Renormalization group approach to two-body scattering in the presence of long-range forces

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We apply renormalization-group methods to two-body scattering by combination of known long-range and unknown short-range potentials. We impose a cutoff in the basis of distorted waves of the long-range potential and identify possible fixed points of the short-range potential as this cutoff is lowered to zero. The expansions around these fixed points define the power countings for the corresponding effective field theories. Expansions around nontrivial fixed points are shown to correspond to distorted-wave versions of the effective-range expansion. These methods applied to scattering the presence of Coulomb, Yukawa, and repulsive inverse-square potentials.

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I. INTRODUCTION

Effective field theories (EFT's) offer the promise of a systematic and model-independent treatment of nuclear and hadronic physics at low energies. In the form of chiral perturbation theory (ChPT), they have been used with some success in both the mesonic and single-nucleon sectors (for reviews, see Refs. [1,2]). Currently, there is much interest in extending these applications to few-nucleon systems, as reviewed in Refs. [3–6].

These theories rely on the existence of a separation of scales between those of the low-energy physics, such as momenta, energies, or the pion mass, and those of the underlying short-distance physics, such the ρ-meson mass, the nucleon mass, or $4\pi f_\pi$. This makes it possible to expand any observable systematically in powers of the ratio Q/Λ_0, where Q denotes a low-energy scale and Λ_0 a scale of the underlying physics. The effective Lagrangian or Hamiltonian used to calculate these observables can also be organized according to a "power counting" in Q/Λ_0. Although an EFT contains an infinite number of these terms, only a finite number of these are needed to calculate observables up a given order. If the separation of scales is wide enough, this expansion will converge rapidly enough to be useful.

For weakly interacting systems the expansion can be organized according to naive dimensional analysis, a term proportional to $(Q/\Lambda_0)^d$ being counted as of order d [7,8]. This "Weinberg" power counting is the one familiar from ChPT in the zero- and one-nucleon sectors. In contrast, for strongly interacting systems there can be new low-energy scales which are generated by nonperturbative dynamics. Important examples for nuclear physics are the very large s-wave scattering lengths in nucleon-nucleon scattering. In such cases we need to resum certain terms in the theory to all orders, and this leads to a new power counting [9–13], often referred to as Kaplan, Savage, and Wise (KSW) power counting. The resulting expansion of the scattering amplitude is, in fact, the effective-range expansion (ERE) [14–17].

A crucial tool for determining the power counting is the renormalization group (RG), which is used to study the scaling behavior of systems in a wide range of areas of physics. In our work we use a Wilsonian version of the RG [18], imposing a momentum cutoff $|\mathbf{k}|<\Lambda$ on the low-energy EFT. So long as there is a clear separation of scales, the cutoff Λ can be chosen to be above all of the low-energy scales of interest, but well below the scales of the underlying physics. Beyond this, the cutoff is arbitrary, and physical observables should be independent of it. This means that physics on momentum scales above the cutoff must be included implicitly in the couplings of the EFT. As a result, all the coupling constants must depend on Λ. Finally, we rescale the theory, expressing all dimensioned quantities in units of Λ. The flow of the resulting rescaled coupling constants with Λ is described by a first-order differential equation: the RG equation.

For a system with a clear separation of scales, the rescaled coupling constants become independent of Λ as $\Lambda\to 0$. This is because, for $\Lambda\ll\Lambda_0$, the only scale left is Λ, and so the rescaled theory becomes independent of Λ. The theory is said to flow towards a fixed point. Small deviations from such a point scale as powers of the cutoff, and this can be used to define the power counting for the EFT, a term that behaves like Λ^ν being assigned an order $d=\nu-1$.

Perturbations around a fixed point are classified as "irrelevant" if $\nu>0$, "marginal" if $\nu=0$, and "relevant" if $\nu>0$. If all these perturbations are irrelevant, then the fixed point is stable: the couplings of any theory close to that point will flow towards it as $\Lambda\to 0$. On the other hand, if there are one or more relevant perturbations, then the fixed point is unstable. If a marginal perturbation is present, we should expect to find logarithmic flow with Λ.

In Ref. [13] these ideas were applied to two-body s-wave scattering by short-range forces and two fixed points were found. One is a trivial fixed point with no scattering. The expansion around it can be organized according to Weinberg power counting. The second is a nontrivial point describing systems with a bound state at zero energy. The appropriate power counting for expanding around it is the KSW counting, which corresponds to the effective-range expansion.

In the present work, we extend this approach to study two-body scattering by a combination of known long-range and unknown short-range forces. One important example of this is proton-proton scattering, where the Coulomb interaction is the long-range force. Another is nucleon-nucleon scattering at higher energies, where the nucleon momenta are comparable to the pion mass. Here one needs to treat explicitly pion-exchange forces, which can be calculated using ChPT [20].

To do this, we apply a cutoff to the basis of distorted waves (DW's) [17] for the known long-range interaction. Applying the RG as outlined above, we identify the possible fixed points of the short-range interaction and establish the corresponding power-counting rules for perturbations around these. In cases where a nontrivial fixed point exists, the terms in the resulting EFT can be directly related to those in a distorted-wave or "modified" effective-range expansion (DWERE) [15,16,21] (see also Refs. [22–24]).

The outline of this paper is as follows. In Sec. II we extend the ideas of Ref. [13] to describe scattering in presence of a long-range potential. The resulting RG equation has a form similar to that in Ref. [13], but which contains an extra term for each low-energy scale associated with the long-range potential.

This method is applied in Sec. III to several examples of long-range forces. To illustrate the method we apply it first, in Sec. III A, to scattering in the presence of a Coulomb potential.

In Sec. III B, we examine $^1S_0$ nucleon-nucleon (NN) scattering in the presence of a one-pion-exchange Yukawa potential. This has the same singularity at $r=0$ as the Coulomb potential, and so the RG behavior is very similar. There has been some debate in the literaure as to how one-pion exchange should be handled in an EFT. One scheme, proposed by KSW [11,25], treats this force perturbatively. The other, suggested by Weinberg [8] and further developed by van Kolck [20,26,3], iterates it to all orders. The RG treatment shows how these two schemes correspond to different choices of low-energy scale. Lastly, in Sec. III C, we consider scattering by an inverse-square potential.

II. RG WITH LONG-RANGE FORCES

The starting point for a derivation of the RG for scattering of two heavy particles is the fully off-shell scattering amplitude $T(k',k,p)$. Here, as throughout this paper, we use k and k' to denote relative momenta, and the energy dependence is expressed in terms of $p=\sqrt{ME}$, the on-shell momentum corresponding to the energy E. For simplicity, we consider only s-wave scattering. In this case, the amplitude is related to the potential $V(k',k,p)$ by the Lippmann-Schwinger (LS) equation [17]

$$T(k',k,p)=V(k',k,p)+\frac{M}{2\pi^2}\int q^2dq\frac{V(k',q,p)T(q,k,p)}{p^2-q^2+i\epsilon}. \tag{1}$$

We are interested here in systems where the potential can be written in the form $V=V_L+V_S$, where V_L is a known long-range potential and V_S is a short-range potential constructed from contact interactions only. In coordinate space these effective interactions can be expressed as δ functions and their derivatives. In momentum space, they have the form

$$V_S(k',k,p,\kappa)=c_{00}(\kappa)+c_{20}(\kappa)(k^2+k'^2)+c_{02}(\kappa)p^2+\cdots. \tag{2}$$

Here κ stands for any of the low-energy scales associated with V_L, such as the pion mass in the NN potential or the inverse Bohr radius for a Coulomb potential. Such scales control the long-range form of the potential and can give rise to nonanalytic energy dependence of the scattering amplitude. This must be separated out if the effective short-range potential is to be expanded in this form. For an EFT to be truly effective, we need to organize the terms in this expansion systematically, according to some power counting. This should include expanding the coefficients in Eq. (2) in powers of the scales κ. The RG provides the framework for doing this.

It is convenient to work in terms of the DW's of the long-range potential. The T matrix describing scattering by V_L alone is

$$T_L=V_L+V_LG_0^+T_L, \tag{3}$$

and the corresponding Green's function is

$$G_L^+=G_0^++G_0^+T_LG_0^+. \tag{4}$$

By using the "two-potential trick" to resum the effects of V_L to all orders [17], the full T matrix can be written in the form

$$T=T_L+(1+T_LG_0^+)\widetilde{T}_S(1+G_0^+T_L), \tag{5}$$

where $\widetilde{T}_S$ satisfies the LS equation

$$\widetilde{T}_S=V_S+V_SG_L^+\widetilde{T}_S. \tag{6}$$

Since

$$\Omega=1+G_0^+T_L \tag{7}$$

is the Moller wave operator which converts a plane wave into a DW of V_L, we can see that $\widetilde{T}_S$ describes the scattering between the DW's as a result of the short-range potential.

The effects of the short-range potential can be expressed in terms of $\widetilde{\delta}_S=\delta-\delta_L$, the difference between the full phase shift, δ, and that due to V_L alone, δ_L. The on-shell matrix elements of $\widetilde{T}_S$ can then be written in the form

$$\langle\psi_L^-(p)|\widetilde{T}_S(p)|\psi_L^+(p)\rangle=-\frac{4\pi}{M}e^{2i\delta_L(p)}\frac{e^{2i\widetilde{\delta}_S(p)}-1}{2ip}, \tag{8}$$

where $\psi_L^+(p,r)$ is the outgoing DW of V_L with energy $E=p^2/M$ and ψ_L^- is the corresponding incoming wave. This can be used as the starting point for the DWERE [15,16,21],

$$e^{2i\delta_L(p)}\left(\frac{|\psi_L^+(p,0)|^2}{\langle\psi_L^-|\tilde{T}_S(p)|\psi_L^+\rangle}\right)+\mathcal{M}_L(p)$$
$$=|\psi_L^+(p,0)|^2 p[\cot\tilde{\delta}_S(p)-i]+\mathcal{M}_L(p)$$
$$=-\frac{1}{\tilde{a}}+\frac{1}{2}\tilde{r}_e p^2+\cdots. \qquad (9)$$

The function $\mathcal{M}_L(p)$ can be written as the logarithmic derivative at the origin of the Jost solution to the Schrödinger equation with the potential V_L [21]. In Eq. (9) all rapid, and possibly nonanalytic, dependence on the energy is removed in $\mathcal{M}_L(p)$ and $e^{2i\delta_L(p)}|\psi_L^+(p,0)|^2$. This leaves an amplitude which can be expanded as a power series in the energy, with coefficients whose scale is set by the underlying short-distance physics. This expansion has long been used to extract low-energy properties of the strong interaction between two protons [16,19,21,22] and more recently to remove the effects of one-pion exchange from *NN* scattering [23,24].

A. RG equation

In order to derive the RG equation for the short-range potential, it is more convenient to work with a reactance matrix $\tilde{K}_S$. This satisfies a LS equation that is very similar to that for $\tilde{T}_S$, Eq. (6), except that the Green's function $G_L^{\mathcal{P}}$ obeys standing-wave boundary conditions. This means that $\tilde{K}_S$ is Hermitian below all thresholds for particle production. On shell, $\tilde{K}_S$ matrix is related to the additional phase shift $\tilde{\delta}_S$ by

$$\langle\psi_L(p)|\tilde{K}_S|\psi_L(p)\rangle=-\frac{4\pi}{Mp}\frac{1}{\cot\tilde{\delta}_S}. \qquad (10)$$

The effective potential V_S has zero range, and so the loop integrals in the LS equation are divergent. To regulate this equation, and as a first step in deriving a Wilsonian RG equation, we apply a cutoff to G_L in the DW basis:

$$G_L^{\mathcal{P}}=\frac{M}{2\pi^2}\mathcal{P}\int_0^\Lambda dq\,q^2\frac{|\psi_L(q)\rangle\langle\psi_L(q)|}{p^2-q^2}(+\text{bound states}), \qquad (11)$$

where $\mathcal{P}$ denotes the principal value of the integral, corresponding to the standing-wave boundary conditions.

The LS equation for $\tilde{K}_S$ is

$$\tilde{K}_S=V_S(\Lambda)+V_S(\Lambda)G_L^{\mathcal{P}}(\Lambda)\tilde{K}_S. \qquad (12)$$

As in Ref. [13] we demand that the off-shell amplitude $\tilde{K}_S$ be independent of the cutoff Λ. The potential V_S must depend on Λ to compensate for the dependence in the loop integrals. Differentiating the LS equation with respect to Λ and eliminating $\tilde{K}_S$ we obtain the differential equation

$$\frac{\partial V_S}{\partial\Lambda}=-V_S\frac{\partial G_L}{\partial\Lambda}V_S. \qquad (13)$$

By letting V_S vary with Λ in this way, we are eliminating high-energy modes from the theory and parametrizing their effects in the effective potential. If we did not apply the cutoff to the DW basis but instead applied it in the basis of free states, V_S would have to change with Λ not only to incorporate the physics that has been integrated out but also to correct for the modification of V_L by the cutoff. The resulting equation would then have had a much more complicated form.

To simplify the analysis, we assume here that V_S depends only on energy, not on momenta. As shown in Ref. [13], the momentum-dependent solutions to the RG equation are not needed to describe on-shell scattering. We need to be careful in defining the short-range interaction since a simple δ function cannot be used in combination with most of the long-range potentials of interest. These potentials are sufficiently singular that their DW's either vanish or diverge as $r\to 0$. As a result, the right-hand side of Eq. (13) is either zero or ill defined for a contact interaction.

Instead of a contact interaction, we choose a spherically symmetric potential with a short but nonzero range. By taking the range R to be much smaller than $1/\Lambda$, we ensure that any additional energy or momentum dependence associated with it is no larger than that of the physics which has been integrated out, and hence the power counting is not altered by it. This scale R separates the region of low-momentum physics, which we wish describe with an EFT, from a region of high momentum physics, which is nonperturbative as a result of the singular behavior of the long-range potential. The precise value of this scale is arbitrary, and so observables should not depend on it.

A simple and convenient choice for the form of this regulator is the "δ-shell" potential,

$$V_S(p,\kappa,\Lambda;r)=V_S(p,\kappa,\Lambda)\frac{\delta(r-R)}{4\pi R^2}, \qquad (14)$$

where κ denotes a generic low-energy scale associated with V_L. With this choice, Eq. (13) for V_S becomes

$$\frac{\partial V_S}{\partial\Lambda}=-\frac{M}{2\pi^2}|\psi_L(\Lambda,R)|^2\frac{\Lambda^2}{p^2-\Lambda^2}V_S^2(p,\kappa,\Lambda), \qquad (15)$$

where the DW $\psi_L(k,R)$ satisfies the Schrödinger equation,

$$\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)\psi_L(k,r)-MV_L(r,\kappa)\psi_L(k,r)+k^2\psi_L(k,r)=0, \qquad (16)$$

and is normalized so that

$$\int_0^\infty dr k^2 r^2\psi_L(k,r)\psi_L(k',r)=\frac{\pi}{2}\delta(k-k'). \qquad (17)$$

The final step in obtaining a Wilsonian RG equation is to rescale all dimensioned quantities, expressing them in terms of the cutoff Λ. Dimensionless variables corresponding to the momenta and other low-energy scales are defined by $\hat{p}$

$=p/\Lambda$, etc., along with a rescaled potential. Unlike the simpler case studied in Ref. [13], the rescaling of the potential depends on the behavior of the wave functions in the limit $R \ll 1/p$. We assume that these have the separable form

$$|\psi_L(p,R)|^2 = |\mathcal{N}(\kappa/p)|^2 f(p) F(R), \tag{18}$$

which is general enough to cover all the examples studied here and also the attractive inverse-square potential. Here $\mathcal{N}(\kappa/p)$ is a normalization factor which depends on the low-energy scales κ. This factor communicates information about the long-distance physics to the short-distance physics. The appropriate rescaled potential is

$$\hat{V}_S(\hat{p},\hat{\kappa},\Lambda) = \frac{M\Lambda}{2\pi^2} F(R) f(\Lambda) V_S(\Lambda\hat{p},\Lambda\hat{\kappa},\Lambda), \tag{19}$$

which satisfies the distorted-wave renormalization-group (DWRG) equation,

$$\Lambda\frac{\partial \hat{V}_S}{\partial\Lambda} = \hat{p}\frac{\partial \hat{V}_S}{\partial \hat{p}} + \hat{\kappa}\frac{\partial \hat{V}_S}{\partial\hat{\kappa}} + \left(1 + \Lambda\frac{f'(\Lambda)}{f(\Lambda)}\right)\hat{V}_S + \frac{|\mathcal{N}(\hat{\kappa})|^2}{1-\hat{p}^2}\hat{V}_S^2. \tag{20}$$

The boundary conditions on solutions to Eq. (20) follow from the fact that the effective potential should describe scattering at energies below all thresholds for production of other particles. If all the nonanalytic energy dependence generated by long-range physics has been factored out into the DW's, then the effective potential V_S should be an analytic function of the energy. Similarly, V_S should be a well-behaved function of κ, a scale associated with V_L, as $\kappa \to 0$. In most cases we need to demand only that V_S is analytic in κ. An example is the inverse Bohr radius which forms a low-energy scale for the Coulomb potential, and which is proportional to α. An important exception is the pion mass which is proportional to the square root of the strength of the chiral symmetry breaking (the current quark mass), and so, as in ChPT, the effective potential should be analytic in m_π^2. Under these requirements, we see that $\hat{V}_S$ should have an expansion in non-negative, integer powers of $\hat{p}^2$ and either $\hat{\kappa}$ or $\hat{\kappa}^2$.

We are interested in the fixed points of the renormalization group, that is, solutions of the DWRG equation (20) that are independent of Λ. These solutions are important because studying the scaling behavior of the RG flow around them leads to power-counting schemes. It is clear that there is always a trivial fixed-point solution, $\hat{V}_S = 0$. However nontrivial fixed points can only exist if the right-hand side is independent of Λ. This occurs if the distorted waves have a power-law form close to the origin,

$$|\psi_L(p,R)|^2 = |\mathcal{N}(\kappa/p)|^2 (pR)^{\sigma-1}, \tag{21}$$

where σ is a real number. This is the case for all the examples considered here, but not for the attractive inverse-square potential. Taking $f(\Lambda) = \Lambda^{\sigma-1}$, the DWRG equation can be written as

$$\Lambda\frac{\partial}{\partial\Lambda}\left(\frac{1}{\hat{V}_S}\right) = \hat{p}\frac{\partial}{\partial\hat{p}}\left(\frac{1}{\hat{V}_S}\right) + \hat{\kappa}\frac{\partial}{\partial\hat{\kappa}}\left(\frac{1}{\hat{V}_S}\right) - \sigma\frac{1}{\hat{V}_S} - \frac{|\mathcal{N}(\hat{\kappa})|^2}{1-\hat{p}^2}, \tag{22}$$

where we have taken advantage of the fact that, for momentum-independent potentials, the RG equation can be divided by $\hat{V}_S^2$ to obtain a more practical linear equation for $1/\hat{V}_S$.

B. Trivial fixed point

As already noted, a trivial fixed point, $\hat{V}_S = 0$, always exists for Eq. (22). It describes a system with no scattering between the distorted waves of the long-range potential. Perturbations about this point can be used to describe systems where the short-range interactions provide small corrections to the scattering by V_L. They correspond to an expansion of the DW K matrix $\tilde{K}_S$ of Eq. (12) in powers of energy and κ. We can find perturbations which scale with definite powers of Λ by linearizing the RG equation (22) about the fixed point and looking for solutions of the form

$$\hat{V}_S(\hat{p},\hat{\kappa},\Lambda) = C\Lambda^\nu \phi(\hat{p},\hat{\kappa}), \tag{23}$$

where the functions ϕ satisfy the eigenvalue equation

$$\hat{p}\frac{\partial\phi}{\partial\hat{p}} + \hat{\kappa}\frac{\partial\phi}{\partial\hat{\kappa}} + \sigma\phi = \nu\phi. \tag{24}$$

Solving, we find that

$$\phi(\hat{p},\hat{\kappa}) = \hat{\kappa}^m \hat{p}^{2n}, \tag{25}$$

with RG eigenvalues $\nu = m + 2n + \sigma$. The boundary conditions demand that m and n are non-negative integers. In addition, if κ is a scale like the pion mass, m must be even. Hence, near the fixed point, we may write

$$\hat{V}_S(\hat{p},\hat{\kappa},\Lambda) = \sum_{n,m} C_{mn}\Lambda^{2n+m+\sigma}\hat{\kappa}^m \hat{p}^{2n}. \tag{26}$$

The stability of the fixed point depends upon the value of σ, which describes the power-law behavior of the DW's near the origin. For positive σ the fixed point is stable, while for negative σ there is one or more unstable perturbation, with $m + 2n + \sigma < 0$. If $\sigma = 0$, then the perturbation with $m = n = 0$ is marginal. A marginal eigenfunction may also exist when σ is a negative integer, if $m + 2n = -\sigma$ is allowed. These marginal eigenfunctions have no power-law evolution with Λ. Instead, we expect to find a logarithmic dependence which will allow us to classify such perturbations as marginally stable or unstable.

C. Nontrivial fixed point

A nontrivial fixed point of Eq. (22), if one exists, satisfies the equation

$$\hat{p}\frac{\partial}{\partial \hat{p}}\left(\frac{1}{\hat{V}_0}\right)+\hat{\kappa}\frac{\partial}{\partial \hat{\kappa}}\left(\frac{1}{\hat{V}_0}\right)=\sigma\frac{1}{\hat{V}_0}+\frac{|\mathcal{N}(\hat{\kappa})|^2}{1-\hat{p}^2}. \tag{27}$$

It is useful to note that this equation is satisfied by the loop integral

$$\hat{J}_0(\hat{p},\hat{\kappa})=\mathcal{P}\int_0^1 \hat{q}^{\sigma+1}\mathrm{d}\hat{q}\frac{|\mathcal{N}(\hat{\kappa}/\hat{q})|^2}{\hat{p}^2-\hat{q}^2}. \tag{28}$$

However, we cannot directly identify $1/\hat{V}_0=\hat{J}_0$ as a fixed-point solution of the RG equation since it is not in general analytic as $\hat{p},\hat{\kappa}\to 0$. Nonetheless, we can use $\hat{J}_0$ as a starting point for finding such solutions. The method will be described in more detail in the examples below, but the basic idea is to subtract from $\hat{J}_0$ a solution to the homogeneous version of Eq. (27) to cancel all its nonanalytic behavior. We can then write

$$\frac{1}{\hat{V}_0}=\hat{J}_0(\hat{p},\hat{\kappa})-\hat{\mathcal{M}}(\hat{p},\hat{\kappa}), \tag{29}$$

where $\hat{\mathcal{M}}(\hat{p},\hat{\kappa})$ is a homogenous function of order σ in $\hat{p}$ and $\hat{\kappa}$.

Perturbations around a nontrivial fixed-point solution to Eq. (27) can be found as above, so that near the fixed point the potential takes the form

$$\frac{1}{\hat{V}_S}=\frac{1}{\hat{V}_0}-\sum_{n,m} C_{mn}\Lambda^{m+2n-\sigma}\hat{\kappa}^m\hat{p}^{2n}, \tag{30}$$

where n and m satisfy the conditions described above in the case of the trivial fixed point. If σ is positive then the fixed point is unstable, the eigenfunctions with $m+2n-\sigma<0$ corresponding to the unstable directions. If σ is negative, then this fixed point is stable. Marginal eigenvectors may exist for integer σ if $m+2n=\sigma$ is allowed. In particular, for $\sigma=0$, the fixed point is marginal with the $m=n=0$ perturbation having zero eigenvalue.

D. DW effective-range expansion

Having obtained a solution to the full RG equation near the nontrivial fixed point, we can use it in the DW Lippmann-Schwinger equation for $\hat{K}_S$ (12) in order to connect the potential to scattering observables. We find a direct connection to a DW effective-range expansion, Eq. (9).

The LS equation can be solved by expanding the Green's functions in terms of a complete set of DW's and iterating to get a geometric series. This series can then be summed, giving

$$\langle\psi_L(p)|\hat{K}_S|\psi_L(p)\rangle = \frac{V_S(p,\kappa,\Lambda)|\psi_L(p,R)|^2}{1-\frac{V_S(p,\kappa,\Lambda)M}{2\pi^2}\mathcal{P}\int_0^\Lambda q^2dq\frac{|\psi_L(q,R)|^2}{p^2-q^2}}. \tag{31}$$

Note that the integral in the denominator is just $\Lambda^\sigma R^{\sigma-1}\hat{J}_0(\hat{p},\hat{\kappa})$, where $\hat{J}_0$ is defined in Eq. (28).

This equation can be rewritten in terms of the cotangent of the additional phase shift as

$$|\mathcal{N}(\kappa/p)|^2\frac{\pi p^\sigma}{2}\cot\tilde{\delta}_S=\Lambda^\sigma\left(\hat{J}_0(\hat{p},\hat{\kappa})-\frac{1}{\hat{V}_S(\hat{p},\hat{\kappa})}\right). \tag{32}$$

This result is independent of R, as anticipated. Despite initial appearances it is also independent of Λ. The difference between $1/\hat{V}_0$ and $\hat{J}_0$ is just the homogeneous function $\hat{\mathcal{M}}(\hat{p},\hat{\kappa})$ introduced in Eq. (29). The corresponding piece of the right-hand side is a homogeneous function of order σ in the physical variables p and κ. Including perturbations of the form given in Eq. (30), we can therefore rewrite Eq. (32) as

$$|\mathcal{N}(\kappa/p)|^2\frac{\pi p^\sigma}{2}\cot\tilde{\delta}_S=-\mathcal{M}(p,\kappa)+\sum_{n,m} C_{mn}\kappa^m p^{2n}, \tag{33}$$

where $\mathcal{M}(p,\kappa)=\Lambda^\sigma\hat{\mathcal{M}}(\hat{p},\hat{\kappa})$.

This resulting equation (33) has exactly the form of the DWERE, Eq. (9). The only difference is that, for sufficiently singular potentials, the wave functions must be evaluated close to, but not exactly at, the origin. However, provided the wave functions have a power-law form in this region, the R dependence cancels, leaving Eq. (33). All nonanalytic effects of the long-range force have been factored or subtracted out in the functions $\mathcal{N}$ and $\mathcal{M}$. The remainder can be written as a power-series expansion in p and κ, which corresponds directly to the expansion of the short-range effective potential.

III. EXAMPLES

A. Coulomb potential

Scattering from a combination of Coulomb and short-range potentials has already been studied from an EFT viewpoint by Kong and Ravndal [22]. Here we examine it using the RG and show how a power counting can be established, which matches exactly with the DWERE.

Since the fine structure constant α is small, the inverse of the Bohr radius, $\kappa=\alpha M/2$, provides a low-energy scale associated with the Coulomb potential. The Coulomb wave function tends to a finite, nonzero value at the origin, and so it has the form assumed in the preceding section, Eq. (21), with $\sigma=1$ and the square of the wave function given by the well-known Sommerfeld factor

$$|\mathcal{N}(\kappa/p)|^2=\lim_{R\to 0}|\psi_L(p,R)|^2=\mathcal{C}(\kappa/p)=\frac{2\pi\kappa/p}{e^{2\pi\kappa/p}-1}. \tag{34}$$

A trivial fixed point always exists and the power counting for the expansion around it can be found using the general analysis in Sec. II B. Of more interest are possible nontrivial solutions to the RG and the expansions around them. We take the basic loop integral of Eq. (28) as our starting point for the construction of these solutions. This satisfies Eq. (27), the fixed-point version of the DWRG equation (22), but it contains nonanalytic terms which should not be present. To identify these, we follow Kong and Ravndal [22] and identify the following terms in $\hat{J}_0$:

$$\hat{J}_0(\hat{p},\hat{\kappa})=-1-\pi\hat{\kappa}(\ln\hat{\kappa}+\gamma)-\pi\hat{\kappa}\mathrm{Re}\left\{H\left(\frac{\hat{\kappa}}{\hat{p}}\right)\right\} + \text{terms analytic in } \hat{p},\hat{\kappa}, \tag{35}$$

where γ is Euler's constant and the function H is given by

$$H(x)=\psi(ix)+\frac{1}{2ix}-\ln(ix), \tag{36}$$

in terms of the logarithmic derivative of the Γ function, denoted by ψ.

A potential which is analytic as $\hat{p}^2,\hat{\kappa}\to 0$ can be built from $\hat{J}_0$ by cancelling the terms proportional to $\hat{\kappa}\ln\hat{\kappa}$ and $\hat{\kappa}\mathrm{Re}\{H(\hat{\kappa}/\hat{p})\}$. The latter of these does not present a problem, however, the first cannot be removed within the confines of a Λ-independent solution. The potential,

$$\frac{1}{\hat{V}_0(\hat{p},\hat{\kappa},\Lambda)}=\hat{J}_0(\hat{p},\hat{\kappa})+\pi\hat{\kappa}\left\{\mathrm{Re}\left[H\left(\frac{\hat{\kappa}}{\hat{p}}\right)\right]+\ln\frac{\hat{\kappa}\Lambda}{\mu}\right\}, \tag{37}$$

which depends logarithmically on Λ and some arbitrary scale μ, satisfies the analyticity boundary conditions and is a solution of the DWRG equation. Although this potential does evolve slowly with Λ, we can still expand around it and use the DWRG equation (22) to determine the forms of the perturbations around it. These are given above in Eq. (30), and so the full short-range potential is given by

$$\frac{1}{\hat{V}_S}=\frac{1}{\hat{V}_0}-\sum_{n,m=0}^{\infty}C_{mn}\Lambda^{m+2n-1}\hat{\kappa}^m\hat{p}^{2n}. \tag{38}$$

This provides the power counting for a strong short-range potential in the presence of the Coulomb potential. Note that for the $\hat{\kappa}$-independent terms this is just KSW counting, which corresponds to the ERE. If we invert Eq. (38), we see that the potential we have obtained resums the leading logarithms ($\kappa\ln\Lambda$) to all orders.

The presence of a marginal perturbation ($m=1$, $n=0$) in the expansion about $\hat{V}_0$ leads to the logarithmic dependence on Λ noted above, and so explains our inability to find a true fixed point. Such a pertubation cannot be separated unambiguously from a fixed-point potential. By letting its coefficient $C_{10}(\mu)$ depend logarithmically on the arbitrary scale μ, we can ensure the full potential, and hence all observables are independent of μ.

Substituting the effective potential into Eq. (32), we get the DWERE for the Coulomb potential [15,16,22],

$$\mathcal{C}(\kappa/p)p(\cot\tilde{\delta}_S-i)+\alpha MH(\kappa/p)=-\frac{1}{\tilde{a}}+\frac{1}{2}\tilde{r}_e p^2+\cdots. \tag{39}$$

Apart from the logarithmic term, the nonanalytic effects of the long-range Coulomb potential are contained in the functions $\mathcal{C}(\kappa/p)$ and $H(\kappa/p)$.

There is a direct correspondence between the expansion of the potential in powers of the energy and the terms in the ERE. The full effective potential gives a further expansion of this in powers of κ, or equivalently α. We get the following expressions for the Coulomb-modified scattering length and effective range,

$$\frac{1}{\tilde{a}}=\alpha M\ln\left(\frac{\alpha M}{2\mu}\right)-\frac{2}{\pi}\sum_{m=0}^{\infty}C_{m0}\left(\frac{\alpha M}{2}\right)^m, \tag{40}$$

$$\tilde{r}_e=\frac{4}{\pi}\sum_{m=0}^{\infty}C_{m1}\left(\frac{\alpha M}{2}\right)^m. \tag{41}$$

The scattering length defined in this way still contains a logarithmic dependence on α, and so it is not a pure short-range effect. However, any attempt to define a strong-interaction scattering length by subtracting off the logarithmic term will introduce an arbitrary scale (cf. Ref. [19] where a particular choice is made for μ).

B. Yukawa potential

The renormalization of scattering in the presence of a Yukawa potential is similar to the Coulomb case since the distorted waves have the same short-distance behavior. We consider in particular OPE between nucleons in the $^1S_0$ channel,

$$V_L(r)=-\alpha_\pi\frac{e^{-m_\pi r}}{r}, \tag{42}$$

where the strength of the potential is $\alpha_\pi=g_A^2m_\pi^2/16\pi f_\pi^2$, in terms of the pion mass $m_\pi=140$ MeV, the pion decay constant $f_\pi=93$ MeV, and the axial coupling of the nucleon, $g_A=1.26$.

There has been some debate in the literature as to how OPE should be handled in an EFT. One scheme, proposed by KSW [11,25], treats this force perturbatively. The other, suggested by Weinberg [8] and further developed by van Kolck [20,26,3], iterates it to all orders. The RG treatment shows how these two schemes correspond to different choices of low-energy scales.

In the KSW scheme [11,25], only the pion mass is treated as a low-energy scale and the rescaled OPE potential in momentum space is

$$\hat{V}_L(\hat{\mathbf{k}}',\hat{\mathbf{k}},\hat{m}_\pi,\Lambda)=-\Lambda\,\frac{Mg_A^2}{8\pi^2 f_\pi^2}\,\frac{\hat{m}_\pi^2}{|\hat{\mathbf{k}}'-\mathbf{k}|^2+\hat{m}_\pi^2}. \quad (43)$$

This is proportional to the cutoff Λ, and so its effect on the RG flow vanishes as $\Lambda\to 0$. This means that the fixed points will be the same as in the pure short-range case [13]. The DW Green's function G_L can be expanded in powers of the long-ranged potential and pion exchange treated as perturbative corrections to the ordinary ERE.

Although the KSW scheme allows ChPT to be extended to the two-nucleon sector, the resulting expansion turns out to be, at best, slowly convergent [23,27,24,28]. The problem is that the pion-nucleon coupling is large, or equivalently that the scale which sets the strength of the potential,

$$\lambda_{NN}=\frac{16\pi f_\pi^2}{Mg_A^2}\simeq 300 \ \text{MeV}, \quad (44)$$

is small. There is thus no good separation of scales, and so it is unsurprising that the corresponding EFT shows poor convergence.

In the alternative scheme of Weinberg and van Kolck (WvK), the effective potential is expanded using Weinberg power counting, and its leading terms are then resummed to all orders.[1] This approach treats λ_{NN} or equivalently the inverse "pionic Bohr radius,"

$$\kappa_\pi=\frac{\alpha_\pi M}{2}, \quad (45)$$

as an additional low-energy scale. The rescaled OPE potential is then independent of Λ and so has been promoted to form part of any fixed point.

The effects of the distortion show up in the normalization of the DW's near the origin. Although it is not possible to write this normalization in analytic form, from dimensional analysis, it must be of the form

$$|\psi_L(p,R\to 0)|^2=\mathcal{C}_\pi(\kappa_\pi/p,m_\pi/p). \quad (46)$$

If we rescale both κ_π and m_π, as just discussed, the normalization of the DW with $p=\Lambda$ is independent of Λ. The resulting RG equation is

$$\Lambda\frac{\partial\hat{V}_S}{\partial\Lambda}=\hat{p}\frac{\partial\hat{V}_S}{\partial\hat{p}}+\hat{\kappa}_\pi\frac{\partial\hat{V}_S}{\partial\hat{\kappa}_\pi}+\hat{m}_\pi\frac{\partial\hat{V}_S}{\partial\hat{m}_\pi}+\hat{V}_S+\frac{\mathcal{C}_\pi(\hat{\kappa}_\pi,\hat{m}_\pi)}{1-\hat{p}^2}\hat{V}_S^2. \quad (47)$$

The expansion around the trivial fixed point is

$$\hat{V}_S=\sum_{l,m,n} C_{lmn}\Lambda^\nu\hat{m}_\pi^{2l}\hat{\kappa}_\pi^m\hat{p}^{2n}, \quad (48)$$

where the eigenvalues are $\nu=2l+m+2n+1=1,3,5,\ldots,$ as in the Coulomb case.

The more interesting expansion around a nontrivial fixed point takes the form

$$\frac{1}{\hat{V}_S}=\frac{1}{\hat{V}_0}-\sum_{l,m,n} C_{lmn}\Lambda^\nu\hat{m}_\pi^{2l}\hat{\kappa}_\pi^m\hat{p}^{2n}, \quad (49)$$

where $\nu=2l+m+2n-1=-1,0,+1,\ldots.$ This includes a marginal perturbation which is linear in the inverse Bohr radius, and leads to logarithmic evolution of the potential $\hat{V}_0$ with Λ. We can resum this using the RG method described above in Sec. III A.

The expansion of the potential corresponds to a DWERE of the form

$$\mathcal{C}_\pi\left(\frac{\kappa_\pi}{p},\frac{m_\pi}{p}\right)\cot\tilde{\delta}_S=\kappa_\pi H_\pi\left(\frac{\kappa_\pi}{p},\frac{m_\pi}{p}\right)+\kappa_\pi\ln\frac{\kappa_\pi}{\mu}+\frac{2}{\pi}\sum_{l,m,n} C_{lmn}m_\pi^{2l}\kappa_\pi^m p^{2n}, \quad (50)$$

where the dependence on the arbitrary scale cancels between the first term and $C_{010}(\mu)$. All nonanalytic behavior is contained in the functions $\mathcal{C}_\pi(\kappa_\pi/p,m_\pi/p)$ and $H_\pi(\kappa_\pi/p,m_\pi/p)$. For the Yukawa potential, these must be calculated using numerical methods. This DWERE has been applied to nucleon-nucleon scattering in Refs. [23,24].

The expansion corresponds to an EFT in which OPE and the leading short-range force are iterated to all orders. This WvK scheme has been developed and applied by van Kolck and others [20,26,3,29], with some success. The RG analysis shows that the expansion around the logarithmically evolving potential defines the power counting for this scheme, a term $m_\pi^{2l}\kappa_\pi^m p^{2n}$ being of order $d=2l+m+2n-2$. Because of the appearance of κ_π as an extra low-energy scale in this expansion, terms of the same order in m_π occur at different orders here. Hence in contrast to the the KSW scheme, the direct link to ChPT has been lost [30,31].

C. Repulsive inverse-square potential

As a final example we consider the inverse-square potential $V_L(r)=\beta M^{-1}r^{-2}$, which is of interest because of its relevance to the three-body problem. Efimov [32] has shown that this potential can describe the scattering of three particles in the limit where the two-body potential has zero range and infinite scattering length. The renormalization of the short-ranged three-body forces which appear in EFT treatments of such systems is currently the object of keen study [33,34].

In the repulsive case, $\beta>-\frac{1}{4}$, the DWs are given by

$$\psi_L(p,r)=\sqrt{\frac{\pi}{2pr}}J_\nu(pr), \quad (51)$$

[1] Note that we distinguish here between Weinberg power counting for the terms in the potential and the scheme based on it for treating pion exchange.

where $\nu=\sqrt{\beta+(1/4)}$, and which go to

$$|\psi_L(p,R)|^2=\frac{1}{\Gamma(1+\nu)^2}\left(\frac{pR}{2}\right)^{2\nu-1} \tag{52}$$

in the limit of small R. This is of the form in Eq. (21) with $\sigma=2\nu$ and $\mathcal{N}^2=1/[2^{2\nu-1}\Gamma(1+\nu)^2]$. For scattering in a partial wave with $l>0$, the centrifugal barrier provides a $1/r^2$ potential with $\beta=l(l+1)$, and hence $\nu=l+\frac{1}{2}$.

Since $\mathcal{N}^2$ is simply a constant in this case, it is convenient to absorb it into the rescaling of the potential. The resulting RG equation is then [refer Eq. (20)]

$$\Lambda\frac{\partial\hat{V}_S}{\partial\Lambda}=\hat{p}\frac{\partial\hat{V}_S}{\partial\hat{p}}+2\nu\hat{V}_S+\frac{1}{1-\hat{p}^2}\hat{V}_S^2. \tag{53}$$

From the general analysis in Sec. II, we see that perturbations around the trivial fixed are of the form

$$\hat{V}_S=C_{2n}\Lambda^{2n+2\nu}p^{2n}. \tag{54}$$

Since $\nu>0$, all of the eigenvalues are positive and the fixed point is stable as $\Lambda\to 0$. The term proportional to p^{2n} is of order $d=2n+2\nu-1$ in the corresponding power counting. For nucleon-nucleon scattering in partial waves with $l>0$, there are no bound states or resonances close to threshold and so this fixed point is the appropriate one. The power counting for this case is given by $d=2(n+l)$.

Also of interest is the nontrivial fixed point, which corresponds to a DWERE. To construct it, we start from the basic loop integral of Eq. (28). The cases of integer and noninteger ν behave differently and need to be considered separately. The loop integral can be evaluated to give

$$\begin{aligned}\hat{J}_0(\hat{p})&=\mathcal{P}\int_0^1\hat{q}^{2\nu+1}d\hat{q}\frac{1}{\hat{p}^2-\hat{q}^2}\\&=\frac{1}{2}\sum_{n=0}^{\infty}{}'\,\frac{\hat{p}^{2n}}{n-\nu}+\frac{\pi}{2}\hat{p}^{2\nu}H(\hat{p},\nu),\end{aligned} \tag{55}$$

where

$$H(\hat{p},\nu)=\begin{cases}\cot\pi\nu, & \nu\notin\mathbb{N}\\ 2\ln\hat{p}, & \nu\in\mathbb{N}.\end{cases} \tag{56}$$

The prime on the sum here indicates that the term with $n=\nu$ term must be omitted when ν is an integer.

To obtain a well-behaved short-range potential, we apply the boundary condition of analyticity by cancelling the final, nonanalytic term from $\hat{J}_0$. When ν is noninteger we can do this using the fact that $p^{2\nu}$ satisfies the homogeneous version of RG equation (53). In the case of integer ν, the logarithm of $\hat{p}$ can be cancelled in a manner similar to the logarithm of $\hat{\kappa}$, which appears for the Coulomb potential. This leads to a potential with a logarithmic dependence on Λ, in which the leading logarithms are resummed to all orders. The result in either case can be expressed in the form

$$\frac{1}{\hat{V}_0(\hat{p})}=\hat{J}_0(\hat{p})-\frac{1}{2}\hat{p}^{2\nu}H(\hat{p}\Lambda/\mu,\nu). \tag{57}$$

Once again the full solution to the RG equation is obtained by adding perturbations around this fixed point,

$$\frac{1}{\hat{V}_S}=\frac{1}{\hat{V}_0}-\sum_{n=0}^{\infty}C_{2n}\Lambda^{2n-2\nu}\hat{p}^{2n}. \tag{58}$$

This fixed point is unstable, with the number of negative eigenvalues being governed by ν. If ν lies between the integers $N-1$ and N, the first N perturbations are unstable. If $\nu=N$, then there is also a marginal eigenvector $\hat{p}^{2N}$, which is the origin of the logarithmic behavior. The corresponding coefficient $C_{2N}(\mu)$ depends on the arbitrary scale μ, so that the full potential is μ independent.

The power counting around the nontrivial fixed point is $d=2n-2\nu-1$ for a term proportional to p^{2n}. This is quite different from the counting found for scattering in the presence of the Coulomb potential. Since the inverse-square potential is scale-free, its strength does not provide an expansion parameter in the low-energy EFT. Instead, it appears in the energy power counting itself, determining the number of relevant (unstable) perturbations. In the limit where this strength vanishes, and $\nu\to\frac{1}{2}$, we have precisely the power counting established earlier for a pure short-range potential.

The scattering amplitude for this potential can be calculated as in Sec. II D. The result can then be expanded the form of a DWERE as

$$p^{2\nu}\left(\cot\tilde{\delta}_S-\frac{1}{\pi}H(p/\mu,\nu)\right)=\frac{2}{\pi}\sum_{n=0}^{\infty}C_{2n}p^{2n}. \tag{59}$$

This is an expansion in powers of the energy, which is the only scale in this system. In general, the coefficients have unusual, noninteger dimensions, as a result of the noninteger power of the energy on the left-hand side. For example, the leading coefficient, which is the analog of a modified scattering length, has a dimension of 2ν.

In the case of scattering of a particle with angular momentum l by a short-range potential, we have $\nu=l+\frac{1}{2}$, and there is no nonanalytic energy dependence in $\cot\tilde{\delta}_S=\cot(\delta+l\pi/2)$. The ERE becomes

$$p^{2l+1}\cot\left(\delta+\frac{l\pi}{2}\right)=\frac{2}{\pi}\sum_{n=0}^{\infty}C_{2n}p^{2n}. \tag{60}$$

For $l=0$, this is the familiar s-wave ERE. For $l=1$, we get the p-wave expansion, whose leading term is a scattering volume rather than a length.

IV. DISCUSSION

The techniques described here extend the RG ideas which underline EFT's for low-energy scattering of two heavy particles to systems where the particles interact by a combination of known long-range and unknown short-range forces.

They provide a framework for constructing EFT's for such systems.

A crucial feature of our approach is that we regulate the loop integrals by cutting them off in the basis of DW's for the long-range potential. This ensures that the long-range potential is not modified by the cutoff. As a result, the RG equation has a simple form. In constructing this, it is important to identify all low-energy scales associated with the long-range potential. The method can be applied when the resulting rescaled potential is independent of the cutoff, and so can be treated as part of a fixed point, its effects being resummed to all orders in the DW's.

As in the case of a pure short-range potential, we always find a trivial fixed point. The expansion around this point can be used to describe systems where the additional scattering by the short-range potential is small. The terms in this potential correspond to an expansion of the DW K matrix in powers of the energy and any other low-energy scales.

In some cases we also find an energy-dependent potential which forms a nontrivial fixed point of the RG. In other cases, such a fixed point would have a marginal perturbation, and instead, the potential evolves logarithmically with the cutoff. These potentials describe systems with bound states at threshold, and the expansions around them correspond to DW versions of the effective-range expansion.

We have applied this method to several examples. In the case of the Coulomb potential our method reproduces the well-known Coulomb DWERE. The terms in the effective potential correspond to a double expansion in powers of the energy and the inverse Bohr radius.

The RG analysis in the presence of a Yukawa potential is similar to that for the Coulomb case. In the specific example of the OPE potential in $^1S_0$ nucleon-nucleon scattering, we have to choose whether to treat the "inverse Bohr radius"

$$\kappa_\pi = \frac{g_A^2 M m_\pi^2}{32\pi f_\pi^2} \tag{61}$$

as an additional low-energy scale. In strict chiral counting, the pion mass provides the only low-energy scale and κ_π is of second order in m_π. Treated in this way, the rescaled OPE potential forms a perturbation in an EFT based on a fixed point of a pure short-range potential. This is the basis for the KSW scheme for treating pion-exchange forces.

The alternative WvK scheme treats κ_π as an additional low-energy scale and resums the effects of the OPE potential. The resulting EFT is equivalent to a DWERE. However, although this scheme results in a consistent EFT, the treatment of κ_π as a quantity of first order in low-energy scales means that the connection with the chiral expansion has been lost.

Our final example is the repulsive inverse-square potential, which is relevant to three-body systems such as neutron-deuteron scattering with $J=\frac{3}{2}$, and also to two-body scattering in higher partial waves. This potential is scale-free, and so its strength shows up in the power counting through the RG eigenvalues. Although it is possible to find a DWERE fixed point, the number of unstable perturbations increases with increasing strength of the potential. This implies that delicate fine tuning would be needed for a bound state at threshold. In general, one would expect scattering in such systems to be weak, and to be described by EFT's based on the trivial fixed point.

We are currently extending this approach to describe scattering in the presence of an attractive inverse-square potential, which is more complicated because of the rapid oscillations of the DW's near the origin. The resulting EFT's will be relevant to three-body systems such as neutron-deuteron scattering with $J=\frac{1}{2}$. It will also be interesting to explore more singular potentials, such as OPE in the $^3S_1$-$^sD_1$ channel. Finally, the real power of the EFT's is their ability to form direct connection between observables for different processes. Doing this will require enlarging the present treatment to include couplings to external electromagnetic and weak currents.

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