Extended random phase approximation in a solvable model

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We propose an extended Lipkin model that can describe decay processes of particle-hole states and is still solvable. We examine several RPA-type theories using the model. We show explicitly the roles played by the self-energies of particle-hole propagators, and clarify how the extended RPA theory describes collective states better than other RPA-type theories.

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I. INTRODUCTION

The nuclear collective states has been studied extensively by the random phase approximation (RPA) [1,2] for a long time. The essential features of the RPA theory can be demonstrated by the Lipkin model [3], which was proposed by Lipkin, Meshkov, and Glick. The Lipkin model still maintains its importance in many fields as a solvable but nontrivial model. The model has enjoyed several extensions, e.g., to incorporate two types of fermions [4] or to treat three energy levels [5,6]. Note that even the original version of the model is still in service, e.g., in the analysis of two phonon states [7].

There have been many attempts to generalize the RPA theory, especially to treat explicitly not only the particle-hole (ph) but also the two-particle-two-hole (2p2h) degrees of freedom in the play. In this work, we concentrate on the following two generalizations. The first one is the Second RPA (SRPA) theory, which takes into account the mixing of the 2p2h states in the ph components [8,9]. The second one is the extended RPA (ERPA) theory, which handles the correlations in the ground and the excited states in a consistent way [10–13].

There have been some efforts to compare these extended versions of the RPA-type theories in real nuclei [14]. It is, however, obviously desirable to have a solvable model at hand to discuss such theories, as we have the Lipkin model for the RPA. In this paper we propose a new solvable model, which is suitable to discuss these improved versions of the RPA. The model is a natural extension of the Lipkin model, which takes into account the decay processes of the ph states into 2p2h states, and is still solvable as the original Lipkin model.

The plan of this paper is the following. In Sec. II, we introduce the solvable model (extended Lipkin model). In the construction of the model, we pay special attention to the Hartree-Fock condition. In Sec. III, we briefly review the ERPA theory in the context of the extended Lipkin model. We first give the ERPA eigenvalue problem explicitly. Then the ph response function in the ERPA is presented in the field theoretical method. In Sec. IV, we examine the excitation energies of the collective states in the RPA, SRPA, and ERPA in the extended Lipkin model, putting emphasis on the roles of self-energies. In Sec. V, we investigate the transition strengths. Here the important role of the energy-weighted sum rule is stressed. We shall see how these physical quantities are well described by the ERPA theory, compared with other RPA-type theories. Finally in Sec. VI we give a summary.

II. SOLVABLE MODELS

In this section we first review the original Lipkin model briefly to fix the notation, and then explain how to extend the model.

A. Lipkin model

The Lipkin model describes an N-fermion system with two N-fold degenerate levels shown in Fig. 1. The upper and the lower levels are indicated by $\sigma=+1$ and $\sigma=-1$, respectively, and the N degenerate levels are distinguished by an additional quantum number $p=1,2,\ldots,N$. The model is described by the following Hamiltonian:

$$H_L=\varepsilon J_z+\tfrac{1}{2}V(J_+^2+J_-^2), \tag{1}$$

where the second-quantized operators

$$J_z=\frac{1}{2}\sum_{p\sigma}\sigma a_{p\sigma}^{\dagger}a_{p\sigma}, \tag{2}$$

$$J_+=\sum_p a_{p+1}^{\dagger}a_{p-1},\quad J_-=\sum_p a_{p-1}^{\dagger}a_{p+1}, \tag{3}$$

satisfy the familiar $SU(2)$ algebra. We have N particles in the model system. Then the $SU(2)$ symmetry of the model shows that the natural basis states are given by the eigenstates of the *angular momentum* as $|J,J_z\rangle$.

Let us first consider the true ground state $|0\rangle$ of the system. If there were not the two-body interaction ($V=0$), then the ground state $|0\rangle$ would be the state $|N/2,-N/2\rangle$ with all the N particles being in the lower level $\sigma=-1$. Let us assume that the true ground state $|0\rangle$ can be generated adiabatically from $|N/2,-N/2\rangle$ by the two-body interaction in

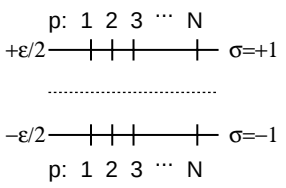

FIG. 1. Graphical representation of the Lipkin model.

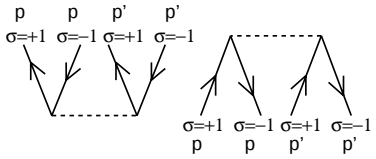

FIG. 2. V interaction in the Lipkin model.

the model, which are shown diagrammatically in Fig. 2. Then the Hamiltonian guarantees that the true ground state $|0\rangle$ of the system belongs to the $J=N/2$ multiplet as the unperturbed ground state $|N/2,-N/2\rangle$.

Next let us come to the excited states of the system. In the study of the excitations, we take J_+ and J_- as the external excitation operators. Therefore, we can limit ourselves to the $J=N/2$ multiplet throughout this paper.

Note that the Lipkin model can be generalized to include an interaction operator that is proportional to J_+J_-. We do not consider such a generalization here because the main effect of the term can be absorbed in the definition of the ph energy ε.

B. Extended Lipkin model

Even though the RPA has been successful as a first step in the description of nuclear collective states, it cannot account for the decay processes that have been drawing much attention. These decay processes are generated by the couplings of the ph to the 2p2h degrees of freedom, and lead to a width and an energy shift of a collective state.

In order to simulate such processes, we extend the Lipkin model so that it allows the interaction between ph↔pp and ph↔hh, which are shown in Fig. 3. Such an interaction can be generated keeping the $SU(2)$ symmetry of the Lipkin model as follows. Let us first consider the following two-body operator

$$J_zJ_+=\frac{1}{2}\sum_p\ (a^\dagger_{p+1}a_{p+1}-a^\dagger_{p-1}a_{p-1})\sum_k\ a^\dagger_{k+1}a_{k-1}, \tag{4}$$

which apparently represents the ph→pp and hh→ph transitions. We cannot adopt, however, the above operator as it is as the new interaction. A two-body interaction of this form necessarily violates the Hartree-Fock condition, i.e., the ground state $|N/2,-N/2\rangle$ of the noninteracting system is unstable with respect to the creation of ph pairs shown in Fig. 4. Then $|N/2,-N/2\rangle$ is not any more the Hartree-Fock ground state, and cannot be used as a first approximation to the real ground state [15].

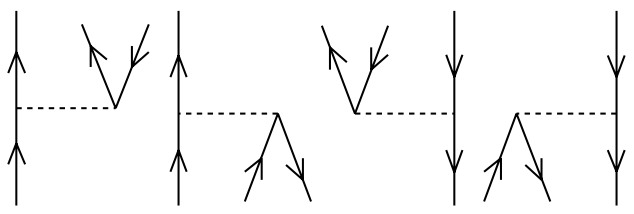

FIG. 3. Two-body interaction representing four types of transitions ph↔pp and ph↔hh. They are realized as the U interaction in the extended Lipkin model.

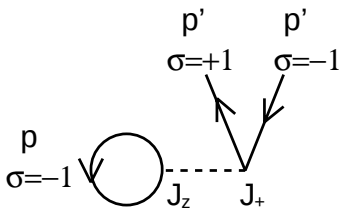

FIG. 4. Diagrammatic representation of the violation of the Hartree-Fock condition by a two-body interaction J_zJ_+. Summation over the quantum number p is assumed. Exchange process is not drawn explicitly.

It is, however, possible to cancel the process in Fig. 4 by replacing the above operator J_zJ_+ by its normal ordered counterpart $:J_zJ_+:$. Then it is clear that the state $|N/2,-N/2\rangle$ remains to be the Hartree-Fock ground state. The same discussion obviously applies to J_zJ_-, J_+J_z, and J_-J_z. We adopt, therefore, the following operator as an additional interaction:

$$:J_z(J_++J_-)+(J_++J_-)J_z:$$
$$=(N-1)(J_++J_-)+J_z(J_++J_-)+(J_++J_-)J_z,$$

which generates the four processes in Fig. 3, but does not create the ph states as in Fig. 4. After the above consideration, we arrive at the extended Lipkin model, which is described by the following Hamiltonian:

$$H=H_0+H_1=\varepsilon J_z+\frac{1}{2}V(J_+^2+J_-^2)+\frac{N-1}{2}U(J_++J_-)$$
$$+\frac{1}{2}U[J_z(J_++J_-)+(J_++J_-)J_z], \tag{5}$$

where the strength of the new interaction is denoted by U. It can be easily shown that the third term guarantees the Hartree-Fock condition

$$\langle N/2,-N/2|HJ_+|N/2,-N/2\rangle=0, \tag{6}$$

for $|N/2,-N/2\rangle$. We shall refer to $H_0=\varepsilon J_z$ as the unperturbed Hamiltonian and to the rest as the residual interaction H_1.

At the end of this subsection, we summarize as follows the extended Lipkin model described by the Hamiltonian H of Eq. (5). First, the Hamiltonian maintains the $SU(2)$ symmetry of the Lipkin model, and therefore it can be solved as easily as the Lipkin model. Second, the Hamiltonian can describe decay processes of ph states. Third, $|N/2,-N/2\rangle$ is the Hartree-Fock ground state.

C. Other extensions

Here we discuss other extensions [16,17] of the Lipkin model to include the interaction between ph↔pp and ph↔hh. In these works, they investigated the systems with new interaction terms that are similar to ours. The essential

difference between these works and the present one is in the treatment of the Hartree-Fock condition, as explained below. To be concrete, let us look at the following interaction term used in Ref. [17]:

$$\frac{1}{2}u\sum_{pk\sigma}(a^{\dagger}_{p\sigma}a^{\dagger}_{k\sigma}a_{k-\sigma}a_{p\sigma}+a^{\dagger}_{p\sigma}a^{\dagger}_{k-\sigma}a_{k\sigma}a_{p\sigma})$$

$$=\frac{1}{2}u(\hat{n}-1)(J_{+}+J_{-}), \tag{7}$$

where $\hat{n}=\Sigma_{p,\sigma}a^{\dagger}_{p\sigma}a_{p\sigma}$ is the number operator. It can be easily seen that, by replacing $\hat{n}$ by J_z, Eq. (7) reduces to the U interaction in the preceding subsection, but without the normal ordering operation. Note that the following discussions apply also to the interaction introduced in Ref. [16].

It is easily seen that the new interaction of Eq. (7) violates the Hartree-Fock condition in the same way as explained in the preceding subsection. There are evidently two approaches to restore the Hartree-Fock condition; one is the method taken in Refs. [16,17], and the other is the one that led to Eq. (5). We now compare these two approaches.

In the first approach [16,17], they look for the Hartree-Fock solution of the given Hamiltonian by a unitary transformation of the single particle basis functions, so that the Hamiltonian satisfies the Hartree-Fock condition in terms of the new single particle states. The transformed Hamiltonian obviously contains the interaction between ph↔pp and ph↔hh in the new basis functions. However, the new coupling strengths are complicated functions of the original ones and the number of particles, and are not under control any more. This approach is excellent to discuss, for example, the ground state properties of a given Hamiltonian. It is, however, obviously inappropriate to examine the system properties with changing the coupling strength between ph↔pp and ph↔hh for fixed Hartree-Fock single particle states, which is our purpose.

The second approach adopted in the preceding subsection can be put in contrast with the first one as follows. In the present work, we want to keep the Hartree-Fock single particle states to be independent of the coupling strength, which we are going to vary as an independent variable at hand. We change, therefore, the Hamiltonian by introducing the one-body field, so that the original basis functions satisfy the Hartree-Fock condition for an arbitrary coupling strength. This corresponds to taking the normal ordered form of the interaction, as explained in the preceding subsection.

At the end, we make a comment on the interaction of Eq. (7), which one might think that we could have adopted. It can be shown, however, that the normal ordered product of Eq. (7) reduces to

$$:\tfrac{1}{2}u(\hat{n}-1)(J_{+}+J_{-}):=\tfrac{1}{2}u(\hat{n}-N)(J_{+}+J_{-}), \tag{8}$$

which obviously leads to a trivial model that does not introduce new physical effects, because $\hat{n}-N$ vanishes identically for the system with a fixed number of particles, N. In other words, the two-body interaction of Eq. (7) presents a nontrivial and interesting problem for a fixed Hamiltonian in the first approach [16,17], but does not introduce anything new to the original Lipkin model in the second approach. This is the reason why we cannot adopt the interaction of Eq. (7) in the present work.

III. EXTENDED RPA

In this section, we explain the ERPA in terms of the extended Lipkin model of the preceding section. We present the ERPA first as an eigenvalue problem and second as a theoretical framework to calculate the response function using the diagrammatical method [10].

A. Eigenvalue problem

Let us start with a general N-fermion system, which is described by a Hamiltonian H. The eigenstates of H are given by $\{|\nu\rangle,\nu=0,1,2,\ldots\}$, which satisfy

$$H|\nu\rangle=E_{\nu}|\nu\rangle. \tag{9}$$

We define an operator $Q^{\dagger}_{\nu}$ that generates the νth excited state $|\nu\rangle$ from the ground state $|0\rangle$ as

$$Q^{\dagger}_{\nu}|0\rangle=|\nu\rangle,\quad Q_{\nu}|0\rangle=0,\quad \nu=1,2,\ldots\ . \tag{10}$$

Then it is straightforward to show that $Q^{\dagger}_{\nu}$ satisfies the following variational equation for an arbitrary δQ [1,2]:

$$\langle 0|[\delta Q,[H,Q^{\dagger}_{\nu}]]|0\rangle=(E_{\nu}-E_{0})\langle 0|[\delta Q,Q^{\dagger}_{\nu}]|0\rangle. \tag{11}$$

Now we take the specific case of the extended Lipkin model described by the Hamiltonian H of Eq. (5). As the first step in the ERPA, we express $Q^{\dagger}_{\nu}$ in terms of the creation and the annihilation operators of ph and 2p2h states. In our case, we need to consider only four operators $\{J_{+},J_{+}J_{+},J_{-},J_{-}J_{-}\}$ because of the symmetry of the model. Then we expand $Q^{\dagger}_{\nu}$ as

$$Q^{\dagger}_{\nu}=x^{(\nu)}_{1}J_{+}-y^{(\nu)}_{1}J_{-}+x^{(\nu)}_{2}J_{+}J_{+}-y^{(\nu)}_{2}J_{-}J_{-}\ . \tag{12}$$

After a straightforward calculation of Eq. (11), we can show that the coefficients in Eq. (12) satisfy the following eigenvalue equation:

$$\begin{pmatrix} A_{11} & B_{11} & A_{12} & \\ B_{11} & A_{11} & & A_{12} \\ A_{12} & & A_{22} & \\ & A_{12} & & A_{22} \end{pmatrix}\begin{pmatrix} x^{(\nu)}_{1} \\ y^{(\nu)}_{1} \\ x^{(\nu)}_{2} \\ y^{(\nu)}_{2} \end{pmatrix}$$

$$=(E_{\nu}-E_{0})\begin{pmatrix} S_{11} & & & \\ & -S_{11} & & \\ & & S_{22} & \\ & & & -S_{22} \end{pmatrix}$$

$$\times\begin{pmatrix} x^{(\nu)}_{1} \\ y^{(\nu)}_{1} \\ x^{(\nu)}_{2} \\ y^{(\nu)}_{2} \end{pmatrix}, \tag{13}$$

where the matrix elements are given by

$$A_{11}=\langle 0|[J_-,[H,J_+]]|0\rangle \simeq N\varepsilon\left[1+\frac{1}{2}\left(\frac{V}{\varepsilon}\right)^2(N-1)\right],$$

$$A_{12}=\langle 0|[J_-,[H,J_+J_+]]|0\rangle \simeq 2N(N-1)U,$$

$$A_{22}=\langle 0|[J_-J_-,[H,J_+J_+]]|0\rangle \simeq 4N(N-1)\varepsilon,$$

$$B_{11}=-\langle 0|[J_-,[H,J_-]]|0\rangle \simeq N(N-1)V,$$

$$S_{11}=\langle 0|[J_-,J_+]|0\rangle \simeq N-\frac{1}{2}\left(\frac{V}{\varepsilon}\right)^2 N(N-1),$$

$$S_{22}=\langle 0|[J_-J_-,J_+J_+]|0\rangle \simeq 2N(N-1). \tag{14}$$

The second step of the ERPA is in the above evaluation of the commutators [10]. A_{11}, B_{11}, and S_{11} are calculated exactly up to the second order in powers of the residual interaction H_1. In the same way we calculate A_{12} up to the first order, and A_{22} and S_{22} up to the zeroth order. This procedure can be shown to give the response function of the system for the external one-body operators $\{J_+,J_-\}$ exactly up to the second order in the residual interaction. Note that the second order term is missing in the expression for B_{11}, because of the special form of the model interaction. This can be understood most easily by noticing the fact that there is no way to construct the second order processes of B_{11} diagrammatically [10] with the interaction terms in Eq. (5).

Here we can briefly mention the derivation of the SRPA theory. If we had evaluated the above commutators by replacing the real ground state $|0\rangle$ by the Hartree-Fock ground state $|N/2,-N/2\rangle$, we would have arrived at the SRPA eigenvalue equation [8]. It is written in the same form as Eqs. (13) and (14), but with the replacement $A_{11}\rightarrow N\varepsilon$, $S_{11}\rightarrow N$ in Eq. (14). It should be noted here that the SRPA was the first attempt toward the microscopic description of decay processes of collective states.

As in the usual RPA theory, the solutions of the above eigenvalue problem of Eq. (13) appear pairwise, i.e., a positive energy solution with $E_\nu>0$ is accompanied by a negative energy solution with $E_{-\nu}=-E_\nu<0$. In our case of the extended Lipkin model, there are only two positive energy solutions with $\nu=1,2$.

It is convenient to normalize the eigenvectors in Eq. (13) as

$$S_{11}(x_1^{(\nu)2}-y_1^{(\nu)2})+S_{22}(x_2^{(\nu)2}-y_2^{(\nu)2})=\mathrm{sgn}(\nu). \tag{15}$$

Then it can be shown that the transition amplitudes for the operators J_+ and J_- are given by

$$\begin{pmatrix}\tilde{x}_1^{(\nu)}\\ \tilde{y}_1^{(\nu)}\end{pmatrix}=\begin{pmatrix}S_{11}x_1^{(\nu)}\\ S_{11}y_1^{(\nu)}\end{pmatrix}=\begin{pmatrix}\langle 0|J_-|\nu\rangle\\ \langle 0|J_+|\nu\rangle\end{pmatrix}. \tag{16}$$

B. ERPA response function

Here we explain the response function for the external operators J_+ and J_- in the ERPA. A general theory to construct the response function from the RPA-type eigenvalue equation is given in Ref. [10].

We define the response function $D(t)$ as a 2×2 matrix using the T-product (time-ordered product) as

$$D(t)=\frac{1}{i}\begin{pmatrix}\langle 0|T[J_-(t),J_+]|0\rangle & \langle 0|T[J_-(t),J_-]|0\rangle\\ \langle 0|T[J_+(t),J_+]|0\rangle & \langle 0|T[J_+(t),J_-]|0\rangle\end{pmatrix}. \tag{17}$$

Then its Fourier transform in the ERPA is given by

$$D(\omega)=\frac{N}{\begin{pmatrix}\omega-\varepsilon & \\ & -\omega-\varepsilon\end{pmatrix}-\begin{pmatrix}\Sigma(\omega) & (N-1)V\\ (N-1)V & \Sigma(-\omega)\end{pmatrix}}, \tag{18}$$

where the ph self-energy $\Sigma(\omega)$ stands for all the second order processes, and shall soon be explained below. The above matrix form of $D(\omega)$ can obviously be expressed using the diagrammatical method in the perturbation expansion, which allows an intuitive understanding of the dynamical processes. Note that the above expression for $D(\omega)$ may be derived directly also from Eq. (13) using the general theory [10], first by deriving the 4×4 full response function matrix for the operator set $\{J_+,J_+J_+,J_-,J_-J_-\}$, and then by taking the 2×2 submatrix corresponding to $\{J_+,J_-\}$.

It is convenient to express the ph self-energy as

$$\Sigma(\omega)=\Sigma_{2p2h}(\omega)+\Sigma_{3p3h}(\omega), \tag{19}$$

where $\Sigma_{2p2h}(\omega)$ and $\Sigma_{3p3h}(\omega)$ represent the self-energies with 2p2h and 3p3h intermediate states, respectively, and are given by

$$\Sigma_{2p2h}(\omega)=H_1\frac{1}{\omega-H_0}H_1=2(N-1)U^2\frac{1}{\omega-2\varepsilon}, \tag{20}$$

$$\Sigma_{3p3h}(\omega)=-\frac{1}{-H_0}H_1(\omega-H_0)H_1\frac{1}{-H_0}$$
$$=-\frac{N-1}{2}\left(\frac{V}{\varepsilon}\right)^2(\omega-3\varepsilon). \tag{21}$$

The above self-energies are shown in Figs. 5 and 6, and the middle expressions of Eqs. (20) and (21) represent the corresponding processes in a symbolic way.

Here the difference between the ERPA and the SRPA can be stated most clearly in terms of the response function; the self-energy of the ERPA response is given by Eq. (19), while the SRPA takes into account only $\Sigma_{2p2h}(\omega)$ and neglects $\Sigma_{3p3h}(\omega)$. In the following sections, we shall examine in detail what roles are played by these two self-energies.

By definition, the response function $D(\omega)$ can be written using the transition amplitudes $\tilde{x}_1^{(\nu)}$ and $\tilde{y}_1^{(\nu)}$ of Eq. (16) as follows:

$$D(\omega)=\sum_\nu \frac{\mathrm{sgn}(\nu)}{\omega-E_\nu-i\eta_\nu}\begin{pmatrix}\tilde{x}_1^{(\nu)}\\ \tilde{y}_1^{(\nu)}\end{pmatrix}(\tilde{x}_1^{(\nu)},\tilde{y}_1^{(\nu)}), \tag{22}$$

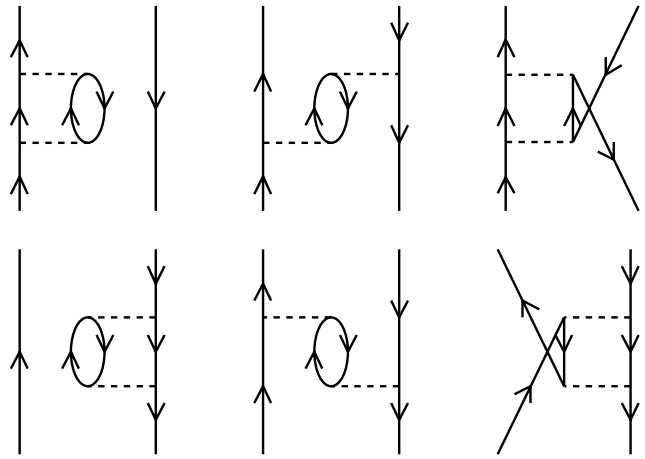

FIG. 5. Self-enegy $\Sigma_{2p2h}(\omega)$ with 2p2h intermediate states. These processes express the couplings of the ph to the 2p2h states, which can be referred to as the correlation in the excited (ph) states. All the interaction vertices are assumed to be antisymmetrized.

where the infinitesimal imaginary part $i\eta_\nu = i\eta\ \mathrm{sgn}(\nu)$ is written explicitly. By comparing Eqs. (18) and (22), it can be shown that the transition amplitudes satisfy the following normalization [10]:

$$(\tilde{x}_1^{(\nu)}, \tilde{y}_1^{(\nu)}) M_\nu \begin{pmatrix} \tilde{x}_1^{(\nu)} \\ \tilde{y}_1^{(\nu)} \end{pmatrix} = \mathrm{sgn}(\nu) N, \tag{23}$$

where

$$M_\nu = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - \frac{\partial}{\partial \omega} \begin{pmatrix} \Sigma(\omega) & \\ & \Sigma(-\omega) \end{pmatrix} \Bigg|_{E_\nu}. \tag{24}$$

Note that the unusual factor N in the right-hand side (rhs) of Eq. (23) originates from the normalization of the operators in the expansion of Eq. (12); J_+, for example, is not so normalized that the norm of $J_+ |N/2, -N/2\rangle$ is unity. We also mention here that a different expression for the normalization of the transition amplitudes is discussed in Ref. [18].

By taking the imaginary part of $D(\omega)$ of Eq. (22) for $\omega > 0$, we can obtain the spectrum of the transition strengths of J_+ and J_- as

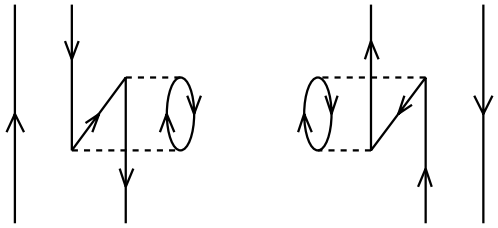

FIG. 6. Self-enegy $\Sigma_{3p3h}(\omega)$ with 3p3h intermediate states. These processes express the effects of the ground state correlation on the ph states through the Pauli principle. All the interaction vertices are assumed to be antisymmetrized.

$$-\frac{1}{\pi} \mathrm{Im}\, D(\omega) = \sum_{\nu > 0} \delta(\omega - E_\nu) \begin{pmatrix} \tilde{x}_1^{(\nu)} \\ \tilde{y}_1^{(\nu)} \end{pmatrix} (\tilde{x}_1^{(\nu)}, \tilde{y}_1^{(\nu)}), \tag{25}$$

which can be compared directly with *experiments*.

IV. EXCITATION ENERGY

In this section, we calculate the excitation energy of the first excited state (collective state) of the extended Lipkin model in the following RPA-type theories.

(1) ERPA: the response function is given by Eq. (18).

(2) SRPA: the response function is obtained from the ERPA response by neglecting $\Sigma_{3p3h}(\omega)$.

(3) RPA: the response function is obtained from the ERPA response by neglecting the whole self-energy $\Sigma(\omega)$.

(4) Ring approximation: the response function is obtained from the RPA response by neglecting the exchange processes, i.e., by the replacement $(N-1)V \to NV$. Response diagrams are composed of simple rings that are connected by the direct processes of the ph interaction V.

The exact eigenenergies $E_0, E_1, \ldots$ can be calculated by diagonalizing the $(N+1)\times(N+1)$ Hamiltonian matrix for the $J = N/2$ multiplet. In the RPA-type theories, excitation energies are obtained from the positions of the poles in the complex ω plane of the response function $D(\omega)$.

We first examine the case with $V \neq 0$, $U = 0$, which reduces to the original Lipkin model. We explain the idea of $1/N$ expansion and clarify the roles of $\Sigma_{3p3h}(\omega)$. Second, we study the case with $V = 0$, $U \neq 0$ to demonstrate the roles of $\Sigma_{2p2h}(\omega)$. Finally, we investigate the general case $V \neq 0$, $U \neq 0$ of the extended Lipkin model.

A. Case 1: $V \neq 0$, $U = 0$

This case corresponds to the original Lipkin model, with which we can discuss the ERPA, RPA, and the ring approximation. Note that there is no way to make $\Sigma_{2p2h}(\omega)$ without U, and therefore the SRPA reduces to the RPA in this case. Here we first explain (i) the idea of $1/N$ expansion [3], and then (ii) the role of $\Sigma_{3p3h}(\omega)$ [12].

In Fig. 7, we show the excitation energy $E_1 - E_0$ of the first excited state as a function of the number of the particles N for $NV/\varepsilon = 0.8$, 0.6, and 0.4.

We immediately recognize in Fig. 7 that the excitation energy in the ring approximation is independent of N, and is given by [3]

$$\left(\frac{E_1 - E_0}{\varepsilon} \right)_{\mathrm{ring}} = \left[1 - \left(\frac{NV}{\varepsilon} \right)^2 \right]^{1/2}. \tag{26}$$

Note that for stronger interaction strengths $NV/\varepsilon > 1$, Eq. (26) gives imaginary excitation energies. This means that the Hartree-Fock ground state $|N/2, -N/2\rangle$ does not correspond any more to the local minimum of the energy surface, and therefore it is unstable [1,2] in the ring approximation. In such very strong coupling regimes, it is possible to discuss a phase transition from a *spherical* to a *deformed* ground state

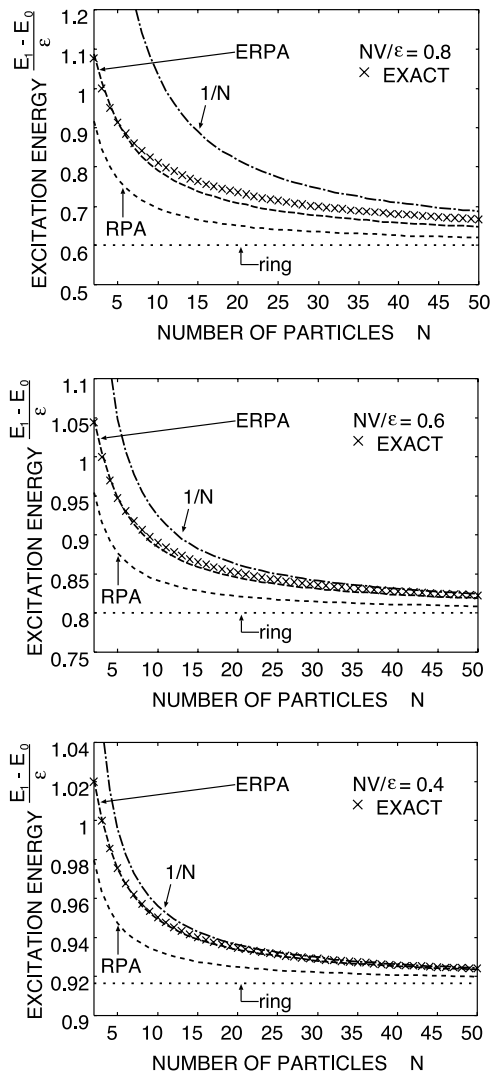

FIG. 7. Excitation energy of the first excited state in units of ε in several theories for $V\neq 0$, $U=0$. Exact values are denoted by crosses. The interaction strength is given by $NV/\varepsilon=0.8$ (top), $NV/\varepsilon=0.6$ (middle), $NV/\varepsilon=0.4$ (bottom).

[2,19]. We shall remain, however, in the regime $NV/\varepsilon<1$ in this paper to discuss the collective states in *spherical* systems.

Let us take into account the exchange processes of all the ph interaction in the ring approximation. This procedure amounts to replacing NV by $(N-1)V$ in Eq. (26), leading to the following RPA result:

$$\left(\frac{E_1-E_0}{\varepsilon}\right)_{\mathrm{RPA}}=\left[1-\left(\frac{(N-1)V}{\varepsilon}\right)^2\right]^{1/2}. \tag{27}$$

Figure 7 clearly shows that the RPA is much better than the ring approximation, especially in the small N region.

Now we explain the idea of the $1/N$ expansion [3]. It is visible in Fig. 7 that the ring approximation reproduces the exact results in the large N limit. Let us think of a Goldstone diagram that represents a response process to an external operator J_+. Because each fermion loop gives a factor N due to the trace over the quantum number p, the ring approximation collects those terms with the most factors of N at each order in powers of the residual interaction V. This explains why the ring approximation is exact in the large N limit. The above idea is completed as the $1/N$ expansion of the response function, which can be stated most easily using the language of the diagrammatical method; the leading term of the expansion is given by the ring approximation, in which the nth order processes (in powers of the residual interaction V) of the response are composed of the diagrams with $n+1$ fermion loops. The second term of the expansion is the sum of all the nth order processes with n fermion loops, which obviously gives a correction of order $1/N$ to the ring approximation. In the same way, the third term is the collection of the diagrams with $n-1$ fermion loops. This procedure can be continued to give an expansion of the response function in powers of $1/N$. Then it is straightforward to derive an expansion of the excitation energy in powers of $1/N$.

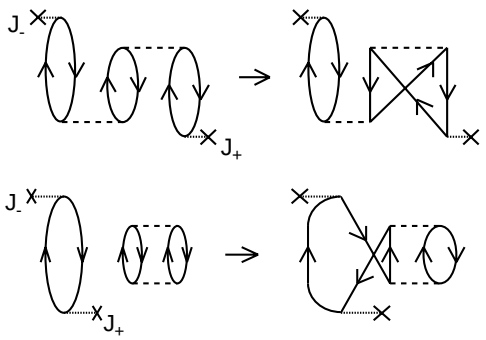

FIG. 8. Diagrammatic explanation of proceeding to higher orders in the $1/N$ expansion of the response function for the excitation operator J_+. The figure shows how to make the second order ($n=2$) diagrams with two ($=n$) fermion loops. Top: replacing a direct ph interaction by an exchange interaction in the left process reduces the number of the fermion loops from $n+1=3$ to $n=2$. Bottom: exchange process of the free process with the ground state correlation diagram. This procedure gives the right diagram with two fermion loops for $n=2$. Note that the unlinked part (ground state correlation) of the left diagram does not appear in the expression for the response function, because it is canceled by virtue of the linked cluster theorem.

Let us briefly explain how to perform the $1/N$ expansion in the case of $n=2$. We can notice that there are two possible ways to decrease the number of the fermion loops of a given process for a given order: one is to open a fermion loop by taking the exchange diagram in the process; the other is to consider an exchange process with a ground state correlation diagram, which has been canceled away by the linked cluster theorem [20]. This situation is shown in Fig. 8 for the second order ($n=2$) processes.

In Fig. 7, we have shown as "$1/N$" the sum of the first (ring approximation) and the second (leading correction) term of the $1/N$ expansion of the excitation energy of the first excited state. The figure shows clearly (i) that the leading correction reproduces the exact results very well for large N, and also (ii) that the convergence of the expansion is slow in the small N region.

Now we compare the RPA and the $1/N$ expansion. We can

see in Fig. 7 that the line denoted by "$1/N$" approximates the exact results to a better extent with decreasing interaction strength NV/ε. On the other hand, the RPA improves only slowly with decreasing NV/ε. Consequently, the RPA is better (for $N \lesssim 20$) in the strong coupling region ($NV/\varepsilon = 0.8$), while the $1/N$ expansion is better in the weak coupling region ($NV/\varepsilon = 0.4$). This can be understood in the following way. Let us expand the excitation energy in powers of NV/ε and $1/N$ as

$$\begin{aligned}\frac{E_1 - E_0}{\varepsilon} &= \sum_{i,j} c_{i,j} \left(\frac{NV}{\varepsilon}\right)^i \frac{1}{N^j} \\ &= 1 + \left(\frac{NV}{\varepsilon}\right)^2 \left(-\frac{1}{2} + 2\frac{1}{N} - \frac{3}{2}\frac{1}{N^2} \right) \\ &\quad + \left(\frac{NV}{\varepsilon}\right)^4 \left(-\frac{1}{8} + \frac{5}{2}\frac{1}{N} - \frac{47}{4}\frac{1}{N^2} + \cdots \right) + \cdots . \end{aligned} \tag{28}$$

The explicit form of the above double expansion can be obtained, e.g., from the results of the fourth order perturbation theory [3]. Here the coefficient $c_{i,j}$ represents the ith order in NV/ε and the jth order in $1/N$, and the indices satisfy $i \geqslant j$. We recognize that the perturbation expansion corresponds to looking the above series from the top to the bottom, and that the $1/N$ expansion is obtained by proceeding from the left to the right. The above double expansion shows that every term of the $1/N$ expansion contains terms up to the infinite order in powers of NV/ε, while the ith order term in the perturbation expansion can be expressed with only finite powers $\{1, 1/N, \ldots, 1/N^i\}$ of $1/N$.

We notice that $c_{i,j}$ is a rapidly increasing function of j for $i \gg j$. This is essentially because the number of ways to decrease the fermion loops grows very rapidly with increasing j. Consequently, $c_{i,j}$ for large values of j can be important for large values of i, while only a few terms with small j (such as $c_{i,0}$ and $c_{i,1}$) can be meaningful for small values of i.

The line "$1/N$" in Fig. 7 collects the first and the second terms in the double expansion of Eq. (28) for each power of NV/ε. On the other hand, the RPA sums up the exchange processes to the infinite order, and therefore takes into account a part of the coefficients $c_{i,j}$ of any power $1/N^j$, but misses several important processes that contribute to $c_{i,j}$ with small j. In particular, the line "$1/N$" gives exactly $c_{2,1} = 2$, while the RPA value for $c_{2,1}$ is 1.

In the weak interaction region ($NV/\varepsilon = 0.4$) where the perturbation expansion is good, only the coefficients $c_{i,j}$ with small i are important in Eq. (28), and therefore we can limit ourselves to the first few terms in the $1/N$ expansion (note $i \gg j$). This explains why the line "$1/N$" is better than the RPA, because it gives the exact expression for the coefficients $c_{i,1}$, while the RPA expression for $c_{i,1}$ is wrong.

In the strong coupling region ($NV/\varepsilon = 0.8$), on the other hand, many coefficients $c_{i,j}$ with large i and j are important. In order to obtain a convergent result in the $1/N$ expansion in this case, we need to evaluate exactly many terms with large powers of $1/N$. Then we can realize that it is more meaningful to sum up the important exchange processes to the infinite order, rather than to collect all the corrections of order $1/N$.

Finally we come to the results of the ERPA in Fig. 7 to discuss the roles of $\Sigma_{3p3h}(\omega)$. We recognize that the ERPA reproduces exact results very well in all cases shown in the figure. Note that the difference between the ERPA and the RPA in the figure originates solely from the self-energy $\Sigma_{3p3h}(\omega)$. One can show that the role of the self-energy $\Sigma_{3p3h}(\omega)$ in the ph propagator is twofold [12]. One is to increase the excitation energy of the ph state, and the other is to reduce the amplitude of the bare ph component in the excited state. This can be understood as follows. First, the energy of the Hartree-Fock ground state is lowered by the second order processes (in powers of V) of the ground state correlation. Second, in the presence of the ph state, a part of the above processes is forbidden because of the Pauli blocking. These two effects amount to increasing the energy of the ph state measured from the ground state. This is exactly what $\Sigma_{3p3h}(\omega)$ expresses. If we look at the same effect from the ph state, we can easily understand that the bare ph amplitude should be reduced by the ground state correlation, which inhibits partly the presence of the bare ph state.

At the end, let us look at Fig. 7 from the viewpoint of improving the ring approximation. First, by taking into account the exchange processes, we obtain the RPA, which increases the excitation energy by reducing the repulsive ph interaction. Second, by including $\Sigma_{3p3h}(\omega)$, the RPA excitation energies are further increased to give the ERPA results, which almost agree with the exact values in the whole range of N. This means that most of the effects of the V interaction beyond the ring approximation can be expressed by the exchange and the 3p3h self-energy $\Sigma_{3p3h}(\omega)$.

B. Case 2: $V = 0$, $U \neq 0$

In order to examine the roles played by the self-energy $\Sigma_{2p2h}(\omega)$, we calculate the excitation energy of the lowest excited state for $V = 0$, $U \neq 0$. In this case, the ring approximation reduces to the free response, and the ERPA and the SRPA coincide, because there is no ph interactions.

The numerical results are given in Fig. 9 for $NU/\varepsilon = 0.6$ and 0.4. It can be seen that the ERPA reproduces the exact values very well in the whole range of N. This means that most of the effects of the U-interaction term on the response function can be represented by $\Sigma_{2p2h}(\omega)$.

The role of the self-energy $\Sigma_{2p2h}(\omega)$ on the ph propagator is twofold. One is to lower the energy of the ph state, and the other is to reduce the amplitude of the bare ph component in the excited state [12]. This can be understood easily in the second order perturbation theory, because $\Sigma_{2p2h}(\omega)$ stands for the couplings of the excited ph states to the 2p2h states.

From the figure, we can see that the ERPA and the free responses reproduce the exact results in the large N limit. This is understandable from the viewpoint of the $1/N$ expansion of the response; with only the U interaction at hand, one needs at least two factors of U to gain a factor N by making a fermion loop. Therefore, the leading contribution of the U

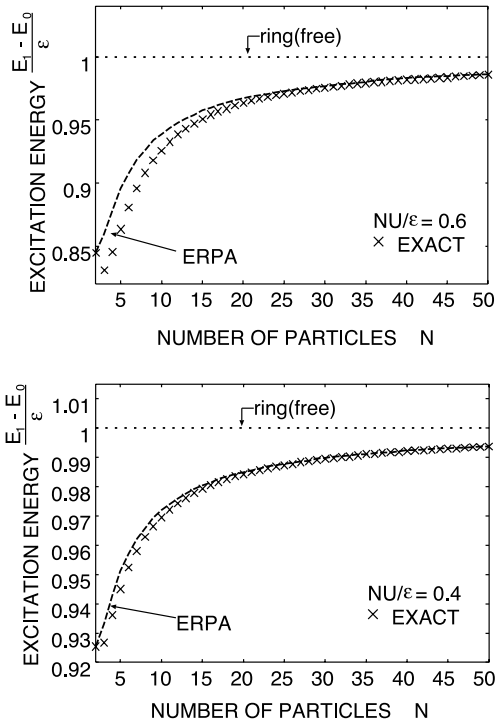

FIG. 9. Excitation energy of the first excited state in units of ε for $V=0$, $U\neq 0$. Exact values are denoted by crosses. The interaction strength is given by $NU/\varepsilon=0.6$ (top) and $NU/\varepsilon=0.4$ (bottom).

interaction is of the order of $1/N\times(NU/\varepsilon)^2$, and vanishes as $N\rightarrow\infty$. This explains clearly that both the ERPA and the exact results should converge to the free value as $N\rightarrow\infty$. This is in contrast with the V interaction in the ring approximation, where the number of the factors of V is identical with that of the fermion loops.

We can also recognize in the figure that the ERPA coincides with the exact results for $N=2$ (smallest number of particles in the figure). It can be shown easily that all the dynamical processes in the response for $N=2$ are taken into account in the framework of the ERPA, and therefore the ERPA becomes exact.

C. Case 3: $V\neq 0$, $U\neq 0$

Finally we examine the general case with $V\neq 0$, $U\neq 0$. The numerical results are shown in Fig. 10. Here we take the interaction strength U, which is slightly weaker than the ph interaction strength V, which is the usual situation in actual nuclei.

We can confirm in the figure the roles played by $\Sigma_{2p2h}(\omega)$ and $\Sigma_{3p3h}(\omega)$ explained in the preceding subsections. Let us start with the RPA results. Then, by including $\Sigma_{2p2h}(\omega)$ in the RPA response, we obtain the SRPA response, which gives lower excitation energies than the RPA results. Finally, by adding $\Sigma_{3p3h}(\omega)$ to the SRPA response, we increase the excitation energies from the SRPA values to the ERPA ones, which are close to the exact results.

We can also recognize that the difference between the ERPA and the exact values are slightly larger than that in case 1 ($V\neq 0$, $U=0$) or in case 2 ($V=0$, $U\neq 0$). This is

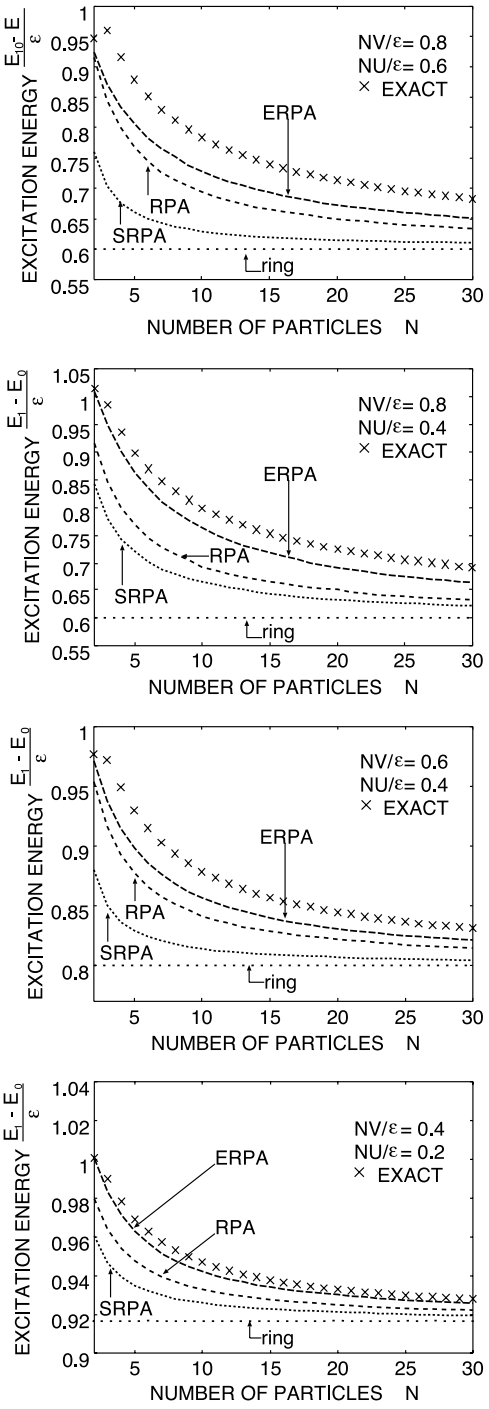

FIG. 10. Excitation energy of the first excited state in units of ε for $V\neq 0$, $U\neq 0$.

understandable because all the processes in the response interfere in the case with $V\neq 0$, $U\neq 0$.

From Fig. 10, we can see that the ERPA reproduces the exact results both in the large and the small N limits. The situation in the large N limit is understandable in the $1/N$ expansion. In the small N region, the system can rarely experience more complicated processes than are described by $\Sigma_{2p2h}(\omega)$ and $\Sigma_{3p3h}(\omega)$, because of the small number of particles. This explains why the ERPA is good also in the small N region.

Let us summarize this section as follows. We have clarified explicitly the effects of the two self-energies $\Sigma_{2p2h}(\omega)$ and $\Sigma_{3p3h}(\omega)$ using the extended Lipkin model. We have demonstrated that it is important to include both self-energies $\Sigma_{2p2h}(\omega)$ and $\Sigma_{3p3h}(\omega)$ in the ph propagator, i.e., to use the framework of the ERPA, to reproduce the exact excitation energies in a satisfactory way.

V. TRANSITION STRENGTH

In this section we study the transition strengths of a Hermitian operator $\mathcal{O}=J_+ + J_-$, using the first and the zeroth moments, which are the basic quantities to characterize the spectrum of the transition strengths.

A. Energy-weighted sum rule

Generally, the kth moment of the transition strengths is defined as

$$m_k = \sum_{\nu>0} (E_\nu - E_0)^k |\langle \nu|\mathcal{O}|0\rangle|^2. \tag{29}$$

In particular, the first moment m_1 (energy-weighted sum rule) is important theoretically because it can be given by the ground state expectation value of a simple commutator as follows [21]:

$$m_1 = \sum_{\nu>0} (E_\nu - E_0)|\langle \nu|\mathcal{O}|0\rangle|^2 = \frac{1}{2}\langle 0|[\mathcal{O},[H,\mathcal{O}]]|0\rangle. \tag{30}$$

Accordingly it can be shown [22,23] that m_1 is sensitive to the ground state correlation, and is only weakly affected by the correlation in the excited states.

In the case of the extended Lipkin model, we can demonstrate the above statement explicitly. By substituting J_+ and J_- in the operator $\mathcal{O}$ in Eq. (30), we can show that m_1 can be written as

$$m_1 = \frac{N}{2} \times (1,1) \begin{pmatrix} A_{11} & -B_{11} \\ -B_{11} & A_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{31}$$

where the definitions of A_{11} and B_{11}, and their ERPA values are given in Eq. (14). We can easily see that the ERPA expression for m_1 is given solely in terms of V, and is not dependent on U. This is the consequence of the fact that the U interaction enters only the correlations in the excited states $[\Sigma_{2p2h}(\omega)]$, while the V interaction causes the ground state correlations $[\Sigma_{3p3h}(\omega)]$. We also notice from Eq. (31) that the SRPA and the RPA values for m_1 are identical, because these two approximations give the same A_{11} and B_{11} [10,22].

We start with the transition strengths to the first excited state, of which the excitation energy has been examined in detail in Sec. IV. The transition strengths per particle to the first excited state, $|\langle 1|\mathcal{O}|0\rangle|^2/N$, are shown in Fig. 11. We adopt here the interaction strengths $NV/\varepsilon = 0.8$ and $NU/\varepsilon = 0.4$, which we think realistic. It can be seen that the ERPA approximates the exact values very well, and the SRPA is even worse than the RPA. Note that this is the same situation as for the excitation energies shown in Fig. 10. This fact shows explicitly that $\Sigma_{3p3h}(\omega)$ plays an important role not only in the calculation of the excitation energies but also in the description of the state vectors.

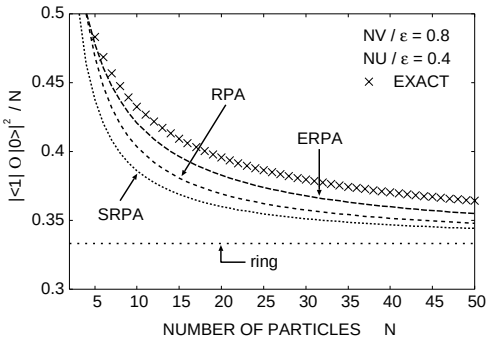

FIG. 11. Transition strengths to the first excited state per particle, $|\langle 1|\mathcal{O}|0\rangle|^2/N$, for $NV/\varepsilon = 0.8$, $NU/\varepsilon = 0.4$. Results in the ERPA, SRPA, RPA, and the ring approximation are presented with the exact values.

In Fig. 12 we present $m_1^{(1)}/m_1$, the ratio to m_1 of the contribution $m_1^{(1)}$ of the first excited state in the ERPA and the SRPA theories, together with the exact results. The figure clearly shows that the first excited states in the ERPA and the SRPA are as collective as in the exact results. It is also visible that the collectivity of the first excited state is enhanced with increasing number of the particles N, as is naturally anticipated.

Now we examine the energy-weighted sum rule m_1 of the system. In order to show the absolute values of m_1 in the ERPA and the SRPA theories, we present in Fig. 13 their ratios $m_1^{\mathrm{ERPA}}/m_1^{ex}$ and $m_1^{\mathrm{SRPA}}/m_1^{ex}$ to the exact value m_1^{ex} of the energy-weighted sum rule. First, we notice in Fig. 13 that both the ERPA and the SRPA values for m_1 are close to the exact values in the small and the large N limits. This situation can be explained in the same way as for the excitation energy. Second, both theories differ from the exact results in the intermediate N region. However, the error in the ERPA is much less than in the SRPA.

Let us take a different point of view, i.e., we can use the value of m_1 given by an approximation to measure how properly the ground state correlation is taken into account by

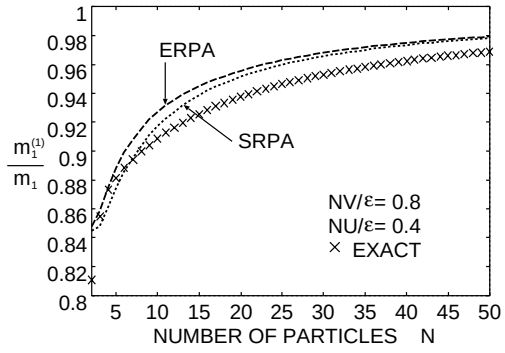

FIG. 12. Contribution $m_1^{(1)}$ of the first excited state to m_1 for $NV/\varepsilon = 0.8$, $NU/\varepsilon = 0.4$.

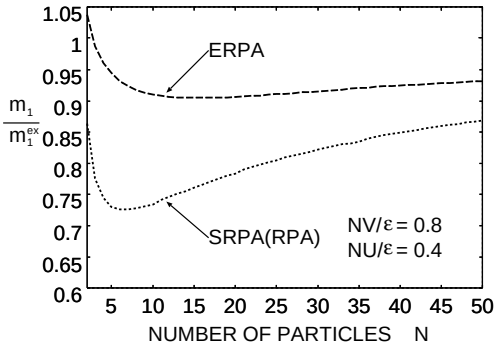

FIG. 13. The first moment m_1 in the ERPA and the SRPA normalized by the exact value m_1^{ex} for $NV/\varepsilon=0.8$, $NU/\varepsilon=0.4$. The RPA value is the same as the SRPA one.

the approximation. Then Fig. 13 exhibits first that the SRPA treats the ground state correlation in the same way as the RPA, and second that the ERPA makes a significant improvement on the SRPA (RPA) in treating the ground state correlation.

At the end, let us stress again that the difference between the ERPA and the RPA (SRPA) stems solely from $\Sigma_{3p3h}(\omega)$. This fact shows clearly that $\Sigma_{3p3h}(\omega)$ is very important in the calculation not only of the excitation energies but also of the transition strengths.

B. Non-energy-weighted sum rule

Finally, we come to the zeroth moment (non-energy-weighted sum rule) m_0, which is given by

$$m_0=\sum_{\nu>0}|\langle\nu|\mathcal{O}|0\rangle|^2. \tag{32}$$

Note that we do not have a simple expression for m_0 as Eq. (31) for m_1.

We present the numerical results for m_0 in Fig. 14 in the same way as in Fig. 13 for m_1. Contrary to the case of m_1, correlations in the excited as well as in the ground states, i.e., $\Sigma_{2p2h}(\omega)$ and $\Sigma_{3p3h}(\omega)$, take place in the description of m_0. The difference between the RPA and the SRPA can be traced back to $\Sigma_{2p2h}(\omega)$, and the difference between the SRPA and the ERPA originates from $\Sigma_{3p3h}(\omega)$.

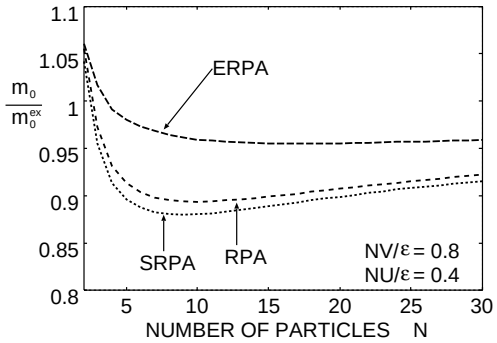

FIG. 14. m_0/m_0^{ex} for $NV/\varepsilon=0.8$, $NU/\varepsilon=0.4$ in the ERPA, SRPA, and RPA. m_0^{ex} represents the exact value for m_0.

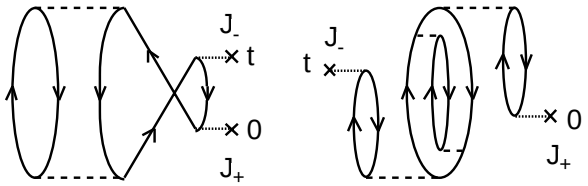

FIG. 15. Goldstone diagrams of $D_+(t)$ with $t>0$ that has no interaction line in the time interval $(0,t)$, and therefore contribute to m_0. The external excitation operator is taken to be J_+. Left: lowest order $[O(V^2)]$ process with Σ_{3p3h}. Right: lowest order $[O(V^2U^2)]$ process with Σ_{2p2h}.

We can see in the figure that $\Sigma_{3p3h}(\omega)$ has important effects in shifting the RPA value of m_0 toward the exact one, while $\Sigma_{2p2h}(\omega)$ has little effect. This can be understood using the idea of $1/\omega$ expansion [21,23]. Let split the response function $D(t)$ of Eq. (17) as

$$D(t)=\theta(t)D(t)+\theta(-t)D(t)=D_+(t)+D_-(t), \tag{33}$$

according to $t>0$ or $t<0$. It is easy to see that $D_+(\omega)$, the Fourier transform of $D_+(t)$, is analytic in the upper half plane of ω. Suppose we expand $D_+(\omega)$ in powers of $1/\omega$. Then it can be proven that the kth moment m_k can be obtained from the coefficient of $1/\omega^{k+1}$. In order to evaluate m_0, therefore, we need to look for the processes of the response that contribute to the coefficient of $1/\omega$. It is straightforward to demonstrate that a Goldstone diagram of $D_+(t)$ with no interaction line in the time interval $(0,t)$ behaves as $O(1)$ (constant) as $t\to+0$. Then we can see that its Fourier transform goes as $1/\omega$ as $\omega\to\infty$, and therefore it contributes to m_0.

Now we consider the response to the external operator J_+ for simplicity, and look for the Goldstone diagrams that have no interaction line in $(0,t)$. Let us start with the V-interaction term. We can immediately recognize that there is a contribution of order V^2 through $\Sigma_{3p3h}(\omega)$, as shown in Fig. 15. Next we come to the U interaction term. It can be immediately recognized that the U interaction alone cannot make a process, which is of the order of $1/\omega$, and also that the lowest order contribution of the U interaction to m_0 through $\Sigma_{2p2h}(\omega)$ is of the order of V^2U^2, as shown in Fig. 15. The above observation tells that $\Sigma_{2p2h}(\omega)$ enters m_0 only in the fourth and the higher order perturbation theory, while $\Sigma_{3p3h}(\omega)$ comes into play at the second order. This explains why $\Sigma_{3p3h}(\omega)$ is much more important than $\Sigma_{2p2h}(\omega)$ in the description of m_0, as is visible in Fig. 14. Note that contributions of order U^2 appear only in higher moments m_k, $k=2,3,\ldots$.

Let us summarize this section as follows. The transition strengths are investigated in terms of the first and the zeroth order moments m_1 and m_0. It has been demonstrated that $\Sigma_{3p3h}(\omega)$ is much more important than $\Sigma_{2p2h}(\omega)$ in the calculation of the transition strengths, which is emphasized, especially in m_1. This fact shows explicitly that the ERPA gives much better results than the SRPA not only in the excitation energies but also in the transition strengths.

VI. SUMMARY

In this paper we have proposed an extended Lipkin model, which has the $SU(2)$ symmetry of the original Lipkin model, and therefore is solvable. It incorporates the two-body interaction between ph↔pp and ph↔hh, which describes decay processes of collective states. The ground state of the noninteracting system satisfies the Hartree-Fock condition, and therefore the model can be used to examine the RPA-type theories.

We have studied the ERPA, using the newly proposed extended Lipkin model in terms of the excitation energies and the transition strengths. The self-energy of the ph propagator in the ERPA is given by $\Sigma_{2p2h}(\omega)+\Sigma_{3p3h}(\omega)$, each of which has been explained in detail. It has been demonstrated with the extended Lipkin model that it is very important to take into account not only $\Sigma_{2p2h}(\omega)$ but also $\Sigma_{3p3h}(\omega)$ in the ph propagator, i.e., to adopt the framework of the ERPA, to reproduce the exact excitation energies and transition strengths.

We believe that the model offers a very useful laboratory to examine a variety of many-body systems.

ACKNOWLEDGMENTS

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