Regularized Legendre series of improved nearside-farside decomposition for charged particle scattering

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A new regularization procedure is proposed for Coulombic Legendre series of improved nearside-farside subamplitudes. The procedure is the extension of the standard one that defines the partial wave series for the scattering amplitude in the presence of a long range Coulomb term in the potential, and it provides the same convergence rate. The new method is applied to a pure Coulomb scattering case and to a $^{16}O+^{16}O$ optical potential at $E_{\rm lab}=145$ MeV.

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In the optical potential analyses of α particles, light and heavy ions, the Fuller nearside-farside (NF) method [1] is an effective tool to separate the full elastic scattering amplitude $f(\theta)$, where θ is the scattering angle, into simpler subamplitudes [2,3]. The Fuller NF subamplitudes are usually more slowly varying and less structured than $f(\theta)$. This allows one to explain the complicated patterns appearing in some cross sections, $\sigma(\theta)=|f(\theta)|^2$, as interference effects between simpler nearside (N) and farside (F) subamplitudes. These subamplitudes can often be interpreted as contributions from simple scattering mechanisms allowing a physical understanding of the scattering process [2].

Sometimes, particularly when applied to scattering of α particles and light heavy ions at intermediate and high energies, the Fuller NF subamplitudes are biased by the presence of unphysical contributions, making the NF subamplitudes more structured than desired. Recently an improved NF method has been proposed [4,5] to further extend the effectiveness of the original Fuller technique. The improved NF method is based on a modified [6] Yennie, Ravenall, and Wilson (YRW) [7] resummation identity, which holds for Legendre polynomial series. The increased effectiveness descends from using resummation parameters with values reducing the unphysical contributions to the Fuller NF subamplitudes.

The Legendre function series (LFS) defining the improved NF subamplitudes are, however, not convergent in the usual sense. A resummation technique [8], named in the following extended YRW (EYRW) resummation, was used in Refs. [4,5] to obtain convergent series. At forward angles, however, the rate of convergence of the EYRW series is not satisfactory in the presence of a long range Coulomb term in the potential. For α particles, light and heavy ions scattering, this fact is disturbing, because it compels one to use more partial waves than necessary in standard optical potential calculations and in the usual Fuller NF method. Here we present a new regularization procedure that, if applied to LFS of improved NF subamplitudes, makes these series as rapidly convergent as those of the conventional Fuller technique.

The starting point for the improved NF method is the quantum mechanical partial wave series of the elastic scattering amplitude

$$f(\theta)=\sum_{l=0}^{\infty} a_l P_l(\cos x), \tag{1}$$

where $x=\cos\theta$, $P_l(x)$ is the Legendre polynomial of degree l, $x\neq 1$, and a_l is given in terms of the scattering matrix element S_l by

$$a_l=\frac{1}{2ik}(2l+1)S_l, \tag{2}$$

where k is the wave number.

We note that the improved NF method uses, on the right-hand side (rhs) of Eq. (2), S_l in place of the usual (S_l-1). The dropped term ensures the convergence of Eq. (1) for scattering by short range potentials, for which $S_l\to 1$ exponentially for $l\to\infty$ (Ref. [9], p. 82). In this case, having dropped from the series a term $\propto\delta(1-x)$, where δ indicates the Dirac distribution (e.g., see Ref. [10], p. 52), the sum in Eq. (1) is defined only in a distributional sense. In the presence of a long range Coulomb term in the potential, the dropped term 1 is not relevant for convergence. With or without 1, the sum in Eq. (1) is convergent only in a distributional sense. The standard trick used to reduce the calculation of $f(\theta)$ to a convergent series is based on adding to Eq. (1) the analytical expression of the Rutherford scattering amplitude and subtracting its *formal* partial wave expasion (e.g., see Ref. [11]). In Refs. [12–16], and in references therein, one can find more or less recent discussions on the convergence of the Coulombic partial wave series, and of other techniques (Padè approximants, Abel summation, or different regularization procedures) solving the problem without using the standard trick.

The improved NF subamplitudes are obtained by using for $f(\theta)$, in place of (1), its resummed form

$$f(\theta)=\left(\prod_{i=0}^{r}\frac{1}{1+\beta_i x}\right)\sum_{n=0}^{\infty}\alpha_n^{(r)}P_n(x), \tag{3}$$

where $r=0,1,2,\ldots,$ and

$$\alpha_n^{(i)}=\beta_i\frac{n}{2n-1}\alpha_{n-1}^{(i-1)}+\alpha_n^{(i-1)}+\beta_i\frac{n+1}{2n+3}\alpha_{n+1}^{(i-1)}, \tag{4}$$

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with $\beta_0=0$, $\alpha_n^{(0)}=a_n$, and $\alpha_{-1}^{(i)}=0$. The resummed form (3) is an *exact mathematical identity* deriving from the recurrence property of the Legendre polynomials. It holds for real or complex values of the resummation parameters $\beta_i(i\neq 0)$, restricted only by the condition $1+\beta_i x\neq 0$, for $-1\leqslant x<1$. The integer index *r* is the order of the resummation, and $r=0$ means no resummation of the original series. In Eq. (3) we changed the index of sum (1) (from *l* to *n*) to remark that the index of the resummed Legendre polynomial series in Eq. (3) has not, for $r\neq 0$, the physical meaning of orbital quantum number, differently from the index of the original series (1). Similarly the coefficients $\alpha_n^{(i)}$ do not have the physical meaning of partial wave amplitudes. The YRW resummed form [7] for $f(\theta)$ is obtained by setting $\beta_i=-1$ $(i\neq 0)$ in Eq. (3).

We note that for pure Coulomb scattering, for which

$$\alpha_n^{\mathrm{C}(0)}\equiv a_n^{\mathrm{C}}=\frac{1}{2ik}(2n+1)\frac{\Gamma(n+1+i\eta)}{\Gamma(n+1-i\eta)}, \tag{5}$$

where η is the Sommerfeld parameter, by using Eq. (4) one obtains, for large *n* values,

$$\alpha_n^{\mathrm{C}(1)}=[1+\beta_1+O(n^{-2})]\alpha_n^{\mathrm{C}(0)}. \tag{6}$$

This means that for $\beta_i\neq -1$ the asymptotic Coulombic behavior of $\alpha_n^{\mathrm{C}(r)}$ does not depend, apart from a renormalization factor, on the resummation order *r*. On the other hand, given a resummed series of Legendre polynomials, of order *r* (eventually 0), by applying one additional YRW $(\beta_{r+1}=-1)$ resummation a convergent series is obtained for asymptotically Coulombic $\alpha_n^{(r)}$. Any successive YRW resummation improves the convergence of the series by a factor $O(n^{-2})$.

The improved NF subamplitudes are obtained by splitting in Eq. (3) the $P_n(x)$ into traveling angular components

$$P_n(x)=Q_n^{(-)}(x)+Q_n^{(+)}(x), \tag{7}$$

where (for $x\neq\pm 1$)

$$Q_n^{(\mp)}(x)=\frac{1}{2}\left[P_n(x)\pm\frac{2i}{\pi}Q_n(x)\right], \tag{8}$$

with $Q_n(x)$ being the Legendre function of the second kind of degree *n*. By inserting Eq. (7) into Eq. (3), $f(\theta)$ is separated into the sum of two subamplitudes

$$f(\theta)=f_{\{\beta\}}^{(-)}(\theta)+f_{\{\beta\}}^{(+)}(\theta), \tag{9}$$

with

$$f_{\{\beta\}}^{(\mp)}(\theta)=\left(\prod_{i=0}^{r}\frac{1}{1+\beta_i x}\right)\sum_{n=0}^{\infty}\alpha_n^{(r)}Q_n^{(\mp)}(x). \tag{10}$$

In Eq. (10), the subscript $\{\beta\}$ indicates that the N ($f_{\{\beta\}}^{(-)}$) and F ($f_{\{\beta\}}^{(+)}$) subamplitudes depend, differently from $f(\theta)$, on the resummation order *r* and parameters β_i. This occurs because the resummed form of the series of linear combination of the first and second kind Legendre functions, of integer degree (LFS), is different from Eq. (3). In fact, let us indicate with

$$\mathcal{F}(\theta)=\sum_{n=0}^{\infty}d_n\mathcal{L}_n(x) \tag{11}$$

a LFS in $\mathcal{L}_n(x)=pP_n(x)+qQ_n(x)$, with *p* and *q* independent of *n*. Owing to the property $nQ_{n-1}(x)\to 1$ as $n\to 0$ [8], the resummed form of $\mathcal{F}(x)$, of order *s* and parameters γ_i, is (see Eq. (19) of Ref. [5])

$$\mathcal{F}(\theta)=\left(\prod_{i=0}^{s}\frac{1}{1+\gamma_i x}\right)\sum_{n=0}^{\infty}\delta_n^{(s)}\mathcal{L}_n(x)+q\sum_{i=0}^{s}\gamma_i\delta_0^{(i-1)}\prod_{j=0}^{i}\frac{1}{1+\gamma_j x}. \tag{12}$$

Equation (12) is an *exact mathematical identity* extending the validity of Eq. (3) to more general LFS, and it reduces to Eq. (3) for Legendre polynomials series $(q=0)$. The conditions of validity of Eq. (12), and the recurrence relation for the resummed coefficients, are the same as those for Eq. (3), after substituting r,β,α, and *a* with s,γ,δ, and *d*, respectively.

Because the $Q_n^{(\mp)}(x)$ used to split $P_n(x)$ in Eq. (7) are a particular case of the more general $\mathcal{L}_n(x)$ (with $p=1/2$, and $q=\pm i/\pi$), the presence of the last term in Eq. (12) is responsible for the dependence of $f_{\{\beta\}}^{(\mp)}(\theta)$ on *r* and β_i. The last term on the rhs of Eq. (12) gives a contribution if splitting (7) is inserted in Eq.(1). This contribution is absent if the splitting is inserted in Eq. (3).

In Refs. [4,5] it was observed that unphysical contributions, when appearing in the Fuller NF subamplitudes [$r=0$ in Eq. (10)], decrease by increasing *r* in Eq. (10) (the values $r=1$, and 2 were tested), if the β_i are selected to make null the coefficients $\alpha_0^{(r)},\alpha_1^{(r)},\ldots,\alpha_{r-1}^{(r)}$ of the resummed LFS ($\alpha_0^{(1)}$ and $\alpha_{0,1}^{(2)}$ for the cases tested). In this way one drops the contributions to the NF resummed LFS (10) from low *n* values, for which the splitting (7) (though exact by construction) is not expected to be physically meaningful.

The $\alpha_n^{(r)}$ in Eqs. (3) and (10) go asymptotically to a constant for short range potentials, or are Coulombic in the presence of a Coulomb term in the potential. Because of this the series defining the full, N, and F amplitudes are not convergent in the usual sense. In Refs. [4,5] the convergence of Eqs. (3) and (10) was forced, and accelerated, by applying to the improved LFS a final EYRW resummation (12), of order $s\geqslant 1$, with $d_n=\alpha_n^{(r)}$, $\gamma_i=-1$, and $i\neq 0$. The final EYRW resummation ensures the numerical convergence of the LFS, with a convergence rate increasing with *s*. The increased rate of convergence costs, however, the cancellation of significant digits (see Ref. [8] for details), and numerically the procedure may be not convenient or, in some cases, even impossible, using arithmetic with a fixed digit number.

These troubles can be avoided by investigating the properties of the resummation identity (12) with d_n equal to the

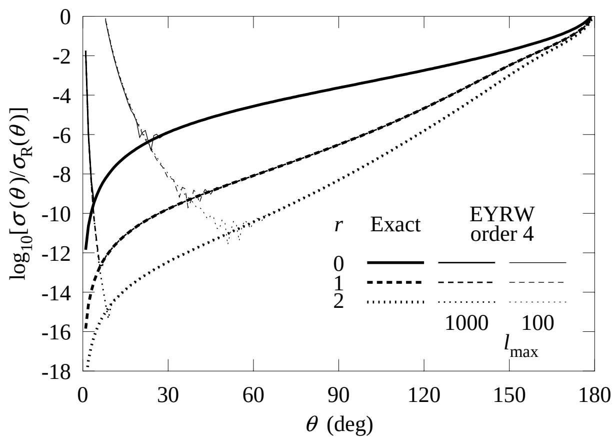

FIG. 1. The thick curves show the order $r=0$, 1, and 2 improved F pure Coulomb cross sections calculated using Eq. (13). The legend ''Exact'' indicates that in this case the sum over n in Eq. (13) is null. The other curves show the results obtained using 1000 and 100 partial waves (medium thickness and thin curves, respectively) and applying an EYRW resummation Eq. (12) of order 4 to Eq. (10), as explained in the text.

pure Coulomb a_n^{C} given by Eq. (5). In this case we explicitly know the lhs of Eq. (12) for the physically interesting p and q values. In fact, if $p=1$ and $q=0$, one obtains the Rutherford scattering amplitude $f_{\mathrm{R}}(\theta)$, while for $p=1/2$ and $q=\pm i/\pi$ one obtains the Fuller-Rutherford NF subamplitudes $f_{\mathrm{FR}}^{(\mp)}(\theta)$ [Ref. [1] Eqs. (14a) and (14b)]. This is because the series in the rhs of Eq. (11) are the *formal* LFS of these amplitudes.

Because Eq. (12) is exact, it holds for arbitrary γ_i, and therefore also for $\gamma_i=\beta_i$, with β_i obtained by applying the improved resummation method to any exact optical potential S_l with a Coulombic asymptotic behavior. With this choice, the pure Coulomb resummed coefficients $\alpha_n^{\mathrm{C}(r)}$ asymptotically approach $\alpha_n^{(r)}$ as rapidly as the pure Coulomb S-matrix elements S_l^{C} approach S_l in the usual optical potential calculations.

With the change of notation $f^{(0)}\equiv f$, $f^{(\mp 1)}\equiv f_{\{\beta\}}^{(\mp)}$, $f_{\mathrm{R}}^{(0)}\equiv f_R$, $f_{\mathrm{R}}^{(\mp 1)}\equiv f_{\mathrm{FR}}^{(\mp)}$, $\mathcal{L}_n^{(0)}\equiv P_n$, and $\mathcal{L}_n^{(\mp 1)}\equiv Q_n^{(\mp)}$, by subtracting from Eq. (3), or Eq. (10), the corresponding resummed forms, Eq. (12), applied to pure Coulomb scattering ($s=r, \gamma_i=\beta_i$, $\delta_n^{(s)}=\alpha_n^{\mathrm{C}(r)}$ and $q=0,\pm i/\pi$), one obtains the final result

$$f^{(m)}(\theta)=\left(\prod_{i=0}^{r}\frac{1}{1+\beta_i x}\right)\sum_{n=0}^{\infty}[\alpha_n^{(r)}-\alpha_n^{\mathrm{C}(r)}]\mathcal{L}_n^{(m)}(x)+f_{\mathrm{R}}^{(m)}(\theta)+m\frac{i}{\pi}\sum_{i=0}^{r}\beta_i\alpha_0^{\mathrm{C}(i-1)}\prod_{j=0}^{i}\frac{1}{1+\beta_j x},\tag{13}$$

with $m=0$ for the full amplitude and $m=\mp 1$ for the NF subamplitudes.

For $r=0$ and $m=0$, or $m=\mp 1$, Eq. (13) is the standard regularization procedure defining the rhs of Eq. (3), or Eq. (10), in the presence of a long range Coulomb term in the potential. This procedure is based on adding, to the original *formal* series, the explicit expression of $f_{\mathrm{R}}(\theta)$, or $f_{\mathrm{FR}}^{(\mp)}(\theta)$, and subtracting its *formal* Legendre polynomials, or traveling angular waves, series for the full amplitude [11], or the Fuller NF subamplitudes [1]. One obtains a convergent series by combining the two *formal* series together.

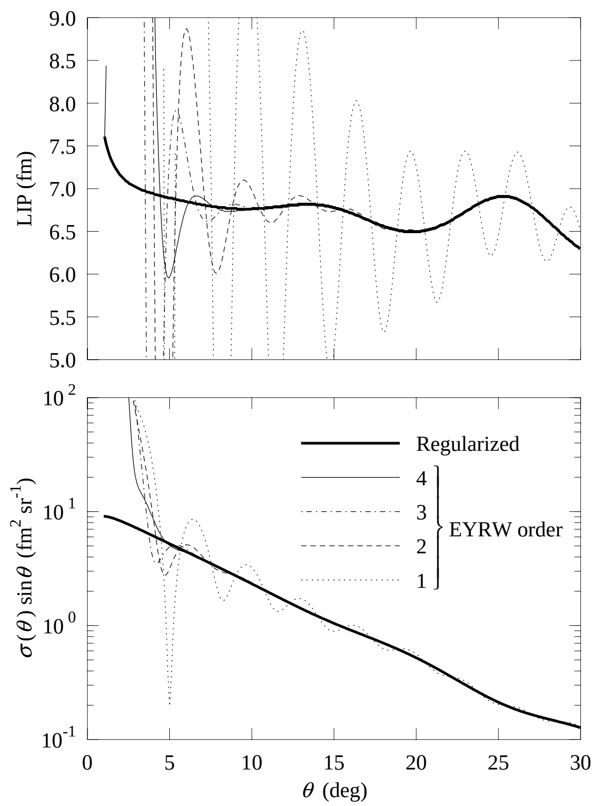

FIG. 2. First order ($r=1$) improved F cross section (lower panel) and LIP (upper panel) for the $^{16}O+^{16}O$ case. The calculations were done using 150 partial waves with our regularization procedure (13) (thick curves) and with different orders of the final EYRW resummation (12) (thin curves).

For $r\geqslant 1$, Eq. (13) is the generalization of this regularization procedure to resummed forms of the full amplitude, or NF subamplitudes. The sum appearing in Eq. (13) is, for $r\geqslant 1$, as rapidly convergent as the sum obtained with the standard regularization procedure for $r=0$.

Before showing the effectiveness of our regularization procedure in a physically interesting case, we show the difficulties met by the EYRW technique [8] to ensure, and speed up, the convergence of improved, or not, LFS for pure Coulomb scattering. In this case $a_n\equiv a_n^{\mathrm{C}}$, and each term of the LFS on the rhs of Eq. (13) is identically null, with an arbitrary choice of β_i. For $r=0$ (no resummation), Eq. (13) trivially states that the scattering amplitude ($m=0$) is the Rutherford amplitude, and the NF subamplitudes ($m=\mp 1$) are the Fuller-Rutherford ones. For $r>0$, by choosing β_i according to the improved resummation method, Eq. (13) gives the explicit expressions for the improved NF subamplitudes ($m=\mp 1$) in terms of the Fuller-Rutherford ones, and of simple functions depending on β_i and $\alpha_0^{\mathrm{C}(i-1)}$. For simplicity, we will name *exact* the explicit expression for pure Coulomb improved NF subamplitudes.

In Fig. 1 the thick curves show the ratio to the Rutherford cross section, $\sigma_R(\theta)$, of the exact pure Coulomb improved F cross sections, of order $r=0$, 1, and 2 ($r=0$ meaning the original Fuller method). In the same figure the thin curves show the F cross sections obtained by forcing, and accelerating, the convergence of Eq. (10) with an additional EYRW resummation of order $s=4$, and fixing the maximum number of the summed partial waves to $l_{max}=100$ and 1000. The results were obtained with $\eta=10$, which is a typical value of the Sommerfeld parameter for heavy-ion scattering. For this η value the improved resummation parameters are $\beta_1=0.9802+0.1980i$ (for $r=1$), $\beta_1=1.0072+0.1166i$ and $\beta_2=0.7804+0.6052i$ (for $r=2$). Figure 1 shows that the final EYRW resummation [8] ensures the convergence of the LFS (10), but the convergence rate is low. For $\theta\lesssim5°$ a numerically satisfactory result is not obtained even with $l_{max}=1000$. By fixing l_{max} and the final resummation order, the angle at which the truncated LFS disagrees with the exact result increases with the improved resummation order.

Figure 1 also shows that the improved resummation method reduces, particularly at forward angles, the unphysical F contribution present in the original Fuller NF method. However, the unphysical contribution is not suppressed. Also the improved method is rather ineffective at $\theta\approx180°$. This is an insurmountable difficulty connected with the NF splitting (7), mathematically continuing (at $\theta=180°$) the N subamplitude into a F one, or vice versa. This holds also in the absence of physically meaningful subamplitudes justifying this continuation. In these situations the only practical suggestion we can give is to not take seriously the NF subamplitudes at $\theta\approx180°$, if in the neighborhood of this angle the cross section and the LIP (local impact parameter) of the full amplitude have a nonoscillatory behavior, suggesting the dominance of a *single side* (positive LIP for F and negative for N) contribution. We recall that the LIP is defined [4] as the derivative of the argument of the scattering amplitude with respect to the scattering angle divided by the wave number k.

As a second example of the effectiveness of our regularization procedure, we consider the first order improved F cross section and LIP of the phenomenological optical potential WS2, used to fit [17] the $^{16}O+^{16}O$ elastic cross section at $E_{lab}=145$ MeV. The improved resummation parameter is in this case $\beta_1=-0.9997-0.0798i$ [5]. The upper panel in Fig. 2 shows, for $1°\leqslant\theta\leqslant30°$ and $l_{max}=150$, the F LIP calculated using our regularization procedure (thick curve) and different order (thin lines) EYRW resummations. The lower panel shows the corresponding F cross sections. Symmetrization effects were ignored.

Note that 150 partial waves are more than really necessary to obtain reliable scattering amplitudes using our regularization procedure. Using an EYRW resummation of order 1 (thin dotted curves), this partial wave number is not sufficient to obtain a satisfactory result. By increasing the EYRW resummation order it decreases the angular width of the region where the thin curves differ from the corresponding thick ones. However, for $\theta\lesssim5°$, the 150 partial waves used are not enough, even using a fourth order final EYRW resummation.

The new regularization procedure in Eq. (13) is very effective in ensuring a rapid convergence of the LFS, which define the improved NF subamplitudes for charged particles scattering. On the contrary, for asymptotically Coulombic S_l, the effectiveness of the EYRW method is computationally poor. The regularization procedure described here can be easily modified to make the LFS in Eqs. (3) and (10) rapidly convergent for scattering by short range potentials. In these cases, however, also an additional first order EYRW resummation makes the LFS convergent with the same rapidity.

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