Two-body Dirac equations for nucleon-nucleon scattering

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We investigate the nucleon-nucleon interaction by using the meson exchange model and the two-body Dirac equations of constraint dynamics. This approach to the two-body problem has been successfully tested for QED and QCD relativistic bound states. An important question we wish to address is whether or not the two-body nucleon-nucleon scattering problem can be reasonably described in this approach as well. This test involves a number of related problems. First we must reduce our two-body Dirac equations exactly to a Schrödinger-like equation in such a way that allows us to use techniques to solve them already developed for Schrödinger-like systems in nonrelativistic quantum mechanics. Related to this, we present a new derivation of Calogero's variable phase shift differential equation for coupled Schrödinger-like equations. Then we determine if the use of nine meson exchanges in our equations gives a reasonable fit to the experimental scattering phase shifts for *n-p* scattering. The data involve seven angular momentum states including the singlet states $^1S_0$, $^1P_1$, $^1D_2$ and the triplet states $^3P_0$, $^3P_1$, $^3S_1$, $^3D_1$. Two models that we have tested give us a fairly good fit. The parameters obtained by fitting the *n-p* experimental scattering phase shift give a fairly good prediction for most of the *p-p* experimental scattering phase shifts examined (for the singlet states $^1S_0$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$). Thus the two-body Dirac equations of constraint dynamics present us with a fit that encourages the exploration of a more realistic model. We outline generalizations of the meson exchange model for invariant potentials that may possibly improve the fit.

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I. INTRODUCTION

In this paper [1], we obtain a semiphenomenological relativistic potential model for nucleon-nucleon interactions by using two-body Dirac equations of constraint dynamics [2–7] and Yukawa's theory of meson exchange. In previous work Long and Crater [8] have derived the two-body Dirac equations for all nonderivative Lorentz invariant interactions acting together or in any combination. They also reduced the two-body Dirac equations to coupled Schrödinger-like equations in which the potentials appear as covariant generalizations of the standard spin dependent interactions appearing in the early phenomenological works in this area [9–15] based on the nonrelativistic Schrödinger equation. This allows us to take advantage of earlier work done by other people on the nonrelativistic Schrödinger equation. In particular, we use the variable phase method developed by Calogero and Degasparis [16,17] for computation of the phase shift from the nonrelativistic Schrödinger equation, presenting a new derivation for the case of coupled equations. Our potentials for different angular momentum states are constructed from combinations of several different meson exchanges. Furthermore, our potentials, as well as the whole equations, are local, yet at the same time covariant. This contrasts our approach with other relativistic schemes such as those by Gross and others [18–21]. It is the aim of this paper to see if the meson exchanges we use are adequate to describe the elastic nucleon-nucleon interactions from low energy to high energy (<350 MeV) when using them together with two-body Dirac equations of constraint dynamics.

Although numerous relativistic approaches have been used in the nucleon-nucleon scattering problem, none of the other approaches have been tested nonperturbatively in both QED and QCD as they have been with the two-body Dirac equations of constraint dynamics [4,6,22–24]. Unlike the earlier local two-body approaches of Breit [25–27], the relativistic spin corrections need not be treated only perturbatively. This means that we can use nonperturbative methods (numerical methods) to solve the two-body Dirac equations. This is a very important advantage of the constraint two-body Dirac equations (CTBDE). The successful numerical tests in QED and QCD give us confidence that they may be appropriate relativistic equations for phase shift analysis of nucleon-nucleon scattering.

In Sec. II we introduce the two-body Dirac equations of constraint dynamics. In Sec. III we obtain the Pauli reduction of the two-body Dirac equations to coupled Schrödinger-like equations. We go a step further than that achieved in the paper of Long and Crater in that we eliminate the first derivative terms that appear in the Schrödinger-like equation. This is relatively simple for the case of uncoupled equations, but not so for the case of coupled Schrödinger-like equation. The reason we perform this extra reduction is that the formulas we use for the phase shift analysis, the variable phase method developed by Calogero, have been worked out already for coupled equations, but ones in which the first derivative terms are absent. This step then becomes an important part of the formalism, allowing us to take advantage of previous work. In Sec. IV, we discuss the phase shift methods used in our numerical calculations, which include phase shift equations for uncoupled and coupled states and the phase shift equations with Coulomb potential. In Sec. V we present the models used in our calculations, including the expressions for the scalar, vector, and pseudoscalar interactions, and the way they enter into our two-body Dirac equa-

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tions with the mesons used in our fits. In Sec. VI we present the results we have acheived and in Sec. VII are the summaries and conclusions of our work.

II. REVIEW OF CONSTRAINT TWO-BODY DIRAC EQUATIONS

The two-body Dirac equations that we will use for studying nucleon-nucleon interaction bear a close relation to the single particle equation proposed by Dirac in 1928 [28].

$$[\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m+V(r)]\psi=E\psi. \tag{2.1}$$

For interactions that transforms as a time component of a four vector and world scalar we have $V(r)=A(r)+\beta S(r)$. Of course, the single particle Dirac equation is not suitable to describe systems such as the mesons (quarkonium), muonium, positronium, the deuteron, and nucleon-nucleon scattering because the particles may have equal or near equal masses.

The earliest attempt at putting both particles on an equal footing was in 1929 by Breit [25–27]. However, the Breit equations do not retain manifest covariant form and in QED the equations cannot be treated nonperturbatively beyond the Coulomb term [26,29]. There have been many attempts to bypass the problems of the Breit equation and also of the full four-dimensional Bethe-Salpeter equation. These are discussed in a number of different contexts in Refs. [3–6]. The approach of the CTBDE provides a manifestly covariant yet three-dimensional detour around many of the problems that hamper the implementation and application of Breit's two-body Dirac equations as well as the full four-dimensional Bethe-Salpeter equation (see also Ref. [30]). In addition, as mentioned above, the approach can by a Pauli reduction, give us a local Schrödinger-like equation.

The CTBDE make use of Dirac's relativistic Hamiltonian formalism. In a series of papers (in addition to those cited above see Refs. [31,32]) Crater and Van Alstine have incorporated Todorov's effective particle idea developed in his quasipotential approach [33] into the framework of Dirac's Hamiltonian constraint mechanics [34] for a description of two-body systems. Their approach yields manifestly covariant coupled Dirac equations. The standard reduction of the Breit equation to a Schrödinger-like equation for QED yields highly singular operators (like δ functions and attractive $1/r^3$ potentials) that can only be treated perturbatively. In the treatment of the CTBDE for QED [22,32], for example, one finds that all the operators are quantum mechanically well defined so that one can therefore use nonperturbative techniques (analytic as well as numerical) to obtain solutions of bound state problems and scattering. (A quantum mechanically well defined, potential is one no more singular than $-1/4r^2$. If it is not quantum mechanically well defined, it can only be treated perturbatively.) Although it is encouraging that good results have been obtained for QED and QCD meson spectroscopy, that is no guarantee that the formalism so developed will lead to effective potentials in the case of nucleon-nucleon scattering that render reasonable fits to the phase shift data.

Using techniques developed by Dirac to handle constraints in quantum mechanics and the method developed by Crater and Van Alstine, one can derive the two-body Dirac equations for eight nonderivative Lorentz invariant interactions acting separately or together [35,8]. These include world scalar, four vector, and pseudoscalar interactions among others. We can also reduce the two-body Dirac equations to coupled Schrödinger-like equations even with all these interactions acting together. Before we test this method in nuclear physics in the phase shift analysis of the nucleon-nucleon scattering problems, we review highlights of the constraint formalism and the form of the two-body Dirac equations.

A. Hamiltonian formulation of the two-body problem from constraint dynamics

Dirac [34] extended Hamiltonian mechanics to include conjugate variables related by constraints of the form $\phi(q,p)=0$. For N constraints, we may write

$$\phi_n(q,p)\approx 0,\quad n=1,2,3,\ldots,N. \tag{2.2}$$

With these constraints the Hamiltonian of the system (with sum over repeated indices)

$$H=\dot{q}_n p_n-L \tag{2.3}$$

is not unique. The Dirac Hamiltonian $\mathcal{H}$ includes the constraints

$$\mathcal{H}=H+\lambda_n\phi_n, \tag{2.4}$$

in which H is the Legendre Hamiltonian obtained from the Lagrangian by means of a Legendre transformation. The λ_n may be functions of conjugate variables q's and p's. The equation of motion for any arbitrary function g (without explicit time dependence) of the conjugate variables q's and p's is then

$$\dot{g}=[g,\mathcal{H}]. \tag{2.5}$$

Dirac called the conditional equality $\approx$ a "weak" equality meaning the constraints $\phi_n\approx 0$ must not be applied before working out the Poisson brackets. Dirac called $=$ a nonconditional equality or a "strong" equality. The equations of motion are

$$\begin{aligned}\dot{g}&=[g,\mathcal{H}]=[g,H+\lambda_n\phi_n]\\&=[g,H]+\lambda_n[g,\phi_n]+[g,\lambda_n]\phi_n\\&\approx[g,H]+\lambda_n[g,\phi_n]\end{aligned} \tag{2.6}$$

for $\phi_n\approx 0$.

In the two-body system, we have two constraints $\phi_n(q,p)\approx 0$, $n=1,2$. For spinless particles they are taken to be the generalized mass shell constraints of the two particles [32,37], namely,

$$\mathcal{H}_1 = p_1^2 + m_1^2 + \Phi_1(x,p_1,p_2) \approx 0,$$

$$\mathcal{H}_2 = p_2^2 + m_2^2 + \Phi_2(x,p_1,p_2) \approx 0, \tag{2.7}$$

where

$$x = x_1 - x_2. \tag{2.8}$$

Dirac extended his idea of handling constraints in classical mechanics to quantum mechanics by replacing the classical constraints $\phi_n(q,p) \approx 0$ with quantum wave equations $\phi_n(q,p)|\psi\rangle = 0$, where q and p are conjugate variables. Thus the quantum forms for each individual particle constraint become Schrödinger-type equations [36]

$$\mathcal{H}_i|\psi\rangle = 0 \quad \text{for} \quad i = 1,2. \tag{2.9}$$

The total Hamiltonian $\mathcal{H}$ from these constraints alone is

$$\mathcal{H} = \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2 \tag{2.10}$$

(with λ_i as Lagrange multipliers). In order that each of these constraints be conserved in time we must have

$$[\mathcal{H}_i, \mathcal{H}]|\psi\rangle = i \frac{d\mathcal{H}_i}{d\tau}|\psi\rangle = 0, \tag{2.11}$$

so that

$$[\mathcal{H}_i, \lambda_1\mathcal{H}_1 + \lambda_2\mathcal{H}_2]|\psi\rangle$$

$$= \{[\mathcal{H}_i, \lambda_1]\mathcal{H}_1|\psi\rangle + \lambda_1[\mathcal{H}_i, \mathcal{H}_1]|\psi\rangle$$

$$+ [\mathcal{H}_i, \lambda_2]\mathcal{H}_2|\psi\rangle + \lambda_2[\mathcal{H}_i, \mathcal{H}_2]\}|\psi\rangle = 0. \tag{2.12}$$

Using Eq. (2.9), the above equation leads to this compatibility condition between the two constraints,

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 0. \tag{2.13}$$

This condition guarantees that with the Dirac Hamiltonian, the system evolves such that the "motion" is constrained to the surface of the mass shell described by the constraints of $\mathcal{H}_1$ and $\mathcal{H}_2$ (Refs. [37,32,31]). As described most recently in Ref. [37], this requires that

$$\Phi_1 = \Phi_2 = \Phi(x_\perp) \tag{2.14}$$

(a kind of relativistic Newton's third law) with the transverse coordinate defined by

$$x_{\nu\perp} = x_{12}^{\mu}(\eta_{\mu\nu} - P_\mu P_\nu / P^2), \tag{2.15}$$

and total momentum by

$$P = p_1 + p_2. \tag{2.16}$$

To complete our review of the spinless case (Ref. [37]) and establish notation we introduce the transverse relative momentum

$$p = \frac{\varepsilon_2}{w} p_1 - \frac{\varepsilon_1}{w} p_2, \tag{2.17}$$

$$P \cdot p = 0, \tag{2.18}$$

where the center of momentum (CM) energy eigenvalue w is defined from

$$\{P^2 + w^2\}|\psi\rangle = 0. \tag{2.19}$$

Taking the difference of the two constraints,

$$(p_1^2 - p_2^2)|\psi\rangle = -(m_1^2 - m_2^2)|\psi\rangle, \tag{2.20}$$

we can show that the longitudinal or timelike components of the momenta in the CM system have the invariant forms

$$\varepsilon_1 = \frac{w^2 + m_1^2 - m_2^2}{2w},$$

$$\varepsilon_2 = \frac{w^2 + m_2^2 - m_1^2}{2w}. \tag{2.21}$$

Thus, on these states $|\psi\rangle$ we obtain

$$\{p^2 + \Phi(x_\perp) - b^2(w^2, m_1^2, m_2^2)\}|\psi\rangle = 0, \tag{2.22}$$

where

$$b^2(w^2, m_1^2, m_2^2) = \varepsilon_1^2 - m_1^2 = \varepsilon_2^2 - m_2^2$$

$$= \frac{1}{4w^2}\{w^4 - 2w^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\} \tag{2.23}$$

and indicates the presence of exact relativistic two-body kinematics. (By this statement we mean that classically $p^2 - b^2 = 0$ would imply $w = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}$.) Note that both of the constituent invariant CM energies ε_1 and ε_2 are positive for positive total CM energy w greater than the square root of $|m_1^2 - m_2^2|$. This is a direct consequence of Eq. (2.20), which in turn depends on the "third law" condition necessary for compatibility. In our scattering applications below, this guarantees that nucleons cannot scatter into a final state having an overall positive energy but with constituent positive and negative energy nucleons.

In the center-of-momentum system, $p = p_\perp = (0, \mathbf{p})$, $x_\perp = (0, \mathbf{r})$, and the relative energy and time are removed from the problem. The equation for the relative motion is then

$$\{\mathbf{p}^2 + \Phi(\mathbf{r}) - b^2\}|\psi\rangle = 0, \tag{2.24}$$

which is in the form of a nonrelativistic Schrödinger equation (with $2mV \to \Phi$, $2mE_{NR} \to b^2$). Thus the relativistic treatment of the two-body problem for spinless particles gives a form that has the simplicity of the ordinary nonrelativistic two-body Schrödinger equation and yet maintains relativistic covariance. Spin and different types of interactions can be included in a more complete framework [8,30,35,38,] and will be reviewed later in this section.

In addition to exact relativistic kinematical corrections, Eq. (2.24) displays through the potential Φ relativistic dynamical corrections. These corrections include dependence of the potential on the CM energy w and on the nature of the interaction. For spinless particles interacting by way of a world scalar interaction S, one finds [31,32,39,40]

$$\Phi = 2m_w S + S^2, \tag{2.25}$$

where

$$m_w = \frac{m_1 m_2}{w}, \tag{2.26}$$

while for (timelike) vector interactions (described by $\mathcal{A}$), one finds [31,33,39,40]

$$\Phi = 2\varepsilon_w \mathcal{A} - \mathcal{A}^2, \tag{2.27}$$

where

$$\varepsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w}. \tag{2.28}$$

For combined spacelike and timelike vector interactions (which reproduce the correct energy spectrum for scalar QED [32])

$$\Phi = 2\varepsilon_w \mathcal{A} - \mathcal{A}^2 + \frac{1}{2}\boldsymbol{\nabla}^2 \ln(1 - 2\mathcal{A}/w) + \frac{1}{4}[\boldsymbol{\nabla} \ln(1 - 2\mathcal{A}/w)]^2. \tag{2.29}$$

The variables m_w and ε_w (both of which approach the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ in the nonrelativistic limit) are called the relativistic reduced mass and energy of the fictitious particle of relative motion. These were first introduced by Todorov [33] in his quasipotential approach. Thus, in the nonrelativistic limit, Φ approaches $2\mu(S + \mathcal{A})$ for combined interactions. In the relativistic case, the dynamical corrections to Φ referred to above include both quadratic additions to S and $\mathcal{A}$ as well as CM energy dependence through m_w and ε_w. The two logarithm terms at the end of Eq. (2.29) are due to the transverse or spacelike part of the potential. Without those terms, spectral results would not agree with the standard (but more complex) spinless Breit and Darwin approaches (see references in Ref. [32] including Ref. [33]).

Equation (2.24) provides a useful way to obtain the solution of the relativistic two-body problem for spinless particles in scalar and vector interactions and, as reviewed below, has been extended to include spin. In that case they have been found to give a very good account of the bound state spectrum of both light and heavy mesons using reasonable input quark potentials.

These ways of putting the invariant potential functions for scalar S and vector $\mathcal{A}$ interactions into Φ will be used in this paper for the case of two spin-one-half particles [see Eqs. (2.67) to (2.71) and (4.1) to (4.3)]. These exact forms are not unique but were motivated by work of Crater and Van Alstine in classical field theory and Sazdjian in quantum field theory [40,41]. Other closely related structures will also be used. These structures play a crucial role in this paper since they give us a nonperturbative framework in which S and $\mathcal{A}$ appear in the equations we use. This structure has been successfully tested (numerically) in QED (positronium and muonium bound states) and is found to give excellent results when applied to the highly relativistic circumstances of QCD (quark model for mesons). An important question we wish to answer in this paper is whether such structures are also valid in the two-body nucleon-nucleon problem. This is an important test since the quadratic forms [see, e.g., Eqs. (2.25) and (2.27)] that appear could very well distort possible fits based on Yukawa-type potentials with strong couplings.

Before going on to describe the constraints for two spin-one-half particles we mention an important but often overlooked aspect of the foundations of the generalized mass shell contraint equations given in Eq. (2.9). It involves their derivation from an alternative starting point. In addition to the connection with the Bethe-Salpeter equation described in Ref. [36], there exists a connection between constraint dynamics and Wigner's early formulation of relativistic quantum mechanics [42]. In particular, Polyzou [43] has demonstrated that the assumption of both Poincaré invariance and manifest Lorentz covariance forces the scalar product for quantum mechanical state vectors to be interaction dependent. So, whereas for a free particle the kernal involved in the scalar product has the form $\delta(p^2 + m^2)\theta(p^0)$, in cases of interactions the self-adjoint nature of the kernal demands the forms $\delta(\mathcal{H}_1)\delta(\mathcal{H}_2)$ with compatible constraints $\mathcal{H}_1$ and $\mathcal{H}_2$ (a related use of such delta functions to construct the state vectors themselves is discussed in Ref. [37]).

B. Two spin-one-half particles

We continue our review in this section by introducing the two-body Dirac equations of constraint dynamics. The Dirac equations for two free spin-one-half particles are

$$\mathcal{S}_{10}\psi = (\theta_1 \cdot p_1 + m_1 \theta_{51})|\psi\rangle = 0,$$
$$\mathcal{S}_{20}\psi = (\theta_2 \cdot p_2 + m_2 \theta_{52})|\psi\rangle = 0, \tag{2.30}$$

where ψ is the product of the two single-particle Dirac wave functions (these equations are equivalent to the free one-body Dirac equation). The "theta" matrices are related to the ordinary gamma matrices by

$$\theta_i^\mu = i\sqrt{\frac{1}{2}}\gamma_{5i}\gamma_i^\mu, \quad \mu = 0,1,2,3, \quad i = 1,2,$$
$$\theta_{5i} = i\sqrt{\frac{1}{2}}\gamma_{5i} \tag{2.31}$$

and satisfy the fundamental anticommutation relations

$$[\theta_i^\mu, \theta_i^\nu]_+ = -\eta^{\mu\nu},$$
$$[\theta_{5i}, \theta_i^\mu]_+ = 0,$$
$$[\theta_{5i}, \theta_{5i}]_+ = -1. \tag{2.32}$$

It is much more convenient to use the "theta" matrices instead of the Dirac gamma matrices for working out the compatibility conditions. In the reduction of complicated commutators to simpler form one uses reduction brackets that involve anticommutators for odd numbers of theta matrices and commutators for even numbers of theta matrices and coordinate and momentum operators. This property follows from the relation of the theta matrices to the Grassmann variables used in the pseudoclassical form of the constraints (see Refs. [2,3,5]). These fundamental anticommutation relations guarantee that the Dirac operators $\mathcal{S}_{10}$ and $\mathcal{S}_{20}$ are the square root of the mass shell operators $-\frac{1}{2}(p_1^2+m_1^2)$ and $-\frac{1}{2}(p_2^2+m_2^2)$. Differencing these implies that the relative momentum p in Eq. (2.18) satisfies $P\cdot p|\psi\rangle=0$.

Writing p_1 and p_2 in terms of the total and relative momenta we obtain

$$\mathcal{S}_{10}\psi=(\theta_{1\perp}\cdot p+\epsilon_1\theta_1\cdot\hat{P}+m_1\theta_{51})|\psi\rangle=0,$$

$$\mathcal{S}_{20}\psi=(-\theta_{2\perp}\cdot p+\epsilon_2\theta_2\cdot\hat{P}+m_2\theta_{52})|\psi\rangle=0. \quad (2.33)$$

The projected theta matrices then satisfy

$$[\theta_i\cdot\hat{P},\theta_i\cdot\hat{P}]_+=1,$$

$$[\theta_i\cdot\hat{P},\theta_{i\perp}^{\mu}]_+=0, \quad (2.34)$$

where

$$\theta_{\nu\perp}^{\mu}=\theta_{i\nu}(\eta^{\mu\nu}+\hat{P}^{\mu}\hat{P}^{\nu}). \quad (2.35)$$

Defining $\alpha_{i\perp}^{\mu}=2\theta_i\hat{P}\theta_{i\perp}^{\mu}$ and $\beta_i=2\theta_i\hat{P}\theta_{5i}$, the above two-body Dirac equations become

$$(\alpha_1\cdot p+\beta_1 m_1)\psi=\epsilon_1\psi,$$

$$(-\alpha_2\cdot p+\beta_2 m_2)\psi=\epsilon_2\psi, \quad (2.36)$$

which have the form of single free particle Dirac equations.

Recall that in the spinless case we had the compatibility condition

$$[\mathcal{H}_1,\mathcal{H}_2]|\psi\rangle=0. \quad (2.37)$$

It was a requirement that followed in the classical case (or the Heisenberg picture in the quantum case) from the individual constraints $\mathcal{H}_i$ being conserved in time. Similarly here with $\mathcal{S}_i$ designating the form of the Dirac constraint with intereactions present, the commutator condition guaranteeing that the Dirac equations for two spin $\frac{1}{2}$ particles form a compatible set is

$$[\mathcal{S}_1,\mathcal{S}_2]|\psi\rangle=0. \quad (2.38)$$

[It would follow from an $\mathcal{H}$ as in Eq. (2.10) composed of a sum of the $\mathcal{S}_i$.]

We found that even for the simplest interaction, a Lorentz scalar, the naive replacement such as making the minimal substitutions [corresponding in the case of the single particle Dirac equation to Eq. (2.1) with $V(r)=\beta S(r)$],

$$m_i\rightarrow M_i(r)=m_i+S_i \quad i=1,2, \quad (2.39)$$

does not lead to compatible constraints. Rather than detailing here the earlier work steps that were taken to make the interactions meet the compatibility condition for scalar interactions [2,3,35] we present here the form of the compatible constraints for general covariant interactions,

$$\mathcal{S}_1|\psi\rangle=[\cosh(\Delta)\mathbf{S}_1+\sinh(\Delta)\mathbf{S}_2]|\psi\rangle=0,$$

$$\mathcal{S}_2|\psi\rangle=[\cosh(\Delta)\mathbf{S}_2+\sinh(\Delta)\mathbf{S}_1)|\psi\rangle=0, \quad (2.40)$$

where the operators $\mathbf{S_1}$ and $\mathbf{S_2}$ are auxiliary constraints of the form

$$\mathbf{S}_1|\psi\rangle=[\mathcal{S}_{10}\cosh(\Delta)+\mathcal{S}_{20}\sinh(\Delta)]|\psi\rangle=0,$$

$$\mathbf{S}_2|\psi\rangle=[\mathcal{S}_{20}\cosh(\Delta)+\mathcal{S}_{10}\sinh(\Delta)]|\psi\rangle=0. \quad (2.41)$$

Both of these sets of constraints [7,30,35] are compatible

$$[\mathcal{S}_1,\mathcal{S}_2]|\psi\rangle=0, \quad (2.42)$$

$$[\mathbf{S}_1,\mathbf{S}_2]|\psi\rangle=0, \quad (2.43)$$

provided only that

$$\Delta(x)=\Delta(x_\perp). \quad (2.44)$$

Furthermore,

$$P\cdot p|\psi\rangle=0, \quad (2.45)$$

the same constraint equation on the relative momentum p as in the spinless case.

The covariant potentials are divided into two categories, four "polar" and four "axial" interactions. The four polar interactions (or tensors of rank 0,1,2) are the following: scalar

$$\Delta_L=-L\theta_{51}\theta_{52}=-\frac{L}{2}\mathcal{O}_1,\mathcal{O}_1=-\gamma_{51}\gamma_{52}, \quad (2.46)$$

timelike vector

$$\Delta_J=J\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}\equiv\mathcal{O}_2\frac{J}{2}=\beta_1\beta_2\frac{J}{2}\mathcal{O}_1, \quad (2.47)$$

spacelike vector

$$\Delta_{\mathcal{G}}=\mathcal{G}\theta_{1\perp}\cdot\theta_{2\perp}\equiv\mathcal{O}_3\frac{\mathcal{G}}{2}=\gamma_{1\perp}\cdot\gamma_{2\perp}\frac{\mathcal{G}}{2}\mathcal{O}_1, \quad (2.48)$$

and tensor (polar)

$$\Delta_{\mathcal{F}}=4\mathcal{F}\theta_{1\perp}\cdot\theta_{2\perp}\theta_{52}\theta_{51}\theta_1\cdot\hat{P}\theta_2\cdot\hat{P}\equiv\mathcal{O}_4\frac{\mathcal{F}}{2}=\alpha_1\cdot\alpha_2\frac{\mathcal{F}}{2}\mathcal{O}_1. \quad (2.49)$$

We may use each equation in Eqs. (2.40) and (2.41) separately or as a sum,

$$\Delta_p=\Delta_L+\Delta_J+\Delta_{\mathcal{G}}+\Delta_{\mathcal{F}}, \quad (2.50)$$

to generate the sets of two-body Dirac equations with corresponding interactions. A particularly important combination occurs for electromagnetic inteactions. While timelike and spacelike vector interactions are characterized by the respective matrices $\beta_1\beta_2$ and $\gamma_{1\perp}\cdot\gamma_{2\perp}$, a potential proportional to $\gamma_1\gamma_2$ would correspond to an electromagneticlike interaction and would require that $J=-\mathcal{G}$,

$$\Delta_{\mathcal{EM}}=\frac{(\mathcal{O}_3-\mathcal{O}_2)\mathcal{G}(x_\perp)}{2}=\frac{\gamma_1\cdot\gamma_2\mathcal{G}(x_\perp)}{2}\mathcal{O}_1. \quad (2.51)$$

The four "axial" interactions (or pseudotensors of rank 0,1,2) are the following:

pseudoscalar

$$\Delta_C=\frac{C}{2}\equiv\mathcal{E}_1\frac{C}{2}=-\gamma_{51}\gamma_{52}\frac{C}{2}\mathcal{O}_1, \quad (2.52)$$

timelike pseudovector

$$\Delta_H=-2H\theta_1\cdot\hat{P}\,\theta_2\cdot\hat{P}\,\theta_{51}\theta_{52}\equiv-\mathcal{E}_2\frac{H}{2}=\beta_1\gamma_{51}\beta_2\gamma_{52}\frac{H}{2}\mathcal{O}_1, \quad (2.53)$$

spacelike pseudovector

$$\Delta_I=-2I\theta_{1\perp}\cdot\theta_{2\perp}\theta_{51}\theta_{52}\equiv-\mathcal{E}_3\frac{I}{2}=-\gamma_{51}\gamma_{1\perp}\cdot\gamma_{52}\gamma_{2\perp}\frac{I}{2}\mathcal{O}_1, \quad (2.54)$$

and tensor (axial)

$$\Delta_Y=-2Y\theta_{1\perp}\cdot\theta_{2\perp}\theta_1\cdot\hat{P}\,\theta_2\cdot\hat{P}\equiv-\mathcal{E}_4\frac{Y}{2}=-\sigma_1\cdot\sigma_2\frac{Y}{2}\mathcal{O}_1. \quad (2.55)$$

Crater and Van Alstine found [35] that these and their sum,

$$\Delta_a=\Delta_C+\Delta_H+\Delta_I+\Delta_Y, \quad (2.56)$$

would be used in Eqs. (2.40) and Eqs. (2.41) but with the $\sinh(\Delta_a)$ terms in Eqs. (2.40) appearing with a negative sign instead of the plus sign as in the case of polar interactions. There is no sign change in Eqs. (2.41) for Δ_a.

For systems with both polar and axial interactions [35], one uses $\Delta_p-\Delta_a$ to replace Δ in Eqs. (2.40), and $\Delta_p+\Delta_a$ to replace Δ in Eqs. (2.41). L, J, $\mathcal{G}$, $\mathcal{F}$, C, H, I, Y are arbitrary invariant functions of $x_\perp$. In this paper, we include only mesons corresponding to the interactions L, J, $\mathcal{G}$ ($J=-\mathcal{G}$), and C. Thus we are ignoring tensor and pseudovector mesons, limiting ourselves to vector, scalar, and pseudoscalar mesons, all having masses less than or about 1000 MeV. We are also ignoring possible pseudovector couplings of the pseudoscalar mesons.

For computational convenience we have found it necessary to transform the Dirac equations to "external potential" form. We obtain these forms by combining the two sets of equations

$$\mathcal{S}_1|\psi\rangle=\{\cosh(\Delta)[\mathcal{S}_{10}\cosh(\Delta)+\mathcal{S}_{20}\sinh(\Delta)]+\sinh(\Delta)\times[\mathcal{S}_{20}\cosh(\Delta)+\mathcal{S}_{10}\sinh(\Delta)]\}|\psi\rangle=0,$$

$$\mathcal{S}_2|\psi\rangle=\{\cosh(\Delta)[\mathcal{S}_{20}\cosh(\Delta)+\mathcal{S}_{10}\sinh(\Delta)]+\sinh(\Delta)\times[\mathcal{S}_{10}\cosh(\Delta)+\mathcal{S}_{20}\sinh(\Delta)]\}|\psi\rangle=0. \quad (2.57)$$

and bringing the $\mathcal{S}_{i0}$ operators through to the right. References [8,35] gives the "external potential" forms of the constraint two-body Dirac equations for each of the eight interaction matrices, Δ_L, Δ_J, $\Delta_{\mathcal{G}}$, $\Delta_{\mathcal{F}}$, Δ_C, Δ_H, Δ_I, Δ_Y acting alone. These forms are similar in appearance to individual Dirac equations for each of the particles in an external potential. In Ref. [8] appeared also the form with all eight interactions acting simultaneously,

$$\begin{aligned}\mathcal{S}_1|\psi\rangle=\Big\{&\exp(\mathcal{G}+\mathcal{F}\mathcal{E}_2+I\mathcal{O}_1+Y\mathcal{O}_2)\Big[\theta_1\cdot p-\frac{i}{2}\theta_2\cdot\partial(L\mathcal{O}_1\\&-J\mathcal{O}_2-\mathcal{G}\mathcal{O}_3-\mathcal{F}\mathcal{O}_4-C\mathcal{E}_1+H\mathcal{E}_2+I\mathcal{E}_3+Y\mathcal{E}_4)\Big]\\&+\epsilon_1\cosh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_1\cdot\hat{P}\\&+\epsilon_2\sinh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_2\cdot\hat{P}\\&+m_1\cosh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{51}\\&+m_2\sinh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{52}\Big\}|\psi\rangle=0,\end{aligned} \quad (2.58)$$

$$\begin{aligned}\mathcal{S}_2|\psi\rangle=\Big\{&-\exp(\mathcal{G}+\mathcal{F}\mathcal{E}_2+I\mathcal{O}_1+Y\mathcal{O}_2)\Big[\theta_2\cdot p-\frac{i}{2}\theta_1\partial(L\mathcal{O}_1\\&-J\mathcal{O}_2-\mathcal{G}\mathcal{O}_3-\mathcal{F}\mathcal{O}_4-C\mathcal{E}_1+H\mathcal{E}_2+I\mathcal{E}_3+Y\mathcal{E}_4)\Big]\\&+\epsilon_1\sinh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_1\cdot\hat{P}\\&+\epsilon_2\cosh(J\mathcal{O}_2+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+Y\mathcal{E}_4)\theta_2\cdot\hat{P}\\&+m_1\sinh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{51}\\&+m_2\cosh(-L\mathcal{O}_1+\mathcal{F}\mathcal{O}_4+H\mathcal{E}_2+I\mathcal{E}_3)\theta_{52}\Big\}|\psi\rangle=0.\end{aligned} \quad (2.59)$$

What is remarkable is that the above hyperbolic and exponential structures account for all of the "interference" terms between the various interactions. The interactions acting separately or in subgroupings are simple reductions of the above. For example, in the case of the combined scalar, timelike, spacelike, and pseudoscalar interactions used in this paper,

$$\Delta=\Delta_J+\Delta_L+\Delta_{\mathcal{G}}+\Delta_C, \quad (2.60)$$

and the two-body Dirac equations (2.58),(2.59) reduce to

$$\mathcal{S}_1|\psi\rangle=\bigg(\exp(\mathcal{G})\theta_1\cdot p+E_1\theta_1\cdot\hat{P}+M_1\theta_{51}+i\frac{\exp(\mathcal{G})}{2}\theta_2\partial(\mathcal{G}\mathcal{O}_3+J\mathcal{O}_2-L\mathcal{O}_1+C\mathcal{E}_1)\bigg)|\psi\rangle=0, \tag{2.61}$$

$$\mathcal{S}_2|\psi\rangle=\bigg(-\exp(\mathcal{G})\theta_2\cdot p+E_2\theta_2\cdot\hat{P}+M_2\theta_{52}-i\frac{\exp(\mathcal{G})}{2}\theta_1\cdot\partial(\mathcal{G}\mathcal{O}_3+J\mathcal{O}_2-L\mathcal{O}_1+C\mathcal{E}_1)\bigg)|\psi\rangle=0, \tag{2.62}$$

where

$$M_1=m_1\cosh(L)+m_2\sinh(L),$$

$$M_2=m_2\cosh(L)+m_1\sinh(L), \tag{2.63}$$

$$E_1=\epsilon_1\cosh(J)+\epsilon_2\sinh(J),$$

$$E_2=\epsilon_2\cosh(J)+\epsilon_1\sinh(J). \tag{2.64}$$

In the limit $m_1\to\infty$ (or $m_2\to\infty$) (when one of the particles becomes infinitely massive), the extra terms $\partial\mathcal{G}$, $\partial J,\partial L$, and ∂C in Eqs. (2.61) and (2.62) vanish, and one recovers the expected one-body Dirac equation in an external potential. The above two-body Dirac equations (without pseudoscalar interactions) have been tested successfully in quark model calculations of the meson spectra [4,5,23,24].

We may rewrite the ''left external potential form'' of the CTBDE for two relativistic spin-one-half particle interacting through scalar and vector potentials as [see Eqs. (2.61) and (2.62) without the pseudoscalar interaction]

$$\mathcal{S}_1|\psi\rangle\equiv\gamma_{51}[\gamma_1\cdot(p_1-A_1)+m_1+S_1]|\psi\rangle=0, \tag{2.65}$$

$$\mathcal{S}_2|\psi\rangle\equiv\gamma_{52}[\gamma_2\cdot(p_2-A_2)+m_2+S_2]|\psi\rangle=0. \tag{2.66}$$

A_i^μ and S_i introduce the interactions that the ith particle experiences due to the presence of the other particle and are both spin dependent [2–6]. In order to identify these potentials we use Eqs. (2.61) and (2.62), and Eqs. (2.63) and (2.64). Then we find that the momentum dependent vector potentials A_i^μ are given in terms of three invariant functions [5,6] G, E_1,E_2,

$$A_1^\mu=\bigg((\epsilon_1-E_1)-i\frac{G}{2}\gamma_2\cdot\frac{\partial E_1}{E_2}\gamma_2\hat{P}\bigg)\hat{P}^\mu+(1-G)p^\mu-\frac{i}{2}\partial G\cdot\gamma_{2\perp}\gamma_{2\perp}^\mu, \tag{2.67}$$

$$A_2^\mu=\bigg((\epsilon_2-E_2)-i\frac{G}{2}\gamma_1\cdot\frac{\partial E_2}{E1}\gamma_1\cdot\hat{P}\bigg)\hat{P}^\mu+(1-G)p^\mu-\frac{i}{2}\partial G\gamma_{1\perp}\gamma_{1\perp}^\mu, \tag{2.68}$$

where

$$G=\exp(\mathcal{G}) \tag{2.69}$$

(with $\hat{P}^2=-1$, where $\hat{P}\equiv P/w$) while the scalar potentials S_i are given in terms of three invariant functions [3,5,6] G, M_1, M_2,

$$S_1=M_1-m_1-\frac{i}{2}G\gamma_2\cdot\frac{\partial M_1}{M_2}, \tag{2.70}$$

$$S_2=M_2-m_2-\frac{i}{2}G\gamma_1\cdot\frac{\partial M_2}{M_1}. \tag{2.71}$$

In QCD, the scalar potentials S_i are semiphenomenological long range interactions. The vector potentials A_i^μ are semiphenomenological in the long range while in the short range are closely related to perturbative quantum field theory [44]. Of course, this does not change the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ still satisfy the compatibility condition Eq. (2.42).

III. PAULI REDUCTION

Now one can use the complete hyperbolic constraint two-body Dirac equations (2.58) and (2.59) to derive the Schrödinger-like eigenvalue equation for the combined interactions: $L(x_\perp),J(x_\perp),H(x_\perp),C(x_\perp),\mathcal{G}(x_\perp),\mathcal{F}(x_\perp),I(x_\perp)$, $Y(x_\perp)$ [8]. In this paper, however, we include only mesons corresponding to the interactions L, J, $\mathcal{G}(J=-\mathcal{G})$, C, thus limiting ourselves to vector, scalar, and pseudoscalar interactions. The basic method we use here has some similarities to the reduction of the single particle Dirac equation to a Schrödinger-like form (the Pauli reduction) and to related work by Sazdjian [7,38].

The state vector $|\psi\rangle$ appearing in the two-body Dirac equations (2.58) and (2.59) is a Dirac spinor written as

$$|\psi\rangle=\begin{bmatrix}|\psi\rangle_1\\|\psi\rangle_2\\|\psi\rangle_3\\|\psi\rangle_4\end{bmatrix}, \tag{3.1}$$

where each $|\psi\rangle_i$ is itself a four component spinor. $|\psi\rangle$ has a total of 16 components and the matrices $\mathcal{O}_i$'s, $\mathcal{E}_i$'s are all 16×16. We use the block forms of the gamma matrices given by Eq. (4.2) in Ref. [8] and

$$\Sigma_i^\mu=\gamma_{5i}\beta_i\gamma_{\perp i}^\mu,\quad i=1,2. \tag{3.2}$$

The Σ_i^μ are four-vector generalizations of the Pauli matrices of particles one and two. In the CM frame, the time component is zero and the spatial components are the usual Pauli matrices for each particle. Appendix A details the procedure that leads to a second-order Schrödinger-like eigenvalue equation for the four component wave function $|\phi_+\rangle=|\psi\rangle_1+|\psi\rangle_4$ in the general form

$$[\mathbf{p}^2+\Phi(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,w)]|\phi_+\rangle=b^2(w)|\phi_+\rangle. \quad (3.3)$$

Below we display all the general spin dependent structures in $\Phi(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,w)$ explicitly, ones very similar to those appearing in nonrelativistic formalisms such as seen in the older Hamada-Johnson and Yale group models (as well as the nonrelativistic limit of Gross's equation). By simplification of the final result in Appendix A by using identities involving $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ and grouping by the $\mathbf{p}^2$ term, Darwin term $(\hat{\mathbf{r}}\cdot\mathbf{p})$, spin-orbit angular momentum term $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$, spin-orbit angular momentum difference term $\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$, spin-spin term $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, tensor term $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})\times(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})$, additional spin dependent terms $\mathbf{L}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)$, and $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_1\cdot\mathbf{p})$, and spin independent terms, we obtain

$$\begin{aligned}&\Bigg\{\mathbf{p}^2-i\left[2\mathcal{G}'-\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'\right]\hat{\mathbf{r}}\cdot\mathbf{p}-\frac{1}{2}\nabla^2\mathcal{G}-\frac{1}{4}\mathcal{G}'^2-\frac{1}{4}(C+J-L)'(-C+J-L)'+\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'\\&+\frac{\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)}{r}\left[\mathcal{G}'-\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'\right]-\frac{\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)}{r}\frac{1}{2}\frac{E_2M_2-M_1E_1}{\mathcal{D}}(J+L)'+(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\left[\frac{1}{2}\nabla^2\mathcal{G}+\frac{1}{2}\mathcal{G}'^2\right.\\&\left.-\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}\mathcal{G}'(J+L)'-\frac{1}{2}\mathcal{G}'C'-\frac{1}{2}\frac{\mathcal{G}'}{r}-\frac{1}{2}\frac{(-C+J-L)'}{r}\right]+(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\left[-\frac{1}{2}\nabla^2(-C+J-L)\right.\\&\left.-\frac{1}{2}\nabla^2\mathcal{G}-\mathcal{G}'(-C+J-L)'-\mathcal{G}'^2+\frac{3}{2r}\mathcal{G}'+\frac{3}{2r}(-C+J-L)'+\frac{1}{2}\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'(\mathcal{G}-C+J-L)'\right]\\&+\frac{\mathbf{L}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)}{r}\frac{i}{2}\frac{M_2E_1-M_1E_2}{\mathcal{D}}(J+L)'-\frac{i(J-L)'}{2}[(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_1\cdot\mathbf{p})]\Bigg\}|\phi_+\rangle=\exp(-2\mathcal{G})\mathcal{B}^2|\phi_+\rangle, \quad (3.4)\end{aligned}$$

where

$$\mathcal{D}\equiv E_1M_2+E_2M_1,$$

$$\begin{aligned}\mathcal{B}^2=E_1{}^2-M_1^2=E_2^2-M_2^2=b^2(w)+(\epsilon_1^2+\epsilon_2^2)\sinh^2(J)\\+2\epsilon_1\epsilon_2\sinh(J)\cosh(J)-(m_1^2+m_2^2)\sinh^2(L)\\-2m_1m_2\sinh(L)\cosh(L). \quad (3.5)\end{aligned}$$

$E_i, M_i, C, J, L, \mathcal{G}$ are all functions of the invariant r. We point out that Eq. (3.4) differs from the forms presented in Ref. [8]. Whereas the above equation involves four component spinor wave functions, the ones given in Ref. [8] are obtained in terms of matrix wave functions involving one component scalar and three component vector wave functions. The form we choose in this paper is easier to compare with the earlier existing nonrelativistic forms.

All of the above equations when reduced to radial form have first derivative terms [from the $\hat{\mathbf{r}}\cdot\mathbf{p}$ and $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_1\cdot\mathbf{p})$ terms]. These can be easily eliminated for the uncoupled equations but are problematic for the coupled equations. The variable phase method developed by Calogero [16] for computation of phases shifts starts with coupled and uncoupled stationary state nonrelativistic Schrödinger equations that do not include the first derivative terms in their radial forms. An advantage of the above for the relativistic case is that they are Schrödinger-like equations. Before we can apply the techniques for phase shift calculations which have been already developed for the Schrödinger-like system in nonrelativistic quantum mechanics, we must get rid of these first derivative terms. In terms of the above equations, we seek a matrix transformation that eliminates the terms first order in $\mathbf{p}$.

The general form of the eigenvalue equation given in Eq. (3.4) is

$$\begin{aligned}\Big[\mathbf{p}^2-ig'\hat{\mathbf{r}}\cdot\mathbf{p}+\frac{g'}{2r}\vec{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}\\+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})+k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+l\vec{L}\cdot(\boldsymbol{\sigma}_1\\-\boldsymbol{\sigma}_2)+ij\vec{L}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)+m\Big]|\phi_+\rangle=\mathcal{B}^2e^{-2\mathcal{G}}|\phi_+\rangle.\end{aligned} \quad (3.6)$$

The m term is the spin independent part involving derivatives of the potentials. For the equal mass case, two terms drop out [see Eq. (3.4)], and the above equation becomes

$$\begin{aligned}\Big[\mathbf{p}^2-ig'\hat{\mathbf{r}}\cdot\mathbf{p}+\frac{g'}{2r}\vec{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}\\+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})+k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+m\Big]|\phi_+\rangle\\=\mathcal{B}^2e^{-2\mathcal{G}}|\phi_+\rangle. \quad (3.7)\end{aligned}$$

We introduce the spin-dependent scale change

$$|\phi_+\rangle \equiv \exp(F+K\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle \equiv (A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle, \quad (3.8)$$

with F,K,A,B to be determined. We find that

$$\mathbf{p}|\phi_+\rangle = (A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\mathbf{p}|\psi_+\rangle - i(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}|\psi_+\rangle - i\frac{B}{r}[(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+(\boldsymbol{\sigma}_2-\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}|\psi_+ \quad (3.9)$$

and

$$\frac{g'}{2r}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|\phi_+\rangle = (A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\frac{g'}{2r}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)|\psi_+\rangle + \frac{g'}{2r}B[2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-4ir\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p} + 2ir(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})-6\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]|\psi_+\rangle. \quad (3.10)$$

We thus find that

$$-ig'\hat{\mathbf{r}}\cdot\mathbf{p}|\phi_+\rangle = (A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(-ig'\hat{\mathbf{r}}\cdot\mathbf{p})|\psi_+\rangle + C|\psi_+\rangle \quad (3.11)$$

and

$$-ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})|\phi_+\rangle = (A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(-ih'[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p} + \boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p}])|\psi_+\rangle + D|\psi_+\rangle, \quad (3.12)$$

and finally

$$\mathbf{p}^2|\phi_+\rangle = (A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\mathbf{p}^2|\psi_+\rangle - 2i(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\cdot\mathbf{p}|\psi_+\rangle + i\frac{2B}{r}[2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}-(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p} + \boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p}]|\psi_+\rangle + E|\psi_+\rangle, \quad (3.13)$$

where C, D, and E do not involve $\mathbf{p}$ and are given by

$$C = -g'(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}), \quad (3.14)$$

$$D = -2h'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}A'+B') - 2h'\frac{B}{r}[\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+2 - \boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2], \quad (3.15)$$

and

$$E = -(A''+B''\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}) - \frac{2}{r}(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}) - 2\frac{B}{r^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}). \quad (3.16)$$

The general form of the eigenvalue equation then becomes after some detail [1]

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\left[\mathbf{p}^2-ig'\hat{\mathbf{r}}\cdot\mathbf{p}+\frac{g'}{2r}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-ih'(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})\right]|\psi_+\rangle$$
$$+\left(\frac{g'}{2r}B[2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-4ir\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}+2ir(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})-6\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]\right.$$
$$-2i(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\cdot\mathbf{p}+i\frac{2B}{r}[2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}-(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})]$$
$$\left.+(k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})+R+m\right)|\psi_+\rangle = \mathcal{B}^2\exp(-2\mathcal{G})(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle \quad (3.17)$$

in which $R=C+D+E$.

Now, to bring this equation to the desired Schrödinger-like form with no linear $\mathbf{p}$ term we multiply both sides by

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1} = \frac{(A-B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})}{A^2-B^2} \quad (3.18)$$

and find, using the exponential form above that appears in Eq. (3.8) (and some detail [1])

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}[-2i(A'+B'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})]\hat{\mathbf{r}}\cdot\mathbf{p} = -2i(F'+K'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\cdot\mathbf{p}, \quad (3.19)$$

and

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}i\frac{2B}{r}[2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}-(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})]$$
$$=\frac{2i\sinh(K)\cosh(K)}{r}[2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}-(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})]+G, \tag{3.20}$$

where (Ref. [1])

$$G=-\frac{2\sinh^2(K)}{r^2}\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2), \tag{3.21}$$

and

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}\frac{g'}{2r}B[2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-4ir\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}+2ir(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})-6\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]$$
$$=\frac{ig'\sinh(K)\cosh(K)}{2r}[-4r\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}+2r(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p})-2i\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+6i\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]+H, \tag{3.22}$$

where (Ref. [1])

$$H=\frac{g'\sinh^2(K)}{2r}[2\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)-2\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+4].$$

Note that G and H do not contain linear **p** type of terms. Now collect the three different linear **p** type of terms in Eq. (3.17):

$$(-2iF'-ig')\hat{\mathbf{r}}\cdot\mathbf{p}, \tag{3.23}$$

$$\left(-2i\frac{\sinh(K)\cosh(K)}{r}-ih'+ig'\sinh(K)\cosh(K)\right)\times(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\mathbf{p}+\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_1\cdot\mathbf{p}), \tag{3.24}$$

$$\left(4i\frac{\sinh(K)\cosh(K)}{r}-2i\sinh(K)\cosh(K)g'-2iK'\right)\times\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\cdot\mathbf{p}. \tag{3.25}$$

If we set the first of the above equations to 0, we obtain the expected result (for the uncoupled portion of the equation)

$$F'=-g'/2. \tag{3.26}$$

If we set $h'=-K'$ and use $\mathbf{p}=\hat{\mathbf{r}}(\hat{\mathbf{r}}.\mathbf{p})-\hat{\mathbf{r}}\times L/r$ then the two expressions (3.24) and (3.25) combine to form

$$\left(2\frac{\sinh(K)\cosh(K)}{r}+h'-g'\sinh(K)\cosh(K)\right)\times\frac{\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\vec{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)}{r}, \tag{3.27}$$

which contains no $\hat{\mathbf{r}}\cdot\mathbf{p}$. Thus the matrix scale change

$$|\phi_+\rangle=\exp(-g/2)\exp(-h\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle \tag{3.28}$$

eliminates the linear **p** terms.

Further note that

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}(k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})\times(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})=(k\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+n\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}),$$

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}C|\psi_+\rangle=-g'(F'+K'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle, \tag{3.29}$$

and (after some algebraic detail [1])

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}D|\psi_+\rangle=-2h'(K'+F'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})|\psi_+\rangle-2h'\frac{\cosh(K)\sinh(K)}{r}[\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+2-\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]|\psi_+\rangle+2h'\frac{\sinh^2(K)}{r}[\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\widehat{\mathbf{r}L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)+3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]|\psi_+\rangle. \tag{3.30}$$

Also,

$$(A+B\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})^{-1}E|\psi_+\rangle=-[F''+F'^2+K'^2+(2F'K'+K'')\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]-\frac{2}{r}[F'+K'\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}]-2\frac{\cosh(K)\sinh(K)}{r^2}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-3\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})+2\frac{\sinh^2(K)}{r^2}(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-2). \tag{3.31}$$

So combining all terms and grouping by $\mathbf{p}^2$ term , spin independent terms, spin-orbit angular momentum term $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$, spin-spin term $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, tensor term $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})$, and additional spin independent term, we have our Schrödinger-like equation

$$\Bigg\{\mathbf{p}^2+\frac{2g'\sinh^2(K)}{r}-g'F'-2h'K'-4h'\frac{\cosh(K)\sinh(K)}{r}-F''-F'^2-K'^2-\frac{2}{r}F'-4\frac{\sinh^2(K)}{r^2}+\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$$
$$\times\left[\frac{g'}{2r}+\frac{g'\sinh^2(K)}{r}-\frac{2\sinh^2(K)}{r^2}-2h'\frac{\cosh(K)\sinh(K)}{r}\right]+\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\cdot\mathbf{L}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$$
$$\times\left(2h'\frac{\sinh^2(K)}{r}+2\frac{\sinh(K)\cosh(K)}{r^2}+\frac{h'}{r}-\frac{g'\sinh(K)\cosh(K)}{r}\right)$$
$$+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\left[k+\frac{g'\cosh(K)\sinh(K)}{r}+\frac{g'\sinh^2(K)}{r}-2h'\frac{\cosh(K)\sinh(K)}{r}-2h'\frac{\sinh^2(K)}{r}-2\frac{\cosh(K)\sinh(K)}{r^2}-2\frac{\sinh^2(K)}{r^2}\right]$$
$$+\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}\Bigg[n-\frac{3g'\cosh(K)\sinh(K)}{r}-\frac{g'\sinh^2K}{r}-g'K'-2h'F'+\frac{2h'\cosh K\sinh K}{r}+6h'\frac{\sinh^2(K)}{r}-(2F'K'+K'')$$
$$-\frac{2}{r}K'+6\frac{\cosh(K)\sinh(K)}{r^2}+2\frac{\sinh^2(K)}{r^2}\Bigg]+m\Bigg\}|\psi_+\rangle=\mathcal{B}^2e^{-2\mathcal{G}}|\psi_+\rangle. \tag{3.32}$$

Comparing Eq. (3.7) with Eq. (3.4), we find

$$g'=2\mathcal{G}'-\frac{E_2M_2+M_1E_1}{\mathcal{D}}(J+L)'=2\mathcal{G}'-\ln'\mathcal{D}=-2F', \tag{3.33}$$

$$h'=\frac{(J-L)'}{2}=-K', \tag{3.34}$$

$$k=\frac{1}{2}\nabla^2\mathcal{G}+\frac{1}{2}\mathcal{G}'^2-\frac{1}{2}\mathcal{G}'\ln'\mathcal{D}-\frac{1}{2}\mathcal{G}'C'-\frac{1}{2}\frac{\mathcal{G}'}{r}-\frac{1}{2}\frac{(-C+J-L)'}{r}, \tag{3.35}$$

$$n=-\frac{1}{2}\nabla^2(-C+J-L)-\frac{1}{2}\nabla^2\mathcal{G}-\mathcal{G}'(-C+J-L)'-\mathcal{G}'^2+\frac{3}{2r}\mathcal{G}'+\frac{3}{2r}(-C+J-L)'+\frac{1}{2}\ln'\mathcal{D}(\mathcal{G}-C+J-L)', \tag{3.36}$$

$$m=-\frac{1}{2}\nabla^2\mathcal{G}-\frac{1}{4}\mathcal{G}'^2-\frac{1}{4}(C+J-L)'(-C+J-L)'+\frac{1}{2}\mathcal{G}'\ln'\mathcal{D}. \tag{3.37}$$

Equation (3.32) and its derivation is an important part of this paper. It will provide us with a way to derive phase shift equations using work by other authors who developed methods for the nonrelativistic Schrödinger equation. First we need the radial form of the coordinate space form of this equation.

The following are the radial eigenvalue equations for singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$ corresponding to Eq. (3.32) with the above substitutions. We emphasize that unlike the potentials used by Reid, Hamada-Johnson, and the Yale group [12,14,15], our potentials are fixed by the structures of the relativistic two-body Dirac equations and we do not have the freedom of choosing different potentials for different angular momentum states.

${}^1S_0$, ${}^1P_1$, ${}^1D_2$ (*a general singlet* ${}^1J_j$): For these states $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=0$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=-3$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=-1$. There is no off diagonal term. We find (adding and subtracting the b^2 term)

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\Phi(r)\right\}v(r)=b^2v(r),$$

where our effective potential for above equation is

$$\Phi(r)=\frac{[2\mathcal{G}-\ln(\mathcal{D})-J+L]'^2}{4}+\frac{[2\mathcal{G}-\ln(\mathcal{D})-J+L]''}{2}+\frac{[2\mathcal{G}-\ln(\mathcal{D})-J+L]'}{r}+\frac{1}{2}\nabla^2(-C+J-L-3\mathcal{G})-\frac{1}{4}(C+J-L-\mathcal{G}+2\ln D)'(-C+J-L-3\mathcal{G})'-\mathcal{B}^2e^{-2\mathcal{G}}+b^2(w). \tag{3.38}$$

Our radial eigenvalue equations for singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ have the same potential forms except for the $j(j+1)/r^2$ angular momentum barrier term. Later, we shall show that their potentials actually are different due to the inclusion of isospin $\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2$ terms.

$^3P_1$(*a general triplet* ^3J_j). For these, $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=-2$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=1$. For the $^3P_1$ state the radial eigenvalue equation is

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\Phi(r)\right\}v(r)=b^2v(r)$$

with

$$\Phi(r)=\frac{[2\mathcal{G}-\ln(\mathcal{D})+J-L]'^2}{4}+\frac{[2\mathcal{G}-\ln(\mathcal{D})+J-L]''}{2}+\frac{(\mathcal{G}+J-L-C)'}{r}-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})+\frac{1}{4}[2\ln(\mathcal{D})-(C+J-L+3\mathcal{G})]'(J-L-C+\mathcal{G})'-\mathcal{B}^2\exp(-2\mathcal{G})+b^2(w). \tag{3.39}$$

The $^3S_1$ and $^3D_1$ are coupled states described by $u_-(r)$ and $u_+(r)$ and their radial eigenvalue equations are [using $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=2(j-1)$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=1/(2j+1)$ (diagonal term), and $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=2\sqrt{j(j+1)}/2j+1$(off diagonal term)] in the form

$$\left\{-\frac{d^2}{dr^2}+\Phi_{11}(r)\right\}u_-+\Phi_{12}(r)u_+=b^2u_-, \tag{3.40}$$

$$\left\{-\frac{d^2}{dr^2}+\frac{6}{r^2}+\Phi_{22}(r)\right\}u_++\Phi_{21}(r)u_-=b^2u_+, \tag{3.41}$$

where

$$\Phi_{11}(r)=\left\{\frac{8}{3}\frac{(2\mathcal{G}'-\ln'\mathcal{D})\sinh^2(h)}{r}+\frac{8}{3}\frac{(J-L)'\cosh(h)\sinh(h)}{r}-\frac{16}{3}\frac{\sinh^2(h)}{r^2}+\frac{[2\mathcal{G}'-\ln'(\mathcal{D})]^2}{4}+\frac{(J-L)'^2}{4}+\frac{[2\mathcal{G}'-\ln'(\mathcal{D})](J-L)'}{6}+\frac{[2\mathcal{G}''-\ln''(\mathcal{D})]}{2}+\frac{(J-L)''}{6}+\frac{[2\mathcal{G}'-\ln'(\mathcal{D})]}{r}+\frac{(J-L)'}{3r}+\frac{1}{3}\left[-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)'\right]+\frac{1}{4}\mathcal{G}'^2-\frac{1}{2}\mathcal{G}'C'-\frac{1}{4}(C+J-L)'(-C+J-L)'-\mathcal{B}^2\exp(-2\mathcal{G})+b^2(w)\right\}, \tag{3.42}$$

$$\Phi_{12}(r)=\frac{2\sqrt{2}}{3}\left\{(2\mathcal{G}'-\ln'\mathcal{D})\left(\frac{3\cosh(h)\sinh(h)}{r}-\frac{\sinh^2(h)}{r}\right)+(J-L)'\left(\frac{3\sinh^2(h)}{r}-\frac{\cosh(h)\sinh(h)}{r}\right)-\frac{6\cosh(h)\sinh(h)}{r^2}+\frac{2\sinh^2(h)}{r^2}-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'+\frac{3(\mathcal{G}+J-L-C)'}{2r}+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)'+\frac{[2\mathcal{G}'-\ln'(\mathcal{D})](J-L)'}{2}+\frac{(J-L)''}{2}+\frac{(J-L)'}{r}\right\}, \tag{3.43}$$

$$\Phi_{22}(r)=\left\{-\frac{8}{3}\frac{(2\mathcal{G}'-\ln'\mathcal{D})\sinh^2(h)}{r}-\frac{8}{3}\frac{(J-L)'\cosh(h)\sinh(h)}{r}+\frac{16}{3}\frac{\sinh^2(h)}{r^2}+\frac{[2\mathcal{G}'-\ln'(\mathcal{D})]^2}{4}+\frac{(J-L)'^2}{4}-\frac{[2\mathcal{G}'-\ln'(\mathcal{D})](J-L)'}{6}+\frac{[2\mathcal{G}''-\ln''(\mathcal{D})]}{2}-\frac{(J-L)''}{6}-\frac{2[2\mathcal{G}'-\ln'(\mathcal{D})]}{r}+\frac{2(J-L)'}{3r}-\frac{(\mathcal{G}+J-L-C)'}{r}-\frac{1}{3}\left[-\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\mathcal{G}'(J-L-C+\mathcal{G})'+\frac{1}{2}\ln'(\mathcal{D})(\mathcal{G}+J-L-C)'\right)\right]+\frac{1}{4}\mathcal{G}'^2-\frac{1}{2}\mathcal{G}'C'-\frac{1}{4}(C+J-L)'(-C+J-L)'-\mathcal{B}^2\exp(-2\mathcal{G})+b^2(w), \tag{3.44}$$

$$\Phi_{21}(r)=\Phi_{12}-4\sqrt{2}\left[(J-L)'\left(\frac{\sinh^2(h)}{r}+\frac{1}{2r}\right)-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{(2\mathcal{G}'-\ln'\mathcal{D})\cosh(h)\sinh(h)}{r}\right] \tag{3.45}$$

(Note that because of the spin-orbit-tensor term, the potential is not symmetric.) In Appendix B we give the coupled equations for triplet ${}^3j_{j-1}$ and ${}^3j_{j+1}$ for general j. The remaining special case is for the ${}^3P_0$ state and has the form

$$\left\{-\frac{d^2}{dr^2}+\frac{2}{r^2}+\Phi(r)\right\}v=b^2(w)v,$$

where

$$\Phi(r)=\frac{[2\mathcal{G}-\ln(\mathcal{D})-J+L]'^2}{4}+\frac{[2\mathcal{G}-\ln(\mathcal{D})-J+L]''}{2}+\frac{[\ln(\mathcal{D})-(4\mathcal{G}+J-L-2C)]'}{r}+\frac{1}{2}\nabla^2(-C+J-L+\mathcal{G})-\frac{1}{2}\mathcal{G}'C'+\frac{1}{4}[C'^2-(J-L)'^2]+\mathcal{G}'\left(\frac{5}{4}\mathcal{G}+J-L-C\right)'-\frac{1}{2}\ln'(\mathcal{D})(J-L-C+\mathcal{G})'-\mathcal{B}^2\exp(-2\mathcal{G})+b^2(w). \tag{3.46}$$

Now we can apply the techniques already developed for the radial Schrödinger equation

$$\left(-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+2mV_{lsj}(r)\right)v=2mEv \tag{3.47}$$

in nonrelativistic quantum mechanics to the above radial equations by the substitutions

$$2mV_{lsj}(r)\rightarrow\Phi_{lsj}(r),\quad 2mE\rightarrow b^2(w). \tag{3.48}$$

By comparing Φ and $2m$ one could determine whether our Φ is similar to standard type of phenomenological potentials such as Reid's potentials. But first, in the following section, we discuss the models we used in our calculation. This includes how we choose the $\mathcal{G}$, L, and C invariant potential functions, the mesons we used in our calculation, and the way they enter into the two-body Dirac equations.

IV. THE INVARIANT INTERACTION FUNCTIONS

A. The $\mathcal{G}$ and L interaction functions

Our dynamics depends on how we parametrize the invariant interaction functions $\mathcal{G}$, L, and C. We first consider how to model $\mathcal{G}$ and L, corresponding to vector and scalar interactions. As we have seen, in order that Eq. (2.65) and Eq. (2.66) satisfy Eq. (2.42), it is necessary that the invariant functions G, E_1,E_2, M_1, and M_2 depend on the relative separation, $x=x_1-x_2$, only through the spacelike coordinate four vector $x_\perp^\mu=x^\mu+\hat{P}^\mu(\hat{P}\cdot x)$, perpendicular to the total four momentum P. For QCD and QED applications, G, E_1,E_2 are functions [4], [6] of an invariant $\mathcal{A}$. The explicit forms for functions E_1,E_2, G are

$$E_1=G(\epsilon_1-\mathcal{A}),$$

$$E_2=G(\epsilon_2-\mathcal{A}), \tag{4.1}$$

and

$$G^2=\frac{1}{\left(1-\frac{2\mathcal{A}}{w}\right)}. \tag{4.2}$$

The function $\mathcal{A}(r)$ is responsible for the covariant electromagnetic like A_i^μ. Even though the dependencies of E_1,E_2,G on $\mathcal{A}$ are not unique, they are constrained by the requirement that they yield an effective Hamiltonian with the correct nonrelativistic and semirelativistic limits (classical and quantum mechanical [40,41]). For QCD and QED application, M_1 and M_2 are functions of two invariant functions [3], [6], $\mathcal{A}(r)$ and $S(r)$,

$$M_1^2(\mathcal{A},S)=m_1^2+G^2(2m_wS+S^2),$$

$$M_2^2(\mathcal{A},S)=m_2^2+G^2(2m_wS+S^2). \tag{4.3}$$

The invariant function $S(r)$ is responsible for the scalar potential since $S_i=0$, if $S(r)=0$, while $\mathcal{A}(r)$ contributes to the S_i [if $S(r)\neq 0$] as well as to the vector potential A_i^μ. So, finally, the five invariant functions G, E_1, E_2, M_1, and M_2 (or $\mathcal{G}=-J,L$) depend on two independent invariant potential functions S and $\mathcal{A}$. [Compare also the spin independent portions to Eqs. (2.25) and (2.27) through calculation of $E_i^2-M_i^2-b^2$.]

Expressing G, E_1, E_2, M_1, and M_2 in terms of S and $\mathcal{A}$ is important for semiphenomenological and other applications that emphasize the relationship of the interactions to effective external potentials of the two associated one-body problems. However, the five invariants G, E_1, E_2, M_1, and M_2 can also be expressed in the hyperbolic representation [35] in terms of the three invariants L, J, and $\mathcal{G}$ [see Eqs. (2.63), (2.64), and (2.69)]. L, J, and $\mathcal{G}$ generate scalar, timelike vector, and spacelike vector interactions, respectively, and enter into our Dirac equations via the sum $\Delta_L+\Delta_J+\Delta_{\mathcal{G}}$ where Eqs. (2.46), (2.47), and (2.48) define Δ_L, Δ_J, $\Delta_{\mathcal{G}}$.

We may use Eqs. (2.41) to relate the matrix potentials Δ to a given field theoretical or semiphenomenological Feynman amplitude. As mentioned earlier, a matrix amplitude proportional to $\gamma_1^\mu\gamma_{2\mu}$ corresponding to an electromagnetic-like interaction would require [22] $J=-\mathcal{G}$. Matrix amplitude proportional to either I_1I_2 or $\gamma_1\cdot\hat{P}\gamma_2\cdot\hat{P}$ would correspond to semiphenomenological scalar or timelike vector interactions. The two-body Dirac equations in the hyperbolic form of Eq. (2.41) give a simple version [35] for the norm of the sixteen component Dirac spinor. The two-body Dirac equations in "external potential" form, Eq. (2.65) and Eq. (2.66), [or more generally Eqs. (2.61) and (2.62)], are simpler to reduce to the Schrödinger-like form and are useful for nu-

TABLE I. Data on mesons (T represents isospin, G represents parity, J represents spin, and π represents parity).

Particles	Mass (MeV)	T^G	J^π	Width (MeV)
$\pi^\pm$	139.570 18±0.000 35	1^-	0^-	
π^0	134.9766±0.0006	1^-	0^-	
η	547.3±0.12	0^+	0^-	$(1.18\pm0.11)\times10^{-3}$
ρ	769.3±0.8	1^+	1^-	150.2±0.8
ω	782.57±0.12	0^-	1^-	8.44±0.09
η'	957.78±0.14	0^+	0^-	0.202±0.016
ϕ	1019.417±0.014	0^-	1^-	4.458±0.032
f_0	980±10	0^+	0^+	40 to 100
a_0	984.8±1.4	1^-	0^+	50 to 100
σ	500–700	0^+	0^+	600 to 1000

merical calculations (see Sazdjian [38] for a related reduction). We describe the parametrization of the pseudoscalar interaction C in Eq. (4.5).

B. Mesons used in the phase shift calculations

We obtain our semiphenomenological potentials for two nucleon interactions by incorporating the meson exchange model and the two-body Dirac equations. Because the pion is the lightest meson, its exchange is associated with the longest range nuclear force. The shortest range behaviors of our semiphenomenological potentials are modified by the form factors, which are treated purely phenomenologically. We exclude heavy mesons that mediate the ranges shorter than that modified by the form factors. The intermediate range part of our semiphenomenological potentials comes from exchange of mesons that are heavier than the pion. We use a total of nine mesons in our fits. These include scalar mesons σ, a_0, and f_0, vector mesons ρ, ω, and ϕ; and pseudoscalar mesons π, η and η'. In this paper, we are ignoring tensor and pseudovector interactions, limiting ourselves to vector, scalar and pseudoscalar interactions, all with masses less than about 1000 MeV. See Table I for detailed features of the mesons we used [45] .

C. Modeling the invariant interaction functions

We initially assume the following introduction of scalar interactions into two-body Dirac equations [see Eqs. (2.70), (2.71), and (4.3)]:

$$S=-g_\sigma^2\frac{e^{-m_\sigma r}}{r}-(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)g_{a_0}^2\frac{e^{-m_{a_0}r}}{r}-g_{f_0}^2\frac{e^{-m_{f_0}r}}{r}, \quad (4.4)$$

where g_σ^2, $g_{a_0}^2$, $g_{f_0}^2$ are coupling constants for the σ, a_0, and f_0 mesons and m_σ, m_{a_0} and m_{f_0} the corresponding masses. $(\boldsymbol{\tau}_1\boldsymbol{\tau}_2)$ is 1 or -3 for isospin triplet or singlet states.

Pseudoscalar interactions are assumed to enter into two-body Dirac equations in the form [see Eq. (3.4)]

$$C=(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)\frac{g_\pi^2}{w}\frac{e^{-m_\pi r}}{r}+\frac{g_\eta^2}{w}\frac{e^{-m_\eta r}}{r}+\frac{g_{\eta'}^2}{w}\frac{e^{-m_{\eta'}r}}{r}, \quad (4.5)$$

where $w=\epsilon_1+\epsilon_2$ is the total energy of two nucleon system. g_π^2, g_η^2, $g_{\eta'}^2$ are coupling constants for mesons π, η, and η', respectively, and m_π, m_η, and $m_{\eta'}$ the corresponding masses. This form for C yields the correct limit at low energy.

We also initially assume that our vector interactions enter into two-body Dirac equations in the form [see Eqs. (4.1) and (4.2)]

$$A=(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)g_\rho^2\frac{e^{-m_\rho r}}{r}+g_w^2\frac{e^{-m_w r}}{r}+g_\phi^2\frac{e^{-m_\phi r}}{r}, \quad (4.6)$$

where g_ρ^2, g_ω^2, g_ϕ^2 are coupling constants for mesons ρ, ω, and ϕ and m_ρ, m_ω and m_ϕ are the corresponding masses.

We use form factors to modify the small r behaviors in S, C, and A, that is, the shortest range part of nucleon-nucleon interaction. We choose our form factors by replacing r in S, C, and A with

$$r\to\sqrt{r^2+r_0^2}. \quad (4.7)$$

In our first model, we just use two different r_0's to fit the experimental data, one r_0 for the pion, one for all the other eight mesons which are heavier than the pion. We set these two r_0's as two free parameters in our fit. These form factors are different from the conventional choices, usually given in momentum space, but the effects are similar.

In the constraint equations, A and S are relativistic invariant functions of the invariant separation $r=\sqrt{x_\perp^2}$ (see below for the distinction between $\mathcal{A}$ and A). Since it is possible that A and S, as identified from the nonrelativistic limit, can take on large positive and negative values, it is necessary to modify G, E_1, E_2, M_1, and M_2 so that the interaction functions remain real when A become large and repulsive [24]. These modifications are not unique but must maintain correct limits.

We have tested several models, two of which can give us fair to good fit to the experimental data.

(a) *Model 1.* For $E_i=G(\epsilon_i-\mathcal{A})$ to be real, we only require that G be real or $\mathcal{A}<w/2$. This restriction on $\mathcal{A}$ is enough to ensure that $M_i=G\sqrt{m_i^2(1-2\mathcal{A}/w)+2m_wS+S^2}$ be real as well (as long as $S\geqslant0$). In order that $\mathcal{A}$ be so restricted we choose to redefine it as

$$\mathcal{A}=A, \quad A\leqslant0, \quad (4.8)$$

$$\mathcal{A}=\frac{A}{\sqrt{4A^2+w^2}}, \quad A\geqslant0. \quad (4.9)$$

This parametrization gives an $\mathcal{A}$ that is continuous through its second derivative.

We next consider the problems that may arise in the limit when one of the masses becomes very large [24]. Even though both our masses used in this paper are equal, we demand that our equations display correct limits. We must modify M_1 and M_2 so that it has the correct static limit (say $m_2\to\infty$). It does appear that $M_1\to m_1+S$ when $m_2\to\infty$.

However, this is only true if $m_1+S\geqslant 0$. In other words, in the limit $m_2\to\infty$, the two-body Dirac equations would reduce to

$$(\gamma\cdot p_1+|m_1+S|)|\psi\rangle=0. \tag{4.10}$$

This would deviate from the standard one-body Dirac equation in the region of strong attractive scalar potential $(S<-m_1)$. In order to correct this problem, we take advantage of the hyperbolic parametrization. We desire a form for M_i that has the expected behavior ($M_i\to m_i+S$ in the limit when S becomes large and negative and one of the masses is large). So we modify our L in the following way [24]

$$\sinh L=\frac{SG^2}{w}\left(1+\frac{G^2(\epsilon_w-A)S}{m_w\sqrt{w^2+S^2}}\right),\quad S<0 \tag{4.11}$$

and for

$$S>0,$$

$$M_1^2=m_1^2+G^2(2m_wS+S^2),$$

$$M_2^2=m_2^2+G^2(2m_wS+S^2) \tag{4.12}$$

with Eqs. (2.63).

A crucial feature of this $\sinh L$ extrapolation is that for fixed S, the static limit($m_2\gg m_1$) form is $\sinh L\to S/w$, which leads to $M_1\to m_1+S$. The above modifications are not unique, given the correct semirelativistic limits [24].

(b) *Model 2*. This model comes from the work of Sazdjian [41]. Using a special technique of amplitude summation, he was able to sum an infinite number of Feynman diagrams (of the ladder and cross ladder variety). For the vector interactions, he obtained results that correspond to Eqs. (2.27)–(2.29) and Eqs. (4.1)–(4.2) [modified here in Eq. (4.9) for $A\geqslant 0$]. For scalar interactions $[L(S,A)]$ he obtained two results. One again agrees with Eq. (2.25) and Eqs. (4.3). As we have seen above, this must be modified [see Eq. (4.11)] for $S\leqslant 0$. His second result is the one we use here for our second model for $[L(S,A)]$. That replaces Eq. (4.11) and Eq. (4.12) with the model

$$S+A>0,$$

then

$$S\to-\mathcal{A}+\frac{(S+A)w}{\sqrt{4(S+A)^2+w^2}}, \tag{4.13}$$

while if

$$S+A<0$$

we let

$$S\to-\mathcal{A}+S+A. \tag{4.14}$$

In both cases we let

$$\sinh L=\sinh\left[-\frac{1}{2}\ln\left(1-\frac{2(S+\mathcal{A})}{w}\right)-\mathcal{G}\right]. \tag{4.15}$$

D. Nonminimal coupling of vector mesons

The coupling of the vector mesons in Eq. (4.6) corresponds in quantum field theory to the minimal coupling $g_\rho V_\mu\bar{\psi}\gamma^\mu\psi$ analogous to $eA_\mu\bar{\psi}\gamma^\mu\psi$ in QED. In our model, we are not concerned about renormalization, since the quantum field theory is not fundamental, so that we cannot rule out the nonminimal coupling of the ρ, ω, ϕ analogous to

$$i\frac{e}{2M}\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi F_{\mu\nu}. \tag{4.16}$$

We can convert the above expressions to something simpler by integration by parts and using the free Dirac equation for the spinor field. This nonrenormalizable interaction becomes

$$i\frac{e}{2M}\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi F_{\mu\nu}$$
$$\to -i\frac{4em_N}{M}\bar{\psi}\gamma^\mu\psi A_\mu-i\frac{2e}{M}[\bar{\psi}\partial^\mu\psi-(\partial^\mu\bar{\psi})\psi]A_\mu. \tag{4.17}$$

The first term can be absorbed into the standard minimal coupling while the second term gives rise to an amplitude written below. Changing from photon to vector mesons (ρ) and using on shell features, we find

$$\frac{4f_\rho^2\left(\eta_{\mu\nu}+\dfrac{q_\mu q_\nu}{m_\rho^2}\right)(p+p')^\mu(p+p')^\nu}{M^2(q^2+m_\rho^2-i\varepsilon)}$$
$$=\frac{4f_\rho^2(p+p')^2}{M^2(q^2+m_\rho^2-i\varepsilon)}=\frac{-4f_\rho^2(4m_N^2+q^2)}{M^2(q^2+m_\rho^2-i\varepsilon)}, \tag{4.18}$$

where $q=p-p'$. The mass M is a mass scale for the interaction, m_N is the fermion (nucleon) mass, and m_ρ is the ρ meson mass.

How does this interaction modify our Dirac equations? Which of the eight or so invariants are affected [see Eqs. (2.46)–(2.56)]? In terms of its matrix structure, the above would appear to contribute to what we called Δ_L [see Eq. (2.46)]. It is as if we include an additional scalar interaction with an exchanged mass of a ρ and subtract from it the Laplacian [the q^2 terms in Eq. (4.18)]. That is,

$$S\to S+S'-\nabla^2S'/4m_N^2, \tag{4.19}$$

where

$$S'=-\frac{16m_N^2}{M^2}\frac{f_\rho^2\exp(-m_\rho r)}{r}, \tag{4.20}$$

so that the modification is rather simple. It has the opposite sign as the vector interaction. That is, it would produce an attractive interaction for pp scattering. But to lowest order, its attractive effects are canceled by the contribution of the first term on the right hand side of Eq. (4.17). In our application, this means that Eq. (4.4) and Eq. (4.6) are replaced [including the r_0 by Eq. (4.7)] by

$$S=-g_\sigma^2\frac{e^{-m_\sigma\bar{r}}}{\bar{r}}-\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2 g_{a_0}^2\frac{e^{-m_{a_0}\bar{r}}}{\bar{r}}-g_{f_0}^2\frac{e^{-m_{f_0}\bar{r}}}{\bar{r}}-S', \tag{4.21}$$

where

$$S'=(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)g_\rho'^2\left(1-\frac{\nabla^2}{4m_N^2}\right)\frac{e^{-m_\rho\bar{r}}}{\bar{r}}+g_w'^2\left(1-\frac{\nabla^2}{4m_N^2}\right)\frac{e^{-m_w\bar{r}}}{\bar{r}}+g_\phi'^2\left(1-\frac{\nabla^2}{4m_N^2}\right)\frac{e^{-m_\phi\bar{r}}}{\bar{r}} \tag{4.22}$$

and

$$A=(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)(g_\rho^2+g_\rho'^2)\frac{e^{-m_\rho\bar{r}}}{\bar{r}}+(g_w^2+g_w'^2)\frac{e^{-m_w\bar{r}}}{\bar{r}}+(g_\phi^2+g_\phi'^2)\frac{e^{-m_\phi\bar{r}}}{\bar{r}}, \tag{4.23}$$

where $g_\rho'^2$, $g_\omega'^2$, $g_\phi'^2$ are also coupling constants we will fit.

V. VARIABLE PHASE APPROACH FOR CALCULATING PHASE SHIFTS

In this section, we discuss and review the phase shift methods that we used in our numerical calculations, which include phase shift equations for uncoupled and coupled states and the phase shift equations with Coulomb potentials. The variable phase approach developed by Calogero has several advantages over the traditional approach. In the traditional approach, one integrates the radial Schrödinger equation from the origin to the asymptotic region where the potential is negligible, and then compares the phase of the radial wave function with that of a free wave and thus obtain the phase shift. In the variable phase approach we need only integrate a first-order nonlinear differential equation from the origin to the asymptotic region, thereby obtaining directly the value of the scattering phase shift.

This method is very convenient for us since we can reduce our two-body Dirac equations to a Schrödinger-like form for which the variable phase approach was developed. Thus, we can conveniently use this variable phase method to compute the phase shift for our relativistic two-body equations.

A. Phase shift equation for uncoupled Schrödinger equation

Reference [16] gives a derivation of a nonlinear equation for the phase shift for the scattering on a spherically symmetrical potential with the boundary condition

$$u_l(0)=0 \tag{5.1}$$

of the radial uncoupled Schrödinger equation

$$u_l''(r)+\left[k^2-\frac{l(l+1)}{r^2}-V(r)\right]u_l(r)=0. \tag{5.2}$$

The radial wave function is real, and it defines the "scattering phase shift" δ_l through the comparison of its asymptotic behavior with that of the sine function:

$$u_l(r)\overset{r\to\infty}{\to}\text{const}\cdot\sin\left(kr-\frac{l\pi}{2}+\delta_l\right). \tag{5.3}$$

The equation that Calogero derives is

$$t_l'(r)=-\frac{1}{k}V(r)[\hat{j}_l(kr)-t_l(r)\hat{n}_l(kr)]^2, \tag{5.4}$$

where $t_l(r)$ has the limiting value $\tan\delta_l$ with the boundary condition $t_l(0)=0$. This is a first-order nonlinear differential equation and can be rewritten [16] in terms of another function $\delta_l(r)$ defined by

$$t_l(r)=\tan\delta_l(r) \tag{5.5}$$

with the boundary condition

$$\delta_l(r)\overset{r\to 0}{\to}0 \tag{5.6}$$

and limiting value

$$\lim_{r\to\infty}\delta_l(r)\equiv\delta_l(\infty)=\delta_l. \tag{5.7}$$

The differential equation for $\delta_l(r)$ is [16]

$$\delta_l'(r)=-k^{-1}V(r)[\cos\delta_l(r)\hat{j}_l(kr)-\sin\delta_l(r)\hat{n}_l(kr)]^2. \tag{5.8}$$

The solution of this first-order nonlinear differential equation yields asymptotically the value of the scattering phase shift. The function $\delta_l(r)$ is named the "phase function" and Eq. (5.8) is called the "phase equation." It is our main tool for studying the properties of scattering phase shifts. Equation (5.8) becomes particularly simple in the case of S waves,

$$\delta_0'(r)=-k^{-1}V(r)\sin^2[kr+\delta_0(r)]. \tag{5.9}$$

Now, since our Schrödinger-like equation in CM system has the form

$$[\boldsymbol{\nabla}^2-b^2-\Phi]|\psi\rangle=0, \tag{5.10}$$

we can directly follow the above steps to obtain the phase shift by swapping $k\to b$, and $V\to\Phi$. There is no change in the phase shift equation, even though our quasipotential Φ depends on the CM system energy w.

We have found it convenient to put all the angular momentum barrier terms in the potentials, and change all the phase shift equations to the form of S-state-like phase shift equations [16]. This puts our phase shift equations in a much simpler form. For spin singlet states, our phase shift equations become just

$$\delta_l'(r) = -b^{-1}\Phi_l(r)\sin^2[br+\delta_l(r)]. \tag{5.11}$$

This equation is similar to the $^1S_0$ state phase equation [see Eq. (5.9)], but it works well for all the singlet states when the angular momentum barrier term $[l(l+1)/r^2]$ is included in $\Phi_l(r)$,

$$\Phi_l(r) = \Phi(r) + \frac{l(l+1)}{r^2}. \tag{5.12}$$

Because the nucleon-nucleon interactions are short range, we integrate our phase shift equations (for both the singlet and triplet states) to a distance [for example, 6 fermis] where the nucleon-nucleon potential becomes very weak. Then the angular momentum barrier terms $l(l+1)/r^2$ dominate the potential $\Phi_l(r)$ and we let our potential $\Phi_l(r) = l(l+1)/r^2$ and integrate our phase shift equations from 6 fm to infinity to get our phase shift. (This can be done analytically in the case of the uncoupled equations [16].)

Because of the modification of our phase shift equations, we also need to modify our boundary conditions for phase shift equations. For the uncoupled singlet states $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$, the modified boundary conditions are [16]

$$\delta_l'(0) = -\frac{l}{l+1}b. \tag{5.13}$$

This is implemented numerically by an additional boundary condition at $r=h$, so our boundary conditions for uncoupled singlet states $^1P_1$, $^1D_2$ and triplet states $^3P_0$, $^3P_1$ are

$$\delta_l(h) = -\frac{l}{l+1}bh, \tag{5.14}$$

where h is the step size in our calculation, $b=\sqrt{b^2}$, and, of course $\delta_l(0)=0$. So for P and D states, the new boundary conditions are $\delta_1(h) = -\frac{1}{2}bh$ and $\delta_2(h) = -\frac{2}{3}bh$, respectively.

B. Phase shift equation for coupled Schrödinger equations

For coupled Schrödinger-like equations, the phase shift equation involves coupled phase shift functions. We discuss an approach here different from that originally presented in Ref. [17]. The key idea for this new derivation is taken from the one presented in a well known quantum text [46]. We present an appropriate adaptation of this idea here in the uncoupled case to demonstrate the general idea and then extend it to the coupled case. Consider a radial equation of the form

$$\left(-\frac{d^2}{dr^2} + \Phi_l(r)\right)u = b^2 u.$$

Following Ref. [46] we assume

$$u(r) = A(r)\sin[br+\delta_l(r)], \tag{5.15}$$

$$u'(r) = bA(r)\cos[br+\delta_l(r)]. \tag{5.16}$$

Taking the derivative of the first equation we find that

$$A' = -A\delta_l'\cot(br+\delta) \tag{5.17}$$

and then using this and Eq. (5.16) the above radial Schrödinger equation reduces to Eq. (5.11).

The coupled radial Schrödinger equation has the form

$$U'' = -b^2 U + \frac{1}{2}(\Phi_L U + U\Phi_L), \tag{5.18}$$

where both U and Φ_L are 2×2 matrices. The effective quasipotential matrix is of the form

$$\Phi_L = \begin{pmatrix} \Phi_{11} + \frac{l_1(l_1+1)}{r^2} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} + \frac{l_2(l_2+1)}{r^2} \end{pmatrix}, \tag{5.19}$$

while the matrix wave function is assumed to be of the form

$$U = \frac{1}{2}[\mathcal{A}\sin(br+\mathcal{D}) + \sin(br+\mathcal{D})\mathcal{A}], \tag{5.20}$$

$$U' = \frac{b}{2}[\mathcal{A}\cos(br+\mathcal{D}) + \cos(br+\mathcal{D})\mathcal{A}], \tag{5.21}$$

with (using Pauli matrices to designate the matrix structure)

$$\mathcal{D} = \delta + \mathbf{D}\cdot\boldsymbol{\sigma},$$

$$\mathcal{A} = a + \mathbf{A}\cdot\boldsymbol{\sigma}. \tag{5.22}$$

(The functions $\mathcal{D}$, $\mathcal{A}$ are not related to earlier functions that use the same symbols.) We further assume (for real and symmetric potentials) that both the phase and amplitude functions are diagonalized by the same orthogonal matrix

$$\tilde{U} = RUR^{-1} = (a + A\sigma_3)\sin(br+\delta+D\sigma_3). \tag{5.23}$$

Combining Eqs. (5.20) and (5.21) together with Eq. (5.18) so as to produce the analog of the phase shift Eq. (5.17) requires we use the following properties of the orthogonal matrix R:

$$R = \begin{pmatrix} \cos\varepsilon(r) & \sin\varepsilon(r) \\ -\sin\varepsilon(r) & \cos\varepsilon(r) \end{pmatrix},$$

$$R'R^{-1} = \varepsilon'\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon' i\sigma_2. \tag{5.24}$$

In Appendix C we derive the coupled phase shift equations $[\delta = (\delta_1+\delta_2)/2,\ D = (\delta_1-\delta_2)/2]$ below:

$$\delta_1'(r) = -\frac{1}{b}\left[\left(\Phi_{11} + \frac{l_1(l_1+1)}{r^2}\right)\cos^2\varepsilon(r) + \left(\Phi_{22} + \frac{l_2(l_2+1)}{r^2}\right)\sin^2\varepsilon(r) + \Phi_{12}\sin 2\varepsilon(r)\right] \times \sin^2[br + \delta_1(r)], \tag{5.25}$$

$$\delta_2'(r) = -\frac{1}{b}\left[\left(\Phi_{22} + \frac{l_2(l_2+1)}{r^2}\right)\cos^2\varepsilon(r) + \left(\Phi_{11} + \frac{l_1(l_1+1)}{r^2}\right)\sin^2\varepsilon(r) - \Phi_{12}\sin 2\varepsilon(r)\right] \times \sin^2[br + \delta_2(r)], \tag{5.26}$$

$$\varepsilon'(r) = \frac{1}{b\sin[\delta_1(r) - \delta_2(r)]}\left\{\frac{1}{2}\left[\Phi_{11} + \frac{l_1(l_1+1)}{r^2} - \Phi_{22}\frac{l_2(l_2+1)}{r^2}\right]\sin 2\varepsilon(r) - \Phi_{12}\cos 2\varepsilon(r)\right\} \times \sin[br + \delta_1(r)]\sin[br + \delta_2(r)]. \tag{5.27}$$

Similar coupled equations are derived in Ref. [17] for coupled *S*-wave equations. Since our potentials include the angular momentum barrier terms we use simple trigonometric functions in place of spherical Bessel and Hankel functions. This requires a modification of the boundary conditions just as in the uncoupled case. To this end, we find it most convenient to rewrite the above three equations in the matrix form

$$T_L' = -\frac{1}{b}[\sin^2(br)\Phi_L + \sin(br)\cos(br)(\Phi_L T_L + T_L \Phi_L) + \cos^2(br) T_L \Phi_L T_L] \tag{5.28}$$

in which the matrix T_L has eigenvalues of $\tan\delta_1$ and $\tan\delta_2$. The actual phase shifts are

$$\delta_1 = \delta_1(r \to \infty),$$
$$\delta_2 = \delta_2(r \to \infty), \tag{5.29}$$
$$\varepsilon = \varepsilon(r \to \infty).$$

The first boundary conditions on the above equations is

$$T_L(0) = 0. \tag{5.30}$$

The further numerical boundary condition that we need for $\varepsilon(h)$, $\delta_1(h)$, and $\delta_2(h)$ are from (for small h)

$$T_L(h) = h T_L'(0). \tag{5.31}$$

At small r, we can approximate our Φ_L for coupled *S* and *D* states in terms of their small r behavior. We find [47]

$$\Phi_L = \frac{1}{r^2}\begin{pmatrix} \eta_- & \eta_0 \\ \eta_0 & 6 + \eta_+ \end{pmatrix}. \tag{5.32}$$

Substitute Eq. (5.31) and Eq. (5.32) into Eq. (5.28) and we find

$$T_L(h) = h T_L'(0) = bh\begin{pmatrix} \alpha & \beta \\ \beta & -\frac{2}{3} + \gamma \end{pmatrix}, \tag{5.33}$$

where

$$\alpha = -\eta_-,$$
$$\beta = -\frac{1}{3}\eta_0, \tag{5.34}$$
$$\gamma = -\frac{\eta_+}{45}.$$

Then we can find $\varepsilon(h)$, $\tan\delta_-(h)$, and $\tan\delta_+(h)$ by diagonalizing the matrix $T_L(h)$. The matrix diagonalizing $T_L(h)$ is

$$\begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}.$$

This leads to the initial conditions

$$\tan(2\varepsilon) = \frac{\frac{2}{3}\eta_0}{\eta_- - \left(\frac{2}{3} + \frac{\eta_+}{45}\right)}$$

$$\tan\delta_-(h) = T_{11} = bh\left[-\eta_-\cos^2\varepsilon - \frac{2}{3}\eta_0\cos\varepsilon\sin\varepsilon - \left(\frac{2}{3} + \frac{\eta_+}{45}\right)\sin^2\varepsilon\right], \tag{5.35}$$

$$\tan\delta_+(h) = T_{22} = bh\left[-\eta_-\sin^2\varepsilon + \frac{2}{3}\eta_0\cos\varepsilon\sin\varepsilon - \left(\frac{2}{3} + \frac{\eta_+}{45}\right)\cos^2\varepsilon\right]$$

and from these initial conditions we can then integrate our equations (5.25)–(5.27) for the coupled system. [Note how these reduce to the uncoupled initial condition Eq. (5.14) with no coupling.]

VI. PHASE SHIFT CALCULATIONS

It is our aim to determine if an adequate description of the nucleon-nucleon phase shifts can be obtained by the use of the CTBDE to incorporate the meson exchange model. In contrast to the relativistic equations used in other approaches [21,48–50,18,19], the CTBDE can be exactly reduced to a local Schrödinger-like form. This allows us to gain additional physical insight into the nucleon-nucleon interactions. We test our two models to find which one gives us the best fit to the experimental phase shift data in nucleon-nucleon scattering. These two models are among many that have been tested.

The data set [20] that we used in our test consists of pp and np nucleon-nucleon scattering phase shift data up to $T_{Lab}=350$ MeV published in physics journals between 1955 and 1992. In our fits, we use experimental phase shift data for NN scattering in the singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$. We use our parameter fit results from np scattering to predicate the result in pp scattering. (The variable phase method for potentials including the Coulomb potential is reviewed in Appendix D.) Thus we did not put the pp scattering data of singlet states ${}^1S_0$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$ into our fits. (There is no pp scattering in ${}^1P_1$, ${}^3S_1$, and ${}^3D_1$ states because of the consideration of the Pauli principle).

We use seven angular momentum states in our fit. There are 11 data points for every angular momentum state, in the energy range from 1 to 350 MeV, so the total number of data points in our fits is 77. To determine the free coupling constant (and the sigma mass m_σ) in our potentials, we have to perform a best fit to the experimentally measured phase shift data. The coupling constants are generally searched by minimizing the quantity χ^2. The definition of our χ^2 is

$$\chi^2=\sum_i \left\{\frac{\delta_i^{th}-\delta_i^{\exp}}{\Delta\delta_i}\right\}^2, \tag{6.1}$$

where δ_i^{th} are theoretical phase shifts, $\delta_i^{\exp}$ are experimental phase shifts, and we let $\Delta\delta_i=1°$. (Our model at this stage is too simplified to perform a fit that involves the actual experimental errors).

We have tried several methods to minimize our χ^2: the gradient method, grid method, and Monte Carlo simulations. Our χ^2 drops very quickly at the beginning if we search by the gradient method, then it always hits some local minima and cannot jump out. Obviously, the grid method should lead us to the global minimum. The problem is that if we want to find the best fit parameters we must let the mesh size be small. But then the calculation time becomes unbearably long. On the other hand if we choose a larger mesh, we will miss the parameters that we are looking for.

We found that the Monte Carlo method can solve the above dilemma. We set a reasonable range for all the parameters that we want to fit and generate all our fitting parameters randomly. Initially, the calculation time is also very long for this method, but it can lead us to a rough area where our fitting parameters are located. Then we shrink the range for all our fitting parameters and do our calculation again (or use the gradient method in tandem), our calculation time then being greatly reduced. By repeating several times in the same way, we can finally find the parameters.

To expedite our calculations further, we put restrictions on ${}^1S_0$ and ${}^3S_1$ states. After every set of parameters is generated randomly, we first test it on the ${}^1S_0$ state at 1 MeV. For ${}^1S_0$ state, if

TABLE II. Parameters from fitting experimental data (model 1).

	η	η'	σ	ρ	ω	π	a_0
g^2	2.25	4.80	47.9	11.6	16.5	13.3	0.13
$r_0(\times 10^{-3})$	2.843	2.843	2.843	2.843	2.843	0.645	2.843
	ϕ	f_0	ρ'	ω'	ϕ'	m_σ	
g^2	5.64	19.9	0.34	20.6	3.10	724.1	
$r_0(\times 10^{-3})$	2.843	2.843	2.843	2.843	2.843		

$$|\delta_i^{th}-\delta_i^{\exp}|>0.2|\delta_i^{\exp}| \tag{6.2}$$

we let the computer jump out of this loop and generate another set of parameters and test it again until a set of parameters passes this restriction. Then we test it on the ${}^3S_1$ states at 1 MeV with the same restriction. We only calculate δ_i^{th} at higher energy if a set of parameters passes these two restrictions. Our code can run at least 50 times faster by these two restrictions. After we shrink our parameter ranges 2 or 3 times, all of our parameters are confined in a small region. At this time, we may change our restriction to

$$|\delta_i^{th}-\delta_i^{\exp}|>0.15|\delta_i^{\exp}| \tag{6.3}$$

and put restriction on ${}^1P_1$ states or any other states to let our code run more efficiently.

Using this method we tried several different models to fit the phase shift experimental data of seven different angular momentum states including the singlet states ${}^1S_0$, ${}^1P_1$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$, ${}^3S_1$, ${}^3D_1$. Two models that we discussed above can give us a fairly good fit to the experimental data. The parameters which we obtained for model 1 are listed in Table II, and for model 2 are listed in Table III. For the features of mesons in Tables II and III, please refer to Table I and Eqs. (4.4)–(4.6). The sigma mass is in MeV while the structure parameter r_0 is in inverse MeV.

A. Model 1

The theoretical phase shifts that we calculated by using the parameters for model 1 and the experimental phase shifts for all the seven states are listed in Table IV. We use parameters given above to predict the phase shift of pp scattering. Our prediction for the four pp scattering states that include singlet states ${}^1S_0$, ${}^1D_2$ and triplet states ${}^3P_0$, ${}^3P_1$ are listed in Table V.

The results for np scattering are also presented from Figs. 1–7 and for pp scattering from Figs. 8–11.

TABLE III. Parameters from fitting experimental data (model 2).

	η	η'	σ	ρ	ω	π	a_0
g^2	0.88	1.70	54.7	2.58	18.3	13.6	10.5
$r_0(\times 10^{-3})$	1.336	1.264	3.180	6.640	2.627	1.717	9.282
	ϕ	f_0	ρ'	ω'	ϕ'	m_σ	
g^2	9.12	33.5	5.11	28.6	12.1	694.3	
$r_0(\times 10^{-3})$	11.45	4.447	6.640	2.627	11.45		

TABLE IV. np scattering phase shift of $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3P_1$, $^3S_1$, and $^3D_1$ states (model 1).

Energy (MeV)	$^1S_0$ Expt. Theor.	$^1P_1$ Expt. Theor.	$^1D_2$ Expt. Theor.	$^3P_0$ Expt. Theor.
1	62.07 59.96	−0.187 −0.359	0.00 0.00	0.18 0.00
5	63.63 63.48	−1.487 −1.169	0.04 0.00	1.63 1.55
10	59.96 60.40	−3.039 −2.870	0.16 0.05	3.65 3.57
25	50.90 51.95	−6.311 −6.641	0.68 0.52	8.13 8.72
50	40.54 41.65	−9.670 −10.23	1.73 1.13	10.70 11.62
100	26.78 26.64	−14.52 −13.49	3.90 2.00	8.460 10.17
150	16.94 15.18	−18.65 −15.26	5.79 2.51	3.690 5.688
200	8.940 5.615	−22.18 −16.49	7.29 2.91	−1.44 0.66
250	1.960 −2.719	−25.13 −17.60	8.53 3.11	−6.51 −4.38
300	−4.460 −10.16	−27.58 −18.63	9.69 3.55	−11.47 −9.206
350	−10.59 −16.94	−29.66 −19.68	10.96 3.311	−16.39 −13.81
Energy (MeV)	$^3P_1$ Expt. Theor.	$^3S_1$ Expt. Theor.	$^3D_1$ Expt. Theor.	ε Expt. Theor.
1	−0.11 −0.33	147.747 142.692	−0.005 0.719	0.105 0.287
5	−0.94 −0.88	118.178 112.670	−0.183 −0.176	0.672 1.224
10	−2.06 −2.26	102.611 98.215	−0.677 −0.256	1.159 1.951
25	−4.88 −5.70	80.63 78.38	−2.799 −2.910	1.793 2.587
50	−8.25 −10.18	62.77 62.00	−6.433 −6.947	2.109 2.495
100	−13.24 −16.66	43.23 43.18	−12.23 −13.94	2.420 3.013
150	−17.46 −22.12	30.72 30.64	−16.48 −19.35	2.750 3.562
200	−21.30 −26.98	21.22 20.95	−19.71 −23.78	3.130 4.489
250	−24.84 −31.46	13.39 12.95	−22.21 −27.62	3.560 5.682
300	−28.07 −35.67	6.600 6.127	−24.14 −31.01	4.030 6.982
350	−30.97 −39.58	0.502 0.171	−25.57 −34.15	4.570 8.536

B. Model 2

The theoretical phase shifts that we calculated by using the parameters for model 2 and the experimental phase shifts for all the seven states are listed in Table VI. We also use the parameters for model 2 to predict the phase shift of pp scattering.

The predictions for the four pp scattering states are listed in Table VII. The results of model 2 for np scattering are given in Figs. 12–18 and for pp scattering are given in Figs. 19–22. Our results show for this model an improvement over those of model 1, especially for the the singlet P and D states. However, there is still much to be desired in the fit. One possible cause of this problem is that we did not include tensor and pseudovector interactions in our covariant potentials, limiting ourselves to scalar, vector, and pseudoscalar. Another may be the ignoring of the pseudovector coupling of

TABLE V. pp scattering phase shift of $^1S_0$, $^1D_2$, $^3P_0$, and $^3P_1$ states (model 1).

Energy MeV	$^1S_0$ Expt. Theor.	$^1D_2$ Expt. Theor.	$^3P_0$ Expt. Theor.	$^3P_1$ Expt. Theor.
1	32.68 51.95	0.001 −0.091	0.134 0.381	−0.081 −1.215
5	54.83 55.47	0.043 −0.183	1.582 0.954	−0.902 −2.536
10	55.22 54.45	0.165 −0.270	3.729 1.773	−2.060 −3.864
25	48.67 47.64	0.696 −0.441	8.575 5.422	−4.932 −7.932
50	38.90 37.77	1.711 −0.504	11.47 9.766	−8.317 −13.15
100	24.97 23.63	3.790 0.511	9.450 7.862	−13.26 −18.45
150	14.75 12.37	5.606 1.141	4.740 3.812	−17.43 −24.42
200	6.550 3.024	7.058 2.407	−0.370 −1.178	−21.25 −28.50
250	−0.31 −5.15	8.270 2.994	−5.430 −6.193	−24.77 −33.26
300	−6.15 −12.55	9.420 3.136	−10.39 −10.98	−27.99 −37.63
350	−11.13 −19.27	10.69 2.902	−15.30 −15.42	−30.89 −41.13

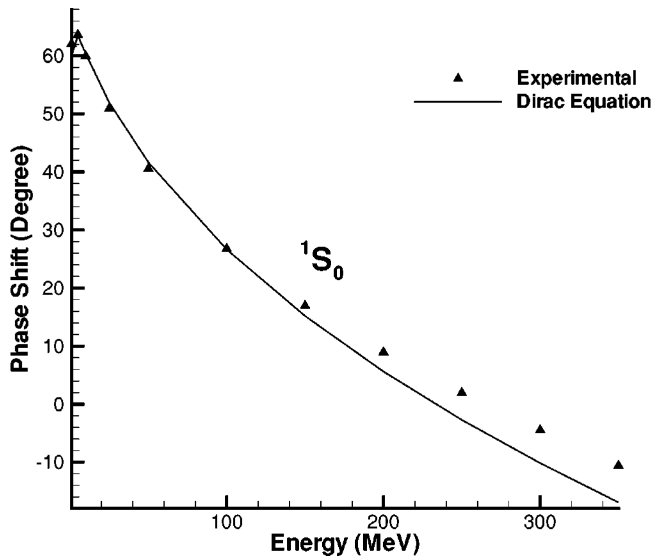

FIG. 1. np scattering phase shift for $^1S_0$ state (model 1).

the pseudoscalar mesons to the nucleon. Our results in pp scattering show that if we obtain a good fit in np scattering our predicted results in pp scattering will also be good. This means that it is unnecessary to include pp scattering in the our fit, we may use the parameters obtained in np scattering to predict the results in pp scattering. Overall, our results are promising and indicate that the two-body Dirac equations of constraint dynamics together with the meson exchange model are suitable to construct semiphenomenological potential models for nucleon-nucleon scattering.

VII. CONCLUSION

The two-body Dirac equations of constraint dynamics constitute the first fully covariant treatment of the relativistic two-body problem that has the following properties.

(a) Includes constituent spin.

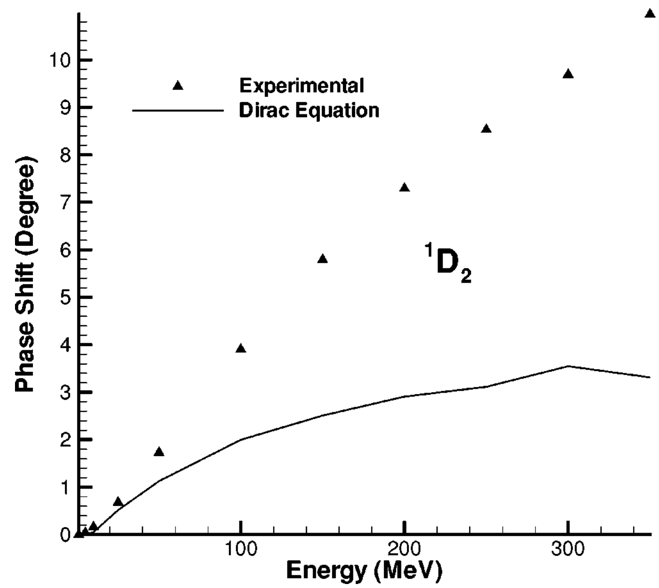

FIG. 3. np scattering phase shift for $^1D_2$ state (model 1).

(b) Regulates the relative time in a covariant manner.

(c) Provides an exact reduction to four decoupled four-component wave equations.

(d) Includes non-perturbative recoil effects in a natural way that eliminates the need for singularity-softening parameters or finite particle size in semiphenomenological applications to QCD.

(e) Is canonically equivalent in the semirelativistic approximation to the Fermi-Breit approximation to the Bethe-Salpeter equation.

(f) Unlike the Bethe-Salpeter equation and most other relativisitc approaches has a local momentum structure as simple as that of the nonrelativistic Schrödinger equation.

(g) Is well defined for zero-mass constituents (hence, permits investigation of the chiral symmetry limit)

(h) Possesses spin structure that yields an exact solution for singlet positronium.

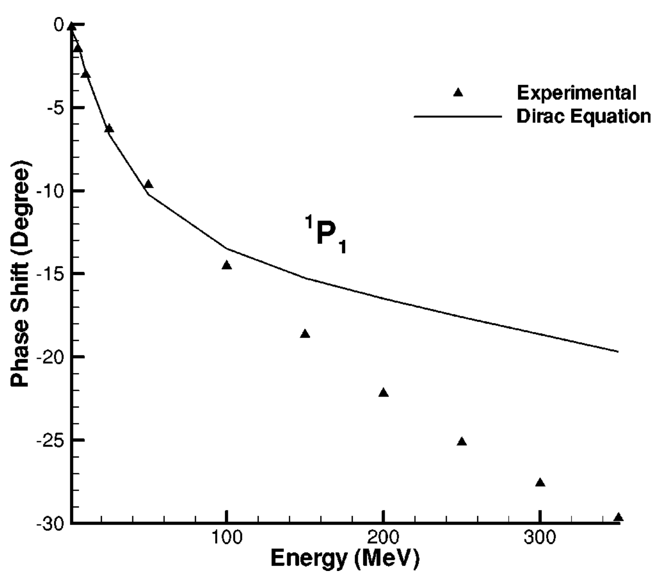

FIG. 2. np scattering phase shift for $^1P_1$ state (model 1).

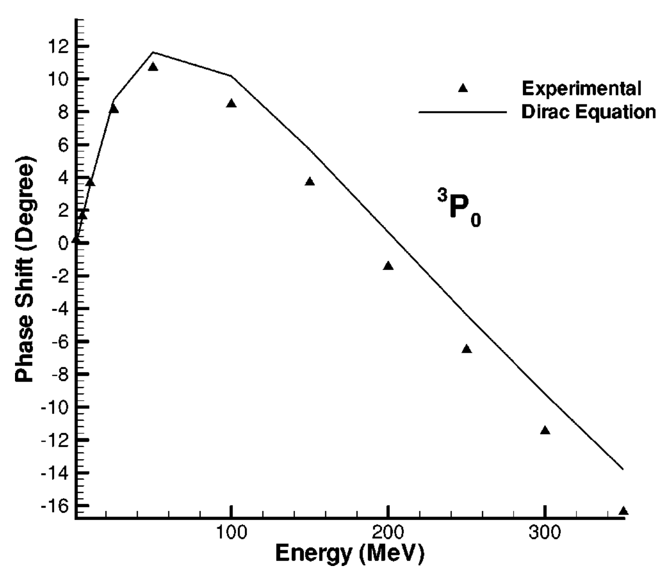

FIG. 4. np scattering phase shift for $^3P_0$ state (model 1).

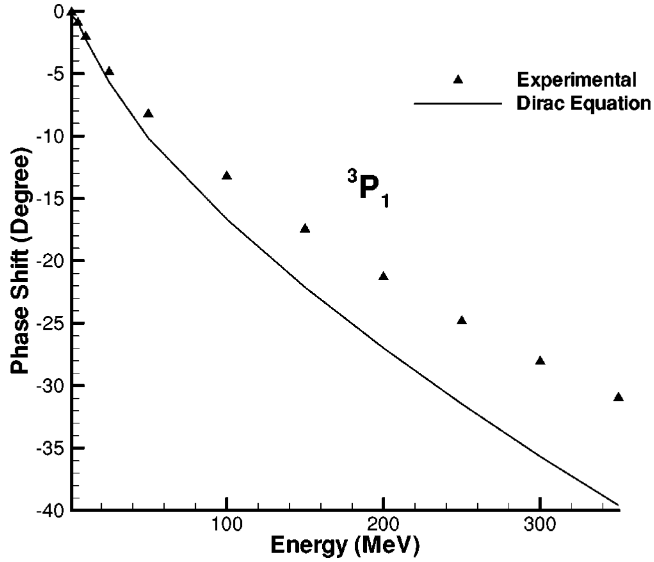

FIG. 5. np scattering phase shift for $^3P_1$ state (model 1).

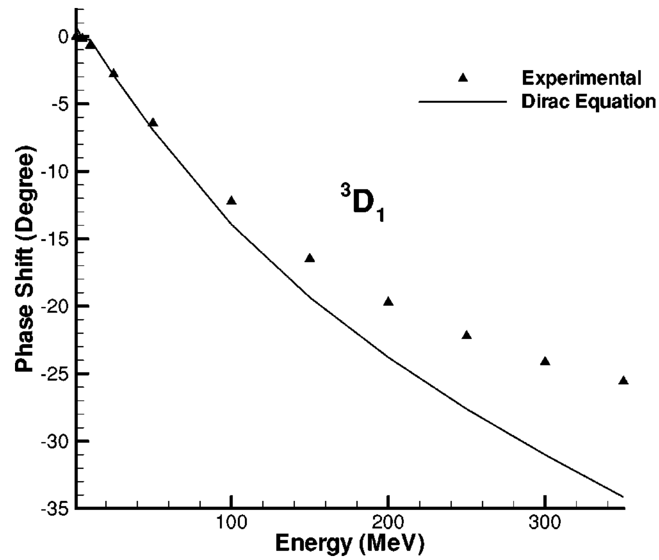

FIG. 7. np scattering phase shift for $^3D_1$ state (model 1).

(i) Has static limits that are relativistic, reducing to the ordinary single-particle Dirac equation in the limit that either particle becomes infinitely heavy.

(j) Possesses a great variety of equivalent forms that are rearrangements of its two coupled Dirac equations (hence is directly related to many previously-known quantum descriptions of the relativistic two-body system).

These structures play an essential role in the success of this approach to both QCD and QED bound states. What is noteworthy in the latter application is that one need only identify the nonrelativistic parts, i.e., the lowest order forms of $\mathcal{A}$ and S. The spin-dependent and covariant structure of the two-body Dirac formalism then automatically stamps out the correct semirelativistic spin dependent and spin independent corrections and provides well defined higher order relativistic corrections as well. In addition, the constraint formalism, although rooted in classical mechanics, has close connections to the Bethe-Salpeter equation of quantum field theory [36] and with Wigner's formulation of relativistic quantum mechanics as a symmetry of quantum theory [43].

In this paper we have shown that these two-body Dirac equations may provide a reasonable account of the nucleon-nucleon scattering data when combined with the meson exchange model. What makes this result important is that it is accomplished with a local and covariant formulation of the two-body problem. What makes this unique is that this approach has been thoroughly tested in a nonperturbative context for both QED and QCD bound states. It is not given that success in one or even both areas would imply that the formalism would do well in another. In particular, the fits could have easily been disastrous, given the minimal coupling idea we have used (based in part on the earlier work on the quasipotential approach of Todorov). The reason for some doubt is that these minimal coupling forms (generalized to the sca-

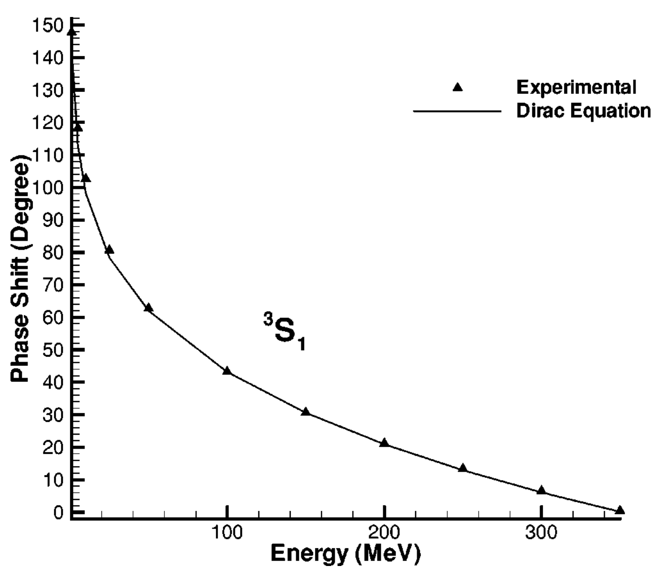

FIG. 6. np scattering phase shift for $^3S_1$ state (model 1).

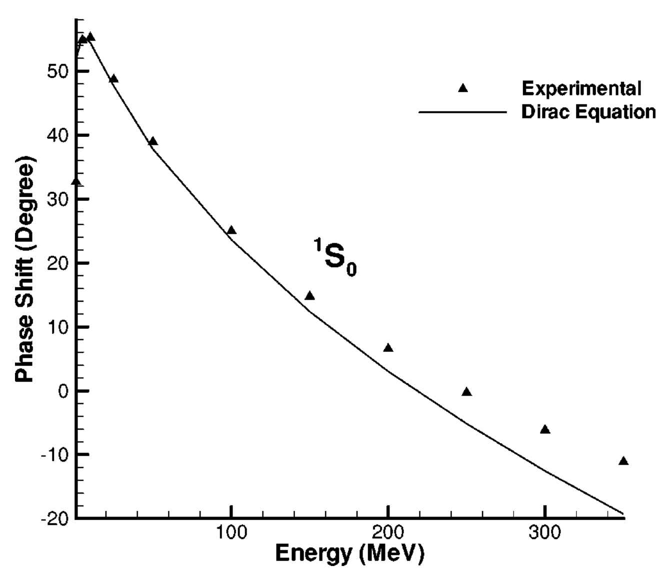

FIG. 8. pp scattering phase shift for $^1S_0$ state (model 1).

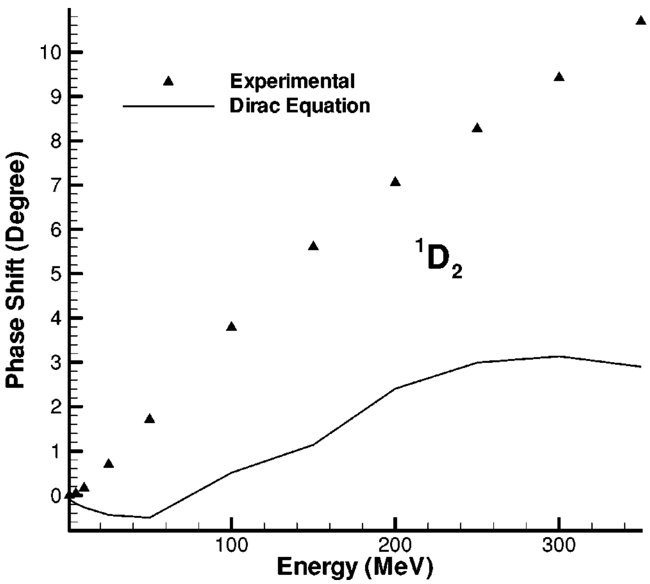

FIG. 9. *np* scattering phase shift for $^1D_2$ state (model 1).

lar interactions as well as the vector) lead to the scalar and vector potentials appearing squared. Because of the size of the coupling constants, the deviation from the standard effective potentials could have been considerable in all cases. There are other nonperturbative structures that appear in the Pauli reduction of our equations to Schrödinger-like form (typical of what appears in the Pauli reduction of the one-body Dirac equation) that could also have prevented any reasonable results. So the general agreement we obtained with the data is very encouraging that this approach could be extended to include more general interactions.

An important step in our reduction was that we put the equation in a form for which we can apply the techniques that have been already developed for the Schrödinger-like system in nonrelativistic quantum mechanics. This required that we get rid of first derivative terms. For the uncoupled states, it is pretty straightforward. For the coupled states we

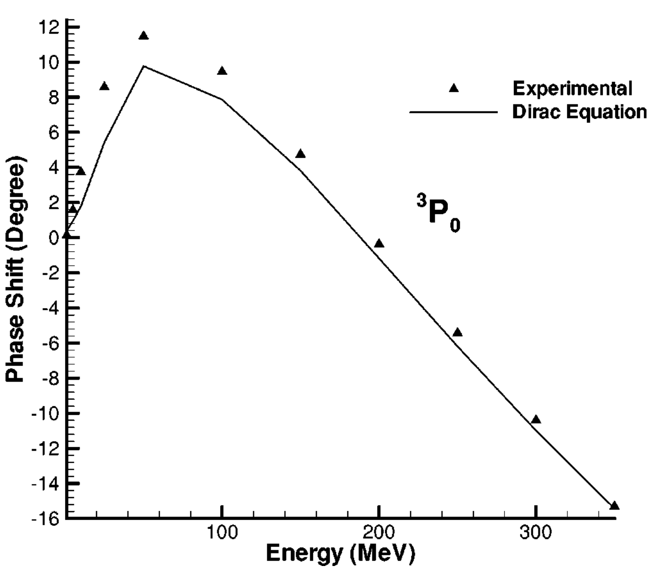

FIG. 10. *pp* scattering phase shift for $^3P_0$ state (model 1).

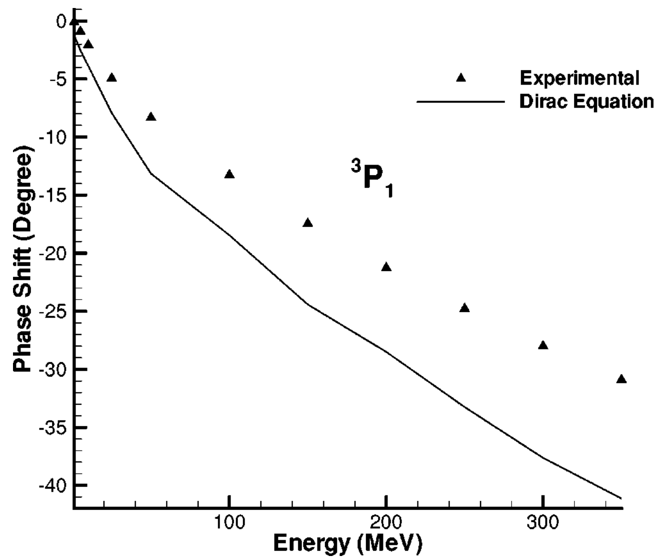

FIG. 11. *pp* scattering phase shift for $^3P_1$ state (model 1).

used a different spin-matrix approach that works for both the uncoupled and coupled states simultaneously.

We then tested several models by using the variable phase methods. We found it most convenient to put all the angular momentum barrier terms in the potentials, and change all the phase shift equations to the form of *S*-state-like phase shift equations [see Eqs. (5.11), (5.25), (5.26), and (5.27)].

After several models and several methods to minimize our χ^2 were tested, we found two models that can lead us to a fairly good fit to the experimental phase shift data.

The most important equation used in our phase shift analysis for nucleon-nucleon scattering is Eq. (3.32) . It is a coupled Schrödinger-like equation derived from two-body Dirac equations with no approximations. All of our radial wave equations for any specific angular momentum state are obtained from this equation.

We use nine mesons in our fit. We summarize the meson-nucleon interactions we used by writing the quantum field theory Lagrange function for their effective interactions,

$$\begin{aligned}\mathcal{L}_I = {} & g_\sigma \bar{\psi}\psi\sigma + g_{f_0}\bar{\psi}\psi f_0 + g_{a_0}\bar{\psi}\boldsymbol{\tau}\psi\cdot\mathbf{a}_0 + g_\rho \bar{\psi}\gamma^\mu \boldsymbol{\tau}\psi\cdot\boldsymbol{\rho}_\mu \\ & + g_\omega \bar{\psi}\gamma^\mu\psi\omega_\mu + g_\phi\bar{\psi}\gamma^\mu\psi\phi_\mu - i g_\pi \bar{\psi}\gamma^5\boldsymbol{\tau}\psi\cdot\boldsymbol{\pi} \\ & - i g_\eta \bar{\psi}\gamma^5\psi\eta - i g_{\eta'}\bar{\psi}\gamma^5\psi\eta', \end{aligned} \tag{7.1}$$

where ψ represent the nucleon field, σ, f_0, ... represent the meson fields.

Several models have been tested by using the variable phase methods, two models can lead us to a fairly good fit to the experimental phase shift data. We use the parameters that give good fits to the *np* scattering data to predict the phase shifts for the *pp* scattering. These lead to a good prediction for the *pp* scattering based on the parameters we obtained (with noted exceptions). This means that our work has shown a promising result. The following are some suggestions to improve our work in the future.

TABLE VI. *np* scattering phase shift of $^1S_0$, $^1P_1$, $^1D_2$, $^3P_0$, $^3P_1$, $^3S_1$, and $^3D_1$ states (model 2).

Energy (MeV)	$^1S_0$ Expt. Theor.	$^1P_1$ Expt. Theor.	$^1D_2$ Expt. Theor.	$^3P_0$ Expt. Theor.
1	62.07 60.60	−0.187 −0.358	0.00 0.02	0.18 0.00
5	63.63 63.50	−1.487 −1.163	0.04 0.15	1.63 1.61
10	59.96 60.20	−3.039 −2.857	0.16 0.39	3.65 3.74
25	50.90 51.44	−6.311 −6.629	0.68 0.40	8.13 9.28
50	40.54 40.91	−9.670 −10.36	1.73 1.37	10.70 12.69
100	26.78 25.86	−14.52 −14.44	3.90 2.42	8.460 11.74
150	16.94 14.62	−18.65 −17.55	5.79 3.62	3.690 7.399
200	8.940 5.435	−22.18 −20.37	7.29 4.55	−1.44 2.36
250	1.960 −2.428	−25.13 −23.15	8.53 5.24	−6.51 −2.78
300	−4.460 −9.330	−27.58 −25.87	9.69 5.34	−11.47 −7.746
350	−10.59 −15.52	−29.66 −28.54	10.96 5.30	−16.39 −12.52
Energy (MeV)	$^3P_1$ Expt. Theor.	$^3S_1$ Expt. Theor.	$^3D_1$ Expt. Theor.	ε Expt. Theor.
1	−0.11 −0.32	147.747 144.797	−0.005 0.719	0.105 0.264
5	−0.94 −0.81	118.178 115.232	−0.183 −0.172	0.672 1.106
10	−2.06 −2.08	102.611 100.668	−0.677 −0.239	1.159 1.723
25	−4.88 −5.07	80.63 80.66	−2.799 −2.834	1.793 2.099
50	−8.25 −8.68	62.77 64.30	−6.433 −6.798	2.109 1.708
100	−13.24 −13.55	43.23 45.68	−12.23 −13.77	2.420 1.663
150	−17.46 −17.74	30.72 33.35	−16.48 −19.34	2.750 1.541
200	−21.30 −21.67	21.22 23.80	−19.71 −24.11	3.130 1.648
250	−24.84 −25.47	13.39 15.90	−22.21 −28.38	3.560 1.834
300	−28.07 −29.14	6.600 9.099	−24.14 −32.29	4.030 1.965
350	−30.97 −32.67	0.502 3.095	−25.57 −36.01	4.570 2.147

VIII. SUGGESTIONS FOR FUTURE WORK

A. Other Model Tests

More model testing is absolutely necessary in the future. By model we mean the way we place the perturbative interactions that arise from Eq. (7.1) into the nonperturbative forms we need for L, C, and $\mathcal{G}$. During our fits, we found that our final results are sensitive to the model we chose ranging from very bad fits to the fits presented here. Changing the way to modify the interactions and the way mesons enter into the two-body Dirac equations may provide a new opportunity to improve our fit.

B. Including World Tensor Interactions

We have included just scalar, pseudoscalar, and vector interactions in our potentials through the invariant forms like L, C, and $\mathcal{G}$. Treating two-body Dirac equations with tensor

TABLE VII. *pp* scattering phase shift of $^1S_0$, $^1D_2$, $^3P_0$, and $^3P_1$ states (model 2).

Energy MeV	$^1S_0$ Expt. Theor.	$^1D_2$ Expt. Theor.	$^3P_0$ Expt. Theor.	$^3P_1$ Expt. Theor.
1	32.68 52.40 0.001 −0.116	0.134 0.417	−0.081 −1.172	
5	54.83 55.48	0.043 −0.232	1.582 1.042	−0.902 −2.434
10	55.22 54.24	0.165 −0.327	3.729 1.934	−2.060 −3.682
25	48.67 47.13	0.696 −0.524	8.575 5.943	−4.932 −7.355
50	38.90 37.04	1.711 −0.505	11.47 10.88	−8.317 −11.57
100	24.97 22.85	3.790 0.994	9.450 9.417	−13.26 −15.41
150	14.75 11.82	5.606 2.036	4.740 5.543	−17.43 −19.97
200	6.550 2.845	7.058 3.211	-−0.370 0.495	−21.25 −23.23
250	−0.31 −4.86	8.270 3.648	−5.430 −4.589	−24.77 −27.28
300	−6.15 −11.72	9.420 3.956	−10.39 −9.516	−27.99 −31.05
350	−11.13 −17.85	10.69 4.014	−15.30 −14.13	−30.89 −34.22

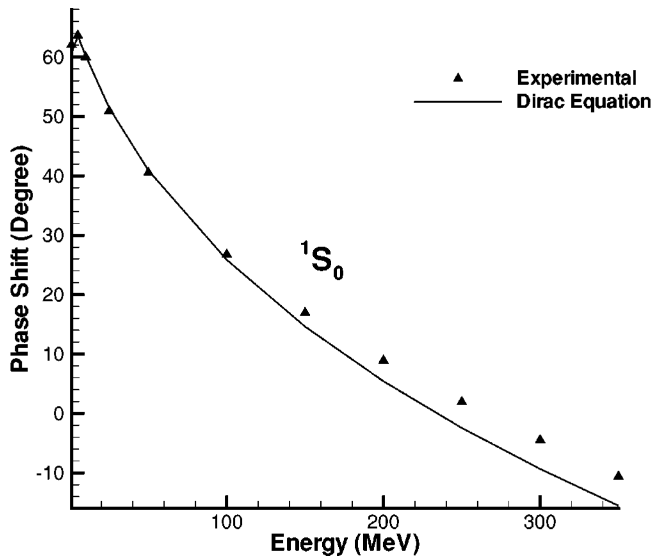

FIG. 12. np scattering phase shift of $^1S_0$ state (model 2).

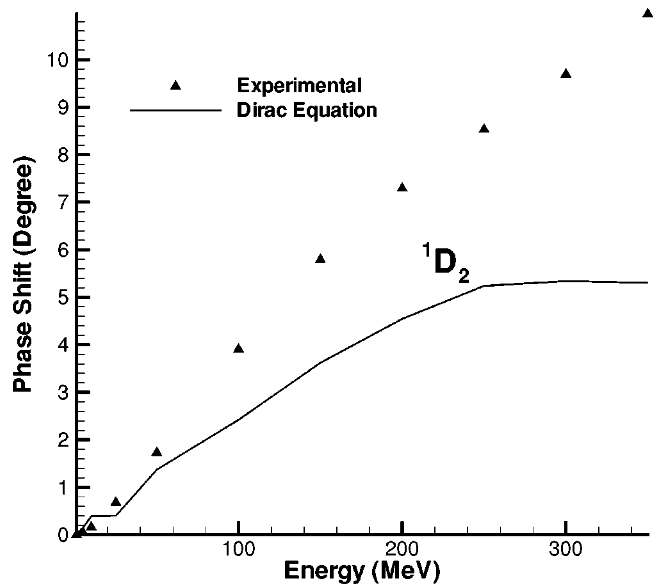

FIG. 14. np scattering phase shift of $^1D_2$ state (model 2).

interactions of the vector meson may improve our fit. These tensor interactions were discussed earlier [see Eq. (4.16)] and correspond to nonminimal coupling of spin one-half particle not present in QED but which cannot be ruled out in massive vector meson-nucleon interactions. The corresponding field theory interaction is

$$\Delta\mathcal{L}_I = g'_\rho \bar{\psi}\sigma^{\mu\nu}\boldsymbol{\tau}\psi\cdot\boldsymbol{\rho}_{\mu\nu} + g'_\omega\bar{\psi}\sigma^{\mu\nu}\psi\omega_{\mu\nu} + g'_\phi\bar{\psi}\sigma^{\mu\nu}\psi\phi_{\mu\nu} \tag{8.1}$$

and would correspond to relaxing the free field equation assumption made in Eq. (4.17).

C. Include pseudovector interactions

Another option is to allow the pseudoscalar mesons (π, η, and η') to interact with the nucleon not only by the pseudoscalar interaction [as in Eq. (7.1)] but also by the way of the pseudovector interactions as below,

$$\Delta\mathcal{L}_I = g'_\pi\bar{\psi}\gamma^\mu\gamma^5\boldsymbol{\tau}\psi\cdot\partial_\mu\boldsymbol{\pi} + g'_\eta\bar{\psi}\gamma^\mu\gamma^5\psi\partial_\mu\eta + g'_{\eta'}\bar{\psi}\gamma^\mu\gamma^5\psi\partial_\mu\eta'. \tag{8.2}$$

D. Include full massive spin-one propagator

We have ignored a portion of the massive spin-one propagator in our fit that is zero for particles on the mass shell. To include this portion of massive spin-one propagator we would have to change the vector propagator as below,

$$\frac{\eta^{\mu\nu}}{q^2+m_\rho^2-i\varepsilon} \to \frac{\eta^{\mu\nu}+\dfrac{q^\mu q^\nu}{m_\rho^2}}{q^2+m_\rho^2-i\varepsilon}. \tag{8.3}$$

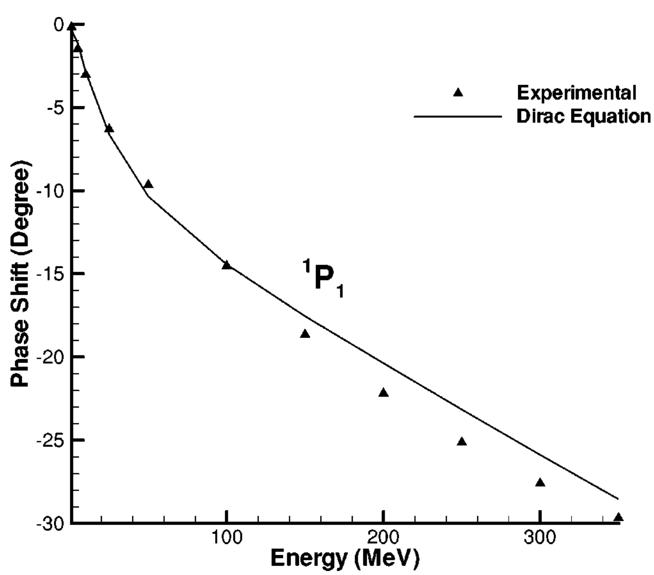

FIG. 13. np scattering phase shift of $^1P_1$ state (model 2).

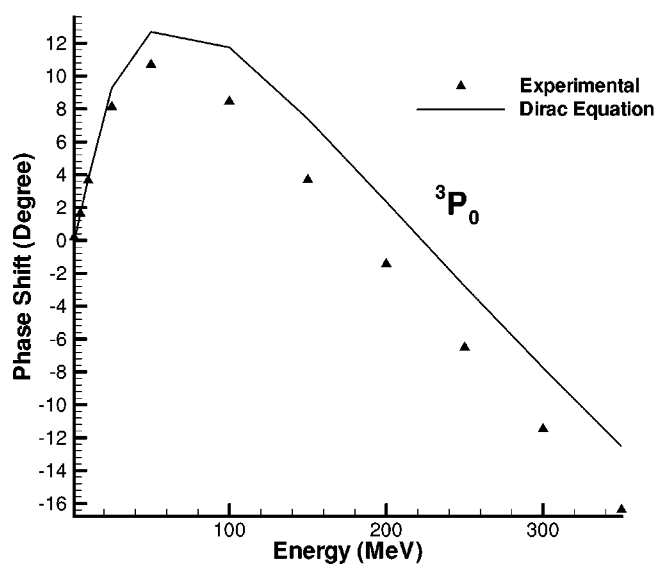

FIG. 15. np scattering phase shift of $^3P_0$ state (model 2).

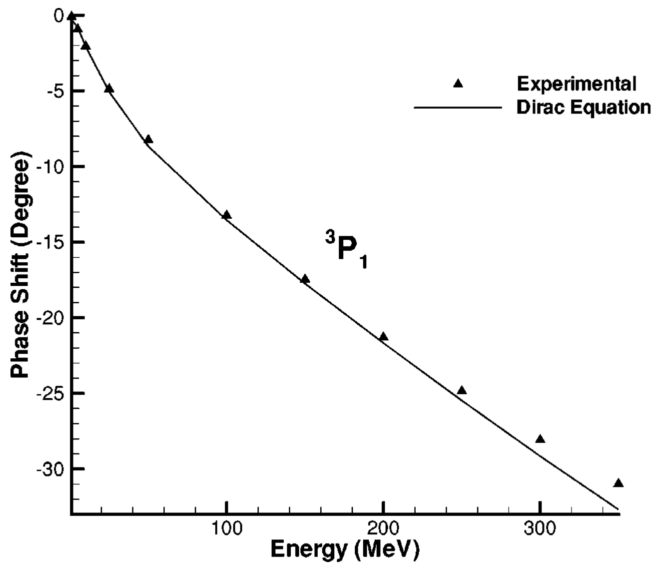

FIG. 16. np scattering phase shift of $^3P_1$ state (model 2).

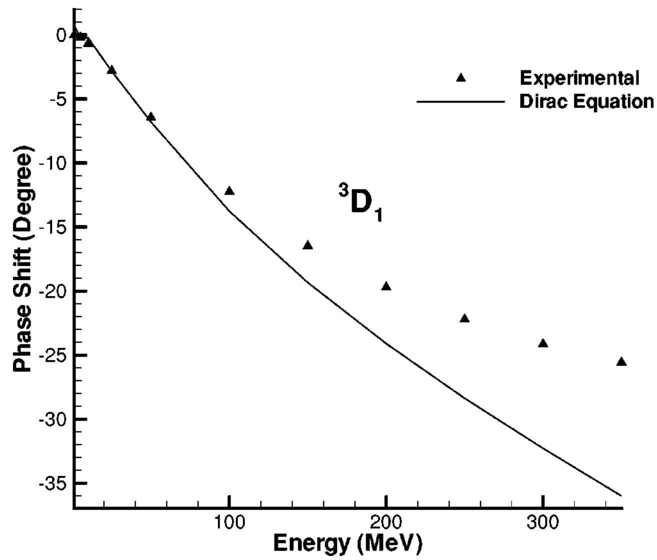

FIG. 18. np scattering phase shift of $^3D_1$ state (model 2).

Among all the four suggestions, the first one would be technically easiest once we find models more general than the two we have presented here. The last three suggestions would involve corresponding additions to the interaction that appear in the two-body Dirac equations. Because the above interactions all involve derivative couplings we will have to examine the CTBDE for the corresponding invariant Δ's. These would include not only the eight invariants listed earlier [see Eqs. (2.46)–(2.56)] but also four additional ones corresponding to $\Delta = R\theta_1\hat{x}_\perp\theta_2\hat{x}_\perp$, $2S\theta_{51}\theta_{52}\theta_1\hat{x}_\perp\theta_2\hat{x}_\perp$, $2T\theta_1\hat{P}\theta_1\hat{P}\theta_1\hat{x}_\perp\theta_2\hat{x}_\perp$, and $4U\theta_{51}\theta_{52}\theta_1\hat{P}\theta_1\hat{P}\theta_1\hat{x}_\perp\theta_2\hat{x}_\perp$. The four functions R,S,T,U are each functions of $x_\perp$ and they represent spacelike interactions paralleling those corresponding to $\mathcal{G},I,Y$, and $\mathcal{F}$, respectively, given earlier. To include all 12 covariant matrix interactions will involve a significant modification of our basic equation, Eq. (3.32), as well as the two-body Dirac equations given in Eqs. (2.58) and (2.59).

E. Extentions to the *N* body problem

Can the constraint formalism be extended to N bodies? There is no solution to the compatibility condition

$$[\mathcal{H}_i,\mathcal{H}_j]|\psi\rangle = 0; \quad i,j=1,..,N \tag{8.4}$$

of generalized mass-shell constraints (or their Dirac counterparts) that has the simplicity of the "third law" and tranversality conditions given in Eqs. (2.14) and (2.15). The difficulty involves satisfying Eq. (8.4) and cluster separability (needed to describe scattering states) at the same time. Rohrlich has shown that this necessarily involves the introduction of N-body forces [51]. If one is willing to limit N-body

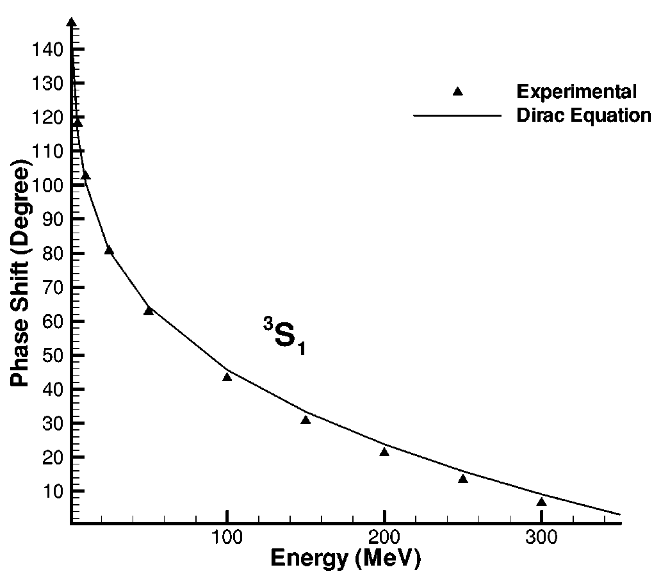

FIG. 17. np scattering phase shift of $^3S_1$ state (model 2).

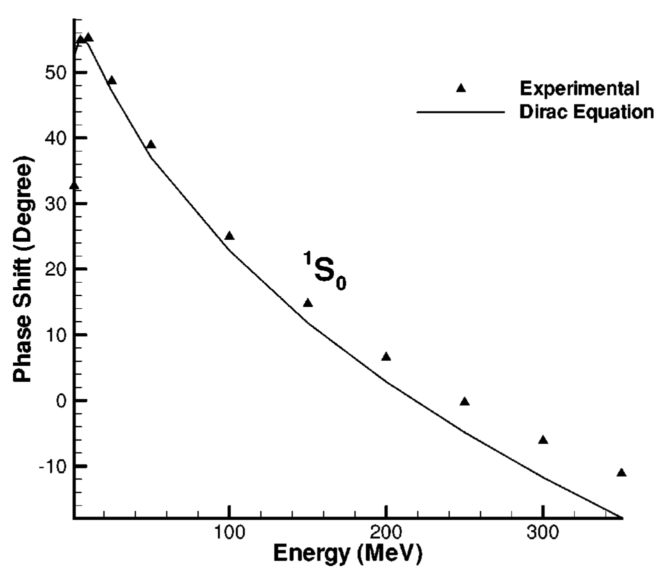

FIG. 19. pp scattering phase shift of $^1S_0$ state (model 2).

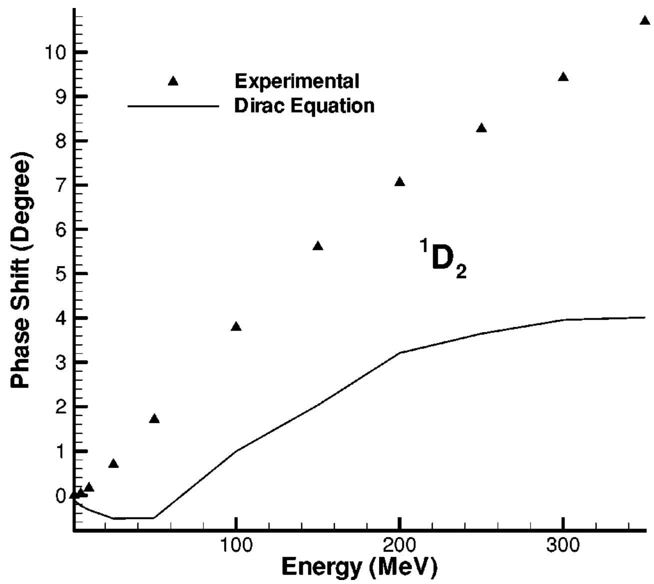

FIG. 20. pp scattering phase shift of $^1D_2$ state (model 2).

considerations to bound states (so that cluster considerations are not important) then Ref. [52] provides a constraint formalism in which a single dynamical wave equation (as in the two-body case) determines the bound state energies. Reference [53] (and references contained therein) provides an N-body constraint formalism that involves particles and fields leading in the end to directly inteacting particles by elimination of the field degrees of freedom by second class contraints.

ACKNOWLEDGMENTS

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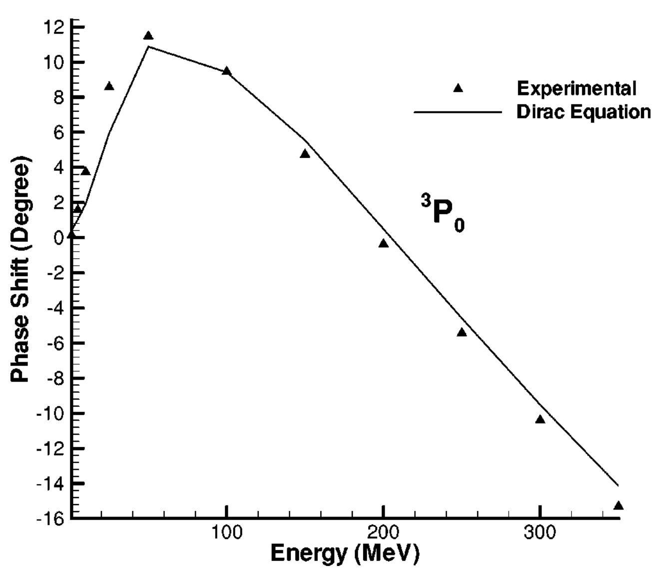

FIG. 21. pp scattering phase shift of $^3P_0$ state (model 2).

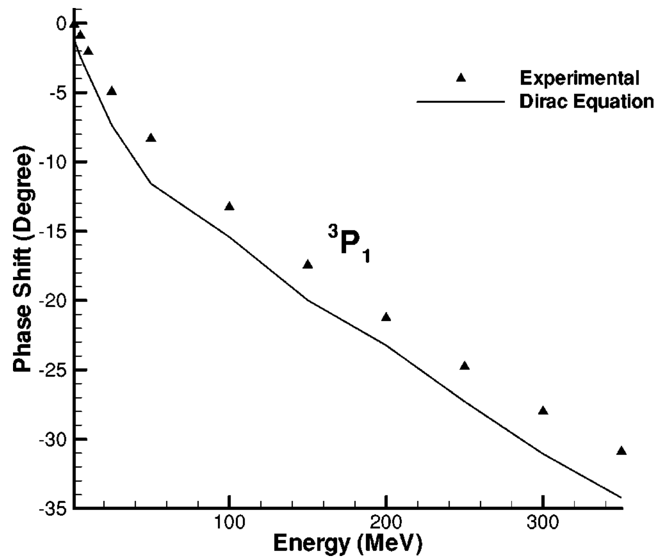

FIG. 22. pp scattering phase shift of $^3P_1$ state (model 2).

APPENDIX A: PAULI FORM OF TWO-BODY DIRAC EQUATIONS

We rewrite Eqs. (2.61) and (2.62) by multiplying the first by $\sqrt{2}i\beta_1$ and the second by $\sqrt{2}i\beta_2$, yielding [8]

$$[T_1(\beta_1\beta_2)+U_1(\beta_1\beta_2)\gamma_{51}\gamma_{52}]|\psi\rangle=(E_1+M_1\beta_1)\gamma_{51}|\psi\rangle,$$

$$-[T_2(\beta_1\beta_2)+U_2(\beta_1\beta_2)\gamma_{51}\gamma_{52}]|\psi\rangle=(E_2+M_2\beta_2)\gamma_{52}|\psi\rangle, \tag{A1}$$

in which the kinetic and recoil terms are

$$T_1(\beta_1\beta_2)=\exp(\mathcal{G})\bigg[\mathbf{\Sigma}_1\cdot\mathbf{p}-\frac{i}{2}\beta_1 \times\beta_2[\mathbf{\Sigma}_2\cdot\nabla(-C+\mathcal{G}\beta_1\beta_2\mathbf{\Sigma}_1\cdot\mathbf{\Sigma}_2)\bigg],$$

$$T_2(\beta_1\beta_2)=\exp(\mathcal{G})\bigg[\mathbf{\Sigma}_2\cdot\mathbf{p}-\frac{i}{2}\beta_1 \times\beta_2[\mathbf{\Sigma}_1\cdot\nabla(-C+\mathcal{G}\beta_1\beta_2\mathbf{\Sigma}_1\cdot\mathbf{\Sigma}_2)]\bigg], \tag{A2}$$

$$U_1(\beta_1\beta_2)=\exp(\mathcal{G})\bigg[-\frac{i}{2}\beta_1\beta_2\mathbf{\Sigma}_2\cdot\nabla(J\beta_1\beta_2-L)\bigg],$$

$$U_2(\beta_1\beta_2)=\exp(\mathcal{G})\bigg[-\frac{i}{2}\beta_1\beta_2\mathbf{\Sigma}_1\cdot\nabla(J\beta_1\beta_2-L)\bigg], \tag{A3}$$

while the timelike and scalar potentials E_i, M_i are given above in Eqs. (2.63) and (2.64).

The final result of the matrix multiplication in Eqs. (A1) is a set of eight simultaneous equations for the Dirac spinors $|\psi\rangle_1, |\psi\rangle_2, |\psi\rangle_3, |\psi\rangle_4$. In an arbitrary frame, the result of the matrix calculation produces the eight simultaneous equations ($\sigma_i^\mu|\psi\rangle\rightarrow\Sigma_i^\mu|\psi\rangle_{1,2,3,4}$) [8]. One then reduces the eight equations to a second-order Schrödinger-like equation by a process of substitution and elimination using the combinations of the four Dirac spinors given below [8]:

$$|\phi_{\pm}\rangle \equiv |\psi\rangle_1 \pm |\psi\rangle_4,$$

$$|\chi_{\pm}\rangle \equiv |\psi\rangle_2 \pm |\psi\rangle_3. \tag{A4}$$

We display all the general spin dependent structures in $\Phi(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,w)$ explicitly, very similar to what appears in nonrelativistic formalisms such as seen in the Hamada-Johnson and Yale group models (as well as the nonrelativistic limit of Gross's equation). We do this by expressing it explicitly in terms of its matrix $(\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2)$, and operator $\mathbf{p}$ structure in the CM system $[\hat{P}=(1,\mathbf{0})]$. We are working in the CM frame [i.e., $x_\perp=(\mathbf{r},0)$], so all the interaction functions $[L(x_\perp),J(x_\perp),C(x_\perp),\mathcal{G}(x_\perp)]$ are functions of $r=\sqrt{x_\perp^2}=|\mathbf{r}|$, $F=F(r)$.

Reference [8] finds the reduction

$$\begin{aligned}
&hE_1[\boldsymbol{\sigma}_1\cdot\mathbf{p}-i\boldsymbol{\sigma}_2\cdot(\mathbf{d}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_1\\
&\quad\times[\boldsymbol{\sigma}_1\cdot\mathbf{p}-i\boldsymbol{\sigma}_2\cdot(\mathbf{z}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle\\
&\quad+hM_1[\boldsymbol{\sigma}_1\cdot\mathbf{p}-i\boldsymbol{\sigma}_2\cdot(\mathbf{o}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_3\\
&\quad\times[\boldsymbol{\sigma}_1\cdot\mathbf{p}-i\boldsymbol{\sigma}_2\cdot(\mathbf{z}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle\\
&\quad-hE_1[\boldsymbol{\sigma}_1\cdot\mathbf{p}-i\boldsymbol{\sigma}_2\cdot(\mathbf{d}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_2\\
&\quad\times[\boldsymbol{\sigma}_2\cdot\mathbf{p}-i\boldsymbol{\sigma}_1\cdot(\mathbf{z}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle\\
&\quad+hM_1[\boldsymbol{\sigma}_1\cdot\mathbf{p}-i\boldsymbol{\sigma}_2\cdot(\mathbf{o}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]hF_4\\
&\quad\times[\boldsymbol{\sigma}_2\cdot\mathbf{p}-i\boldsymbol{\sigma}_1\cdot(\mathbf{z}+\mathbf{k}\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)]|\phi_+\rangle=\mathcal{B}^2|\phi_+\rangle.
\end{aligned} \tag{A5}$$

in which

$$\begin{aligned}
\mathcal{B}^2&=E_1{}^2-M_1{}^2=E_2{}^2-M_2{}^2\\
&=b^2(w)+(\epsilon_1^2+\epsilon_2^2)\sinh^2(J)+2\epsilon_1\epsilon_2\sinh(J)\cosh(J)-(m_1^2\\
&\quad+m_2^2)\sinh^2(L)-2m_1m_2\sinh(L)\cosh(L)
\end{aligned} \tag{A6}$$

and

$$h\equiv\exp(\mathcal{G}),$$

$$\mathbf{k}\equiv\frac{1}{2}\boldsymbol{\nabla}\ln(h),$$

$$\mathbf{z}\equiv\frac{1}{2}\boldsymbol{\nabla}(-C+J-L)$$

$$\mathbf{d}\equiv\frac{1}{2}\boldsymbol{\nabla}(C+J+L)$$

$$\mathbf{o}\equiv\frac{1}{2}\boldsymbol{\nabla}(C-J-L), \tag{A7}$$

with

$$F_1\equiv\frac{M_2}{\mathcal{D}},$$

$$F_2\equiv\frac{M_1}{\mathcal{D}},$$

$$F_3\equiv\frac{E_2}{\mathcal{D}},$$

$$F_4\equiv\frac{E_1}{\mathcal{D}}, \tag{A8}$$

$$\mathcal{D}\equiv E_1M_2+E_2M_1. \tag{A9}$$

Equation (A5) is a second-order Schrödinger-like eigenvalue equation for the newly defined wavefunction $|\phi_+\rangle$ in the form

$$[p_\perp^2+\Phi(\mathbf{r},\mathbf{p},\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,w)]|\phi_+\rangle=b^2(w)|\phi_+\rangle. \tag{A10}$$

Equation (3.5) for $\mathcal{B}^2$ provides us with the primary spin independent part of Φ, the quasipotential. Note that in the CM system $p_\perp^2=\mathbf{p}^2$, $\sigma=(0,\boldsymbol{\sigma})$. For future reference we will refer to the four sets of terms on the left-hand side of Eq. (A5) as (a), the (b), (c), (d) terms.

Now we proceed with a different derivation than Long and Crater's derivation [8]. The aim is to produce a Schrödinger-like form like in Eq. (A10) involving the Pauli matrices for both particles.

Substituting $\mathbf{d}$, h, F_1, $\mathbf{z}$, $\mathbf{k}$'s expressions to (a) term of Eq. (A5), we obtain

$$\begin{aligned}
\text{(a)}\quad \text{term}=&\exp(\mathcal{G})E_1\Big\{\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(C+J+L)\\
&-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Big]\\
&\times\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}\Big[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)\\
&-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Big]\Big\}.
\end{aligned} \tag{A11}$$

Working out the commutation relation of $\boldsymbol{\sigma}_1\cdot\mathbf{p}$ in the above expression, we can find the (a) term is

$$(\mathrm{a})\quad \mathrm{term}=\exp(\mathcal{G})E_1\Bigg\{\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}\Bigg[\mathbf{p}^2-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\{[\mathbf{p}+i(\boldsymbol{\sigma}_1\times\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})-i(\boldsymbol{\sigma}_2\times\mathbf{p})]\}$$

$$+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}\Bigg\{\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Bigg]\Bigg\}$$

$$-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(C+J+L)+\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]$$

$$\times\exp(\mathcal{G})\frac{M_2}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Bigg]\Bigg]\Bigg\}.$$

Likewise, we can the find (b), (c), (d) terms,

$$(\mathrm{b})\quad \mathrm{term}=\exp(\mathcal{G})M_1\Bigg\{\exp(\mathcal{G})\frac{E_2}{\mathcal{D}}\Bigg[\mathbf{p}^2-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\{[\mathbf{p}+i(\boldsymbol{\sigma}_1\times\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})-i(\boldsymbol{\sigma}_2\times\mathbf{p})]\}$$

$$+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}\Bigg\{\exp(\mathcal{G})\frac{E_2}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Bigg]\Bigg\}$$

$$-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(C-J-L)+\boldsymbol{\nabla}\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]$$

$$\times\exp(\mathcal{G})\frac{E_2}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_1\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\Bigg]\Bigg]\Bigg\};$$

$$(\mathrm{c})\quad \mathrm{term}=-\exp(\mathcal{G})E_1\Bigg\{\exp(\mathcal{G})\frac{M_1}{\mathcal{D}}\Bigg[(\boldsymbol{\sigma}_2\cdot\mathbf{p})(\boldsymbol{\sigma}_1\cdot\mathbf{p})-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\{[\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1\cdot\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})+i(\boldsymbol{\sigma}_2\times\mathbf{p})]\}$$

$$+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}\Bigg\{\exp(\mathcal{G})\frac{M_1}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_2\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)\Bigg]\Bigg\}$$

$$-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(C+J+L)+\boldsymbol{\nabla}\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]$$

$$\times\exp(\mathcal{G})\frac{M_1}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_2\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)\Bigg]\Bigg]\Bigg\};$$

$$(\mathrm{d})\quad \mathrm{term}=\exp(\mathcal{G})M_1\Bigg\{\exp(\mathcal{G})\frac{E_1}{\mathcal{D}}\Bigg[(\boldsymbol{\sigma}_2\cdot\mathbf{p})(\boldsymbol{\sigma}_1\cdot\mathbf{p})-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-C+J-L)(\boldsymbol{\sigma}_1\cdot\mathbf{p})$$

$$-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}\{[\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1\cdot\mathbf{p})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{p}+\boldsymbol{\sigma}_1(\boldsymbol{\sigma}_2\cdot\mathbf{p})+i(\boldsymbol{\sigma}_2\times\mathbf{p})]\}$$

$$+\frac{1}{i}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}\Bigg\{\exp(\mathcal{G})\frac{E_1}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_2\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)\Bigg]\Bigg\}$$

$$-\frac{i}{2}[\boldsymbol{\sigma}_2\cdot\boldsymbol{\nabla}(C-J-L)+\boldsymbol{\nabla}\mathcal{G}\cdot(\boldsymbol{\sigma}_1+i\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)]$$

$$\times\exp(\mathcal{G})\frac{E_1}{\mathcal{D}}\Bigg[\boldsymbol{\sigma}_2\cdot\mathbf{p}-\frac{i}{2}\boldsymbol{\sigma}_1\cdot\boldsymbol{\nabla}(-C+J-L)-\frac{i}{2}\boldsymbol{\nabla}\mathcal{G}(\boldsymbol{\sigma}_2+i\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_1)\Bigg]\Bigg]\Bigg\}.$$

We simplify the above expressions by using identities involving $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ and group above equations by the $\mathbf{p}^2$ term, Darwin term $(\hat{\mathbf{r}}\cdot\mathbf{p})$, spin-orbit angular momentum term $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)$, spin-orbit angular momentum difference term $\mathbf{L}\cdot(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)$, spin-spin term $(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)$, tensor term $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})\times(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})$, additional spin dependent terms $\mathbf{L}\cdot(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)$ and $(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_2\cdot\mathbf{p})+(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}})(\boldsymbol{\sigma}_1\cdot\mathbf{p})$, and spin independent terms. Collecting all terms for the (a) + (b) + (c) + (d) terms our Eq. (A5) becomes Eq. (3.4).

APPENDIX B: RADIAL EQUATIONS

The following are radial eigenvalue equations corresponding to Eq. (3.4) after getting rid of the first derivative terms for singlet states $^1S_0$, $^1P_1$, $^1D_2$ (a general singlet ^1J_j), triplet states $^3P_1$ (a general let ^3J_j), a general $s=1$, $j=l+1$ ($^3P_0$, $^3S_1$ states), and a general $s=1$, $j=l+1$ ($^3D_1$ state).

$^1S_0$, $^1P_1$, $^1D_2$ (a general singlet ^1J_j) $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=0$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=-3$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=-1$,

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}-3k-j-g'h' -h''-\frac{2h'}{r}+m\right\}v=\mathcal{B}^2\exp(-2\mathcal{G})v. \tag{B1}$$

$^3P_1$ (a general triplet ^3J_j) $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=-2$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=1$,

$$\left\{-\frac{d^2}{dr^2}+\frac{j(j+1)}{r^2}+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+k+n+g'h'+h'' +m\right\}v=\mathcal{B}^2\exp(-2\mathcal{G})v. \tag{B2}$$

$s=1$, $j=l+1$ ($^3S_1$ states), $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=2(j-1)$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=1/(2j+1)$ (diagonal term), and $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=2\sqrt{j(j+1)}/(2j+1)$(off diagonal term),

$$\begin{aligned}&\left\{-\frac{d^2}{dr^2}+\frac{j(j-1)}{r^2}+\frac{3g'\sinh^2 h}{r}+6h'\frac{\cosh h\sinh h}{r}-\frac{6\sinh^2 h}{r^2}-\frac{g'\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}\right.\\&+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}+k+2(j-1)\left[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}\right]\\&+\frac{2(j-1)}{2j+1}\left[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}\right]+\frac{1}{2j+1}\left[\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}\right.\\&\left.\left.-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+n+g'h'+h''+\frac{2h'}{r}\right]+m\right\}u_+\\&+\frac{2\sqrt{j(j+1)}}{2j+1}\left\{\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+n+g'h'+h''\right.\\&\left.+\frac{2h'}{r}+2(j-1)\left[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}\right]\right\}u_-=\mathcal{B}^2\exp(-2\mathcal{G})u_+,\end{aligned} \tag{B3}$$

$s=1$, $j=l-1$ ($^3P_0$, $^3D_1$ states), $\mathbf{L}\cdot(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)=-2(j+2)$, $\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2=1$, $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=-1/(2j+1)$ (diagonal term), and $\boldsymbol{\sigma}_1\cdot\hat{\mathbf{r}}\cdot\boldsymbol{\sigma}_2\cdot\hat{\mathbf{r}}=2\sqrt{j(j+1)}/(2j+1)$ (off diagonal term),

$$\begin{aligned}&\left\{-\frac{d^2}{dr^2}+\frac{(j+1)(j+2)}{r^2}+\frac{3g'\sinh^2 h}{r}+6h'\frac{\cosh h\sinh h}{r}-\frac{6\sinh^2 h}{r^2}-\frac{g'\cosh h\sinh h}{r}-2h'\frac{\sinh^2 h}{r}+2\frac{\cosh h\sinh h}{r^2}\right.\\&+\frac{g'^2}{4}+h'^2+\frac{g''}{2}+\frac{g'}{r}+k+2(j+2)\left[\frac{g'}{2r}+\frac{g'\sinh^2 h}{r}-2\frac{\sinh^2 h}{r^2}+2h'\frac{\cosh h\sinh h}{r}\right]\\&+\frac{2(j-1)}{2j+1}\left[2h'\frac{\sinh^2 h}{r}-2\frac{\cosh h\sinh h}{r^2}+\frac{h'}{r}+\frac{g'\cosh h\sinh h}{r}\right]-\frac{1}{2j+1}\left[\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}\right.\\&\left.\left.-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+n+g'h'+h''+\frac{2h'}{r}\right]+m\right\}u_-\end{aligned}$$

$$+\frac{2\sqrt{j(j+1)}}{2j+1}\left\{\frac{3g'\cosh h\sinh h}{r}-\frac{g'\sinh^2 h}{r}-2h'\frac{\cosh h\sinh h}{r}+6h'\frac{\sinh^2 h}{r}-6\frac{\cosh h\sinh h}{r^2}+2\frac{\sinh^2 h}{r^2}+n+g'h'+h''\right.$$
$$\left.+\frac{2h'}{r}-2(j+2)\left[\frac{2h'\sinh^2(h)}{r}-\frac{2\cosh(h)\sinh(h)}{r^2}+\frac{h'}{r}+\frac{g'\cosh(h)\sinh(h)}{r}\right]\right\}u_+=\mathcal{B}^2\exp(-2\mathcal{G})u_-. \tag{B4}$$

Substituting for g',h',m,n,k, we obtain the radial equations and potentials Φ given in the text.

APPENDIX C: DERIVATION OF COUPLED PHASE SHIFT EQUATIONS

We have found that we can use the Messiah ansatz [46],

$$u=A\sin(br+\delta),$$
$$u'=bA\cos(br+\delta) \tag{C1}$$

for the solution of

$$\left(-\frac{d^2}{dr^2}+\Phi_L(r)\right)u=b^2u(r) \tag{C2}$$

to yield

$$\delta'=-\frac{\Phi_L}{b}\sin^2(br+\delta). \tag{C3}$$

Next we see how this can be worked out in the case of coupled radial equations of the form

$$-\begin{pmatrix}u_-\\u_+\end{pmatrix}''+\Phi_L\begin{pmatrix}u_-\\u_+\end{pmatrix}=b^2\begin{pmatrix}u_-\\u_+\end{pmatrix}, \tag{C4}$$

where Φ_L is a 2×2 matrix. This equation will have solutions that are S-wave dominant and D-wave dominant. Form them together into a 2×2 matrix U so that the above equation becomes

$$-U''+\Phi_LU=b^2U. \tag{C5}$$

Then take its transpose and add the two. One obtains

$$-(U''+U''^T)+(\Phi_LU+U^T\Phi_L^T)=b^2(U+U^T). \tag{C6}$$

In analogy to the uncoupled case we assume

$$U=A(r)\sin[br+\mathcal{D}(r)], \tag{C7}$$

where

$$\mathcal{D}=\delta(r)+\mathbf{D(r)}\cdot\boldsymbol{\sigma},$$
$$A=a(r)+\mathbf{A(r)}\cdot\boldsymbol{\sigma}. \tag{C8}$$

Let R be a matrix that diagonalizes the phase shift matrix function $\mathcal{D}(r)$ to the form $\delta(r)+D(r)\sigma_3$,

$$\tilde{U}=RUR^{-1}=\tilde{A}\sin(br+\delta+D\sigma_3), \tag{C9}$$

where

$$\tilde{A}=RAR^{-1}. \tag{C10}$$

Continuing the analogy we let

$$U'=bA\cos(br+\mathcal{D}). \tag{C11}$$

Then

$$\begin{aligned}RU'R^{-1}&=b\tilde{A}\cos(br+\delta+D\sigma_3)=R(R^{-1}\tilde{U}R)'R^{-1}\\&=RR^{-1\prime}\tilde{A}\sin(br+\delta+D\sigma_3)+\tilde{A}'\sin(br+\delta\\&\quad+D\sigma_3)+\tilde{A}(b+\delta'+D'\sigma_3)\cos(br+\delta+D\sigma_3)\\&\quad+\tilde{A}\sin(br+\delta+D\sigma_3)R'R^{-1}.\end{aligned} \tag{C12}$$

But $RR^{-1\prime}=-R'R^{-1}$ so that we obtain the condition

$$\tilde{A}'\sin(br+\delta+D\sigma_3)+\tilde{A}(\delta'+D'\sigma_3)\cos(br+\delta+D\sigma_3)$$
$$+[\tilde{A}\sin(br+\delta+D\sigma_3),R'R^{-1}]_-=0. \tag{C13}$$

In general, we would take

$$\tilde{A}\equiv a+\tilde{A}_3\sigma_3+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma}\equiv\tilde{A}_\parallel+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma} \tag{C14}$$

and decompose Eq. (C13) and Eq. (C6) into two sets of four coupled equations.

Give R the following general form:

$$R=\exp[i\varepsilon(r)\sigma_2]\exp[\eta(r)\sigma_1],$$
$$R^{-1}=\exp(-\eta\sigma_1)\exp(-i\varepsilon\sigma_2),$$
$$R'=i\varepsilon'\sigma_2\exp(i\varepsilon\sigma_2)\exp(\eta\sigma_1)$$
$$+\exp(i\varepsilon\sigma_2)\exp(\eta\sigma_1)\eta'\sigma_1,$$
$$R'R^{-1}=i\varepsilon'\sigma_2+\exp(i\varepsilon\sigma_2)\eta'\sigma_1\exp(-i\varepsilon\sigma_2)$$
$$=i\varepsilon'\sigma_2+\eta'\cos(2\varepsilon)\sigma_1+\eta'\sin(2\varepsilon)\sigma_3. \tag{C15}$$

We consider the case in which Φ_L is a symmtric matrix and furthermore that as a result $\mathcal{D}=\mathcal{D}^T$. In that case our matrix is R is orthogonal ($\eta=0$).

Next we examine the three terms of Eq. (C6). We assume that $\tilde{\mathbf{A}}_\perp$ is symmetric so that $\tilde{A}_2=0$ and

$$b^2R(U+U^T)R^{-1}=b^2[(\tilde{A}_{||}+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma})\sin(br+\delta+D\sigma_3)+\sin(br+\delta+D\sigma_3)(\tilde{A}_{||}+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma})]$$
$$=b^2\{2\tilde{A}_\|\sin(br+\delta+D\sigma_3)+2[\tilde{A}_1\sin(br+\delta)\cos D]\sigma_1\} \tag{C16}$$

and

$$R(\Phi U+U^T\Phi^T)R^{-1}=(\tilde{\Phi}_{||}+\tilde{\mathbf{\Phi}}_\perp\cdot\boldsymbol{\sigma})(\tilde{A}_{||}+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma})\sin(br+\delta+D\sigma_3)+(\text{transpose})=(\tilde{\Phi}_\|\tilde{A}_\|+\tilde{\mathbf{\Phi}}_\perp\cdot\tilde{\mathbf{A}}_\perp)\sin(br+\delta+D\sigma_3)$$
$$+(\tilde{\Phi}_\|\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma}+\tilde{\mathbf{\Phi}}_\perp\cdot\boldsymbol{\sigma}\tilde{A}_\|+i\tilde{\mathbf{\Phi}}_\perp\times\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma})\sin(br+\delta+D\sigma_3)+\sin(br+\delta+D\sigma_3)(\tilde{\Phi}_\|\tilde{A}_\|+\tilde{\mathbf{\Phi}}_\perp\cdot\tilde{\mathbf{A}}_\perp)$$
$$+\sin(br+\delta+D\sigma_3)(\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma}\tilde{\Phi}_\|+\tilde{A}_\|\tilde{\mathbf{\Phi}}_\perp\cdot\boldsymbol{\sigma}+i\tilde{\mathbf{\Phi}}_\perp\times\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma}) \tag{C17}$$

The term $\tilde{\mathbf{\Phi}}_\perp\times\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma}$ is zero since $\tilde{A}_2=0=\tilde{\Phi}_2$. The second derivative term is

$$R(U''+U''^T)R^T=R(R^T\tilde{U}'R)'R^{-1}+(\text{transpose})=bR(R^T\tilde{A}\cos(br+\delta+D\sigma_3)R)'R^T+(\text{transpose})$$
$$=b\{[\tilde{A}\cos(br+\delta+D\sigma_3),R'R^T]_-+\tilde{A}'\cos(br+\delta+D\sigma_3)$$
$$-\tilde{A}(b+\delta'+D'\sigma_3)\sin(br+\delta+D\sigma_3)\}+(\text{transpose})$$
$$=b\{i\varepsilon'[(\tilde{A}_\|+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma})\cos(br+\delta+D\sigma_3),\sigma_2]_-+(\tilde{A}'_\|+\tilde{\mathbf{A}}'_\perp\cdot\boldsymbol{\sigma})\cos(br+\delta+D\sigma_3)$$
$$-(\tilde{A}_\|+\tilde{\mathbf{A}}_\perp\cdot\boldsymbol{\sigma})(b+\delta'+D'\sigma_3)\sin(br+\delta+D\sigma_3)\}+(\text{transpose}). \tag{C18}$$

Using properties of the Pauli matrices and dividing Eq. (C13) and Eq. (C6) into $\|$ and $\perp$ components we obtain the following four equations:

$$-\tilde{A}_\|(\delta'+D'\sigma_3)\sin(br+\delta+D\sigma_3)+\tilde{A}'_\|\cos(br+\delta+D\sigma_3)$$
$$-2b\varepsilon'\sigma_3\tilde{A}_1\cos(br+\delta)\cos D$$
$$=\frac{1}{b}(\tilde{\Phi}_\|\tilde{A}_\|+\tilde{\Phi}_1\tilde{A}_1)\sin(br+\delta+D\sigma_3), \tag{C19}$$

$$\tilde{A}'_\|\sin(br+\delta+D\sigma_3)+\tilde{A}_\|(\delta'+D'\sigma_3)\cos(br+\delta+D\sigma_3)$$
$$-2\varepsilon'\sigma_3\tilde{A}_1\sin(br+\delta)\cos D=0, \tag{C20}$$

$$\cos(br+\delta)\cos D\tilde{A}'_1-[\delta'\sin(br+\delta)\cos D$$
$$+D'\cos(br+\delta)\sin D]\tilde{A}_1$$
$$+2\varepsilon'[\tilde{A}_3\cos(br+\delta)\cos D-a\sin(br+\delta)\sin D]$$
$$=\frac{1}{b}[\phi\sin(br+\delta)\cos D\tilde{A}_1-\tilde{\Phi}_3\cos(br+\delta)\sin D\tilde{A}_1]$$
$$+\{a\sin(br+\delta)\cos D+\tilde{A}_3[\cos(br+\delta)\sin D]\tilde{\Phi}_1\}, \tag{C21}$$

$$\tilde{A}'_1\sin(br+\delta)\cos D+\tilde{A}_1[\delta'\cos(br+\delta)\cos D$$
$$-D'\sin(br+\delta)\sin D]+2\varepsilon'[a\cos(br+\delta)\sin D$$
$$+\tilde{A}_3\sin(br+\delta)\cos D]=0. \tag{C22}$$

Combining Eq. (C19) and Eq. (C20) we obtain

$$-\tilde{A}_\|(\delta'+D'\sigma_3)-2\varepsilon'\tilde{A}_1\sin D\cos D$$
$$=\frac{1}{b}(\tilde{\Phi}_\|\tilde{A}_\|+\tilde{\Phi}_1\tilde{A}_1)\sin^2(br+\delta+D\sigma_3). \tag{C23}$$

Combining Eqs. (C21) and (C22) gives

$$\tilde{A}_1\delta'\csc(br+\delta)\cos D-2\varepsilon'\csc(br+\delta)\sin D$$
$$=\frac{1}{b}\{[\phi\sin(br+\delta)\cos D-\tilde{\Phi}_3\cos(br+\delta)\sin D]\tilde{A}_1$$
$$+a\sin(br+\delta)\cos D+\tilde{A}_3[\cos(br+\delta)\sin D]\tilde{\Phi}_1\}. \tag{C24}$$

Rewrite the above two equations as

$$\tilde{A}_\|\left[(\delta'+D'\sigma_3)+\frac{1}{b}\tilde{\Phi}_\|\sin^2(br+\delta+D\sigma_3)\right]$$
$$+\tilde{A}_1\left(2\varepsilon'\sin D\cos D+\frac{1}{b}\tilde{\Phi}_1\sin^2(br+\delta+D\sigma_3)\right)=0, \tag{C25}$$

$$\frac{1}{b}[a\sin(br+\delta)\cos D+\tilde{A}_3\cos(br+\delta)\sin D]\tilde{\Phi}_1$$
$$+2\varepsilon'a\csc(br+\delta)\sin D+\tilde{A}_1\left[\frac{1}{b}[\phi\sin(br+\delta)\cos D\right.$$
$$\left.-\tilde{\Phi}_3\cos(br+\delta)\sin D]-\delta'\csc(br+\delta)\cos D\right]=0. \tag{C26}$$

The first of these two equations is actually two equations

$$a\left[\delta'+\frac{1}{2b}\phi[1-\cos2(br+\delta)\cos(2D)]+\frac{1}{2b}\tilde{\Phi}_3\sin 2\times(br+\delta)\sin(2D)\right]+\tilde{A}_3\left[D'+\frac{1}{2b}\tilde{\Phi}_3[1-\cos2(br+\delta)\times\cos(2D)]+\frac{1}{2b}\phi\sin 2(br+\delta)\sin(2D)\right]+\tilde{A}_1\left(\varepsilon'\sin 2D+\frac{1}{2b}\tilde{\Phi}_1[1-\cos2(br+\delta)\cos 2D]\right)=0 \tag{C27}$$

and

$$a\left[D'+\frac{1}{2b}\tilde{\Phi}_3[1-\cos 2(br+\delta)\cos(2D)]+\frac{1}{2b}\phi\sin 2(br+\delta)\sin(2D)\right]+\tilde{A}_3\left[\delta'+\frac{1}{2b}\phi[1-\cos 2(br+\delta)\cos(2D)]+\frac{1}{2b}\tilde{\Phi}_3\sin 2(br+\delta)\sin(2D)\right]+\tilde{A}_1\frac{1}{2b}\tilde{\Phi}_1\sin 2(br+\delta)\sin 2D=0. \tag{C28}$$

So now together with Eq. (C26),

$$a\left[\frac{1}{b}\sin(br+\delta)\cos(D)\tilde{\Phi}_1+2\varepsilon'\csc(br+\delta)\sin D\right]+\tilde{A}_3\left[\frac{1}{b}\cos(br+\delta)\sin(D)\tilde{\Phi}_1\right]+\tilde{A}_1\left[\frac{1}{b}[\phi\sin(br+\delta)\cos D-\tilde{\Phi}_3\cos(br+\delta)\sin D]-\delta'\csc(br+\delta)\cos D\right]=0, \tag{C29}$$

we have three homogeneous equations in $a,\tilde{A}_3,\tilde{A}_1$. We simplify these equations further by assuming that $\tilde{A}_1=0$,

$$=a\left[\delta'+\frac{1}{2b}\phi[1-\cos2(br+\delta)\cos(2D)]+\frac{1}{2b}\tilde{\Phi}_3\sin 2(br+\delta)\sin(2D)\right]+\tilde{A}_3\left[D'+\frac{1}{2b}\tilde{\Phi}_3[1-\cos 2(br+\delta)\cos(2D)]+\frac{1}{2b}\phi\sin 2(br+\delta)\sin(2D)\right]=0, \tag{C30}$$

$$=a\left[D'+\frac{1}{2b}\tilde{\Phi}_3[1-\cos2(br+\delta)\cos(2D)]+\frac{1}{2b}\phi\sin 2(br+\delta)\sin(2D)\right]+\tilde{A}_3\left[\delta'+\frac{1}{2b}\phi[1-\cos 2(br+\delta)\cos(2D)]+\frac{1}{2b}\tilde{\Phi}_3\sin 2(br+\delta)\sin(2D)\right]=0, \tag{C31}$$

$$a\left[\frac{1}{b}\sin(br+\delta)\cos(D)\tilde{\Phi}_1+2\varepsilon'\csc(br+\delta)\sin D\right]+\tilde{A}_3\left[\frac{1}{b}\cos(br+\delta)\sin(D)\tilde{\Phi}_1\right]=0. \tag{C32}$$

The solution we seek is

$$\delta'+\frac{1}{2b}\phi[1-\cos 2(br+\delta)\cos(2D)]+\frac{1}{2b}\tilde{\Phi}_3\sin2(br+\delta)\sin(2D)=0, \tag{C33}$$

$$D'+\frac{1}{2b}\tilde{\Phi}_3[1-\cos 2(br+\delta)\cos(2D)]+\frac{1}{2b}\phi\sin 2(br+\delta)\sin(2D)=0. \tag{C34}$$

Let

$$\delta=\frac{1}{2}(\delta_1+\delta_2),$$

$$D=\frac{1}{2}(\delta_1-\delta_2), \tag{C35}$$

and that leads to

$$\delta_1'=-\frac{1}{b}(\phi+\tilde{\Phi}_3)\sin^2(br+\delta_1), \tag{C36}$$

$$\delta_2'=-\frac{1}{b}(\phi-\tilde{\Phi}_3)\sin^2(br+\delta_2). \tag{C37}$$

Returning to Eq. (C22) we find it reduces to

$$\tilde{A}_3=-a\cot(br+\delta)\tan D, \tag{C38}$$

and combining that with Eq. (C32) yields

$$\left[\frac{1}{b}\sin(br+\delta)\cos(D)\tilde{\Phi}_1+2\varepsilon'\csc(br+\delta)\sin D\right]-\cot(br+\delta)\tan D\left[\frac{1}{b}\cos(br+\delta)\sin(D)\tilde{\Phi}_1\right]=0,$$

so that

$$\varepsilon' = \frac{1}{2b}\tilde{\Phi}_1[\tan D\cos^2(br+\delta)-\sin^2(br+\delta)\cot(D)]$$
$$= \frac{1}{2\sin D\cos(D)b}\tilde{\Phi}_1[\sin^2 D\cos^2(br+\delta) - \sin^2(br+\delta)\cos^2(D)]$$
$$= \frac{1}{b\sin 2D}\tilde{\Phi}_1\sin(br+\delta+D)\sin(br+\delta-D). \quad \text{(C39)}$$

From the definintion of $\tilde{\Phi}$ we see that

$$\begin{pmatrix} \cos\varepsilon & \sin\varepsilon \\ -\sin\varepsilon & \cos\varepsilon \end{pmatrix}\begin{pmatrix} \Phi_3 & \Phi_1 \\ \Phi_1 & -\Phi_3 \end{pmatrix}\begin{pmatrix} \cos\varepsilon & -\sin\varepsilon \\ \sin\varepsilon & \cos\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} \Phi_3\cos 2\varepsilon+\Phi_1\sin 2\varepsilon & -\Phi_3\sin 2\varepsilon+\Phi_1\cos 2\varepsilon \\ -\Phi_3\sin 2\varepsilon+\Phi_1\cos 2\varepsilon & -\Phi_3\cos 2\varepsilon-\Phi_1\sin 2\varepsilon \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{\Phi}_3 & \tilde{\Phi}_1 \\ \tilde{\Phi}_1 & -\tilde{\Phi}_3 \end{pmatrix}. \quad \text{(C40)}$$

So from this and Eqs. (C36) and (C37) we obtain the phase shift equations Eqs. (5.25) and (5.26) given in the text while Eq. (C39) gives us Eq. (5.27).

APPENDIX D: PHASE SHIFT EQUATION WITH THE COULOMB POTENTIAL

We review here the necessary modification of our phase equations when we consider pp scattering [16,22]. When we study pp scattering, we must consider the influence of the Coulomb potential. The general form of the uncoupled Schrödinger-like equation with Coulomb potential is [22]

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}-\frac{2\epsilon_w\alpha}{r}+\Delta\Phi\right]u(r)=b^2u(r), \quad \text{(D1)}$$

where $\Delta\Phi$ consists of the short range parts of the effective potential, α is the fine structure constant. [Compare the Coulomb term with the first term on the right-hand sides of Eqs. (2.27) and (2.29).] Due to the long range behavior of the potential in above equation, the asymptotic behavior of the wave function is

$$u(r)\overset{r\to\infty}{\to}\text{const}\cdot\sin(br-\eta\ln 2br+\Delta), \quad \text{(D2)}$$

in which

$$\Delta=\delta_l+\sigma_l-\frac{l\pi}{2}, \quad \text{(D3)}$$

where $\sigma_l=\arg\Gamma(l+1+i\eta)$ is the Coulomb phase shift, here $\eta=-\epsilon_w\alpha/b$.

We describe here the variable phase method to calculate the phase shift with the Coulomb potential. Consider the two differential equations

$$u''+(b^2-W-\overline{W})u=0, \quad \text{(D4)}$$

and

$$\overline{u}''+(b^2-\overline{W})\overline{u}=0, \quad i=1,2, \quad \text{(D5)}$$

in which $u(0)=\overline{u}_1(0)=0$. Let

$$\overline{W}(r)=-\frac{2\epsilon_w\alpha}{r},$$
$$W(r)=\frac{l(l+1)}{r^2}+\Delta\Phi, \quad \text{(D6)}$$

so that

$$\overline{u}_1(r)\overset{r\to\infty}{\to}\text{const}\cdot\sin(br-\eta\ln 2br+\overline{\Delta}),$$
$$\overline{u}_2(r)\overset{r\to\infty}{\to}\text{const}\cdot\cos(br-\eta\ln 2br+\overline{\Delta}), \quad \text{(D7)}$$

where $\overline{\Delta}=\sigma_0$.

Just as in the variable phase method, we obtain a nonlinear first-order differential equation for the phase shift function $\delta_l(r)$ such that $\delta_l(\infty)=\delta_l$, and $\delta_l(0)=0$. This is done by rewriting $u(r)$ as

$$u(r)=\alpha(r)[\cos\gamma(r)\overline{u}_1(r)+\sin\gamma(r)\overline{u}_2(r)], \quad \text{(D8)}$$

so that

$$\Delta=\overline{\Delta}+\gamma(\infty). \quad \text{(D9)}$$

Since we have rewritten $u(r)$ in two arbitrary functions, we are free to impose a condition on $u(r)$,

$$u'(r)=\alpha'(r)[\cos\gamma(r)\overline{u}_1'(r)+\sin\gamma(r)\overline{u}_2'(r)]. \quad \text{(D10)}$$

Combining $u(r)$ and $u'(r)$ leads to

$$\gamma(r)=-\tan^{-1}\left[\frac{u(r)\overline{u}_1'(r)-u'(r)\overline{u}_1(r)}{u(r)\overline{u}_2'(r)-u'(r)\overline{u}_2(r)}\right], \quad \text{(D11)}$$

where $\gamma(0)=0$, and $\overline{u}_1(r)=F_0(\eta,br)$ and $\overline{u}_2(r)=G_0(\eta,br)$ are the two Coulomb wave functions. With the Wronskian $F_0G_0'-F_0'G_0=b$, we obtain, by differentiating, the differential equation

$$\gamma'(r)=-W(r)[\cos\gamma(r)F_0(\eta,br)+\sin\gamma(r)G_0(\eta,br)]^2/b. \quad \text{(D12)}$$

Note that for

$$W(r)\overset{r\to 0}{\to}\frac{\lambda(\lambda+1)}{r^2}, \quad \frac{\lambda(\lambda+1)}{r^2}=\frac{l(l+1)}{r^2}-\frac{\alpha^2}{r^2},$$

$$F_0(\eta,br)\overset{r\to 0}{\to} C_0 br,$$

$$G_0(\eta,br)\overset{r\to 0}{\to} \frac{1}{C_0}, \tag{D13}$$

we obtain the relation

$$\gamma'(0)=-\frac{C_0^2 b\lambda}{\lambda(\lambda+1)}. \tag{D14}$$

Letting

$$\gamma(r)=\beta(r)+\eta(r), \tag{D15}$$

where $\beta(r)$ is defined as

$$\beta'(r)=-\frac{l(l+1)}{r^2}[\cos\gamma(r)F_0(\eta,br)$$

$$+\sin\gamma(r)G_0(\eta,br)]^2/b, \tag{D16}$$

$\beta(r)$ has the exact solution

$$\gamma(r)=-\tan^{-1}\left[\frac{F_l(\eta,br)F_0'(\eta,br)-F_l'(\eta,br)F_0(\eta,br)}{F_l(\eta,br)G_0'(\eta,br)-F_l'(\eta,br)G_0(\eta,br)}\right], \tag{D17}$$

with $\beta(0)=0$ and $\beta'(0)=-C_0^2 bl/l(l+1)$ and $\beta(\infty)=\sigma_l-l\pi/2-\sigma_0$ lead to

$$\delta_l=\eta(\infty). \tag{D18}$$

Thus , if we solve

$$\eta'(r)=\left[-\frac{l(l+1)}{r^2}+\Delta\Phi\right]\{\cos[\beta(r)+\eta(r)]F_0(\eta,br)$$

$$+\sin[\beta(r)+\eta(r)]G_0(\eta,br)\}^2/b+\frac{l(l+1)}{r^2}$$

$$\times[\cos\beta(r)F_0(\eta,br)+\sin\beta(r)G_0(\eta,br)]^2/b \tag{D19}$$

with the condition $\eta(0)=0$, we obtain the additional phase shift (above the Coulomb phase shift) by integration to $\eta(\infty)$.

There is no Coulomb scattering for the coupled triplet $^3S_1$ and $^3D_1$ states as a consideration of Pauli principle would show.

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