# Vector mesons in a relativistic point-form approach

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We apply the point form of relativistic quantum mechanics to develop a Poincaré invariant coupled-channel formalism for two-particle systems interacting via one-particle exchange. This approach takes the exchange particle explicitly into account and leads to a generalized eigenvalue equation for the Bakamjian-Thomas type mass operator of the system. The coupling of the exchange particle is derived from quantum field theory. As an illustrative example we consider vector mesons within the chiral constituent quark model in which the hyperfine interaction between the confined quark-antiquark pair is generated by Goldstone-boson exchange. We study the effect of retardation in the Goldstone-boson exchange by comparing with the commonly used instantaneous approximation. As a nice physical feature we find that the problem of a too large $\rho$-$\omega$ splitting can nearly be avoided by taking the dynamics of the exchange meson explicitly into account.

## I. INTRODUCTION

In 1949 Dirac formulated a way of incorporating relativity into quantum theory that differed from quantum field theory [1]. Although Dirac's paper was written in the context of classical mechanics, his methodology—the use of representations of the symmetry group of the theory of special relativity, the Poincaré group—was also applicable to quantum theory *and* to quantum field theory (for a review, see Ref. [2]). In Dirac's original presentation he made evident how to add interactions to a theory of free particles in agreement with the Poincaré algebra, ending up with conditions for the interaction terms that are, in general, nonlinear. In 1953, Bakamjian and Thomas gave a prescription for an explicit construction involving only linear constraints [3].

Of the various forms that Dirac introduced the instant form is the most widely used [4–7], although almost exclusively in the context of quantum field theory. The front form of Hamiltonian dynamics became popular as a natural framework for treating parton phenomena. For a topical review, see Ref. [8]. The point form of relativistic dynamics has also been considered in quantum field theory [9–11], but because of its complicated quantization surface it was not developed further. Only recently has the point form been rediscovered, this time in the context of the quantum mechanics for finite degree-of-freedom systems. Lev has analyzed electromagnetic current operators in the point form and then transformed them to the other two forms [12]. Klink has suggested a basis of states, called velocity states [13], which are suitable for few-body point-form quantum theory. These states have also been introduced by Karmanov in a different context [14,15]. Klink made use of velocity states when treating nuclear physics problems [16–18]. Recently, also the Graz group employed the point form of relativistic dynamics to describe the electroweak structure of baryons within a chiral constituent quark model [19–21].

This paper uses the point form to elaborate on a coupled-channel formalism which is applicable to a wide range of problems. As a first and simple application we have chosen a two-particle system of one (constituent) quark and one (constituent) antiquark which form vector mesons. The hyperfine interaction in this simple system comes from the chiral constituent quark model with pseudoscalar meson exchange. Such an interaction has been used in a semirelativistic form with great success for the calculation of baryon spectra [22,23]. For vector mesons [24] the results within this semirelativistic approach are not as good. In the present paper a fully relativistic calculation with the exchange-meson channel explicitly included is presented and compared to the semirelativistic approach in which the meson exchange is treated in an instantaneous approximation.

The necessary formalism is introduced in detail in Secs. II–V. In Sec. II we summarize relevant features of the Poincaré group. Velocity states are introduced as a suitable basis for the quantum-mechanical treatment of few-particle systems in point form. These states are subsequently used to construct the elementary meson-(anti)quark vertex, which enters the invariant-mass operator. The mass operator is treated in Sec. III; in our example its interacting part arises from a pseudoscalar Hamiltonian density. Our mass operator is of Bakamjian-Thomas type and acts on a Hilbert space that is the direct sum of two-particle and two-plus-one particle Hilbert spaces. Such an ansatz deals with effective degrees of freedom in contrast to a quantum field theory. It leads to the coupled two-channel problem outlined in Sec. IV. The validity of Poincaré invariance for systems with a finite (but not necessarily conserved) number of particles is guaranteed by the Bakamjian-Thomas construction as described in Sec. IV. The fact that quarks and antiquarks are always confined is accounted for by adding harmonic oscillator confinement terms to the square of the kinetic terms of the coupled-channel mass operator. As a first step to solve the eigenvalue problem for the mass operator the two-channel problem is reduced to a one-channel problem with an optical potential that depends on the mass eigenvalue to be determined. The harmonic-oscillator eigenfunctions of the pure confinement problem are then used as a basis for expanding the quark-antiquark wave functions of the full prob-

lem including the hyperfine interaction. As a result, one can discretize the dynamical two-particle equation and obtain a set of coupled algebraic equations that can be solved. The eigenvalues of the mass operator are determined by a resonance condition, which takes into account the nonlinear appearance of eigenvalues in the eigenvalue equation. The structure of the equation and the implementation of confinement are discussed in Sec. V.

Section VI contains some remarks on the numerics and the specific solution method employed in the calculation. Comments on the instantaneous approximation and the choice of the model parameters are given in Secs. VII and VIII, respectively. The results of the calculations are vector-meson masses and some branching ratios of their hadronic decay widths. The numbers are presented in Sec. IX; they are compared with experimental numbers as well as with the instantaneous approximation to elucidate retardation effects coming from the hyperfine interaction. Concluding remarks can be found in Sec. X. Our conventions, normalizations, and matrix elements required for the calculations are summarized in the Appendix.

## II. POINCARÉ GROUP

The starting point for dealing with few-body systems in relativistic quantum mechanics is the set of commutation relations of the Poincaré generators:

$$[P_\mu, P_\nu] = 0, \tag{1}$$

$$[J_{\mu\nu}, P_\kappa] = i(g_{\nu\kappa} P_\mu - g_{\mu\kappa} P_\nu), \tag{2}$$

$$[J_{\mu\nu}, J_{\kappa\lambda}] = -i(g_{\mu\kappa} J_{\nu\lambda} - g_{\nu\kappa} J_{\mu\lambda} + g_{\nu\lambda} J_{\mu\kappa} - g_{\mu\lambda} J_{\nu\kappa}). \tag{3}$$

One can write these relations in a global way by defining $U_\Lambda$ as the unitary operator representing the Lorentz transformation $\Lambda$ on the Hilbert space. In the point form all interactions are contained in the four-momentum operator, so the significant commutation relation is

$$[P^\mu, P^\nu] = 0, \tag{4}$$

which states that the components of the four-momentum commute among each other. The other commutation relation involving $P^\mu$ is written as

$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu{}_\nu P^\nu, \tag{5}$$

which means that the four-momentum operator has to transform as a four-vector under Lorentz transformations. The commutation relations of the Lorentz generators among themselves are unaffected by interactions in the point form. We will refer to Eqs. (4) and (5) later, when we construct mass operators.

Let us start with single-particle states $|p,\sigma\rangle$, e.g., for spin-$\frac{1}{2}$ particles, which transfer irreducibly under the Poincaré group. Defining the action of the four-momentum operator $P^\mu$ on such single-particle states by

$$P^\mu |p,\sigma\rangle = |p,\sigma\rangle p^\mu \tag{6}$$

one can easily show that Eqs. (4) and (5) are satisfied for a single-particle representation. In the following we need also the Poincaré transformation properties of such states:

$$U_b |p,\sigma\rangle = e^{-ibp} |p,\sigma\rangle \quad \text{and} \tag{7}$$

$$U_\Lambda |p,\sigma\rangle = \sum_{\sigma'=\pm 1/2} |\Lambda p,\sigma'\rangle D^{1/2}_{\sigma'\sigma}[R_W(p,\Lambda)]. \tag{8}$$

$U_b$ denotes a space-time translation by a constant four-vector $b$ and the Wigner rotation $R_W(p,\Lambda)$ is given by

$$R_W(p,\Lambda) = B^{-1}\left(\Lambda \frac{p}{m}\right) \Lambda B\left(\frac{p}{m}\right), \tag{9}$$

where $B(v)$ denotes a Lorentz boost with velocity $v$. The $D^{1/2}_{\sigma'\sigma}$ are the matrix elements of the standard Wigner $D$ functions [25].

Concentrating on the single-particle representation of the four-momentum operator we see that Eq. (4) follows immediately from Eq. (6). Applying the left-hand side of Eq. (5) to $|p,\sigma\rangle$ and using Eqs. (6) and (8), we obtain

$$U_\Lambda P^\mu U_\Lambda^{-1}|p,\sigma\rangle = U_\Lambda P^\mu U_{\Lambda^{-1}}|p,\sigma\rangle$$

$$= U_\Lambda P^\mu \sum_{\sigma'} |\Lambda^{-1}p,\sigma'\rangle D^{1/2}_{\sigma'\sigma}[R_W(p,\Lambda^{-1})]$$

$$= U_\Lambda \sum_{\sigma'} |\Lambda^{-1}p,\sigma'\rangle (\Lambda^{-1})^\mu{}_\nu p^\nu$$

$$\times D^{1/2}_{\sigma'\sigma}[R_W(p,\Lambda^{-1})]$$

$$= U_\Lambda U_{\Lambda^{-1}}|p,\sigma\rangle (\Lambda^{-1})^\mu{}_\nu p^\nu$$

$$= U_{\Lambda\Lambda^{-1}}|p,\sigma\rangle (\Lambda^{-1})^\mu{}_\nu p^\nu$$

$$= (\Lambda^{-1})^\mu{}_\nu P^\nu |p,\sigma\rangle, \tag{10}$$

which proves relation (5) for the single-particle representation. In this derivation we have extensively used the representation properties of $U_\Lambda$ and $U_\Lambda^{-1}$, e.g., that

$$U_\Lambda U_{\Lambda^{-1}} = U_{\Lambda\Lambda^{-1}} = U_1 = \mathbf{1}. \tag{11}$$

In the generalization of single-particle states to multiparticle states, it is useful to introduce velocity states, which have simple transformation properties under Lorentz transformations. We start with usual multiparticle momentum states that are tensor products of irreducible representations of the Poincaré group. We observe that under a Lorentz-transformation [see Eq. (7)]

$$U_\Lambda |p_1,\sigma_1,p_2,\sigma_2,\ldots,p_n,\sigma_n\rangle$$

$$= \sum_{\sigma'_1,\sigma'_2,\ldots,\sigma'_n=\pm 1/2} |\Lambda p_1,\sigma'_1,\Lambda p_2,\sigma'_2,\ldots,\Lambda p_n,\sigma'_n\rangle$$

$$\times \prod_{i=1}^n D^{1/2}_{\sigma'_i\sigma_i}[R_W(p_i,\Lambda)], \tag{12}$$

where each of the $D$ functions depends on a *different* Wigner rotation $R_W(p_i,\Lambda)$. This implies that one cannot couple angular momenta in the standard way. So it is desirable to have more uniform transformation properties of $n$-particle states under a Lorentz transformation. This is the case for velocity states. For such states all spin projections and also the individual particle momenta are subject to the *same* Wigner rotation and the effect of the Lorentz transformation goes mainly into the overall velocity $v$. We now show the construction of velocity states in detail and make their Lorentz transformation properties evident.

We consider an $n$-particle system with particle momenta $p_i$ and spin projections $\sigma_i$, $i=1,\ldots,n$, and start by defining internal momenta $k_i$ via

$$k_i = B_c^{-1}(v)p_i, \tag{13}$$

where $B_c(v)$ is a canonical spin boost, i.e., a rotationless Lorentz transformation, which transforms our system from its rest frame to total velocity $\vec{v}$. The momenta $\vec{k}_1,\vec{k}_2,\ldots,\vec{k}_n$ satisfy

$$\sum_{i=1}^{n} \vec{k}_i = 0 \tag{14}$$

and thus only $n-1$ of them are linearly independent. The remaining independent variable is the overall four velocity $v$ of the system.

The construction of a velocity state can be viewed as starting from a multiparticle momentum state in its rest frame. This state is then boosted to overall velocity $v$ by means of the canonical spin boost whose inverse is used in Eq. (13) to yield the velocity state

$$|v,\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\ldots,\vec{k}_n,\mu_n\rangle$$
$$:= U_{B_c(v)}|k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n\rangle \tag{15}$$

$$= \sum_{\sigma_1,\sigma_2,\ldots,\sigma_n=\pm 1/2} |p_1,\sigma_1,p_2,\sigma_2,\ldots,p_n,\sigma_n\rangle$$

$$\times \prod_{i=1}^{n} D_{\sigma_i\mu_i}^{1/2}[R_W(k_i,B_c(v))]. \tag{16}$$

This equation makes evident that a velocity state is a linear combination of multiparticle momentum states. We also note that velocity states transform irreducibly under transformations of the Poincaré group. Concerning notation we, in general, write $\sigma_i$ to denote spin projection variables, but for velocity states and when using internal variables of a system we will instead write $\mu_i$ for spin projections to make a clear distinction between general and internal variables. In this sense the $\sigma_i$ should always appear together with the $p_i$, whereas the $\mu_i$ appear together with the $\vec{k}_i$.

Next we study the Lorentz-transformation properties of a velocity state. We again apply the general boost operator $U_\Lambda$ to the velocity state (15) and combine its action with that of $U_{B_c(v)}$. Using Eq. (9) one obtains

$$U_\Lambda|v,\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\ldots,\vec{k}_n,\mu_n\rangle$$
$$= U_\Lambda U_{B_c(v)}|k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n\rangle$$
$$= U_{\Lambda B_c(v)}|k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n\rangle$$
$$= U_{B_c(\Lambda v)R_W}|k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n\rangle$$
$$= U_{B_c(\Lambda v)}U_{R_W}|k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n\rangle \tag{17}$$

with the Wigner rotation

$$R_W = B_c^{-1}(\Lambda v)\Lambda B_c(v). \tag{18}$$

Evaluating the action of $U_{R_W}$ on the velocity state, we get

$$U_\Lambda|v,\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\ldots,\vec{k}_n,\mu_n\rangle = U_{B_c(\Lambda v)}U_{R_W}|k_1,\mu_1,k_2,\mu_2,\ldots,k_n,\mu_n\rangle$$

$$= U_{B_c(\Lambda v)} \sum_{\mu_1',\mu_2',\ldots,\mu_n'=\pm 1/2} |R_Wk_1,\mu_1',R_Wk_2,\mu_2',\ldots,R_Wk_n,\mu_n'\rangle \prod_{i=1}^{n} D_{\mu_i'\mu_i}^{1/2}[R_W]$$

$$= \sum_{\mu_1',\mu_2',\ldots,\mu_n'=\pm 1/2} |\Lambda v,R_W\vec{k}_1,\mu_1',R_W\vec{k}_2,\mu_2',\ldots,R_W\vec{k}_n,\mu_n'\rangle \prod_{i=1}^{n} D_{\mu_i'\mu_i}^{1/2}[R_W]. \tag{19}$$

In this derivation use has been made of the fact that for canonical spin boosts the Wigner rotation $R_W$ corresponding to a rotation $R$ is the rotation itself. It is helpful to notice here that a rotation is also a Lorentz transformation. One can now clearly see that the rotation appearing in the $D$ functions and in the state is the same for all $\mu_i$ and all $\vec{k}_i$. So one can

couple spins and also orbital angular momenta using the standard addition rules, which was the desired goal.

### III. MASS OPERATOR

In this section we discuss properties of the mass operator and its role in our approach.

In Refs. [2,3] one can find a general procedure for adding interactions to a system of free relativistic particles so that Poincaré invariance is preserved. Such a procedure, called the Bakamjian-Thomas construction, adds an interaction to the free mass operator. The Bakamjian-Thomas construction in the point form involves the (free) four-velocity operator $V_0^\mu$ which is introduced by expressing the free four-momentum operator as

$$P_0^\mu = M_0 V_0^\mu. \tag{20}$$

Interactions are added by perturbing the free mass operator, $M_0 \to M = M_0 + M_I$, in such a way that Eqs. (4) and (5) are satisfied (i.e., the components of the four-momentum must commute with each other and they have to transform as the components of a four-vector under Lorentz transformations). We emphasize once more that this formulation reflects the fact that interactions do not enter the Lorentz generators, but solely the components of the four-momentum. The linear constraints on the interacting part of the mass operator are that it should be a Lorentz scalar and commute with the free four-velocity. The interacting four-momentum operator is then reconstructed by

$$P^\mu = M V_0^\mu, \tag{21}$$

where $M$ contains the interactions and $V_0^\mu$ is still kinematical.

Before starting to construct explicitly an interacting mass operator, we will discuss the free four-momentum and mass operators. Let us first examine the effects of the free four-momentum operator $P_0^\mu$, $\mu = 0,1,2,3$, on the velocity states defined in Sec. II. We consider a system of $n$ free particles with masses $m_i$, internal momenta $\vec{k}_i$, overall four-velocity $v$, and spin projections $\mu_i$. Then we define

$$\mathcal{M}_n := \sum_{i=1}^n \sqrt{m_i^2 + \vec{k}_i^2}, \tag{22}$$

which is the free relativistic mass of the system. Recalling the action of the free four-momentum operator $P_0^\mu$ on usual $n$-particle momentum states, using Eqs. (15), (5), and (22), and evaluating the boost explicitly we get

$$P_0^\mu |v, k_1, \mu_1, k_2, \mu_2, \ldots, k_n, \mu_n\rangle$$

$$= \begin{pmatrix} \mathcal{M}_n \sqrt{1 + \vec{v}^2} \\ \mathcal{M}_n \vec{v} \end{pmatrix} |v, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \ldots, \vec{k}_n, \mu_n\rangle$$

$$= \mathcal{M}_n v^\mu |v, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \ldots, \vec{k}_n, \mu_n\rangle. \tag{23}$$

Hence a velocity state $|v, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \ldots, \vec{k}_n, \mu_n\rangle$ is an eigenstate of $P_0^\mu$ with the eigenvalue $\mathcal{M}_n v^\mu$. Thus we can split $P_0^\mu$ according to Eq. (20) and a velocity state $|v, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \ldots, \vec{k}_n, \mu_n\rangle$ becomes also an eigenstate of $M_0$ and $V_0^\mu$ with the eigenvalues $\mathcal{M}_n$ and $v^\mu$, respectively:

$$M_0 |v, \vec{k}_1, \mu_1, \ldots, \vec{k}_n, \mu_n\rangle = \mathcal{M}_n |v \vec{k}_1, \mu_1, \ldots, \vec{k}_n, \mu_n\rangle, \tag{24}$$

$$V_0^\mu |v, \vec{k}_1, \mu_1, \ldots, \vec{k}_n, \mu_n\rangle = v^\mu |v \vec{k}_1, \mu_1, \ldots, \vec{k}_n, \mu_n\rangle.$$

The decomposition of Eq. (20) is the starting point of the Bakamjian-Thomas construction in point form. This result is also consistent with the common definition of the mass operator

$$M_0^2 = P_0^\mu P_{0\mu}, \tag{25}$$

since the square of the four-velocity is always the identity.

## IV. BAKAMJIAN-THOMAS TYPE VERTEX INTERACTION

In this section we will show how an interacting mass operator that couples $i$ and $(i+1)$ particle channels can be derived from a field theoretical vertex interaction such that it fits into the Bakamjian-Thomas framework. Our presentation takes up the procedure suggested in Ref. [26] to which we also refer for further details. We will set up a model with a finite number of effective degrees of freedom and consider the dynamical equation that describes this system of interacting particles.

On a direct-sum Hilbert space for $i$ and $i+1$ particles the (full interacting) mass operator $M$ becomes a matrix operator

$$M = M_0 + M_I = \begin{pmatrix} \mathcal{D}_i^0 & 0 \\ 0 & \mathcal{D}_{i+1}^0 \end{pmatrix} + \begin{pmatrix} 0 & K^\dagger \\ K & 0 \end{pmatrix}. \tag{26}$$

The two subspaces are coupled by the vertex operator $K$ and $\mathcal{D}_i^0$ denotes a free (indicated by the superscript 0) $i$-particle operator, which corresponds to the relativistic $i$-particle mass (22). In order to obtain an expression for the vertex operator $K$ we consider a field theoretical Hamiltonian density $\mathcal{H}_I(x)$, which describes a vertex interaction and is a polynomial in free fields, meaning that

$$U_\Lambda \mathcal{H}_I(x) U_\Lambda^{-1} = \mathcal{H}_I(\Lambda x). \tag{27}$$

Then $\mathcal{H}_I(0)$ is a Lorentz scalar, since

$$U_\Lambda \mathcal{H}_I(0) U_\Lambda^{-1} = \mathcal{H}_I(0). \tag{28}$$

These properties of $\mathcal{H}_I(0)$ can be used to define $K$. Taking the velocity-state representation and keeping in mind that the whole velocity dependence of a Bakamjian-Thomas-type mass operator in point form is merely a factor $\propto v_0 \delta^3(\vec{v}' - \vec{v})$ (see Appendix) we are led to introduce $K$ via the relation

$$\langle v, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \ldots, \vec{k}_{i+1}, \mu_{i+1} | K | v', \vec{k}_1', \mu_1', \vec{k}_2', \mu_2', \ldots, \vec{k}_i', \mu_i' \rangle$$

$$\propto v_0 \delta^3(\vec{v} - \vec{v}') f(\Delta m) \langle v=0, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \ldots, \vec{k}_{i+1}, \mu_{i+1} | \mathcal{H}_I(0) | v=0, \ \vec{k}_1', \mu_1', \ \vec{k}_2', \mu_2', \ldots, \vec{k}_i', \mu_i' \rangle. \qquad (29)$$

The matrix element on the right-hand side of Eq. (29) has to be understood such that two particles on the "left" and one on the "right" are coupled by $\mathcal{H}_I(0)$ in all possible ways. The remaining particles yield spectator conditions. This means that $M_I$ is constructed from matrix elements of $\mathcal{H}_I(0)$ between velocity states with the *same overall velocity* which furthermore can be taken to be zero since $\mathcal{H}_I(0)$ is a Lorentz scalar. What we have neglected in this kind of construction as compared to the full interacting field theory with vertex interaction $\mathcal{H}_I$ are off-diagonal terms in the overall velocity, which should not occur in a Bakamjian-Thomas-type mass operator. Part of such terms can be simulated with an appropriate choice of the vertex form factor $f[\Delta m]$ which guarantees also that the mass operator is a well defined operator on the Hilbert space. For velocity state matrix elements ($v' = v$) this form factor can be expressed as

$$f[(p' - p)^2] = f[(\mathcal{M}_{i+1}' v' - \mathcal{M}_i v)^2] = f[(\Delta m)^2], \qquad (30)$$

since $v^2 = 1$.

For our subsequent application to vector mesons, matrix element (29) should describe the coupling of a pseudoscalar meson to a quark. It has the particular form

$$\langle v, \vec{k}_1, \mu_1, \vec{k}_2, \mu_2, \vec{k}_3 | K | v', \vec{k}_1', \mu_1', \vec{k}_2', \mu_2' \rangle$$

$$= v_0 \delta^3(\vec{v} - \vec{v}') f[\Delta m] \frac{(2\pi)^3}{\sqrt{(\omega_1 + \omega_2 + \omega_3)^3 (\omega_1' + \omega_2')^3}}$$

$$\times \langle k_1, \mu_1, k_2, \mu_2, k_3 | \mathcal{H}_I(0) | k_1', \mu_1', k_2', \mu_2' \rangle \qquad (31)$$

with $\omega_i = \sqrt{m_i^2 + \vec{k}_i^2}$ denoting the particle energies and $\Delta m = \omega_1' + \omega_2' - \omega_1 - \omega_2 - \omega_3$. $\mathcal{H}_I(0)$ is the pseudoscalar interaction Hamiltonian density

$$-i g_{\text{PS}} \bar{\psi}(0) \gamma_5 \vec{\lambda}_F \psi(0) \cdot \vec{\phi}(0), \qquad (32)$$

evaluated at the space-time point $x = 0$. $\psi$ and $\phi$ are the fermion and boson fields, $\vec{\lambda}_F$ the Gell-Mann flavor matrices, and $g_{\text{PS}}$ is the pseudoscalar quark-meson coupling constant. The kinematical factor in front of the matrix element $\langle k_1, \mu_1, k_2, \mu_2, k_3 | \mathcal{H}_I(0) | k_1', \mu_1', k_2', \mu_2' \rangle$ has been chosen such that $\langle k_1, \mu_1, k_2, \mu_2, k_3 |$ and $| k_1', \mu_1', k_2', \mu_2' \rangle$ can be taken as usual momentum states (with $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$ and $\vec{k}_1' + \vec{k}_2' = 0$).

We now turn to dynamical equations. The bound-state problem of nonrelativistic quantum mechanics is usually reduced to the stationary Schrödinger equation

$$H|\Psi\rangle = |\Psi\rangle E. \qquad (33)$$

In nonrelativistic quantum mechanics the Hamiltonian $H$ is the only generator of the Galilei group that contains interactions, the remaining nine generators are kinematical, i.e., free of interactions. In relativistic quantum mechanics one deals with the Poincaré group instead and due to the different underlying Lie algebra at least three generators contain interaction terms. In the point form, as already mentioned above, interactions are contained in all components of the four-momentum whereas the Lorentz generators are kinematical. This means that one has to solve the system of dynamical equations

$$P^\mu |\Psi\rangle = p^\mu |\Psi\rangle \qquad (34)$$

instead of Eq. (33) to obtain the simultaneous eigenstates of the components of the four-momentum operator $P^\mu$. Within the Bakamjian-Thomas framework the interaction dependence of the four-momentum operator becomes particularly simple [cf. Eq. (21)] and Eq. (34) reduces to

$$M|\Psi\rangle = m|\Psi\rangle. \qquad (35)$$

We note that the $v$ dependence of the total wave function of the system $\Psi(v, \vec{k}_1, \mu_1, \ldots) = \langle v, \vec{k}_1, \mu_1, \ldots | \Psi \rangle$ factors out after projection of Eq. (35) onto velocity states.

Starting from the coupled-channel mass operator of Eq. (26) and eliminating the $i+1$ particle channel one arrives at an equation for only $i$ particles, with one-particle exchange between any two of them. We write the eigenvalue equation for the mass operator of the two-channel problem:

$$\begin{pmatrix} \mathcal{D}_i^0 & K^\dagger \\ K & \mathcal{D}_{i+1}^0 \end{pmatrix} \begin{pmatrix} |\Psi_i\rangle \\ |\Psi_{i+1}\rangle \end{pmatrix} = m \begin{pmatrix} |\Psi_i\rangle \\ |\Psi_{i+1}\rangle \end{pmatrix}, \qquad (36)$$

where $|\Psi_i\rangle$ and $|\Psi_{i+1}\rangle$ are states living on the $i$- and $i+1$-particle subspaces, respectively. $\mathcal{D}_i^0$ and $\mathcal{D}_{i+1}^0$ denote the free $i$- and $i+1$-particle masses, $m$ is the mass eigenvalue of the system, and $K$ is the vertex operator coupling the two channels. From these two coupled equations for $|\Psi_i\rangle$ and $|\Psi_{i+1}\rangle$ the latter can be eliminated to yield

$$K^\dagger (m - \mathcal{D}_{i+1}^0)^{-1} K |\Psi_i\rangle = (m - \mathcal{D}_i^0) |\Psi_i\rangle. \qquad (37)$$

Using velocity states, one can now turn this equation into an integral equation, which is a generalized eigenvalue equation, because $m$ also appears in the propagator ($m - \mathcal{D}_{i+1}^0)^{-1}$. The operator $K^\dagger (m - \mathcal{D}_{i+1}^0)^{-1} K$ acts as an optical (one-particle exchange) potential. This optical potential contains, in principle, also loop contributions in which the exchange particle is reabsorbed by the emitting particle. But since we are interested in studying relativistic few-body systems that describe the dynamics of effective degrees of

freedom we neglect such contributions and assume that they can be absorbed in the (renormalized) mass of the emitting particle.

## V. THE DYNAMICAL EQUATION

So far we have not been very specific about the system we want to investigate. Equation (36) is a general mass eigenvalue equation for any coupled two-channel problem. Up to this point one could use this equation for various different systems as described at the end of this paper. In this section we apply it to a confined system of a constituent quark and a constituent antiquark interacting via pseudoscalar meson exchange, in order to give a description of vector mesons. We make use of the velocity-state representation, so that the Bakamjian-Thomas properties of the interacting mass operator can be fully exploited. Taking only the one-meson exchange dynamics into account, we could immediately start with Eq. (37) with $K$ defined according to Eqs. (31) and (32).

Quarks, however, have not been observed as free particles in nature and are therefore subject to a confining force. For a system of a constituent quark and a constituent antiquark interacting via pseudoscalar meson exchange, $i$ in Eq. (37) is equal to 2 and one has

$$(\mathcal{D}_2^0 - m)|\Psi_2\rangle = K^\dagger(\mathcal{D}_3^0 - m)^{-1}K|\Psi_2\rangle. \qquad (38)$$

The right-hand side now corresponds to an eigenvalue-dependent pseudoscalar meson-exchange potential. In order to introduce confinement in this equation in as simple a manner as possible we modify the relativistic kinetic energy terms $\mathcal{D}_2^0$ and $\mathcal{D}_3^0$ to include harmonic oscillator confinement. As a next step the two-particle wave function is expanded in terms of harmonic oscillator eigenfunctions, which will lead to a discretization of the problem and allow us to apply straightforward techniques for the numerical solution of the equation after a standard partial wave analysis has been carried out.

Confinement is included in Eq. (38) in the diagonal terms in such a way that the two quarks are confined in the two-particle channel as well as in the three-particle channel, whereas the third particle, the exchange boson, is free. As we will argue in the following, this still provides the correct Lorentz-transformation properties. In the two-particle channel one can introduce confinement "by hand" by substituting the free two-particle mass operator by a confinement one, say $\mathcal{D}_2^c$. Such a confinement term *a priori* does not need to be of harmonic oscillator type. The constraint that guarantees Lorentz invariance is that the confinement operator must not depend on the overall velocity and must be a rotational scalar. In the three-particle channel only the two-quark subsystem is confined, which is not at rest and therefore has to be transformed to the correct frame. Using velocity states as a basis, all internal momenta [and also the angular momenta via the corresponding $D$ functions, see Eq. (19)] are rotated by the same rotation as an effect of a Lorentz transformation. Since only scalar products of internal momenta appear in the diagonal terms, nothing changes by an overall rotation and

also the three-particle confinement mass operator has the correct Lorentz-transformation properties.

To introduce confinement in the outlined manner, one has to make the following replacements in Eq. (38):

$$\mathcal{D}_2^0 \rightarrow \mathcal{D}_2^c \text{ and} \qquad (39)$$

$$\mathcal{D}_3^0 \rightarrow \mathcal{D}_3^c. \qquad (40)$$

The operators $\mathcal{D}_2^c$ and $\mathcal{D}_3^c$ are explained in more detail in the Appendix. They are essentially square roots of the usual Schrödinger operator for the three-dimensional isotropic harmonic oscillator. The Appendix contains also the velocity-state representation of these operators as well as our actual notation for the eigenvalues.

## VI. SOLUTION METHOD AND NUMERICS

In the preceding section we encountered an equation that has the structure of an eigenvalue equation, but the eigenvalue also appears in the optical potential. Therefore one cannot directly employ standard techniques for eigenvalue problems. We use the following approach.

One sets the eigenvalue in the optical potential to some preset value $m_{\text{pre}}$ and treats $m_{\text{pre}}$ as a parameter. In this way the equation becomes a linear eigenvalue equation and can be solved using standard techniques to obtain eigenvalues $\lambda_j(m_{\text{pre}})$, $j = 1, 2, \ldots$, (which, of course, depend on the value of $m_{\text{pre}}$).

For different preset values $m_{\text{pre}}$ the resulting spectra will be different.

Interpolating the spectra for different $m_{\text{pre}}$ leads to continuous functions $\lambda_j(m_{\text{pre}})$ of the preset eigenvalue $m_{\text{pre}}$. The graphs of these functions exhibit the $m_{\text{pre}}$ dependence of the spectrum caused by the $m_{\text{pre}}$ dependence of the optical potential.

To find the positions of the generalized eigenvalues $M_j$, $j = 1, 2, \ldots$, one has to solve the equations

$$m = \text{Re}(\lambda_j(m)). \qquad (41)$$

We will call Eq. (41) the "resonance condition."

We take the real part of $\lambda_j(m)$, since the eigenvalues are, in general (at least above the thresholds for the production of the various exchanged mesons), complex numbers. This is justified, because the shifts and widths resulting from the hyperfine interaction are actually perturbations to the pure oscillator spectra. The imaginary part of $\lambda_j(M_j)$ corresponds to the width of the (resonant) state with mass $M_j$, i.e.,

$$\Gamma(M_j) = 2|\text{Im}(\lambda_j(M_j))|. \qquad (42)$$

A procedure of this kind has, e.g., also been adopted in Refs. [27,28].

Our generalized eigenvalue equation for a two-particle system interacting via one-particle exchange without confinement is basically an integral equation. After introducing the confinement terms in the diagonal parts of the coupled-channel mass operator of Eq. (26), it is natural to use a harmonic-oscillator basis for the quark-antiquark wave func-

tion. By expressing the $q$-$\bar{q}$ eigenstate $|\Psi_2\rangle$ in terms of the eigenstates $|v,n,l,s,j,m_j\rangle$ of $\mathcal{D}_2^c$ and taking only a finite number of basis states the eigenvalue equation (38) reduces to a system of coupled algebraic equations from which the mass eigenvalues $m$ and the expansion coefficients $A_{n,l,s,j,m_j}$ [see Eq. (A7)] have to be determined. The matrix elements of the optical potential between the harmonic oscillator eigenstates appear in this set of coupled equations. These are nine-dimensional integrals and involve sums over various quantum numbers. The integrations are done using standard Monte Carlo techniques. Thereby the statistical errors have always been kept smaller than 1 MeV, i.e., smaller than about one per mille. In the course of the numerical calculation the Wigner rotations for the three-particle intermediate-state wave functions [see Eq. (A16)] have been neglected. This approximation seems to be justified by two observations. On the one hand, the pair of three-particle wave functions which shows up in the completeness relation for the three-particle intermediate state must have similar arguments to contribute substantially to the nine-dimensional integral. But this also means that the corresponding Wigner rotations approximately compensate each other. On the other hand, it has been observed in the investigation of baryon form factors within point-form dynamics [19] that Wigner-rotation effects are of minor importance. The numerical effort, by the way, would substantially increase if Wigner rotations were included in our calculations.

## VII. INSTANTANEOUS APPROXIMATION

In order to study the effects of the exchange particle in flight as compared to the standard instantaneous treatment of particle exchange, we perform a nonrelativistic reduction of the optical potential in the point-form mass operator. This is done via standard techniques and goes along with an "instantaneous approximation" of the propagator in the optical potential. "Instantaneous approximation" means that the propagator denominator $(\mathcal{D}_3^c - m)$ is replaced by the energy of the exchanged meson $\sqrt{\vec{q}^2 + m_{\text{Mes}}^2}$. In the nonrelativistic limit the argument of the form factor reduces to the square of the three-momentum of the exchanged meson. Suppressing the flavor part of the hyperfine interaction, one arrives at the well known form for pseudoscalar meson-exchange potential (see, e.g., Ref. [29])

$$V_{\text{NR}}(\vec{k}',\vec{k}) = \frac{g_{\text{ps}}^2}{4\pi} \frac{f^2[\vec{q}^2]}{4m_1 m_2} \frac{(\vec{\sigma}_1 \cdot \vec{q})(\vec{\sigma}_2 \cdot \vec{q})}{\vec{q}^2 + m_{\text{Mes}}^2}, \qquad (43)$$

where $\vec{q}$ is given by

$$\vec{q} = \vec{k}' - \vec{k} \qquad (44)$$

with $\vec{k}$ and $\vec{k}'$ representing center-of-mass momenta of the incoming and outgoing quarks, respectively.

## VIII. MODEL PARAMETERS

We adopt the parametrization of the chiral constituent quark model of Ref. [22] for our actual calculation of the

vector-meson spectrum. In the following the properties and parameters of this model are briefly reviewed. In a constituent-quark model one deals with constituent quarks instead of current quarks; the constituent mass is generated dynamically and is larger than the corresponding current quark mass:

$$m_u = m_d = 340 \text{ MeV} \quad \text{and} \quad m_s = 500 \text{ MeV}$$

turn out to be appropriate mass values for light and strange (constituent) quarks. These numbers can already be obtained approximately from simple quark-model arguments; recent lattice calculations [30,31] also hint at these values. The constituent quarks (and antiquarks) are confined and interact, in addition, via the exchange of the lightest pseudoscalar mesons which are the Goldstone bosons associated with chiral symmetry breaking. The vertex describing this interaction is constructed from the well-known pseudoscalar interaction Hamiltonian density given in Eq. (32). The interaction vertex involves the pseudoscalar coupling constant $g_{\text{PS}}$ between constituent quark and exchange meson and one cutoff parameter $\Lambda_i$, $i = \pi, K, \eta, \eta'$, for each meson. The parameters $\Lambda_i$ occur in the meson–(anti)quark-vertex form factors $f_i(\Delta m)$ [see Eqs. (29) and (30)] that we are using. Following Ref. [22] we have taken

$$f_i(\Delta m) = \sqrt{\frac{\Lambda_i^2 - m_i^2}{\Lambda_i^2 - m_i^2 + \Delta m^2}} \qquad (45)$$

for the functional form of these vertex form factors. The cutoff parameters $\Lambda_i$ are related by

$$\Lambda_i = \Lambda_0 + \kappa m_i, \qquad (46)$$

with $\Lambda_0 = 566.33$ MeV, $\kappa = 0.81$, and $m_i$ being the mass of the pseudoscalar meson of type $i$. These vertex form factors go to one when $\Delta m$ reaches zero[1] and they go to zero like $1/\Delta m$ for $\Delta m \rightarrow \infty$, leading to an additional $1/\Delta m^2$ decay of the exchange potential. In Ref. [22] such a kind of form factor serves to smear out the contact term that occurs when Eq. (43) is transformed to configuration space. The coupling constant $g_8 = g_{\text{PS}}$ for the pseudoscalar octet can be derived from the $N$-$\pi$ coupling constant via the Goldberger-Treiman relation. The value quoted by Glozman *et al.* [22] is $g_{\text{PS}}^2/4\pi = 0.67$. Furthermore, two different coupling constants are used for the pseudoscalar meson octet and singlet, respectively. The ratio of the singlet to octet couplings taken in Ref. [22] is $(g_0/g_8)^2 = 1.34$. For our calculations the charge of the exchange particles is irrelevant; therefore the (small) mass differences between differently charged particles of the same sort, e.g., the $\pi^\pm$ and the $\pi^0$, are neglected. The values

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[1]For $\Delta m = 0$ we have four-momentum conservation at the vertex with all three particles being on-mass-shell. For $t$-channel exchange of massive particles this can, of course, only happen for unphysical momenta, but it is just the kinematical situation (also in instant form) where the influence of the vertex form factor is supposed to vanish and the coupling is supposed to become pointlike.

TABLE I. Parameters for the point form description of vector mesons within the chiral constituent quark model. Apart from $a$ and $V_0$, parameters are taken from Ref. [22].

| Quark masses (MeV) | | | Meson masses (MeV) | | | |
|---|---|---|---|---|---|---|
| $m_u$, $m_d$ | $m_s$ | | $m_\pi$ | $m_K$ | $m_\eta$ | $m_{\eta'}$ |
| 340 | 500 | | 139 | 494 | 547 | 958 |
| Meson-quark coupling | | | | Confinement | | |
| $g_8^2/4\pi$ | $(g_0/g_8)^2$ | $\Lambda_0$ (MeV) | $\kappa$ | $a$ (MeV) | $V_0$ (GeV$^2$) | |
| 0.67 | 1.34 | 566.33 | 0.81 | 312 | $-1.041\,15$ | |

used for the pseudoscalar meson masses are basically the physical masses. As in Ref. [22], we take

$$m_\pi = 140 \text{ MeV}, \quad m_K = 498 \text{ MeV}, \quad m_\eta = 547 \text{ MeV},$$

and $\quad m_{\eta'} = 958$ MeV.

Two more parameters come from the harmonic-oscillator treatment of the quark-antiquark confinement. We denote the eigenvalues of $\mathcal{D}_2^c$, i.e., the square root of the harmonic-oscillator eigenvalues, by

$$M_{nl} = \sqrt{8a^2\left(2n+l+\frac{3}{2}\right)+V_0+4\bar{m}^2}, \qquad (47)$$

where $a$ is the oscillator parameter, $4\bar{m}^2$ contains the rest masses of the quark and antiquark, and $V_0$ leads to an overall shift of the spectrum (for details, see Appendix). Since confinement is introduced in Ref. [22] in a different and not easily comparable way, $a$ and $V_0$ are free parameters. $a$ is fixed in such a way that $M_{00}$ and $M_{10}$ agree with the masses of the ground state and the first excited state of the $\varrho$ spectrum. Doing this we get

$$a \approx 312 \text{ MeV}.$$

This is a reasonable procedure, because the difference ($M_{10} - M_{00}$) is nearly independent of the additional hyperfine interaction. This value for the oscillator parameter $a$ is kept fixed throughout all calculations. From the spectrum of the full calculations including the hyperfine interaction, $V_0$ is fixed to yield the $\varrho$ ground state at 770 MeV. A suitable value for $V_0$ is $V_0 = -1.041\,15$ GeV$^2$. All parameters of the model are summarized in Table I.

We have also done calculations without vertex form factors. For this purpose all parameters are kept the same, only $V_0$ had to be adjusted to yield the $\varrho$ ground state at 770 MeV. One gets a slightly different value, namely, $V_0 = -1.043\,85$ GeV$^2$. The calculations within the instantaneous approximation were performed with the same set of parameters as the corresponding full calculations. Finally we note that the $\omega$ and $\phi$ flavor wave functions used in our calculations are the ones that correspond to ideal mixing of the singlet and octet states of SU(3)$_F$.

## IX. RESULTS AND DISCUSSION

At the beginning of this section we want to emphasize that our primary goal is not an optimal description of the meson spectrum, but rather to demonstrate with a simple model how the multichannel formalism developed works and how it differs from the standard instantaneous treatment of particle exchange. In our calculations we have concentrated on the lowest-lying negative-parity light and strange vector mesons, i.e., mesons with $J^P = 1^-$ ($J$ being the total angular momentum and $P$ the parity of the system). This implies that only $l=0$ and $l=2$ states of the harmonic-oscillator basis can contribute to the $q$-$\bar{q}$ wave function. Whereas $l$ is a good quantum number when taking only the confining interaction into account, $l=0$ and $l=2$ contributions start to mix if the hyperfine interaction is turned on. The numerical analysis, however, reveals that the $l=2$ contributions have practically no effect on the absolute masses (less than or at most 1 MeV, which is also the upper limit for our numerical accuracy). Even if compared to the level shift caused by the hyperfine interaction, the $l=2$ contributions are negligible with the exception of the two excited states of the $\omega$. For these states the $l=2$ contributions amount to 11% (first) and 18% (second excited state) of the total level shift. In all other cases the contributions lack significance since they are smaller than the required numerical accuracy. As already explained in Sec. VI, the solution of the full coupled channel problem involves an expansion of the vector-meson wave functions in terms of harmonic oscillator eigenfunctions. It turns out that already three basis states are enough to obtain convergent results on the per mille level for the ground and the first two excited states. For the instantaneous approximation of the meson exchange the convergence properties are worse. One needs about two times as many basis states as in the calculation with the full optical potential to achieve the required accuracy. It should also be mentioned that at those places where the harmonic oscillator eigenfunctions appear in completeness relations for intermediate states [cf. Eqs. (A4) and (A8)] the upper limits for the main quantum number $n$ and the orbital angular momentum quantum number $l$ have been taken to be the same as in the expansion of the $q$-$\bar{q}$ wave function.

The spectrum of the lowest-lying vector mesons is plotted in Fig. 1. The comparison of the full calculation and the pure confinement result shows that the hyperfine interaction due to (dynamical) Goldstone-boson exchange can be considered as a perturbation. Therefore the qualitative features of the
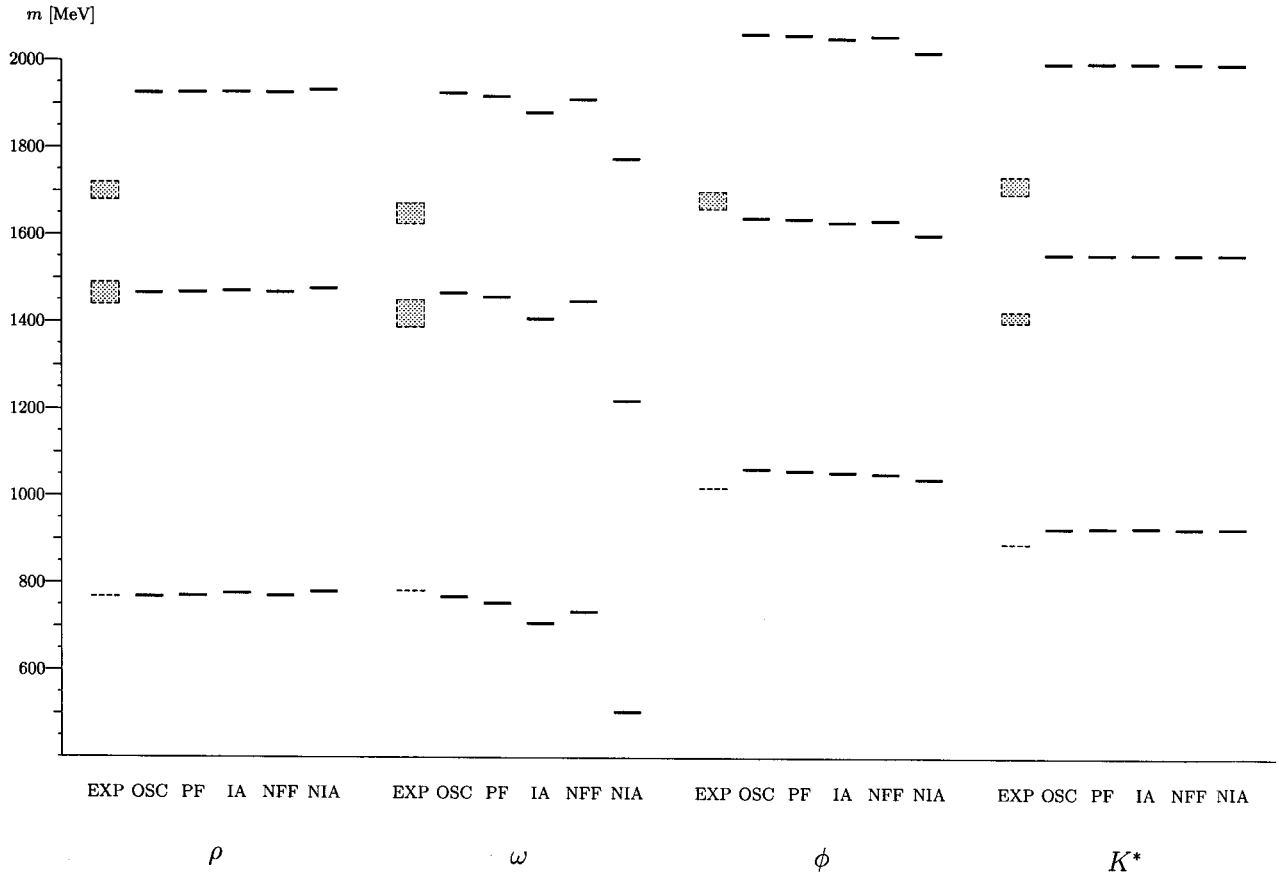
FIG. 1. The spectra for the lowest-lying light and strange vector mesons. The boxes in the columns labeled by "EXP" represent the experimental values with their uncertainties [32]. The other columns give our numerical results for the pure confinement interaction ("OSC"), the full calculation with dynamical meson exchange ("PF"), and the instantaneous approximation to the meson exchange ("IA"). Corresponding results with the meson-quark vertex form factor set to 1 are labeled by "NPF" and "NIA," respectively.

vector-meson spectrum are, in our model, essentially determined by the confinement potential. It is thus not too surprising that only the masses of the ground states and the first excited states are comparable to experiment, whereas the predictions for the second excited states lie already much too high. To obtain also quantitative agreement with experiment, it would certainly be necessary to take a confinement potential which is more sophisticated than our simple harmonic-oscillator confinement. A refined confinement potential which is applicable in momentum-space calculations has, e.g., been suggested in Ref. [33]. But as we said already at the beginning of this section, we rather want to study particle exchange within a relativistic framework and the conclusions about the particle exchange should not depend too much on the specific choice of the additional confinement potential.

The biggest level shifts caused by the hyperfine interaction are detected for the $\omega$ spectrum. This observation can already be anticipated from the fact that the flavor factor at the $\pi$-quark vertex, the pion being also the lightest exchange particle, has its maximum value for the $\omega$ meson. The $\omega$ spectrum is thus also the best place to study the features of our treatment of particle exchange. The differences between the full calculation and the instantaneous approximation are indeed seen to be most prominent in this case. Whereas the usage of a static meson-exchange potential for the hyperfine

interaction leads to an unphysically large splitting of the $\rho$ and $\omega$ ground states, their approximate degeneracy is nearly preserved by our dynamical treatment of the Goldstone-boson exchange. The $\omega$ spectrum is obviously also most sensitive to the choice of the meson-quark vertex form factor. Comparing the results for the standard parametrization of the vertex-form factors (see Sec. VIII) with the outcome for pointlike coupling, i.e., the form factors set to 1, a striking observation can be made: whereas the instantaneous approximation depends very strongly on the form factor, only a mild dependence is seen for the full calculation. The reason for this discrepancy is the difference in the propagators that make up the hyperfine interaction. In the instantaneous approximation it is the (nonrelativistic) meson propagator, in the full optical potential it is rather the propagator of the intermediate $q$-$\bar{q}$-meson state. The $q$-$\bar{q}$ system in the intermediate state is, in addition, subject to confinement that acts as a natural cutoff and damps the dependence on the vertex form factors.

Our approach does not only cover recoil effects in particle exchange, it provides, in principle, also nonperturbative predictions for vector-meson decay widths. As soon as the mass of a vector meson excitation becomes larger than the ground state energy of the confinement potential plus the mass of an exchange meson, the corresponding channel opens and the

pseudoscalar meson can also be emitted leaving a lower lying vector meson. Above such a decay threshold the optical potential and thus the eigenvalues acquire an imaginary part and the width for the decay of the vector-meson resonance into the open two-particle channels can be calculated via Eq. (42). Within our simple two-channel model the decay modes are restricted to $\rho$-$\pi$, $\omega$-$\pi$, and $\rho$-$\eta$. Among the resonances in Fig. 1 there is only one prominent, the $\omega(1420)$, which decays into $\rho$-$\pi$ with a measured width of $174\pm40$ MeV. The experimental information on the other resonance widths for the strong decay into one of the above mentioned two-particle channels is rather poor. Only upper bounds, which are of the order of MeV, are given. Our theoretical results are all below 1 MeV, i.e., below our calculational accuracy. In the outlook we will discuss possible improvements of our model which may also lead to larger decay widths. It seems, however, unlikely that the huge decay width of the $\omega(1420)$ can be explained within a simple two-channel approach. We rather expect that mechanisms other than those included so far, e.g., a strong final-state interaction, have to be taken into account.

## X. SUMMARY AND OUTLOOK

We have presented a Poincaré invariant and Lorentz covariant point-form approach to the dynamical treatment of particle exchange. We have worked within the Bakamjian-Thomas framework, which means that the invariant mass operator takes over the role of the Hamiltonian in nonrelativistic quantum mechanics. Operators and wave functions have been defined with respect to a velocity-state basis. Velocity states are very natural and advantageous for treating relativistic few-body systems within point-form dynamics. The starting point of our approach to particle exchange is a two-channel problem in which the $i$ and $(i+1)$ particle channels are coupled via a vertex interaction that was derived from a field theoretical Hamiltonian density such that the resulting mass operator is of Bakamjian-Thomas type. By reducing the problem to a one-channel problem for the $i$-particle channel we have ended up with an optical potential that describes the dynamics of the particle exchange. The corresponding eigenvalue problem, however, is nonlinear and has to be solved by appropriate means. Since this framework accounts for particle production it is able to provide nonperturbative predictions for (partial) decay widths of resonances.

As a first application of the developed formalism we have investigated vector mesons within the chiral constituent-quark model in which the hyperfine interaction between the confined quark-antiquark pair is mediated by Goldstone-boson exchange, i.e., by the exchange of the lightest pseudoscalar mesons. With a simple harmonic-oscillator confinement and a parametrization of the chiral constituent-quark model that has already been successfully applied for the description of baryon spectra we have found that the hyperfine interaction due to Goldstone-boson exchange causes only small level shifts. Thus it can be considered as a perturbation of the confinement interaction and the confinement potential essentially determines the properties of the mass spectrum.

The comparison of the results for the full optical potential and the standard instantaneous meson-exchange potential revealed sizable differences, in particular for the $\omega$ spectrum. These differences are also reflected in the sensitivity to the parametrization of the meson-quark vertex form factors. Whereas the full calculation depends only mildly on the choice of the vertex form factors, the instantaneous approximation is extremely sensitive to changes in the form factors. Since the meson-quark couplings and the exchange-meson masses are subject to physical constraints, any reasonable parametrization of the Goldstone-boson exchange can thus be expected to provide similar results in the full calculation. Our predictions for vector-meson decay widths lie below the demanded numerical accuracy and thus lack significance.

Our conclusions from the investigation of vector mesons are that a proper relativistic treatment of particle exchange has to go beyond the standard instantaneous approximation and must account for the dynamical behavior of the exchange particle. The predictions for the vector meson spectrum could be improved with a refined confinement interaction. For a reasonable description of resonance widths it may be necessary to extend the optical potential by loop contributions, i.e., contributions in which the emitted meson is again absorbed by the emitting particle. For the present calculation we have assumed that such contributions go as self-energy contributions into the constituent-quark masses. But this is at most an approximation since the (anti)quark in a loop is not free, but confined. Loop contributions have, e.g., also been seen to be important in the semirelativistic treatment of the nucleon-nucleon system if one reaches the pion-production threshold [34]. It will be worthwhile and necessary to investigate their role in our coupled-channel formalism. This formalism should also be useful in treating other relativistic few-body systems that interact via particle exchange. The positronium and hydrogen systems are presently under investigation. They are well studied within instant- and front-form dynamics and would allow for a comparison of the different approaches and forms of relativistic dynamics.

## ACKNOWLEDGMENTS

## APPENDIX: NORMALIZATIONS AND MATRIX ELEMENTS

In this appendix we collect the most important definitions and formulas used in the calculation. We start with some definitions concerning velocity states. Consider a system with overall four-velocity $v$ consisting of $n$ (spin 1/2) fermions with masses $m_i$. Their spins and momenta are uniquely specified by their spin projections $\mu_i$ and momenta $\vec{k}_i$, $i = 1, \ldots, n$ in the overall rest frame of the system. We also define $\omega_i := (m_i^2 + \vec{k}_i^2)^{1/2}$. Then the completeness relation for the $n$-particle velocity states reads

$$\frac{1}{(2\pi)^{3n}}\sum_{\mu_1,\mu_2,\ldots,\mu_n}\int\frac{d^3v}{v_0}\left(\prod_{i=1}^{n-1}d^3k_i\right)\frac{\left(\sum_{i=1}^{n}\omega_i\right)^3}{\prod_{i=1}^{n}2\omega_i}$$

$$\times|v,\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\ldots,\vec{k}_n,\mu_n\rangle$$

$$\times\langle v,\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\ldots,\vec{k}_n,\mu_n|=1; \tag{A1}$$

the corresponding orthogonality relation is

$$\langle v,\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\ldots,\vec{k}_n,\mu_n|v',\vec{k}_1',\mu_1',\ldots,\vec{k}_n',\mu_n'\rangle$$

$$=(2\pi)^{3n}\frac{\prod_{i=1}^{n}2\omega_i}{\left(\sum_{i=1}^{n}\omega_i\right)^3}v_0\delta^3(\vec{v}-\vec{v}')$$

$$\times\prod_{i=1}^{n-1}\delta^3(\vec{k}_i-\vec{k}_i')\prod_{i=1}^{n}\delta_{\mu_i\mu_i'}. \tag{A2}$$

The representation of the *n*-particle free mass operator in the basis of *n*-particle velocity states is

$$\langle v,\vec{k}_1,\mu_1,\ldots,\vec{k}_n,\mu_n|\mathcal{D}_n^0|v',\vec{k}_1',\mu_1',\ldots,\vec{k}_n',\mu_n'\rangle$$

$$=(2\pi)^{3n}\frac{\prod_{i=1}^{n}2\omega_i}{\left(\sum_{i=1}^{n}\omega_i\right)^3}v_0\delta^3(\vec{v}-\vec{v}')$$

$$\times\prod_{i=1}^{n-1}\delta^3(\vec{k}_i-\vec{k}_i')\prod_{i=1}^{n}\delta_{\mu_i\mu_i'}\quad\sum_{j=1}^{n}\sqrt{m_j^2+\vec{k}_j^2}. \tag{A3}$$

A state representing the confined quark-antiquark pair is labeled by the overall velocity $v$ of the pair and the internal (oscillator) quantum numbers $n$ and $l$, the total spin $s$, as well as the total angular momentum $j$ and its projection $m_j$. The completeness relation for such states is

$$\frac{1}{(2\pi)^3}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{s=0}^{1}\sum_{j=|l-s|}^{l+s}\sum_{m_j=-j}^{j}\int\frac{d^3v}{v_0}$$

$$\times\frac{M_{nl}^2}{2}|vnlsjm_j\rangle\langle vnlsjm_j|=1, \tag{A4}$$

where $M_{nl}^2$ are just the harmonic oscillator eigenvalues [see Eq. (A14)] with main quantum number $n$ and orbital angular momentum quantum number $l$. The corresponding orthogonality relation is

$$\langle v'n'l's'j'm_j'|vnlsjm_j\rangle$$

$$=(2\pi)^3\frac{2}{M_{nl}^2}v_0\delta^3(\vec{v}-\vec{v}')\delta_{n'n}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{m_j'm_j}. \tag{A5}$$

The mass operator $\mathcal{D}_2^c$ for the confined pair in the above basis is

$$\langle v'n'l's'j'm_j'|\mathcal{D}_2^c|vnlsjm_j\rangle$$

$$=(2\pi)^3\frac{2}{M_{nl}^2}v_0\delta^3(\vec{v}-\vec{v}')\delta_{n'n}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{m_j'm_j}M_{nl}. \tag{A6}$$

Expansion coefficients $A_{nlsjm_j}$ of the $q\text{-}\bar{q}$ wave function with respect to the harmonic-oscillator basis are defined by

$$\langle vnlsjm_j|\Psi\rangle=\langle vnlsjm_j|V,\Psi_{\text{int}}\rangle$$

$$=(2\pi)^{3/2}\frac{\sqrt{2}}{M_{nl}}v_0\delta^3(\vec{v}-\vec{V})A_{nlsjm_j}. \tag{A7}$$

For our problem of (negative parity) vector mesons $j=1$ and $m_j=-1,0,1$ are fixed. Furthermore, parity restricts spin and orbital angular momentum to $s=1$ and $l=0,2$, so that the coefficients $A_{nlsjm_j}$ depend de facto only on $n$ and $l$. A state describing a system of a quark-antiquark cluster and a free pseudoscalar meson is labeled by the quantum numbers of the cluster, the overall velocity $v$ and the relative momentum between the cluster, and the third particle $\kappa$. The completeness and orthogonality relations for such states are

$$\frac{1}{(2\pi)^6}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{s=0}^{1}\sum_{j=|l-s|}^{l+s}\sum_{m_j=-j}^{j}\int\frac{d^3v}{v_0}$$

$$\times d^3\kappa\frac{(\omega_{\text{cl}}+\omega_3)^3}{2\omega_{\text{cl}}2\omega_3}|v\kappa nlsjm_j\rangle\langle v\kappa nlsjm_j|=1 \tag{A8}$$

and

$$\langle v'\kappa'n'l's'j'm_j'|v\kappa nlsjm_j\rangle$$

$$=(2\pi)^6\frac{2\omega_{\text{cl}}2\omega_3}{(\omega_{\text{cl}}+\omega_3)^3}v_0\delta^3(\vec{v}-\vec{v}')\delta^3(\vec{\kappa}-\vec{\kappa}')$$

$$\times\delta_{n'n}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{m_j'm_j}, \tag{A9}$$

respectively. The factors in the Jacobian are defined by $\omega_{\text{cl}}=\sqrt{M_{nl}^2+\kappa^2}$ and $\omega_3=\sqrt{m_3^2+\kappa^2}$. The mass operator $\mathcal{D}_3^c$ for the confined pair and the free third particle in this basis takes the form

$$\langle v'\kappa'n'l's'j'm_j'|\mathcal{D}_3^c|v\kappa nlsjm_j\rangle$$

$$=(2\pi)^6\frac{2\omega_{cl}2\omega_3}{(\omega_{cl}+\omega_3)^3}v_0\delta^3(\vec{v}-\vec{v}')\delta^3(\vec{\kappa}-\vec{\kappa}')$$

$$\times\delta_{n'n}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{m_j'm_j}(\omega_{cl}+\omega_3).\qquad\text{(A10)}$$

The states defined above can be combined to yield the wave function for the confined quark-antiquark pair. One has

$$\langle\tilde{v},\vec{k}_1,\vec{k}_2,\mu_1,\mu_2|vnlsjm_j\rangle$$

$$=(2\pi)^{9/2}v_0\delta^3(\vec{\tilde{v}}-\vec{v})\frac{\sqrt{2}}{M_{nl}}\sqrt{\frac{2\omega_1 2\omega_2}{(\omega_1+\omega_2)^3}}$$

$$\times\sum_{m_l=-l}^{l}\sum_{m_s=-s}^{s}C_{lm_lsm_s}^{jm_j}C_{(1/2)\mu_1(1/2)\mu_2}^{sm_s}$$

$$\times u_{nl}(|\vec{k}_1|)Y_{lm_l}(\hat{k}_1).\qquad\text{(A11)}$$

The functions $u_{nl}(k)$ are the well-known eigenfunctions of the three-dimensional isotropic harmonic oscillator. Their explicit form is

$$u_{nl}(k)=\frac{1}{\sqrt[4]{\pi}a^{3/2}}\sqrt{\frac{2^{n+l+2}n!}{(2n+2l+1)!!}}L_n^{l+1/2}\left(\frac{k^2}{a^2}\right)\left(\frac{k}{a}\right)^l$$

$$\times e^{-k^2/2a^2},\qquad\text{(A12)}$$

where $L_n^{l+1/2}$ is a generalized Laguerre polynomial. The corresponding normalization integral is

$$\int_0^{\infty}[u_{nl}(k)]^2k^2dk=1.\qquad\text{(A13)}$$

The $Y_{lm_l}$ are the usual spherical harmonic functions. For further details see, e.g., Ref. [35] or [36]. The eigenvalues $M_{nl}$ of $D_2^c$ are the square root of harmonic-oscillator eigenvalues, i.e.,

$$M_{nl}=\sqrt{8a^2(2n+l+3/2)+V_0+4\bar{m}^2}.\qquad\text{(A14)}$$

The well-known oscillator eigenvalues have been modified by adding an overall spectral shift constant $V_0$ and an averaged rest-mass term $\bar{m}^2$ to account for the different masses of the light and strange constituent quarks. The value of $\bar{m}$ in this expression is easily determined for $\varrho$, $\phi$, and $\omega$, since the masses of quark and antiquark are equal in these cases. For the $K^*$ we adopt an averaged mass squared of the form

$$\bar{m}^2=\frac{m_q^2+m_{\bar{q}}^2}{2}.\qquad\text{(A15)}$$

For the system of the confined quark-antiquark pair and the third free particle one can write

$$\langle\tilde{v},\vec{k}_1,\mu_1,\vec{k}_2,\mu_2,\vec{k}_3|v\kappa nlsjm_j\rangle$$

$$=(2\pi)^{15/2}v_0\delta^3(\vec{\tilde{v}}-\vec{v})\delta^3(\vec{\kappa}-\vec{k}_3)\sqrt{\frac{2\omega_{cl}2\omega_3}{(\omega_{cl}+\omega_3)^3}}\sqrt{\frac{2\tilde{\omega}_1 2\tilde{\omega}_2}{2(\tilde{\omega}_1+\tilde{\omega}_2)}}\sqrt{\frac{2\omega_{12}2\omega_3}{(2\omega_{12}+2\omega_3)^3}}$$

$$\times\sum_{m_l=-l}^{l}\sum_{m_s=-s}^{s}\sum_{\tilde{\mu}_1\tilde{\mu}_2=\pm 1/2}C_{lm_lsm_s}^{jm_j}C_{(1/2)\tilde{\mu}_1(1/2)\tilde{\mu}_2}^{sm_s}u_{nl}(|\vec{\tilde{k}}|)Y_{lm_l}(\hat{\tilde{k}})$$

$$\times D_{\mu_1\tilde{\mu}_1}^{1/2}[B_c^{-1}(k_1/m_1)B_c(v_{12})B_c(\tilde{k}_1/m_1)]D_{\mu_2\tilde{\mu}_2}^{1/2}[B_c^{-1}(k_2/m_2)B_c(v_{12})B_c(\tilde{k}_2/m_2)],\qquad\text{(A16)}$$

where $\tilde{k}=B_c^{-1}(v_{12})k_1$,

$$v_{12}=\left(\begin{array}{c}\sqrt{1+\dfrac{\kappa^2}{m_{12}^2}}\\[6pt]-\dfrac{\vec{\kappa}}{m_{12}}\end{array}\right),$$

$\omega_{12}=\sqrt{m_{12}^2+\kappa^2}$, $m_{12}=\tilde{\omega}_1+\tilde{\omega}_2$, $\omega_{cl}=\sqrt{m_{cl}^2+\kappa^2}$, $D\ldots$ are Wigner $D$ functions, and $\tilde{\omega}_i=\sqrt{\tilde{k}_i^2+m_i^2}$.

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