Asymptotic freedom for nonrelativistic confinement

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(Received 22 July 2002; published 23 December 2002)

Some aspects of asymptotic freedom are discussed in the context of a simple two-particle nonrelativistic confining potential model. In this model, asymptotic freedom follows from the similarity of the free-particle and bound state radial wave functions at small distances and for the same angular momentum and the same large energy. This similarity, which can be understood using simple quantum mechanical arguments, can be used to show that the exact response function approaches that obtained when final state interactions are ignored. A method of calculating corrections to this limit is given, and explicit examples are given for the case of a harmonic oscillator.

DOI: 10.1103/PhysRevC.66.065205 PACS number(s): 13.60.Hb, 12.39.Pn, 25.30.Dh, 25.30.Fj

I. INTRODUCTION

Asymptotic freedom was discovered to be a property of non-Abelian gauge theories nearly 30 years ago [1,2], and has been used to explain scaling in electroproduction [3]. A version of asymptotic freedom also holds trivially in nonrelativistic quantum mechanics since the Born series converges more and more rapidly with increasing energy. In particular, final state interactions can be ignored in the response functions for inclusive scattering from condensed matter systems at large momentum transfer [4]. This result follows easily from simple quantum mechanical arguments, although the calculation of corrections can be quite complicated [5]. The case of confined systems is a bit more subtle, however, since the particles in the final state are always in discrete excited states of finite (possibly large) size, and thus never really free. The case of nonrelativistic confinement has been discussed extensively, however, as a toy model for asymptotic freedom and quark-hadron duality [6] in the structure function for electroproduction, where the production of resonances at low energies and momentum transfers averages to the smooth scaling curve observed at large momentum transfers. This is now known to occur quite locally, essentially for each resonance. An explanation in terms of a relativistic quark model has been given recently [7], while Close and Isgur [8] have used a simple nonrelativistic model to explain one aspect of duality: how the square of a sum in the amplitude for production of resonances becomes approximately the sum of squares required for duality. A similar nonrelativistic model had been used earlier by Greenberg [9] to investigate the scaling limit in terms of the Bjorken scaling variable.

In this paper we also use a two-body model to discuss nonrelativistic asymptotic freedom. Such a model provides at best an approximate description of mesons containing heavy quarks [10,11], but it is simple enough that some features of the large momentum transfer limit of the response function can be illustrated very concretely. In the context of the model, asymptotic freedom follows because of a simple relation between the radial wave functions describing bound states in the confining potential and the spherical Bessel functions that describe free particles of the same energy and angular momentum. Not only do these wave functions have the same shape for high enough energy and small enough separation, their relative normalizations are such that a form of duality follows once one properly allows for the comparison of a discrete sum with an integral.

The paper begins with the definition of the response functions of the model, showing how they are related to integrals over the final state radial wave functions. Both the exact case, where the final state wave functions are just the bound state wave functions for the discrete final states, and the approximate case using free-particle (plane-wave) wave functions are considered. The close relation between the two sets of wave functions for the same angular momentum, similar large energies, and small radii are then described in Sec. III for fairly general confining potentials, and the implications of these results for local duality are discussed. A method for calculating corrections to this simple relation is also given, leading to a discussion of the conditions under which asymptotic freedom is a good approximation. The results are applied in Sec. IV to the case of the harmonic oscillator, where many of the expressions become simple analytic functions. The results are summarized in Sec. V, which also contains suggestions for further work in this area.

II. STRUCTURE FUNCTIONS

We use the model of Ref. [8], consisting of two particles of equal mass m and reduced mass $\mu=m/2$ carrying charges e_1 and e_2. This system is initially in its ground state in the spherically symmetric confining potential $V(r)$ and, after being hit by a "scalar photon," transferring momentum $\mathbf{q}$ to one of the particles, it makes the transition to an excited state $|n_r,l,m\rangle$ with energy $E_{n_r,l}$. The probability for this transition is proportional to the response or structure function

$$F_{n_r,l,m}(\mathbf{q})=|\langle n_r,l,m|[e_1e^{i\mathbf{r}\cdot\mathbf{q}/2}+e_2e^{-i\mathbf{r}\cdot\mathbf{q}/2}]|0,0,0\rangle|^2. \tag{1}$$

If the polarizations m of the degenerate final states of a given n_r and l are not measured, one needs only the sum

$$F_{n_r,l}(q)=\sum_{m=-l}^{l}F_{n_r,l,m}(\mathbf{q}). \tag{2}$$

Using completeness it is easy to see that

$$\sum_{n_r,l} F_{n_r,l}(q)=e_1^2+e_2^2+2e_1e_2S(q), \quad (3)$$

where

$$S(q)=\langle 0,0,0|e^{i\mathbf{r}\cdot\mathbf{q}}|0,0,0\rangle \quad (4)$$

is the ground state form factor. Using standard manipulations, one can write $F_{n_r,l}(q)$ in terms of the square of a radial integral involving the bound state radial wave functions $u_{n_r,l}(r)$:

$$F_{n_r,l}(q)=[e_1^2+e_2^2+2e_1e_2(-1)^l](2l+1)r_{n_r,l}(q)^2, \quad (5)$$

where

$$r_{n_r,l}(q)=\int_0^\infty dr u_{n_r,l}(r)j_l(qr/2)u_{0,0}(r), \quad (6)$$

with $j_l(x)$ a spherical Bessel function. The $(-1)^l$ in 5 shows that the interference terms will tend to cancel when states of adjacent values of l are included in a sum [8].

If the interaction of the two particles in the final state is completely ignored, as would follow from the assumption of asymptotic freedom, the final states can be labeled by either the relative momentum $\mathbf{k}$ of the two particles or its magnitude k and the angular momentum l,m. Both will be useful below, so we define

$$F(\mathbf{k},\mathbf{q})=|\langle\mathbf{k}|[e_1e^{i\mathbf{r}\cdot\mathbf{q}/2}+e_2e^{-i\mathbf{r}\cdot\mathbf{q}/2}]|0,0,0\rangle|^2 \quad (7)$$

and

$$F_{l,m}(k,\mathbf{q})=|\langle k,l,m|[e_1e^{i\mathbf{r}\cdot\mathbf{q}/2}+e_2e^{-i\mathbf{r}\cdot\mathbf{q}/2}]|0,0,0\rangle|^2. \quad (8)$$

With these definitions,

$$F(\mathbf{k},\mathbf{q})=e_1^2\phi^2(|\mathbf{k}-\mathbf{q}/2|)+e_2^2\phi^2(|\mathbf{k}+\mathbf{q}/2|) +2e_1e_2\phi(|\mathbf{k}-\mathbf{q}/2|)\phi(|\mathbf{k}+\mathbf{q}/2|), \quad (9)$$

where $\phi(p)$ is the ground state momentum space wave function, which is large only when p is less than $1/r_0$, where r_0 indicates the size of the ground state wave function, and

$$F_l(k,q)=\sum_{m=-l}^{l} F_{l,m}(k,\mathbf{q})=[e_1^2+e_2^2+2e_1e_2(-1)^l] \times(2l+1)r_l(k,q)^2, \quad (10)$$

with

$$r_l(k,q)=2\int_0^\infty dr\hat{j}_l(kr)j_l(qr/2)u_{0,0}(r). \quad (11)$$

Here, to emphasize the similarity to the bound state case, we have introduced the Ricatti-Bessel functions $\hat{j}(x)=xj(x)$. Summing $F_l(k,q)$ over l produces the same structure function obtained by integrating $F(\mathbf{k},\mathbf{q})$ over all directions of $\mathbf{k}$:

$$F(k,q)=\sum_{l=0}^{\infty} F_l(k,q)=(k/2\pi)^2\int d\Omega_{\mathbf{k}}F(\mathbf{k},\mathbf{q}). \quad (12)$$

The completeness relation now take the form

$$\int_0^\infty (dk/2\pi)F(k,q)=e_1^2+e_2^2+2e_1e_2S(q). \quad (13)$$

Several properties of the structure functions can be read off from the expressions above. In the free case, the structure function is large when $\mathbf{k}\approx\pm\mathbf{q}/2$, and, for large q, the interference term cannot be large. The corresponding result for $F_l(k,q)$ can be seen in Eq. (11), since here the two spherical Bessel functions are exactly in phase only if $k\approx q/2$, so that $r_l(k,q)$ will be maximum here and begin to decrease significantly when $|k-q/2|r_0\geqslant 1$, where r_0 indicates the size of the ground state wave function. Using Eq. 9, a simple change of variable in the integral over the direction of $\mathbf{k}$ shows that, in the large q limit, $F(k,q)$ is simply the ground state probability distribution for the component of $\mathbf{k}$ in the $\mathbf{q}$ direction [12]. This means that, in this limit, $F(k,q)$ depends only on the scaling variable $k-q/2\approx(k^2-q^2/4)/q$, which is the nonrelativistic version of Bjorken scaling [3].

In the confinement case, the structure function $F_{n_r,l}$ can be large for large q only if the oscillations of the radial wave function $u_{n_r,l}(r)$ are in phase with those of the spherical Bessel function $j_l(qr/2)$, so that asymptotic freedom will hold only if these radial wave functions have a form similar to that of $\hat{j}(kr)$. In the following section this requirement will be studied in more detail and it will be shown that the two radial wave functions have almost the same shapes for the same energy and angular momentum.

III. RADIAL WAVE FUNCTIONS

Assuming $V(r)\to 0$ smoothly as $r\to 0$, the bound state radial wave function must have the same shape as the free-particle radial wave functions for the same l and for high enough energy and small enough r since they satisfy approximately the same wave equation, and both vanish at $r=0$. In this section we use the WKB approximation to discuss the normalization of the bound state radial wave function and show that the result obtained guarantees asymptotic freedom.

In the WKB approximation, the radial wave function in the classically allowed region is simply

$$u_{n_r,l}(r)\approx\mathcal{N}_{n_r,l}\cos\left[\int_{r_-}^{r} dr' k_{n_r,l}(r')-\pi/4\right]/\sqrt{k_{n_r,l}(r)}, \quad (14)$$

where

$$k_{n_r,l}(r)=\sqrt{k_{n_r,l}^2-2\mu V(r)-[l+(1/2)]^2/r^2} \quad (15)$$

is the classical radial wave number at r for a system with energy

$$E_{n_r,l}=k_{n_r,l}^2/2\mu, \tag{16}$$

with the usual replacement $l(l+1)\rightarrow[l+(1/2)]^2$ [13], and the classical turning points r_- and r_+ are determined by the condition $k_{n_r,l}(r)=0$. The normalization constant $\mathcal{N}_{n_r,l}$ can be estimated by assuming that, for high energy bound states, the average of the square of the cosine is close to 1/2, and that the contributions to the normalization integral from the classically forbidden regions is negligible. Normalization then requires

$$1\approx\mathcal{N}_{n_r,l}^2\int_{r_-}^{r_+}dr/[2k_{n_r,l}(r)]. \tag{17}$$

The integral is clearly proportional to the period of the classical motion, and it is easy to show from the Bohr-Sommerfeld condition that it is inversely proportional to the level splitting. The final result can be written as

$$\mathcal{N}_{n_r,l}^2\approx2\mu(E_{n_r+1,l}-E_{n_r,l})/\pi. \tag{18}$$

(Similar arguments can be used to derive the expression for the local electron density used in Thomas-Fermi theory [14].)

From the discussion above, the shape of $u_{n_r,l}(r)$ must match that of the Ricatti-Bessel function $\hat{j}_l(k_{n_r,l}r)$ at small r. If we assume that this match extends into the classically allowed region and to $kr\gg l$, so that the Ricatti-Bessel function takes on its sinusoidal asymptotic form, it must be approximately true that

$$u_{n_r,l}(r)\approx\mathcal{N}_{n_r,l}\hat{j}_l(k_{n_r,l}r)/\sqrt{k_{n_r,l}}. \tag{19}$$

This relation holds only for small enough radius since the local wave number $k_{n_r,l}(r)$ appearing in the WKB approximation will eventually begin to differ significantly from $k_{n_r,l}$. The radial integrals Eqs. (6) and (11), required for calculating the structure functions, however, involve only values of r such that the ground state radial wave function $u_{0,0}(r)$ is large. As shown in Fig. 1, this can be much smaller than the size of higher energy excited states, and so it might well be possible that the radial integrals in the bound and free cases are essentially identical except for the normalization constant $\mathcal{N}_{n_r,l}$.

Using relation, Eq, (16), between $E_{n_r,l}$ and $k_{n_r,l}$, it is easy to show that

$$\sum_{n_r}\rightarrow\int kdk/(\mu\Delta E), \tag{20}$$

where ΔE is the energy difference appearing in Eq (18) for energies related to k by Eq. (16). Then, using Eqs. (6) and (11),

$$\sum_{n_r}F_{n_r,l}(q)\approx\int dkF_l(k,q)/(2\pi). \tag{21}$$

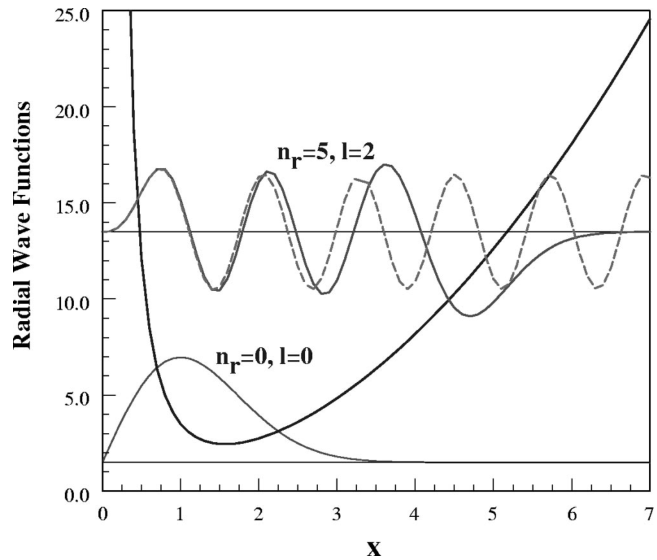

FIG. 1. Oscillator bound state radial wave function for $n_r=5$ and $l=2$ (solid) and the free particle radial wave function (dashed) for the same energy and angular momentum, together with the ground state ($n_r=0,l=0$) radial wave function, plotted against the dimensionless radial variable $x=\sqrt{\alpha}r$. The heavy curve shows the effective potential in the oscillator well for this angular momentum.

These approximate relations hold for any set of n_r's provided the integral on the right is over the corresponding range of k's, in the sense that the range of energies covered on the two sides is the same. This, of course, is just a statement of asymptotic freedom. Note that it holds for groups of states with a single definite angular momentum l, so that the correspondence is local in angular momentum as well as energy.

It has been stated somewhat vaguely above that the above results hold for small enough r and high enough energy. It would be useful to have a more quantitative estimate of the errors and a scheme for correcting them. One method of calculating corrections has been discussed by Gurvits and Rinat [5]. Another, of more direct application to the approach here, using radial wave functions, can be obtained from the integral equation for the radial wave function [15],

$$\hat{u}_{n_r,l}(r)=\hat{j}_l(k_{n_r,l}r)-(1/k_{n_r,l})\int_0^r dr'[\hat{j}_l(k_{n_r,l}r)\hat{n}_l(k_{n_r,l}r')-\hat{j}_l(k_{n_r,l}r')\hat{n}_l(k_{n_r,l}r)]2\mu V(r')\hat{u}_{n_r,l}(r'), \tag{22}$$

where $\hat{n}_l(x)$ is the Ricatti-Bessel function of the second kind and $\hat{u}_{n_r,l}(r)$ is an un-normalized version of u, which equals $\hat{j}_l(k_{n_r,l}r)$ for very small r. (In this equation, $\hat{u}$ is normalizable only if $k_{n_r,l}$ corresponds to an energy eigenvalue.) This integral equation shows clearly that the error in the proportionality between u and $\hat{j}_l$ generally increases with r and the strength of V, but decreases as $k_{n_r,l}$ increases. It can be used to develop a perturbation expansion for the correction, the leading correction being simply the integral term in Eq. (22) with $\hat{u}_{n_r,l}(r')$ replaced by $\hat{j}_l(k_{n_r,l}r')$. This expansion must, of course, diverge at large r, but it can be used to estimate the accuracy of proportionality (19) and thus of local duality itself. A simple change of variables, for example, shows that for an r^n potential the leading correction decreases as $1/k_{n_r,l}^{n+2}$.

IV. HARMONIC OSCILLATOR EXAMPLES

In the case of the harmonic oscillator potential $V(r) = \frac{1}{2}Kr^2$, there are analytic expressions for most of the quantities discussed above. It will be useful to define the usual parameters $\omega = \sqrt{K/\mu}$ and $\alpha = \mu\omega$, in terms of which $E_{n_r,l} = \omega(2n_r + l + 3/2)$. The radial wave functions are then given by

$$u_{n_r,l}(r) = N_{n_r,l}\alpha^{1/4}x^{l+1}L_{n_r}^{l+1/2}(x^2)e^{-x^2/2}, \tag{23}$$

where $L_{n_r}^{l+1/2}$ is an associated Laguerre polynomial, the dimensionless variable $x = \sqrt{\alpha}r$ and the normalization constant is

$$N_{n_r,l} = [2n_r!/\Gamma(n_r + l + 3/2)]^{1/2}. \tag{24}$$

Since in this case $k_{n_r,l} = \sqrt{2\alpha(2n_r + l + 3/2)}$, the relation between the bound and free radial wave functions in terms of dimensionless quantities is

$$u_{n_r,l}(r)/\alpha^{1/4} \approx 2\hat{j}_l(\sqrt{4n_r + 2l + 3}x)/[\sqrt{\pi}(4n_r + 2l + 3)^{1/4}]. \tag{25}$$

An example of this relation is shown in Fig. 1, which shows that the normalized bound state and free-particle radial wave functions for the same l and energy are indeed almost identical near the the origin, in the region where the ground state radial wave function appearing in the radial integrals is large.

For the oscillator the radial integrals have simple analytic expression as:

$$r_{n_r,l}(q) = \sqrt{2^{n_r+l}/[n_r!(2n_r + 2l + 1)!!]}(p/2)^{2n_r+l}e^{-p^2/4}, \tag{26}$$

and

$$r_l(k,q) = 2\sqrt{2}\kappa(\pi/\alpha)^{1/4}e^{-(\kappa^2+p^2)/2}i_l(\kappa p), \tag{27}$$

where $p = q/(2\sqrt{\alpha})$ and $\kappa = k/\sqrt{\alpha}$ are dimensionless momenta and i_l is a modified spherical Bessel function of the first kind. These two functions are plotted in Fig. 2 for the particular case of $n_r = 5$ and $l = 2$.

With these expressions for $r_{n_r,l}(q)$ and $r_l(k,q)$, we can use Eq. (5) and (10) to obtain expressions for $F_{n_r,l}(q)$ and $F_l(k,q)$. These can be summed over l to obtain the structure functions for transitions to all states at a definite energy:

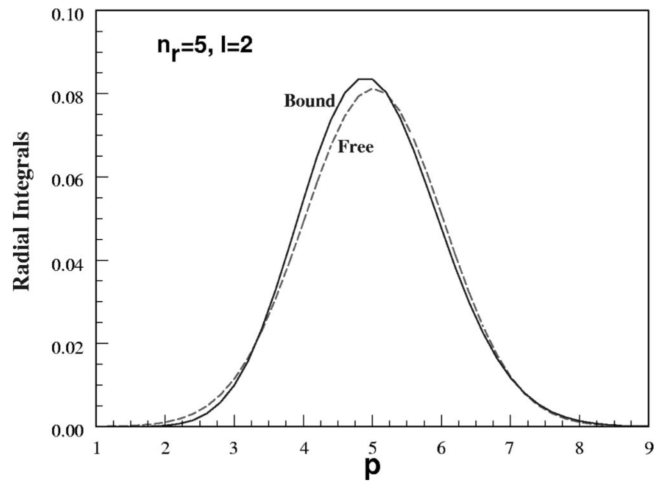

FIG. 2. Radial integrals for $n_r = 5$ and $l = 2$, computed using bound state (solid) and free-particle (dashed) radial wave functions for the same energy ($\kappa = \sqrt{2n_r + l + 3/2} = \sqrt{27} \approx 5.2$), as a function of the dimensionless momentum transfer $p = q/(2\sqrt{\alpha})$. Note that both curves peak near $p = 5.2$.

$$F_n(q) = \sum_{n_r,l}\delta_{n,2n_r+l}F_{n_r,l} = [e_1^2 + e_2^2 + 2e_1e_2(-1)^n](1/n!)(p^2/2)^n e^{-p^2/2} \tag{28}$$

and

$$F(k,q) = \sum_{l=0}^{\infty}F_l(k,q) = [e_1^2 + e_2^2]2\sqrt{\pi/\alpha}(\kappa/p)[e^{-(\kappa-p)^2} - e^{-(\kappa+p)^2}] + 2e_1e_2 8\sqrt{\pi/\alpha}\kappa^2 e^{-p^2-\kappa^2}. \tag{29}$$

It should be noted that for large momentum transfers, the l dependence of the terms in the sums above becomes Gaussian: $(2l+1)\exp(-l^2/2n)$ in Eq. (28) and $(2l+1)\exp(-l^2/\kappa p)$ in (29). This means that the number of terms that contribute significantly to the sums is only a few times $\sqrt{2n}$ or, equivalently, $\sqrt{\kappa p}$, respectively, so that the difference between the finite sum in Eq. (28) and the infinite sum in Eq. (29) is not significant. This limit on the angular momenta of the internal states produced can be easily understood by noting that the impact parameters involved cannot be greater than the size of the initial bound state. The expression for $F_n(q)$ shows how the contributions from alternate energy levels tend to cancel for the e_1e_2 interference term. For $F(k,q)$, on the other hand, the interference term is always small for large values of momentum transfer. Furthermore, for large p, only the $\exp[-(\kappa-p)^2]$ term can be large which, as will become clearer below, gives scaling of the structure function. These structure functions, of course, satisfy the sum rules

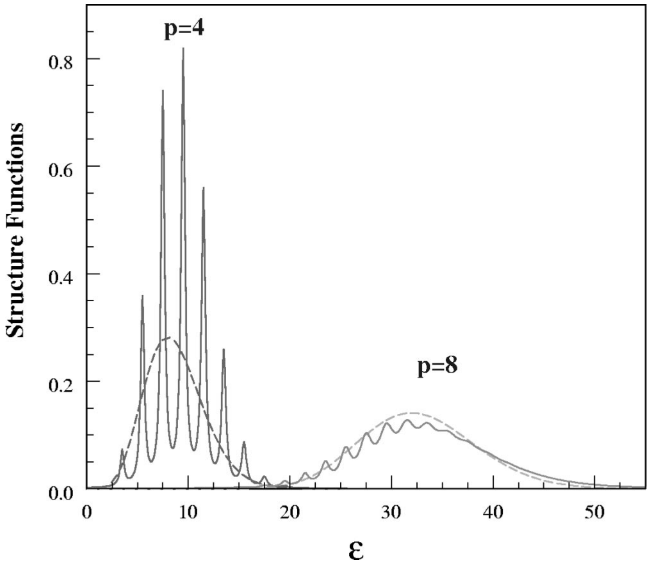

FIG. 3. Structure functions as a function of the dimensionless energy $\epsilon = n+3/2 = k^2/2\alpha$ for dimensionless momentum transfers $p = q/(2\sqrt{\alpha})$ equal to 4 and 8. The smooth curves are for the free case $F(\epsilon,q)$, while the curves with peaks are for the bound case $F(a,\epsilon,q)$. In the latter cases, the sharp energy levels have been given widths a_n that increase from 0.2 to 2 as their energy increases.

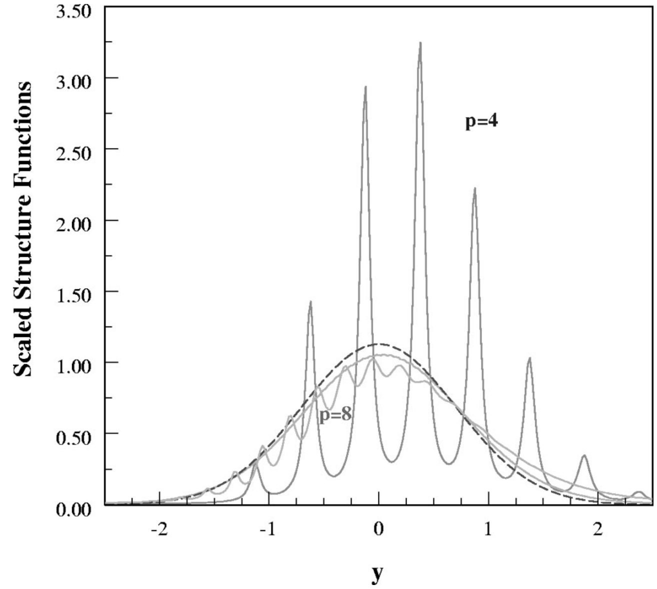

FIG. 4. Structure functions F times p as a function of the scaling variable y for dimensionless momentum transfers p equal to 4 (large peaks), 8 (small peaks), and 32 (smooth curve). The sharp energy levels have been given a width that increases from 0.2 to 2 as their energy increases. The dashed curve is the Gaussian limit of the scaled free particle pF: $(2/\sqrt{\pi})e^{-y^2}$.

$$\sum_{n=0}^{\infty} F_n(q) = \int_0^{\infty} dk/(2\pi) F(k,q) = e_1^2 + e_2^2 + 2e_1e_2 S(q), \tag{30}$$

where here the form factor $S(q) = e^{-p^2}$, showing exactly how the interference term vanishes as q increases [8].

In comparing the two structure functions $F_n(q)$ and $F(k,q)$, the fact that in the former the energy levels are discrete, while in the latter they form a continuum, must be addressed. Here the nth excited discrete level is given a half-width a_n being in ϵ, the energy being in units of ω. This is done by defining

$$F(a,\epsilon,q) = F_0(q)\,\delta(\epsilon-\epsilon_0) + \sum_{n=1}^{\infty} F_n(q)\,\delta_{a_n}(\epsilon-\epsilon_n), \tag{31}$$

where $\epsilon_n = n+3/2$, and δ_a is a finite-width version of the Dirac delta function δ:

$$\delta_a(\epsilon-\epsilon_n) = (a/\pi)/[(\epsilon-\epsilon_n)^2 + a^2]. \tag{32}$$

In general, we expect the widths a_n of the individual resonance peaks to increases with n . Furthermore, if

$$F(\epsilon,q) = [\alpha/(2\pi k)]F(k,q), \tag{33}$$

where ϵ and k are related by $\epsilon = k^2/(2\alpha)$, then the integral over ϵ from zero to infinity for both $F(a,\epsilon,q)$ and $F(\epsilon,q)$ equals the right-hand expression in Eq. (30). [This equality is approximate for $F(a,\epsilon,q)$ because the tails of the $F(a,\epsilon,q)$ extend into the negative ϵ region.] The two F functions are plotted in Fig. 3 for the case where $e_1 = e_2 = e$, for which the square bracket in Eq. (28) is alternately $4e^2$ and 0: the contributions from the two particles cancel or add coherently for odd and even parity states, respectively [8]. Note that the peaks of the smooth(ed) curves appear approximately at $\epsilon = p^2/2$, and that the widths in ϵ increase nearly linearly with p. (The heights of the curves therefore decrease inversely

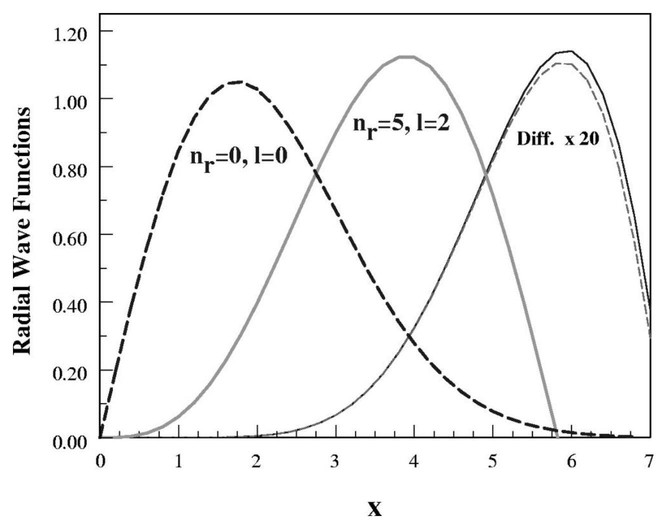

FIG. 5. Un-normalized radial wave functions and corrections for $n_r = 5$ and $l = 2$ versus the dimensionless radius x. The heavy solid curve is the exact radial wave function, while the lighter solid and dashed curve are 20 times the difference between $\hat{j}_2$ and the exact wave function, and the first-order estimate of this difference given by Eq. (36), respectively. The heavy dashed curve is the ground state radial wave function.

with p.) Thus plotting pF versus a scaling variable $y=(\epsilon-p^2/2)/p$ gives curves of a nearly Gaussian shape almost independent of p as $p\to\infty$, as shown in Fig. 4. This scaling variable is, aside from a trivial factor, just the y variable of West [12], and corresponds to the component of a parton's momentum in the **q** direction before the collision.

We can also calculate the first-order correction to the approximations in which the bound state wave functions for the oscillator are proportional to spherical Bessel functions. The first two terms in the expansion can be written as

$$v_{n_r,l}(x)=v^{(0)}_{n_r,l}(x)+v^{(1)}_{n_r,l}(x), \tag{34}$$

where

$$v^{(0)}_{n_r,l}(x)=\hat{j}_l(x) \tag{35}$$

and

$$v^{(1)}_{n_r,l}(x)=-[1/(4n_r+2l+3)^2]\int_0^x dx'[\hat{j}_l(x)\hat{n}_l(x')-\hat{j}_l(x')\hat{n}_l(x)]x'^2\hat{j}_l(x'), \tag{36}$$

with $x=k_{n_r,l}r$. An indication of the size of the errors and the accuracy of the leading correction above for the cases $n_r=5$ and $l=2$ are shown in Fig. 5. It is clear that the corrections are small in the region where the ground state wave function is large, but including them does improve the accuracy of expression (19), especially at larger values of r. Note, however, that the corrections diverge rapidly at very large r.

V. CONCLUSION

In the nonrelativistic case, asymptotic freedom and scaling are automatic for high energy transfers, provided the confining potential approaches zero smoothly as $r\to 0$. This result is almost trivial since the system is effectively free at small separations. Only the relative normalizations of the terms in the sums for the bound states and the integrals in the free case require additional argument from the WKB approximation, or, equivalently, the correspondence principle.

As is well known, the resulting structure functions are essentially the same for the confined and free cases, except for the resonancelike bumps in the former, and for large momentum transfers depends only on the scaling variable $y=(\epsilon-p^2/2)/p\approx\kappa-p$, as shown in Fig. 3.

It would be of interest to extend these results to more general potentials and to semi-relativistic calculations [10]. For example, how would the results be modified for the frequently used Coulomb-plus-linear confining potential [16], or for various relativistic extensions [7,17–19]? Another possible extension would be to many-body systems, including the nucleon and nuclei [12].

ACKNOWLEDGMENTS

The author would like to thank Professor Dieter Drechsel and his colleagues in the Theory Group at the Institut für Kernphysik at the Univerität Mainz, where this work was done, for stimulating his interest in this topic and for their hospitality.

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