Induced pseudoscalar form factor of the nucleon at two-loop order in chiral perturbation theory

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We calculate the imaginary part of the induced pseudoscalar form factor of the nucleon $G_P(t)$ in the framework of two-loop heavy baryon chiral perturbation theory. The effect of the calculated three-pion continuum on the pseudoscalar constant $g_P=(m_\mu/2M)G_P(t=-0.877m_\mu^2)$, measurable in ordinary muon capture $\mu^- p\to\nu_\mu n$, turns out to be negligibly small. Possible contributions from counterterms at two-loop order are numerically smaller than the uncertainty of the dominant pion-pole term proportional to the pion-nucleon coupling constant $g_{\pi N}=13.2\pm0.2$. We conclude that a sufficiently accurate representation of the induced pseudoscalar form factor of the nucleon at low momentum transfers t is given by the sum of the pion-pole term and the Adler-Dothan-Wolfenstein term: $G_P(t)=4g_{\pi N}Mf_\pi/(m_\pi^2-t)-2g_AM^2\langle r_A^2\rangle/3$, with $\langle r_A^2\rangle=(0.44\pm0.02)$ fm^2 the axial mean square radius of the nucleon.

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The structure of the nucleon as probed by charged weak currents is encoded in two form factors, the axial and the pseudoscalar ones. To be specific, consider the matrix element of the isovector axial current between nucleon states:

$$\langle N(p+k)|\bar{q}\gamma^\nu\gamma_5\tau_a q|N(p)\rangle=\bar{u}(p+k)\left[\gamma^\nu G_A(t)+\frac{k^\nu}{2M}G_P(t)\right]\gamma_5\tau_a u(p), \tag{1}$$

where $t=k^2$ denotes the Lorentz-invariant squared momentum transfer and $u(p)$ stands for a Dirac spinor. $M=938.92$ MeV is the (average) nucleon mass. The form in Eq. (1) follows from Lorentz covariance, isospin conservation, and the discrete symmetries *C*, *P*, and *T*. $G_A(t)$ is called the axial form factor of the nucleon and $G_P(t)$ is the induced pseudoscalar form factor of the nucleon. While experimentally much attention has been focussed on the first one, the latter is generally believed to be well understood in terms of pion-pole dominance as indicated from ordinary muon capture experiments $\mu^- p\to\nu_\mu n$ (see, e.g., Refs. [1–3]). The pseudoscalar coupling constant g_P as measured in ordinary muon capture is defined via

$$g_P=\frac{m_\mu}{2M}G_P(t_\mu),$$
$$t_\mu=\frac{M_n^2m_\mu}{M_p+m_\mu}-M_pm_\mu=-0.877m_\mu^2=-0.502m_\pi^2, \tag{2}$$

with t_μ the Lorentz-invariant squared momentum transfer if the proton and muon are initially at rest. $m_\mu=105.66$ MeV is the muon mass, $M_p=938.27$ MeV is the proton mass, $M_n=939.57$ MeV is the neutron mass, and $m_\pi=139.57$ MeV is the charged pion mass.

Chiral perturbation theory allows to calculate systematically the corrections to the dominant pion-pole term in $G_P(t)$ [see Eq. (7) below]. At one-loop order, this correction is uniquely expressed in terms of the mean square axial radius of the nucleon $\langle r_A^2\rangle$ by making use of the chiral Ward identity of QCD [4]. Exactly the same term, derived originally by Adler, Dothan and Wolfenstein [5], is also found in the small scale expansion of Ref. [6] where additional diagrams with intermediate $\Delta(1232)$-isobars contribute. While the one-loop prediction for the pseudoscalar coupling constant $g_P=8.4\pm0.2$ [4,6] is consistent with the earlier result of the Saclay experiment $g_P=8.7\pm1.9$ [2] a reanalysis [1] of that experiment using the modern world average of the muon mean lifetime gives the enhanced value $g_P=10.6\pm2.7$. For further details on that and the conflicting situation concerning radiative muon capture $\mu^- p\to\nu_\mu n\gamma$ see the recent review of Gorringe and Fearing [1] and also Ref. [3].

The purpose of the present short paper is to investigate the two-loop corrections to the induced pseudoscalar form factor $G_P(t)$ in order to clarify whether these could affect (numerically) the theoretical prediction for g_P used so far. The essentially new feature at two-loop order is a nonvanishing imaginary part $\mathrm{Im}\,G_P(t)$ for $t>9m_\pi^2$, which originates from the (direct and indirect) coupling of the isovector axial current to the three-pion intermediate state. The pertinent ten topologically distinct two-loop diagrams generated by leading order vertices of the effective chiral Lagrangian $\mathcal{L}_{\pi\pi}^{(2)}+\mathcal{L}_{\pi N}^{(1)}$ are shown in Fig. 1. The Feynman rules for the relevant interaction vertices can be found in Appendix A of Ref. [7].

Let us now turn to the evaluation of the imaginary part $\mathrm{Im}\,G_P(t)$ from the two-loop diagrams shown in Fig. 1. Application of the Cutkosky cutting rules gives the spectral function $\mathrm{Im}\,G_P(t)$ as an integral of the product of *axial source*$\to3\pi$ and $3\pi\to\bar{N}N$ transition amplitude over the Lorentz-invariant three-pion phase space. Some details about these techniques can be found in Refs. [8,9], where the same method has been used to calculate the (two-loop) spectral functions of the isoscalar electromagnetic nucleon form factors and the 3π-exchange nucleon-nucleon potential. The choices $\epsilon\cdot k=0$ and $\epsilon^\nu=k^\nu$ for the polarization vector ϵ^ν of the external isovector axial source allow a separation of $\mathrm{Im}\,G_A(t)$ and $\mathrm{Im}[G_A(t)+tG_P(t)/4M^2]$. Alternatively, one

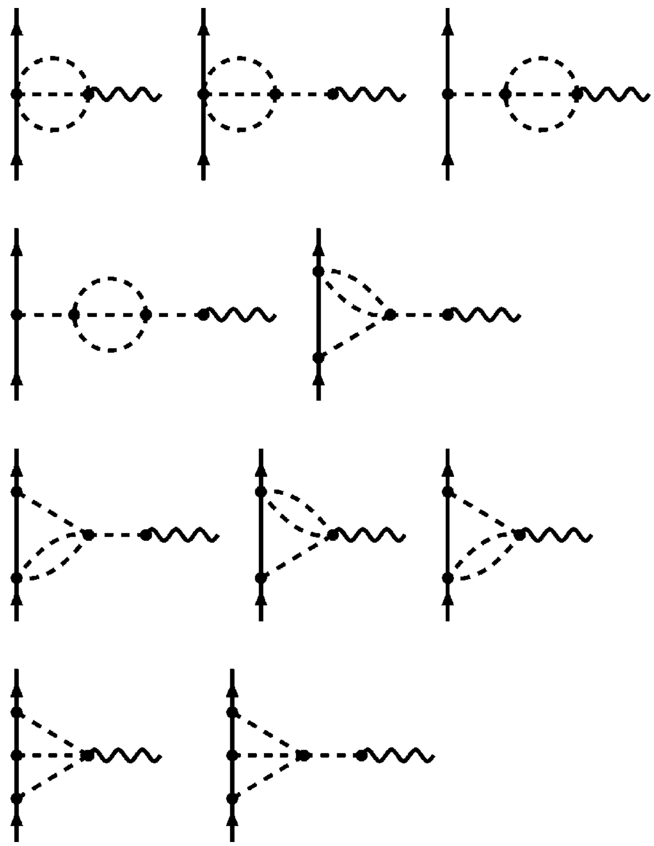

FIG. 1. Two-loop diagrams contributing to the imaginary part of the induced pseudoscalar form factor of the nucleon $G_P(t)$. Dashed and solid lines denote pions and nucleons, respectively. The wiggly line symbolizes the external isovector axial source. The combinatoric factor of the first four diagrams is 1/6 and the next four graphs have the combinatoric factor 1/2. The last two diagrams scale as g_A^3 whereas the other eight graphs scale as g_A.

can use projection operators and tensorial integrals over the 3π-phase space can be reduced to scalar ones (see Eq. (13) in Ref. [8]). The pertinent three-body phase space integrals are most conveniently performed in the three-pion center-of-mass frame. The corresponding on-mass-shell four-momenta of the three pions read in this frame: $k_1^\nu=(\omega_1,\vec{k}_1)$, $k_2^\nu=(\omega_2,\vec{k}_2)$ and $k_3^\nu=(\sqrt{t}-\omega_1-\omega_1,-\vec{k}_1-\vec{k}_2)$. The mass-shell condition $k_3^2=m_\pi^2$ determines the cosine of the angle between $\vec{k}_1$ and $\vec{k}_2$ (called z) as

$$zk_1k_2=\omega_1\omega_2-\sqrt{t}(\omega_1+\omega_2)+\tfrac{1}{2}(t+m_\pi^2),$$
$$k_{1,2}=\sqrt{\omega_{1,2}^2-m_\pi^2}. \tag{3}$$

The ten diagrams in Fig. 1 fall into two classes. The first eight diagrams carrying the common prefactor g_A/f_π^4 give rise altogether to the following contribution to the imaginary part of the induced pseudoscalar form factor of the nucleon:

$$\mathrm{Im}\,G_P^{(1)}(t)=\frac{g_{\pi N}M}{(2\pi f_\pi)^3}\int_{z^2<1}d\omega_1 d\omega_2\left\{\frac{1}{18}-\frac{m_\pi^4}{12(t-m_\pi^2)^2}+\frac{4\omega_1^2-m_\pi^2}{6t}+\frac{\omega_1^2(3m_\pi^2-t)}{(t-m_\pi^2)^2}+\frac{2m_\pi^2\omega_1\omega_2 zk_2}{t(t-m_\pi^2)k_1}\right\}. \tag{4}$$

Here, $f_\pi=92.4$ MeV denotes the pion decay constant and we have employed the Goldberger-Treiman relation: $g_{\pi N}f_\pi=g_AM$. The inequality $z^2<1$ determines the kinematically allowed region in the $\omega_1\omega_2$ plane (which is bounded by a cubic curve) together with the obvious kinematical constraints $m_\pi<\omega_{1,2}<\sqrt{t}-2m_\pi$ and $2m_\pi<\omega_1+\omega_2<\sqrt{t}-m_\pi$. Furthermore, one derives from the last two diagrams in Fig. 1 which are proportional to g_A^3/f_π^4 the following contribution to the imaginary part $\mathrm{Im}\,G_P(t)$:

$$\mathrm{Im}\,G_P^{(3)}(t)=\frac{g_{\pi N}Mg_A^2}{(2\pi f_\pi)^3 t}\int_{z^2<1}d\omega_1 d\omega_2\left\{(m_\pi^2-\sqrt{t}\omega_1)\times\left(z+\frac{k_2}{k_1}\right)\frac{\arccos(-z)}{\sqrt{1-z^2}}+\frac{k_1^2}{3}+\frac{t}{9}+\frac{m_\pi^2}{t-m_\pi^2}\left(\frac{7}{8}\sqrt{t}-\omega_1-\omega_2\right)\times\left[2\omega_1\frac{zk_2}{k_1}+\sqrt{t}+[(t+m_\pi^2)(4\omega_1-\sqrt{t})-4\sqrt{t}\omega_1\omega_2]\frac{\arccos(-z)}{2k_1k_2\sqrt{1-z^2}}\right]\right\}. \tag{5}$$

In the chiral limit $m_\pi=0$ the total (two-loop) spectral function $\mathrm{Im}\,G_P(t)$ shows a simple linear t dependence of the form

$$\mathrm{Im}\,G_P(t)|_{m_\pi=0}=-\frac{4M^2}{t}\mathrm{Im}\,G_A(t)\bigg|_{m_\pi=0}=\frac{g_{\pi N}Mt}{9(8\pi f_\pi)^3}\left[1-g_A^2\left(1+\frac{64\pi^2}{35}\right)\right]\simeq-\frac{3t}{M^2}. \tag{6}$$

The first part of this equation follows from the fact that the combination $G_A(t)+t\,G_P(t)/4M^2$ is the form factor of the divergence of the isovector axial current, which vanishes in the chiral limit $m_\pi=0$ (QCD chiral Ward identity). The (two-loop) result for $\mathrm{Im}\,G_A(t)|_{m_\pi=0}$ has been taken over from Eq. (27) in Ref. [8].

In Fig. 2 we show by the full line the total imaginary part $\mathrm{Im}\,G_P(t)$, calculated from Eqs. (4) and (5) after division by a factor t. The horizontal dashed line in Fig. 2 indicates the asymptotic behavior of $\mathrm{Im}\,G_P(t)/t$ for $t\to\infty$.

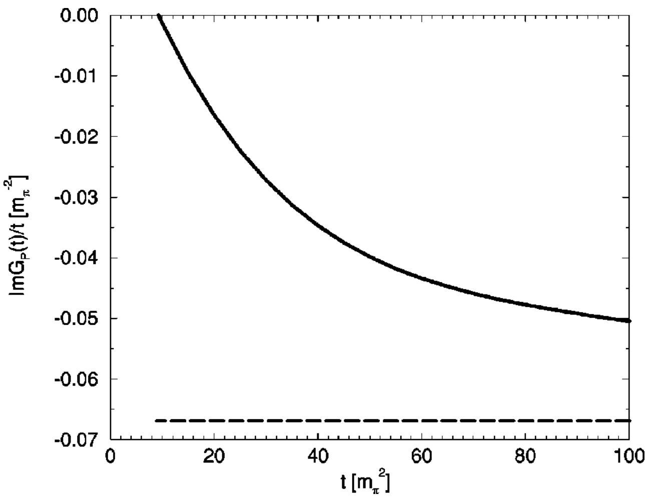

FIG. 2. The spectral function $\mathrm{Im}\, G_P(t)$ of the induced pseudoscalar form factor of the nucleon divided by t. The horizontal dashed line indicates its asymptotic form obtained by taking the chiral limit $m_\pi = 0$.

With the help of the spectral function $\mathrm{Im}\, G_P(t)$ the complete two-loop representation of the induced pseudoscalar form factor of the nucleon can be written as

$$G_P(t) = \frac{4 g_{\pi N} M f_\pi}{m_\pi^2 - t} - \frac{2}{3} g_A M^2 \langle r_A^2 \rangle + \frac{M^2}{(2\pi f_\pi)^4}(\zeta_0 m_\pi^2 + \zeta_1 t) + \frac{t^2}{\pi} \int_{9m_\pi^2}^{\infty} dt' \frac{\mathrm{Im}\, G_P(t')}{t'^2 (t' - t - i0^+)}. \tag{7}$$

The first two terms are the well-known pion-pole term and Adler-Dothan-Wolfenstein term [4,5]. The parameters $g_{\pi N}, f_\pi, m_\pi, \langle r_A^2 \rangle$, etc., are to be understood as the physical ones including their individual one- and two-loop chiral corrections. These (not explicitly calculated) two-loop renormalization effects come along with the real parts of the diagrammatic amplitudes. Note that the dispersion integral in Eq. (7) requires two subtractions because of the asymptotic linear growth of the imaginary part $\mathrm{Im}\, G_P(t)$. The third term in Eq. (7) involving the two dimensionless low-energy constants ζ_0 and ζ_1 subsumes all polynomial contributions which arise from (tadpole-type) loop diagrams and possible chiral-invariant counterterms (beyond renormalizing the Adler-Dothan-Wolfenstein term). The prefactor of this term is chosen such that the negative mass dimension of the counterterm coupling strength is accounted for by appropriate powers of the chiral symmetry breaking scale $\Lambda_\chi = 2\sqrt{2}\pi f_\pi \simeq 0.82$ GeV. Based on naturalness arguments one expects that the dimensionless low-energy constants $\zeta_{0,1}$ are of order one. Indeed the same considerations applied to the Adler-Dothan-Wolfenstein term give for the analogous dimensionless low-energy constant at one-loop order: $-(2\pi f_\pi)^2 g_A \langle r_A^2 \rangle / 3 = -g_A (4\pi f_\pi / M_A)^2 \simeq -1.63$. Here, we have inserted the value of the axial dipole mass $M_A = (1.03 \pm 0.02)$ GeV as extracted in Ref. [10] from (quasi) elastic neutrino and antineutrino scattering experiments.

Let us now turn to numerical results. From the twice-subtracted dispersion integral in Eq. (7) one gets a tiny contribution to the pseudoscalar coupling constant of $\delta g_P \simeq -1 \times 10^{-5}$. The extreme smallness of this number comes partly from the proximity of the chosen subtraction point $t_0 = 0$ to $t_\mu = -0.502 m_\pi^2$. Nevertheless, when varying the subtraction point [via the substitution $t^2/t'^2 \to (t-t_0)^2/(t'-t_0)^2$ in Eq. (7)] in the broad range $-24 m_\pi^2 < t_0 < 9 m_\pi^2$, the contribution of the dispersion integral to the pseudoscalar coupling constant stays in magnitude smaller than 1%, $0 > \delta g_P > -1 \times 10^{-2}$. Note that a change of the subtraction point t_0 is equivalent to changes of the low-energy constants $\zeta_{0,1}$ parametrizing the polynomial piece in Eq. (7). Since we cannot accurately determine the coefficients $\zeta_{0,1}$, we turn here the argument around and ask only for some upper bound. For example, in order to cause a correction of $\delta g_P = 0.09$, corresponding to a relative 1% change of $g_{\pi N}$ in the pion-pole term, the relation $2\zeta_0 - \zeta_1 \simeq 21$ must hold. Such values of $\zeta_{0,1}$ exceed the expectation from naturalness already by one order of magnitude.

The delta-nucleon mass-splitting $\Delta = 293$ MeV introduces another small scale to the problem. The systematic power counting scheme inherent to the small scale expansion of Refs. [3,6] ensures, however, that the contribution to $G_P(t)$ from two-loop diagrams with intermediate $\Delta(1232)$ excitations is a homogeneous function of degree one in the three variables (t, m_π^2, Δ^2). Consequently, possible negative powers of Δ get always overcompensated by two more powers of the numerically smaller scales m_π and/or $\sqrt{|t_\mu|}$ in $G_P(t_\mu)$.

One may, therefore, conclude that all two-loop corrections to the pseudoscalar coupling constant g_P are numerically insignificant. A sufficiently accurate representation of $G_P(t)$ at low momentum transfers t is given by the sum of the pion-pole term and the Adler-Dothan-Wolfenstein term. Using for the πN-coupling constant $g_{\pi N} = 13.2 \pm 0.2$ [11,12], which is consistent with recent results from πN-dispersion relation analyses [12] and $\langle r_A^2 \rangle = 12/M_A^2 = (0.44 \pm 0.02)$ fm^2 [10] for the axial mean square radius one gets in this case:

$$g_P = 8.3 \pm 0.2. \tag{8}$$

The major theoretical uncertainty of g_P comes obviously from the πN-coupling constant $g_{\pi N}$ entering the dominant pion-pole term.

Let us finally consider the form factor of the nucleon related to the divergence of the isovector axial current:

$$G_A(t) + \frac{t}{4M^2} G_P(t) = \frac{g_A m_\pi^2}{m_\pi^2 - t} D(t). \tag{9}$$

The prefactor on the right-hand side expresses the vanishing of this form factor in the chiral limit $m_\pi = 0$ as well as the presence of a pion-pole contribution. The imaginary part $\mathrm{Im}\, D(t)$ (at two-loop order) can be easily constructed from the expressions of $\mathrm{Im}\, G_P(t)$ given here in Eqs. (4) and (5) as well as from the formula of $\mathrm{Im}\, G_A(t)$ written in Eq. (26) of Ref. [8]. As a further nontrivial result we give here only the spectral function $\mathrm{Im}\, D(t)$ in the chiral limit:

$$\mathrm{Im}\, D(t)|_{m_\pi=0} = \frac{2t^2}{9\pi^3(8f_\pi)^4}\left[1+g_A^2\left(5+\frac{68\pi^2}{35}\right)\right], \quad (10)$$

which may be useful for some quick order of magnitude estimates.

In summary, we have calculated in this work the imaginary part of the induced pseudoscalar form factor of the nucleon $\mathrm{Im}\, G_P(t)$ at two-loop order in heavy baryon chiral perturbation theory. Two-loop corrections to the pseudoscalar coupling constant g_P measurable in ordinary muon capture $\mu^- p \to \nu_\mu n$ are numerically unimportant in view of the present uncertainty of the pion-nucleon coupling constant $g_{\pi N}$.

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