\mu p^\mu, \tag{89}$$

$$R_H(\Lambda,p) := H^{-1}(p_\Lambda)\Lambda H(p), \tag{90}$$

where $H(p)$ is the helicity boost [18] given by

$$H(p) := R(\hat{z}\to\hat{p})B_c(|\vec{p}|\hat{z}). \tag{91}$$

The matrix $R(\hat{z}\to\hat{p})$ is the SU(2) matrix corresponding to a rotation about the axis parallel to $\hat{\theta}=\hat{z}\times(\hat{p}/|\hat{z}\times\hat{p}|)$ through an angle θ define by $\cos(\theta)=\hat{z}\cdot\hat{p}$. It rotates $\hat{z}$ in the direction parallel to $\vec{p}$. This matrix has the form

$$R(\hat{z}\to\hat{p}) = \sigma_0\cos\left(\frac{\theta}{2}\right) + i\,\theta\cdot\vec{\sigma}\,\sin\left(\frac{\theta}{2}\right). \tag{92}$$

The matrix $B_c(|\,\vec{p}\,|\hat{z})$ is the $SL(2,C)$ matrix corresponding to a rotationless Lorentz transformation in the $\hat{z}$ direction given by

$$B_c(|\,\vec{p}\,|\hat{z}) = \sigma_0\cosh\left(\frac{\eta}{2}\right) + \sigma_z\sinh\left(\frac{\eta}{2}\right), \tag{93}$$

where the rapidity η satisfies

$$\sinh(\eta) = \frac{|\,\vec{p}\,|}{m_i}, \quad \cosh(\eta) = \frac{\omega_i(p)}{m_i}. \tag{94}$$

The matrix $D^j_{\lambda'\lambda}[R]$ is the SU(2) Wigner D function of R. The tensor product basis is not a useful basis for including interactions. It is more useful to work in an equivalent basis that transforms irreducibly with respect to $U_0[\Lambda,Y]$.

Clebsch-Gordan coefficients [3,5,18] of $ISL(2,C)$ can be used to construct linear combinations of the basis elements on the subspace $\mathcal{H}_1\otimes\mathcal{H}_2$, which transform irreducibly with respect to $U_0[\Lambda,Y]$. The form of the Clebsch-Gordan coefficients in the helicity basis depends on the choice of degeneracy quantum numbers. Wick [18] uses ''body-fixed'' single-particle helicities to label degeneracies. In this model we use ''spin'' and ''orbital'' angular momentum labels that are more natural for formulating two-body interactions. For this choice the Clebsch-Gordan coefficients are

$$\begin{aligned}
&\langle\vec{p}_1,\lambda_1;m_1,j_1:\vec{p}_2,\lambda_2;m_2,j_2|\vec{p}_{12},\lambda_{12};k_{12},j_{12};l_{12},s_{12}\rangle \\
&= \sum_{\lambda_1',\lambda_2',\lambda_{s12}',\mu_l'} \delta(\vec{p}_{12}-\vec{p}_1-\vec{p}_2) \\
&\quad\times\frac{\delta(k_{12}-k(\vec{p}_1,\vec{p}_2))}{k_{12}^2}\left|\frac{\partial(\vec{p}_{12},\vec{k}_1(\vec{p}_1,\vec{p}_2))}{\partial(\vec{p}_1,\vec{p}_2)}\right|^{1/2} \\
&\quad\times Y^l_{\mu_l}(\hat{k}_1)D^{j_1}_{\lambda_1,\lambda_1'}[R_{HMW}(p_{12},k_1)]D^{j_2}_{\lambda_2,\lambda_2'} \\
&\quad\times[R_{HMW}(p_{12},k_2)]\langle j_1,\lambda_1',j_2,\lambda_2'|s_{12},\lambda_{s12'}\rangle \\
&\quad\times\langle s_{12},\lambda_{s12}',l,\mu_l'|j_{12},\lambda_{12}\rangle
\end{aligned} \tag{95}$$

where

$$R_{HMW}(p,k_i) := H^{-1}(p_i)H(p_{12})B_c(k_i), \tag{96}$$

is a rotation obtained by composing a helicity-Melosh rotation with a helicity-Wigner rotation [3]. The other quantities in Eq. (96) are defined by

$$k_i^\nu := \frac{1}{2}\mathrm{Tr}[\sigma_\nu H^{-1}(p_{12})\sigma_\mu(H^{-1}(p_{12}))^\dagger]p_i^\mu, \tag{97}$$

$$B_c(k_i) = \sigma_0\cosh\left(\frac{\eta_i}{2}\right) + \hat{k}_i\cdot\vec{\sigma}\sinh\left(\frac{\eta_i}{2}\right) \tag{98}$$

and

$$\sinh(\eta_i) = \frac{|\vec{k}_i|}{m_i}, \quad \cosh(\eta_i) = \frac{\omega_i(k_i)}{m_i}. \tag{99}$$

In this coefficient the mass m_{12} is related to the continuous variable k_{12}^2 by

$$k_{12}^2 := \frac{m_{12}^4+m_1^4+m_2^4-2m_1^2m_2^2-2m_{12}^2m_1^2-2m_{12}^2m_2^2}{4m_{12}^2}, \tag{100}$$

which has a spectrum $\in[0,\infty]$. The Jacobian is

$$\left|\frac{\partial(\vec{p}_{12},\vec{k}_1(\vec{p}_1,\vec{p}_2))}{\partial(\vec{p}_1,\vec{p}_2)}\right| = \frac{\omega_1(k_1)\omega_2(k_2)[\omega_1(p_1)+\omega_2(p_2)]}{\omega_1(p_1)\omega_p(p_2)[\omega_1(k_1)+\omega_2(k_2)]}. \tag{101}$$

These Clebsch-Gordan coefficients define the irreducible noninteracting eigenstates

$$|\vec{p}_{12},\lambda_{12};k_{12},j_{12};l_{12},s_{12}\rangle \tag{102}$$

as linear superpositions of the tensor product states. The irreducible noninteracting eigenstates transform as

$$\begin{aligned}
&U_1[\Lambda,Y]\otimes U_2[\Lambda,Y]|\vec{p}_{12},\lambda_{12};k_{12},j_{12};l_{12},s_{12}\rangle \\
&= \sum_{\lambda_{12}'=-j_{12}}^{j_{12}} |\vec{p}_{12}{}',\lambda_{12}';k_{12},j_{12};l_{12},s_{12}\rangle \\
&\quad\times\sqrt{\frac{\omega_{12}(p_{12}')}{\omega_{12}(p_{12})}}D^{j_{12}}_{\lambda_{12}',\lambda_{12}}[R_H(\Lambda,p_{12})]e^{ip'\cdot y}.
\end{aligned} \tag{103}$$

This has the same structure as a single-particle transformation, except that the noninteracting two-body invariant mass m_{12} is replaced by the more convenient continuous variable k_{12}^2. The transformed four-momentum p' is related to the original momentum by Eq. (89). The quantum numbers s_{12} and l_{12} are invariant degeneracy quantum numbers, which are needed because multiple copies of the m_{12}, j_{12} representation appear in the tensor product.

The irreducible free eigenstates,

$$\begin{pmatrix} |\vec{p}_{12},\lambda_{12};k_{12},j_{12};l_{12},s_{12}\rangle \\ 0 \end{pmatrix} \tag{104}$$

and

$$\begin{pmatrix} 0 \\ |\vec{p}_c,\lambda_c;m_c,j_c\rangle \end{pmatrix}, \tag{105}$$

are a basis that can be used to solve the two-charge dynamics.

The dynamics is defined by adding an interaction V to the free mass operator M_0 that commutes with and is independent of $\vec{p}_{12}$ and λ_{12}.

In the free-particle irreducible basis the noninteracting mass operator has the form

$$M_0 = \begin{pmatrix} m_{12} & 0 \\ 0 & m_c \end{pmatrix} = \begin{pmatrix} \sqrt{m_1^2+k_{12}^2}+\sqrt{m_2^2+k_{12}^2} & 0 \\ 0 & m_c \end{pmatrix} \tag{106}$$

and the interaction is assumed to have a kernel of the form:

$$\langle \vec{p},\lambda;\vec{j},\ldots|V|\vec{p}\,',\lambda';\vec{j}\,',\ldots\rangle = \delta(\vec{p}-\vec{p}\,')\delta_{jj'}\delta_{\lambda\lambda'}\begin{pmatrix} \langle k_{12},l_{12},s_{12}\|V^j\|k'_{12},l'_{12},s'_{12}\rangle & \langle k_{12},l_{12},s_{12}\|V^j\|m_c\rangle \\ \langle m_c\|V^j\|k'_{12},l'_{12},s'_{12}\rangle & \langle m_c\|V^j\|m_c\rangle \end{pmatrix}. \tag{107}$$

The term $\langle m_c\|V^j\|m_c\rangle$ is a constant which could be absorbed in the bare mass m_c.

For an interaction of the form (107) the dynamical mass operator

$$M := M_0 + V \tag{108}$$

commutes with and is independent of $\vec{p}$ and h. It follows that $M,\vec{p},h,j^2$ can be simultaneously diagonalized by diagonalizing M in the free-particle irreducible basis (104), (105). In this basis the mass eigenfunctions have the form

$$\langle \vec{p},\lambda;j,\ldots|\vec{p}\,'\lambda';m',j'\rangle = \delta(\vec{p}-\vec{p}\,')\delta_{jj'}\delta_{\lambda\lambda'} \times \begin{pmatrix} \langle k_{12},l_{12},s_{12}|m',j\rangle \\ \langle m_c|m',j\rangle \end{pmatrix}, \tag{109}$$

where the components of the reduced wave function, $\langle k_{12},l_{12},s_{12}|m,j\rangle$ and $\langle m_c|m,j\rangle$, are solutions of the coupled equations

$$(m-M_0)\langle k_{12},l_{12},s_{12}|m,j\rangle = \sum_{l'_{12},s'_{12}} \int_0^\infty \langle k_{12},l_{12},s_{12}\|V^j\|k'_{12},l'_{12},s'_{12}\rangle k'^2_{12}dk'_{12} \times \langle k'_{12},l'_{12},s'_{12}|m,j\rangle + \langle k_{12},l_{12},s_{12}\|V^j\|m_c\rangle\langle m_c|m,j\rangle, \tag{110}$$

$$(m-m_c)\langle m_c|m,j\rangle = \sum_{l'_{12},s'_{12}} \int_0^\infty \langle m_c\|V^j\|k'_{12},l'_{12},s'_{12}\rangle k'^2_{12}dk'_{12} \times \langle k'_{12},l'_{12},s'_{12}|m,j\rangle + \langle m_c\|V^j\|m_c\rangle\langle m_c|m,j\rangle \tag{111}$$

for the eigenvalue m. These equations can be combined into a single dynamical equation for $\langle k'_{12},l'_{12},s'_{12}|m,j\rangle$,

$$(m-m_0)\langle k_{12},l_{12},s_{12}|m,j\rangle = \int_0^\infty \sum_{s'_{12},l'_{12}} \langle k_{12},l_{12},s_{12}|K^j|k'_{12},l'_{12},s'_{12}\rangle k'^2_{12}dk'_{12} \times \langle k'_{12},l'_{12},s'_{12}|m,j\rangle, \tag{112}$$

where

$$\langle k_{12},l_{12},s_{12}|K^j|k'_{12},l'_{12},s'_{12}\rangle := \langle k_{12},l_{12},s_{12}\|V^j\|k'_{12},l'_{12},s'_{12}\rangle + \frac{\langle k_{12},l_{12},s_{12}\|V^j\|m_c\rangle\langle m_c\|V^j\|q'_{12},l'_{12},s'_{12}\rangle}{m-m_c-\langle m_c\|V^j\|m_c\rangle}. \tag{113}$$

The component $\langle m_c|m,j\rangle$ can be obtained by quadrature

$$\langle m_c|m,j\rangle = \int_0^\infty \sum_{s'_{12},l'_{12}} \frac{\langle m_c\|V^j\|k'_{12},l'_{12},s'_{12}\rangle}{m-m_c-\langle m_c\|V^j\|m_c\rangle} \times k'^2_{12}dk'_{12}\langle k'_{12},l'_{12},s'_{12}|m,j\rangle. \tag{114}$$

For scattering states, Eqs. (112) and (114) must be solved with incoming or outgoing asymptotic conditions, $m \to m \pm i0+$. This dynamical equation is of comparable difficulty to solving the two-body Lippmann-Schwinger equation.

For a self-adjoint M with a well-behaved short-ranged interaction V the simultaneous eigenstates (109), $|\vec{p},\lambda;m,j\rangle$ of M, $\vec{p}=\vec{p}_0$, $h=h_0$, $j^2=j_0^2$ are a complete set of eigenstates that transform irreducibly with respect to a dynamical representation $U[\Lambda,Y]$ of $ISL(2,C)$. The transformation properties of these eigenstates follow because the operators M_0, $\vec{p}$, $\vec{j}$, $i\vec{\nabla}_p$ and M, $\vec{p}$, $\vec{j}$, $i\vec{\nabla}_p$ have identical commutation relations. Note that in both cases the partial p derivatives are performed holding the helicity constant.

It follows that $U[\Lambda,Y]$ is defined in the basis (109) by

$$U[\Lambda,Y]|\vec{p},\lambda;m,j\rangle = \sum_{\lambda'=-j}^{j} |\vec{p}\,',\lambda';m,j\rangle \times \sqrt{\frac{\omega_m(p')}{\omega_m(p)}} D^j_{\lambda'\lambda}[R_H(\Lambda,p)] e^{ip'\cdot y}, \tag{115}$$

where

$$p'^{\mu} = \frac{1}{2}\mathrm{Tr}(\sigma_\nu \Lambda \sigma_\mu \Lambda^\dagger) p^\mu = \Lambda^\nu_\mu p^\mu \tag{116}$$

with $p^0 = \sqrt{\vec{p}\cdot\vec{p}+m^2}$.

This is identical to the transformation properties of a single particle, except the *mass parameter is the eigenvalue* m of mass operator (108).

The action of $U[\Lambda,Y]$ on an arbitrary state with wave function $\psi(\vec{p},\lambda)$ is given by completeness:

$$U[\Lambda,Y]|\psi\rangle = \sum_j \sum_{\lambda,\lambda'=-j}^{j} \int |\vec{p}\,',\lambda';m,j\rangle \times \sqrt{\frac{\omega_m(p')}{\omega_m(p)}} D^j_{\lambda'\lambda}[R_H(\Lambda,p)] \times e^{ip'\cdot y}\psi_{m,j}(\vec{p},\lambda) d^3p\, dm. \tag{117}$$

Solutions of the dynamical equations with $m < m_1 + m_2$ in the pure point spectrum of M correspond to stable ρ or Δ particles. Solutions with $m \geq m_1 + m_2$ in the absolutely continuous spectrum of M correspond to scattering eigenstates. We assume that all of the eigenstates fall into one of these two classifications. In addition, we assume that the incoming and outgoing wave scattering solutions each span the subspace of $\mathcal{H}$ orthogonal to the bound state subspace.

This representation satisfies algebraic cluster properties because the operator $U[\Lambda,Y]$ is a function of single bare-particle operators and V. In the limit that V vanishes, this representation becomes the noninteracting representation $U_0[\Lambda,Y]$.

This completes the $|\mathcal{N}|=2$ construction for the charge sector (1,1) or (0,2).

X. THE {2,1} SECTOR

The next step in the construction of the dynamics in the three-charge sector is to consider the problem of two interacting charges and a spectator charge. This problem defines the asymptotic behavior of the three-charge system in the limit that one charge is asymptotically separated from an interacting pair. This form is used in the mathematical formulation of cluster properties.

To be specific, consider the three-charge system with $(q_1,q_2)=(1,2)$. For this set of charge quantum numbers the Hilbert space is

$$\mathcal{H} = (\mathcal{H}_N \otimes \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}) \oplus (\mathcal{H}_{\Delta_1} \otimes \mathcal{H}_{\pi_2}) \oplus (\mathcal{H}_{\Delta_2} \otimes \mathcal{H}_{\pi_1}) \oplus (\mathcal{H}_N \otimes \mathcal{H}_\rho). \tag{118}$$

The nontrivial partitions $\mathcal{P}$ of the minimal charges of this system are $a=(N)(\pi_1)(\pi_2)$, $(N,\pi_1)(\pi_2)$, $(N,\pi_2)(\pi_1)$, and $(\pi_1,\pi_2)(N)$. For each partition $a\in\mathcal{P}$ there is a decomposition of the Hilbert space of the form (16). For example, for $a=(N)(\pi_1)(\pi_2)$ the orthogonal subspaces are

$$\mathcal{H}_a = \mathcal{H}_N \otimes \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}, \tag{119}$$

$$\mathcal{H}^a = (\mathcal{H}_{\Delta_1} \otimes \mathcal{H}_{\pi_2}) \oplus (\mathcal{H}_{\Delta_2} \otimes \mathcal{H}_{\pi_1}) \oplus (\mathcal{H}_N \otimes \mathcal{H}_\rho) \tag{120}$$

and for $a=(N,\pi_1)(\pi_2)$ the orthogonal subspaces are

$$\begin{aligned}\mathcal{H}_a &= (\mathcal{H}_N \otimes \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}) \oplus (\mathcal{H}_{\Delta_1} \otimes \mathcal{H}_{\pi_2}) \\ &= [(\mathcal{H}_N \otimes \mathcal{H}_{\pi_1}) \oplus \mathcal{H}_{\Delta_1}] \otimes \mathcal{H}_{\pi_2} \\ &= \mathcal{H}_{(N\pi_1)} \otimes \mathcal{H}_{\pi_2},\end{aligned} \tag{121}$$

$$\mathcal{H}^a = (\mathcal{H}_{\Delta_2} \otimes \mathcal{H}_{\pi_1}) \oplus (\mathcal{H}_N \otimes \mathcal{H}_\rho) \tag{122}$$

with similar expressions for the two other nontrivial partitions a.

The goal of the three-body construction is to find a $U[\Lambda,Y]$ that asymptotically factorizes into a tensor product of subsystem representations when charges in different clusters of a are separated. As mentioned in Sec. V, the formulation of cluster properties differs from the fixed-particle number case because $U[\Lambda,Y]$ acts on $\mathcal{H}$ while $\otimes_i U_{a_i}[\Lambda,Y]$ acts on $\mathcal{H}_a$.

Following Eq. (37) it is useful to extend the asymptotic forms on $\mathcal{H}_a$ to operators on $\mathcal{H}$,

$$U_{(N)(\pi_1)(\pi_2)}[\Lambda,Y] := \begin{pmatrix} U_N[\Lambda,Y]\otimes U_{\pi_1}[\Lambda,Y]\otimes U_{\pi_2}[\Lambda,Y] & 0 \\ 0 & I \end{pmatrix} \tag{123}$$

for $a=(N)(\pi_1)(\pi_2)$ and

$$U_{(N\pi_1)(\pi_2)}[\Lambda,Y] := \begin{pmatrix} U_{(N\pi_1)}[\Lambda,Y]\otimes U_{\pi_2}[\Lambda,Y] & 0 \\ 0 & I \end{pmatrix} \tag{124}$$

for $a=(N\pi_1)(\pi_2)$, with similar expression for the two-cluster partitions $(N\pi_2)(\pi_1)$ and $(N)(\pi_1\pi_2)$. These operators are tensor products of subsystem representations of $ISL(2,C)$ on $\mathcal{H}_a$ and are extended so they are the identity on the subspace $\mathcal{H}^a$.

In the first expression the factors $U_{\pi_i}[\Lambda,Y]$ and $U_N[\Lambda,Y]$ are single-particle irreducible representations. The factor $U_{(N\pi_1)}[\Lambda,Y]$ in the second expression is the *interact-*

ing two-charge representation (117) constructed in the preceding section. It acts on the two-charge space $\mathcal{H}_{(N\pi_1)} := (\mathcal{H}_N \otimes \mathcal{H}_{\pi_1}) \oplus \mathcal{H}_{\Delta_1}$.

To formulate cluster condition (23), for each $a \in \mathcal{P}$, we define Π_a to be the orthogonal projector onto the subspace $\mathcal{H}_a$ of $\mathcal{H}$,

$$\Pi_a := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{125}$$

and the cluster translation operators

$$T_a(Y_1, \ldots, Y_{n_a}) := \begin{pmatrix} U_{a_1}[I, Y_1] \otimes \cdots \otimes U_{a_{n_a}}[I, Y_{n_a}] & 0 \\ 0 & 0 \end{pmatrix}, \tag{126}$$

which independently translate the charges in each cluster of the partition a by displacements $y_1, \ldots, y_{n_a}$ where $(Y_i := y_i^\mu \sigma_\mu)$. In this example $a = (N\pi_1)(\pi_2)$, $(N\pi_2)(\pi_1)$, $(\pi_1\pi_2)(N)$, or $(N)(\pi_1)(\pi_2)$. Formally, for the partition $a = (N, \pi_1)(\pi_2)$, the formulation of cluster properties (23) for the three-charge problem is

$$\lim_{(y_1 - y_2)^2 \to \infty} \| [U[\Lambda, Y] - U_{(N\pi_1)(\pi_2)}[\Lambda, Y]] \times T_{(N\pi_1)(\pi_2)}(Y_1, Y_2) \Pi_{(N\pi_1)(\pi_2)} |\psi\rangle \| = 0, \tag{127}$$

where the limit corresponds to a large spacelike separation.

The projection operator $\Pi_{(N\pi_1)(\pi_2)}$ projects on the subspace of the Hilbert space where independent translations of the $(N\pi_1)$ and π_2 subsystems are defined.

One term that is not eliminated by the projection operator is the part of $U[\Lambda, Y]$ that maps $\mathcal{H}_{(N\pi_1)(\pi_2)}$ to $\mathcal{H}^{(N\pi_1)(\pi_2)}$. This contribution involves the interactions of the general type $\rho \leftrightarrow \pi_1 + \pi_2$ and $\Delta_2 \leftrightarrow N + \pi_2$. These should vanish in the limit that π_2 is asymptotically separated from the $N\pi_1$ system, provided the interactions have sufficiently short range. Condition (127) provides a range condition on these production interactions.

In the three- and many-charge systems it is also necessary to consider cluster properties for sequential limits. In the fixed charge case, sequential limits require care because different cluster operators are defined on different subspaces. To illustrate the problem, consider the following three limits.

(1) First take the limit corresponding to the partition of charges $(N\pi_2)(\pi_1)$, followed by $(N\pi_1)(\pi_2)$.

(2) First take the limit corresponding to the partition of charges $(N\pi_1)(\pi_2)$, followed by $(N\pi_2)(\pi_1)$.

(3) Take the limit corresponding to the partition of charges $(N)(\pi_1)(\pi_1)$.

Intuitively, one expects that all three limits should give the free dynamics on $\mathcal{H}_{(N)(\pi_1)(\pi_2)}$. The problem is that the first two limits are defined on the larger subspaces $\mathcal{H}_{(N\pi_1)(\pi_2)} \supset \mathcal{H}_{(N)(\pi_1)(\pi_2)}$ and $\mathcal{H}_{(N\pi_2)(\pi_1)} \supset \mathcal{H}_{(N)(\pi_1)(\pi_2)}$, respectively.

The projection and translation operators that appear in the asymptotic condition in each of these three cases are

$$T_{(N\pi_2)(\pi_1)}(Y_{N\pi_2}, Y_{\pi_1}) \Pi_{(N\pi_2)(\pi_1)} \times T_{(N\pi_1)(\pi_2)}(Y_{N\pi_1}, Y_{\pi_2}) \Pi_{(N\pi_1)(\pi_2)}, \tag{128}$$

$$T_{(N\pi_1)(\pi_2)}(Y_{N\pi_1}, Y_{\pi_2}) \Pi_{(N\pi_1)(\pi_2)} \times T_{(N\pi_2)(\pi_1)}(Y_{N\pi_2}, Y_{\pi_1}) \Pi_{(N\pi_2)(\pi_1)}, \tag{129}$$

and

$$T_{(N)(\pi_1)(\pi_2)}(Y_N, Y_{\pi_1}, Y_{\pi_2}) \Pi_{(N)(\pi_1)(\pi_2)}, \tag{130}$$

respectively.

In the representation used in this example, if all of the cluster displacements Y_i are space vectors with no time component, then the translation operators have no two-body terms and

$$T_{(N\pi_1)(\pi_2)}(Y_{N\pi_1}, Y_{\pi_2}) T_{(N\pi_2)(\pi_1)}(Y'_{N\pi_2}, Y'_{\pi_1}) = T_{(N)(\pi_1)(\pi_2)}(Y_{N\pi_1} + Y'_{N\pi_2}, Y_{N\pi_1} + Y'_{\pi_1}, Y_{\pi_2} + Y'_{N\pi_2}) \tag{131}$$

and

$$\Pi_{(N\pi_1)(\pi_2)} \Pi_{(N\pi_2)(\pi_1)} = \Pi_{(N)(\pi_1)(\pi_2)}, \tag{132}$$

etc. In this case all three limits have the same form. For the case that the relative displacements are spacelike, but the individual displacements have nonvanishing time components, the space of initial vectors differs in all three cases. For example, $T_{(N\pi_1)(\pi_2)}$ has components that map vectors from $\mathcal{H}_{\Delta_1} \otimes \mathcal{H}_{\pi_2}$ to $\mathcal{H}_{(N)(\pi_1)(\pi_2)}$, while for the other two configurations the ranges of the initial projections are orthogonal to the subspace $\mathcal{H}_{\Delta_1} \otimes \mathcal{H}_{\pi_2}$.

In this case the contribution of the subspace $\mathcal{H}_{\Delta_1} \otimes \mathcal{H}_{\pi_2}$ can be eliminated at the outset by insisting that the sequential limits should give the same result only on the common subspace where all of the limits are defined. Effectively, this means that the limit should only be applied to vectors in the range of $\Pi_{(N)(\pi_1)(\pi_2)}$.

Mathematically, this means that Eq. (128) should be replaced by

$$T_{(N\pi_2)(\pi_1)}(Y_{N\pi_2}, Y_{\pi_1}) \Pi_{(N\pi_2)(\pi_1)} \times T_{(N\pi_1)(\pi_2)}(Y_{N\pi_1}, Y_{\pi_2}) \Pi_{(N\pi_1)(\pi_2)} \to T_{(N\pi_2)(\pi_1)}(Y_{N\pi_2}, Y_{\pi_1}) \Pi_{(N)(\pi_1)(\pi_2)} \times T_{(N\pi_1)(\pi_2)}(Y_{N\pi_1}, Y_{\pi_2}) \Pi_{(N)(\pi_1)(\pi_2)}. \tag{133}$$

Cluster properties in the two-charge sector allow the replacement of

$$T_{(N\pi_2)(\pi_1)}(Y_{N\pi_2},Y_{\pi_1})\Pi_{(N)(\pi_1)(\pi_2)} \times T_{(N\pi_1)(\pi_2)}(Y_{N\pi_1},Y_{\pi_2})\Pi_{(N)(\pi_1)(\pi_2)} \quad (134)$$

by

$$T_{(N)}(Y_{N\pi_2}+Y_{N\pi_1})T_{(\pi_1)}(Y_{\pi_1})T_{(\pi_2)}(Y_{N\pi_2}+Y_{\pi_2}) \times \Pi_{(N)(\pi_1)(\pi_2)}T_{(N\pi_1)}(Y_{N\pi_1})\Pi_{(N)(\pi_1)(\pi_2)}, \quad (135)$$

which is equivalent to formulating the cluster limit with the noninteracting translation operators. Similar remarks apply for representations where the dynamical translation operators have interactions.

This shows that by projecting on the largest subspace where all cluster translations are defined, cluster properties can be formulated in a manner that is similar to the fixed number of particles case. These observations generalize to sequential limits of systems with more than three charges.

The tensor product representations constructed satisfy cluster properties by definition. These tensor product representations are important building blocks for the full three-charge dynamics.

For each of the 2+1 charge problems discussed, there is an alternate construction that leads to the same S matrix.

To see this recall that the irreducible free-particle basis constructed using the Clebsch-Gordan coefficients in Eq. (104) have the form

$$\begin{pmatrix} |p_{12}\lambda_{12};k_{12},j_{12},l_{12},s_{12}\rangle \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ |p_{12},\lambda_{12};,m_c,j_c\rangle \end{pmatrix}. \quad (136)$$

The tensor product of this basis with a spectator basis defines a basis on the subspace $\mathcal{H}_{(12)(3)}$ of $\mathcal{H}$:

$$\begin{pmatrix} |p_{12}\lambda_{12};k_{12},j_{12},l_{12},s_{12}\rangle|p_3\lambda_3;m_3,j_3\rangle \\ 0 \end{pmatrix}, \quad (137)$$

$$\begin{pmatrix} 0 \\ |p_{12},\lambda_{12};,m_c,j_c\rangle|p_3\lambda_3;m_3,j_3\rangle \end{pmatrix}. \quad (138)$$

Since Eqs. (137) and (138) are tensor products of irreducible representations, the Clebsch-Gordan coefficients can be used to transform the tensor product basis to a three-charge irreducible basis:

$$\begin{pmatrix} |p\lambda;q_{(12)(3)},j_{(12)(3)},L_{(12)(3)},S_{(12)(3)},k_{12},j_{12},l_{12},s_{12},m_3,j_3\rangle \\ 0 \end{pmatrix}, \quad (139)$$

$$\begin{pmatrix} 0 \\ |p,\lambda;q_{(12)(3)},j_{(12)(3)},L_{(12)(3)},S_{(12)(3)},m_c,j_c,m_3,j_3\rangle \end{pmatrix}. \quad (140)$$

The free two-charge mass operator is a multiplication operator in each of the representations (137), (138) and (139), (140),

$$M_{0:12}=\begin{pmatrix} \sqrt{m_1^2+k_{12}^2}+\sqrt{m_2^2+k_{12}^2} & 0 \\ 0 & m_c \end{pmatrix}. \quad (141)$$

Natural extensions of the two-charge interactions (107) are defined in each of the representations (137), (138), and (139), (140) by

$$\langle \vec{p}_{12},\lambda_{12};\vec{j}_{12},\ldots,\vec{p}_3,\lambda_3,m_3,j_3|v|\vec{p}_{12}{}',\lambda'_{12};\vec{j}'_{12},\ldots,\vec{p}_3{}',\lambda'_3,m'_3,j'_3\rangle = \delta(\vec{p}_{12}-\vec{p}_{12}{}')\delta_{\lambda_{12}\lambda'_{12}}\delta_{j_{(12)},j'_{(12)}}\delta(\vec{p}_3-\vec{p}_3{}')\delta_{\lambda_3\lambda'_3}\delta_{j_3,j'_3} \times\begin{pmatrix} \langle k_{12},l_{12},s_{12}\|V^{j_{12}}\|k'_{12},l'_{12},s'_{12}\rangle & \langle k_{12},l_{12},s_{12}\|V^{j_{12}}\|m_c\rangle \\ \langle m_c\|V^{j_{12}}\|k'_{12},l'_{12},s'_{12}\rangle & \langle m_c\|V^{j_{12}}\|m_c\rangle \end{pmatrix} \quad (142)$$

and

$$\langle \vec{p},\lambda;\vec{j},\ldots|\overline{v}|\vec{p}\,',\lambda';\vec{j}',\ldots\rangle = \delta(\vec{p}-\vec{p}\,')\delta_{\lambda\lambda'}\delta_{j_{(12)(3)},j'_{(12)(3)}}\frac{\delta(q_{(12)(3)}-q'_{(12)(3)})}{q^2_{(12)(3)}} \quad (143)$$

$$\times\delta_{L_{(12)(3)},L'_{(12)(3)}}\delta_{S_{(12)(3)},S'_{(12)(3)}}\delta_{j_{(12)},j'_{(12)}}\begin{pmatrix} \langle k_{12},l_{12},s_{12}\|V^{j_{12}}\|k'_{12},l'_{12},s'_{12}\rangle & \langle k_{12},l_{12},s_{12}\|V^{j_{12}}\|m_c\rangle \\ \langle m_c\|V^{j_{12}}\|k'_{12},l'_{12},s'_{12}\rangle & \langle m_c\|V^{j_{12}}\|m_c\rangle \end{pmatrix}. \quad (144)$$

Since both of the above interactions only differ in the choice of multiplicative δ functions, which are asymptotically equivalent, it follows that both

$$M_{12} := M_{0:12} + v, \quad \overline{M}_{12} := M_{0:12} + \overline{v} \tag{145}$$

give the same two-charge mass eigenvalues and S-matrix elements. These are *different* operators, because M_{12} commutes with $\vec{p}_3$ and $\overline{M}_{12}$ commutes with $\vec{q}_{(12)(3)}$, which is defined analogously to Eq. (97). The operator $\overline{M}_{12}$ has the additional important property that it commutes with $\vec{p}$, $j_{(12)(3)}$ and $h_{(12)(3)}$ and is independent of $\vec{p}$ and $h_{(12)(3)}$.

The interesting property is that the interaction $\overline{v}$ satisfies $[\overline{v}, p_3] \neq 0$. This means that if $\overline{v}$ is a short-ranged interaction, then it *vanishes* in the limit that the *spectator* particle 3 is translated infinitely far from the interacting 12 pair. This is *not* the expected behavior when a noninteracting spectator is asymptotically separated from an interacting pair of particles. This type of violation of cluster properties is characteristic of how cluster properties typically fail in improperly formulated relativistic models.

The dynamics given by M_{12} and $\overline{M}_{12}$ are both defined on the subspace $\mathcal{H}_a$ of $\mathcal{H}$, give the same spectral properties and S matrix, and are thus related by a unitary scattering equivalence A_a on $\mathcal{H}_a$. Since the interaction v in the tensor product representation satisfies cluster properties by construction, it follows that scattering equivalences do not necessarily preserve cluster properties. The simplest way to construct the unitary transformation A_a is to use the Clebsh-Gordan coefficients of $ISL(2,C)$ to transform the eigenstates of M_{12}, which transform like a tensor product of an interacting two-charge representation and a spectator representation, to a superposition of irreducible representations of $ISL(2,C)$. The eigenstates

$$|p_{12}, \lambda_{12}, m_{12}, j_{12}; \vec{p}, \lambda_3; m_2, j_3\rangle, \tag{146}$$

which transform as a product of irreducible representations with respect to $U_{(12)}[\Lambda, Y] \otimes U_{(3)}[\Lambda, Y]$, are mapped to irreducible eigenstates of the form

$$|\vec{p}, \lambda; \tilde{q}_{(12)(3)}, \vec{j}, \tilde{L}_{(12)(3)}, \tilde{S}_{(12)(3)}, m_{12}, j_{12},\rangle, \tag{147}$$

where $\tilde{q}^2_{(12)(3)}$ is related to the mass operator $M_{(12)(3)}$ of this irreducible representation by

$$\tilde{q}^2_{(12)(3)} = \frac{M^4_{(12)(3)} + {M_{12}}^4 + m_3^4 - 2{M_{12}}^2 m_3^2 - 2M^2_{(12)(3)}{M_{12}}^2 - 2M^2_{(12)(3)} m_3^2}{4M^2_{(12)(3)}}, \tag{148}$$

where M_{12} is the *interacting* two-body mass operator. This irreducible representation is obtained by first solving the two-charge problem followed by using Poincaré Clebsch-Gordan coefficients to construct a superposition of irreducible representations.

The mass operator $M_{(12)(3)}$ has the form

$$M_{(12)(3)} = \sqrt{M^2_{12} + \tilde{q}^2_{(12)(3)}} + \sqrt{m_3^2 + \tilde{q}^2_{(12)(3)}}, \tag{149}$$

which is the invariant mass operator associated with the tensor product $U_{(12)}[\Lambda, Y] \otimes U_3[\Lambda, Y]$ of the interacting two-body representation and the spectator representation on $\mathcal{H}_{(12)(3)}$.

It is also possible to use the operator $\overline{M}_{12}$ to construct a three-charge mass operator in the barred representation,

$$\overline{M}_{(12)(3)} = \sqrt{\overline{M}^2_{12} + q^2_{(12)(3)}} + \sqrt{m_3^2 + q^2_{(12)(3)}}, \tag{150}$$

where in this case $q^2_{(12)(3)}$ is the noninteracting operator that replaces the mass in the Clebsch-Gordan coefficients for the noninteracting three-charge system, Eq. (139). The operators $\overline{M}_{(12)(3)}$, $j^2_{(12)(3)} = j_0^2$, $\vec{p}_{(12)(3)} = \vec{p}_0$, $h_{(12)(3)} = h_0$, satisfy the same commutation relations as $M_{0(12)(3)}$, $j^0_{(12)(3)} = j^0_0$, $\vec{p}_{(12)(3)} = \vec{p}_0$, $h_{(12)(3)} = h_0$ where $M_{0(12)(3)}$ is the noninteracting invariant mass of the three-charge system. It follows that simultaneous eigenstates of both sets of operators have the same Poincaré transformation properties if the eigenvalues of $M_{0(12)(3)}$ are replaced by the eigenvalues of $\overline{M}_{0(12)(3)}$. This leads to a complete set of eigenstates

$$\overline{|\vec{p}, \lambda; q_{(12)(3)}, \vec{j}, \tilde{L}_{(12)(3)}, \tilde{S}_{(12)(3)}, \overline{m}, j_{12}\rangle} \tag{151}$$

that transform irreducibly with respect to the representation $\overline{U}_{(12)(3)}[\Lambda, Y]$.

The difference between the barred representation and the unbarred representation is that *the order of adding interactions and coupling to three-charge irreducible representation is reversed*. In the unbarred representation, interactions are added to the two-charge system. The interacting two-charge system is decomposed into irreducible representation of $ISL(2,C)$ and these are coupled to the spectator representation using $ISL(2,C)$ Clebsch-Gordan coefficients. In the barred representation the spectator is coupled to the *noninteracting* two-charge system using the $ISL(2,C)$ Clebsch-Gordan coefficients. The two-charge interaction $\overline{v}$ is introduced directly in this representation. In the absence of interactions both representations become equivalent.

Both sets of irreducible eigenstates are complete on $\mathcal{H}_{(n\pi_1)(\pi_2)}$ and the operators whose eigenvalues label the irreducible eigenstates in both representations have identical

spectra. If follows that the scattering equivalence $A_a:\mathcal{H}_a \to \mathcal{H}_a$ can be expressed as the identity in this mixed representation,

$$A_a = \sum \int \overline{|\vec{p},\lambda;q_{(12)(3)},\vec{j},L_{(12)(3)},S_{(12)(3)},\overline{m},j_{12}\rangle} dp \times \langle \vec{p},\lambda;\tilde{q}_{(12)(3)},\vec{j},\tilde{L}_{(12)(3)},\tilde{S}_{(12)(3)},m,j_{12},|, \quad (152)$$

where the sum and integral are over the common eigenvalues of the corresponding observables. The sum over the two-charge mass eigenstates includes a bound state sum and an integral over the incoming- or outgoing-wave scattering states in both representations. Either choice of asymptotic condition $(\pm)$ gives the same operator A_a because the representations are scattering equivalent [5]. The operator A_a is nontrivial if it is evaluated in a single representation. By construction, all of the A_a's for two-cluster partitions become the identity when interactions are turned off.

Because the Hilbert space H_a is a proper subspace of the three-charge Hilbert space for each partition a, it is necessary to extend the definitions of $U_a[\Lambda,Y]$, $\overline{U}_a[\Lambda,Y]$, and A_a to all of $\mathcal{H}$ following

$$U_a[\Lambda,Y] \to \begin{pmatrix} U_a[\Lambda,Y] & 0 \\ 0 & I \end{pmatrix}, \quad (153)$$

$$\overline{U}_a[\Lambda,Y] \to \begin{pmatrix} \overline{U}_a[\Lambda,Y] & 0 \\ 0 & I \end{pmatrix}, \quad (154)$$

$$A_a \to \begin{pmatrix} A_a & 0 \\ 0 & I \end{pmatrix}. \quad (155)$$

Thus the solution of the three $2+1$-charge problems gives for each partition a, into at least two clusters, operators

$$U_a[\Lambda,Y], \quad \overline{U}_a[\Lambda,Y], \quad A_a \quad (156)$$

satisfying

$$\overline{U}_a[\Lambda,Y] = A_a U_a[\Lambda,Y] A_a^\dagger, \quad (157)$$

where $U_a[\Lambda,Y]$ is a tensor product of subsystem representations on $\mathcal{H}_a$ and the mass operator $\overline{M}_a$ for $\overline{U}_a[\Lambda,Y]$ commutes with and is independent of the noninteracting three-charge operators, $\vec{p}$ and h, and commutes with the noninteracting j^2. The operators A_a become the identity when the interactions are turned off. Equations (156) and (157) also hold for the extended representations [Eqs. (153), (154), and (155)] on $\mathcal{H}$.

The representations $U_a[\Lambda,Y]$ and $\overline{U}_a[\Lambda,Y]$ are scattering equivalent, but only the unbarred representation satisfies cluster properties.

The computations of $U_a[\Lambda,Y]$, $\overline{U}_a[\Lambda,Y]$, and A_a can all be expressed in terms of the solution to the mass eigenvalue problems in two-charge sectors.

XI. THREE-CHARGE SECTOR

The construction of the dynamics in the three-charge sector is similar to the three-particle dynamics in the fixed N case.

The construction starts with the mass operators in the barred representation. The reason for introducing the barred operators, which violate cluster properties, is that they commute with and are independent of the total three-charge momentum and helicity of the *non-interacting* system. In addition, they commute with the square of the spin of the non-interacting three-charge system, independent of the partition a.

Mass operators $\overline{M}_a$ for each $\overline{U}_a[\Lambda,Y]$ are constructed as discussed in the preceding section. This is done for each of the four partitions of charges,

$$a = (N\pi_1)(\pi_2),(N\pi_2)(\pi_1),(\pi_1\pi_2)(N),(N)(\pi_1)(\pi_2). \quad (158)$$

These operators are easily expressed in terms of their kernels in free three-charge irreducible bases on $\mathcal{H}_a$. Note that even though $U_a[\Lambda,Y]$ was extended to all of $\mathcal{H}$, the generators and mass operators are nonvanishing only on the subspace $\mathcal{H}_a$.

A mass operator on $\mathcal{H}$ is defined by the linear combination of operators using the Möbius function of the lattice of partitions

$$\overline{M} := \sum_{a\in\mathcal{P}}{}' C_a \overline{M}_a + \overline{V}, \quad (159)$$

$$C_a := -\mu_{1\supset a} = -\delta^{-1}_{1\supset a} = -(-1)^{n_a}(n_a-1)!,$$

where the sum is over all partitions with at least two disjoint clusters of charge. For two-cluster partitions a of charges, like $a=(N\pi_1)(\pi_2)$, the relation to the mass operator of the two-charge $N\pi$ system, $\overline{M}_{N\pi}$, is of the general form (150). The combinatorial coefficients ensure the each two-charge interaction appears only once. The three-charge kinetic energy on $\mathcal{H}_{(N)(\pi_1\pi_2)}$ appears once for each of the three two-cluster partitions and is subtracted twice in the three-cluster partition. This ensures that it appears once in the final expression for the mass. The Möbius function is defined so that this property is preserved for any number of charges and any type of interaction.

The operator $\overline{V}$ is an analog of a three-body interaction. It vanishes when any pair of charges is separated. In addition, it commutes with the noninteracting three-body j^2 and commutes with and is independent of the noninteracting three-body $\vec{p}$ and h. In this example, $\overline{V}$ includes the following types of interactions.

(1) Three-body interaction on $\mathcal{H}_N \otimes \mathcal{H}_{\pi_1} \otimes \mathcal{H}_{\pi_2}$.

(2) ρ-N interactions on $\mathcal{H}_N \otimes \mathcal{H}_\rho$.

(3) Δ-π interactions on $\mathcal{H}_{\Delta_i} \otimes \mathcal{H}_{\pi_j}$ for $i \neq j$.

(4) Connected interactions that couple different subspaces in the direct sum, such as ρ-$N \leftrightarrow \pi_2$-Δ_1 interactions, etc.

Since each of the operators $\overline{M}_a$, and $\overline{V}$ commute with j_0^2 for the noninteracting three-charge system and commute with and are independent of the linear momentum $\vec{p}$ and helicity h, for the noninteracting three-charge system, the combined operator $\overline{M}$ also has this property. It follows that simultaneous eigenstates of $\overline{M}$, j^2, $\vec{p}$, and h transform as mass m, spin j irreducible representations of $ISL(2,C)$.

The simultaneous eigenstates of $\overline{M}$, $\vec{p}_0$, j_0^2, and h_0,

$$|\vec{p},\lambda;j,\overline{m}\rangle \tag{160}$$

generally have components in all of the cluster subspaces of $\mathcal{H}$. A dynamical representation $\overline{U}[\Lambda,Y]$ of $ISL(2,C)$ is given by

$$\overline{U}[\Lambda,Y]|\vec{p},\lambda;j,\overline{m}\rangle = \sum_{\lambda'=-j}^{j} |\vec{p}\,',\lambda';\overline{m},j\rangle \times \sqrt{\frac{\omega_{\overline{m}}(p')}{\omega_{\overline{m}}(p)}} D^j_{\lambda'\lambda}[R_H(\Lambda,p)] e^{ip'\cdot y}. \tag{161}$$

As in the three-particle case, the eigenstates of $\overline{M}$ are obtained by solving generalized Faddeev equations. The scattering solutions must be solved with the appropriate asymptotic condition. The two-charge interactions in the three-charge Hilbert space have the form

$$\overline{V}_a := \sqrt{q_a^2+(m_{120}+\overline{v})^2} - \sqrt{q_a^2+m_{120}^2} \tag{162}$$

and the mass operator (159) has the form

$$\overline{M} = \overline{V}_{(N\pi_1)(\pi(2)} + \overline{V}_{(N\pi_2)(\pi_1)} + \overline{V}_{(\pi_1\pi_2)(N)} + \overline{M}_{(N)(\pi_1)(\pi_2)} + \overline{V}. \tag{163}$$

The Faddeev equations have the same form as the corresponding nonrelativistic equations in terms of the internal kinetic energy, the interactions, and three-charge forces. The form of the eigenstates in a noninteracting irreducible basis is

$$\langle \vec{p}_0,\lambda_0;j_0\cdots|\overline{\vec{p},\lambda;m,j}\rangle = \delta(\vec{p}_0-\vec{p})\,\delta_{\lambda_0\lambda}\delta_{j_0 j}\langle\cdots|\overline{m}\rangle. \tag{164}$$

The Faddeev equations (in the absence of $\overline{V}$) have the form

$$|\overline{m}\rangle = \sum_a |\overline{m;a}\rangle, \tag{165}$$

$$|\overline{m;a}\rangle = \frac{1}{\lambda-\overline{M}_a}\overline{V}_a \sum_{b\neq a} |\overline{m;b}\rangle, \tag{166}$$

where the indices a,b correspond to two-cluster partitions.

These equations must be solved in a fixed representation. The representations that are natural for the different partitions differ by the choice of degeneracy parameters, which are dictated by the spectator charge. To diagonalize this operator, the individual mass operators need to be put in a common representation. This is done using the Racah coefficients of $ISL(2,C)$ which can be computed using four $ISL(2,C)$, Clebsch-Gordan coefficients in the same manner as they are used to compute SU(2) Racah coefficients (Refs. [3,5,18]). The Racah coefficients do not depend on $\vec{p}$ or h, they only act on the "$\cdots$" in $\langle\cdots|$ in Eq. (164) above. Faddeev equations with interactions of the general form (162) have been solved numerically for realistic interactions [13].

In representation (161), all of the interactions are in $\overline{M}$. In the limit that a given interaction is simply turned off, we have

$$\overline{M}\to\overline{M}_a = A_a^\dagger M_a A_a, \tag{167}$$

which is related to the mass operator of the desired tensor product representation by the scattering equivalence A_a. Interactions in the operators which have domain or range on $\mathcal{H}^a$ are set to zero.

In order to recover the desired tensor product representation, it is enough to construct an operator A with the property that $A\to A_a$ in the limit that the charges in different clusters of a are asymptotically separated. This can be done following Refs. [3,5], which use Cayley transforms:

$$\alpha_a := i\frac{I-A_a}{I+A_1}, \tag{168}$$

$$\alpha := \alpha_{(N\pi_1)(\pi_2)} + \alpha_{(N\pi_2)(\pi_1)} + \alpha_{(N)(\pi_1\pi_2)}, \tag{169}$$

$$A = \frac{I+i\alpha}{1-i\alpha}. \tag{170}$$

The operator A has the desired algebraic cluster property; which follows because each of the operators α_a vanishes in the limit that charges in the same cluster of a are separated. The individual operators A and α_a can be obtained by solving nonsingular integral equations,

$$\alpha_a = i\frac{I-A_a}{2} + \frac{I-A_a}{2}\alpha_a. \tag{171}$$

The operator A can be obtained by solving the integral equation

$$B_a = \frac{\alpha_a}{I-i\alpha_a}(I+i\alpha) + i\frac{\alpha_a}{I-i\alpha_a}\sum_{b\neq a} B_b, \tag{172}$$

$$A = (I+i\alpha) + i\sum_b B_a. \tag{173}$$

In the case that the α_a's are bounded operators the resulting solution is in the C^* algebra of asymptotic constants, which means that A is a scattering equivalence. While the boundedness of the α_a has not been established in general, this property is strongly suggested by the structure of the expansion of the operators A_a in the N-particle case [2].

The operator

$$U[\Lambda,Y] := A\overline{U}[\Lambda,Y]A^{\dagger} \tag{174}$$

defines the desired solution of the (2,1) charge sector of this model. The dynamics is scattering equivalent to the $\overline{U}[\Lambda,Y]$ dynamics and has the property that when the interactions between charges in different clusters of a partition A are turned off, the result is the tensor product of subsystem representations (on $\mathcal{H}_a$). The effect of the operators A is to introduce nontrivial three-charge interactions into the theory. These interactions will not affect the spectrum or cross sections in the three-charge problem, but they are important contributions when the three-charge dynamics is used as input to a many-charge problem. For example, they generate important exchange currents in electron scattering off of this three-charge system, and these interactions are also needed to embed this model in a four-charge sector. Unlike the interactions $\overline{V}$, the three-charge operators generated by cluster properties are not optional.

XII. CONCLUSION

In this paper it was shown how to extend the construction of Ref. [5] to formulate a class of relativistic theories with cluster properties, a spectral condition, that allow particle production. Particle production requires modifications to the general construction discussed in Ref. [5]. The necessary modifications were discussed in Secs. II–VII. Rather than reproving all of the results of Ref. [5] using this modified framework, the general construction was illustrated using a nontrivial example in Secs. IX–XI.

Cluster properties in relativistic models are not commonly discussed, but constitute a very important topic for the experimental program at laboratories like TJNAF. The reason for emphasizing experiments on few-body systems at such laboratories is that one expects that what is learned from few-body experiments will constrain the structure of theories that can be applied to more complex experiments. This requires that the many-body theories cluster to the few-body theories that are used to model the few-body physics. This expectation is trivially realized in nonrelativistic quantum mechanics. When the reactions have sufficient energy to produce particles, a relativistic treatment is necessary and the realization of cluster properties becomes nontrivial.

Relativistic quantum field theory provides a formal solution to these problems, although it is difficult to find mathematically well-defined examples that have all of the properties that are dictated the physical constraints. This makes it very difficult to find *ab initio* methods to control errors in applications involving strong interactions. In addition, while cluster properties are realized elegantly, there are no few-body problems; even the simplest systems necessarily involve an infinite number of degrees of freedom.

This paper illustrates a large class of theories with all of the desired properties. The underlying assumption in these theories is that number of bare-particle degrees of freedom is bounded. This is achieved by introducing fictitious conserved charges in the theory. The assumption that these charges can take on only non-negative values and each particle has at least one positive charge limits the number of degrees of freedom. If these conditions are relaxed, the resulting theory will involve an infinite number of degrees of freedom.

The theories constructed in this paper have meaningful few-body problems. In the three-charge model it was shown that the two-charge models determine the interactions in the three-charge sector up to a three-charge interaction. In general, K-charge interactions in the N-charge problem are determined from the K-charge problem using cluster properties. Like the fixed particle number case, cluster properties introduce nontrivial many-charge interactions into the dynamics. These interactions are determined recursively by the fewer-charge interactions in the absence of an explicit N-charge interaction like $\overline{V}$. In general, just like with the fixed number of particles case, the few- and many-charge interactions mix under change of representation. The construction in this paper can be used to formulate relativistic isobar models and models with a dynamics dominated by resonances.

It is desirable to go beyond the restrictions imposed by charge conservation. The requirement of having a meaningful few-body problem puts strong constraints on how cluster properties should be implemented in the general theory. One way to control the number of degrees of freedom and have a meaningful few-body problem is to reformulate the theory so the relevant degrees of freedom are physical-particle degrees of freedom. In this way the center of momentum energy controls the number of degrees of freedom. In this picture physical particles play the same role as minimally charged particles. The mechanics of coupling the physics on different energy scales provides an interesting challenge that needs to be addressed to extend the construction of this paper.

The models discussed in this paper are valuable precisely because they are quantum models with an exact Poincaré symmetry which also satisfies cluster properties. In the absence of a more fundamental theory, cluster properties and experiments on subsystems put strong constraints on the relativistic many-charge dynamics, which can then be used to make predictions of the theory.

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