Rotationally invariant Hamiltonians for nuclear spectra based on quantum algebras

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The rotational invariance under the usual physical angular momentum of the $su_q(2)$ Hamiltonian for a description of rotational nuclear spectra is explicitly proved, and a connection of this Hamiltonian to the formalisms of Amal'sky and Harris is provided. In addition, a Hamiltonian for rotational spectra is introduced, based on the construction of irreducible tensor operators (ITO's) under $su_q(2)$ and the use of q-deformed tensor products and q-deformed Clebsch-Gordan coefficients. The rotational invariance of this $su_q(2)$ ITO Hamiltonian under the usual physical angular momentum is explicitly proved, a simple closed expression for its energy spectrum (the "hyperbolic tangent formula") is introduced, and its connection to the Harris formalism is established. Numerical tests in a series of Th isotopes are provided.

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I. INTRODUCTION

Quantum algebras [1–3] have started to find applications in the description of symmetries of physical systems over the last years [4]. In one of the earliest attempts, a Hamiltonian proportional to the second order Casimir operator of $su_q(2)$ was used for a description of rotational nuclear spectra [5], and its relation to the variable moment of inertia model [6] was clarified. However, several open problems remained.

(a) Is the $su_q(2)$ Hamiltonian invariant under the usual su(2) Lie algebra, i.e. under usual angular momentum, or it breaks spherical symmetry and/or the isotropy of space?

(b) How does the physical angular momentum appear in the framework of $su_q(2)$? Is there any relation between the generators of $su_q(2)$ and the usual physical angular momentum operators?

(c) How can one add angular momenta in the $su_q(2)$ framework? In other words, how does angular momentum conservation work in the $su_q(2)$ framework?

Answers to these questions are provided in the present paper, along with connections of the $su_q(2)$ model to other formalisms.

After a brief introduction to the $su_q(2)$ formalism in Sec. II, we prove explicitly in Sec. III that the $su_q(2)$ Hamiltonian does commute with the generators of su(2), i.e., with the generators of usual physical angular momentum. Therefore, the $su_q(2)$ Hamiltonian does not violate the isotropy of space and does not destroy spherical symmetry. The generators of $su_q(2)$ are expressed in terms of the generators of su(2). In addition, it turns out that the angular momentum quantum numbers appearing in the description of the irreducible representations (irreps) of $su_q(2)$ are exactly the same as the ones appearing in the irreps of su(2), establishing a one-to-one correspondence between the two sets of irreps (in the generic case in which the deformation parameter q is not a root of unity).

Taking advantage of the results of Sec. III, in Sec. IV we write the eigenvalues of the $su_q(2)$ Hamiltonian as an exact power series in $l(l+1)$ (where l is the usual physical angular momentum). An approximation to this expansion, studied in Sec. V, leads to a closed energy formula for rotational spectra introduced by Amal'sky [7]. The study of analytic expressions for the moment of inertia and the rotational frequency based on the closed formula of Sec. V leads, in Sec. VI, to a connection between the present approach and the Harris formalism [8].

We then turn in Sec. VII to the study of irreducible tensor operators under $su_q(2)$ [9,10], constructing the irreducible tensor operator of rank 1 corresponding to the $su_q(2)$ generators. We also define tensor products in the $su_q(2)$ framework and construct the scalar square of the angular momentum operator, a task requiring the use of q-deformed Clebsch-Gordan coefficients [9]. In addition to exhibiting explicitly how the addition of angular momenta works in the $su_q(2)$ framework, this exercise leads to a Hamiltonian built out of the components of the above mentioned irreducible tensor operator (ITO), which can also be applied to a description of rotational spectra. We are going to refer to this Hamiltonian as the *$su_q(2)$ ITO Hamiltonian*.

The fact that the $su_q(2)$ ITO Hamiltonian does commute with the generators of the usual su(2) algebra is shown explicitly in Sec. VIII. Based on the results of Sec. VIII in Sec. IX we express the eigenvalues of the $su_q(2)$ ITO Hamiltonian as an exact power series in $l(l+1)$, where l is the usual physical angular momentum. An approximation to this series, studied in Sec. X, leads to a simple closed formula for the spectrum (the "hyperbolic tangent formula"), which is used in Sec. XI in order to obtain analytic expressions for the moment of inertia and the rotational frequency, leading to a connection of the present results to the Harris formalism [8].

Finally in Sec. XII all the exact and closed approximate energy formulas obtained above are compared to the experimental spectra of a series of Th isotopes, as well as to the

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results provided by the usual rotational expansion and by the Holmberg-Lipas formula [11], which is probably the best two-parameter formula for the description of rotational nuclear spectra [12]. A discusion of the present results and plans for future work are given in Sec. XIII.

II. QUANTUM ALGEBRA $\mathbf{su}_q(2)$

The quantum algebra $\mathrm{su}_q(2)$ [13–15] is a q deformation of the Lie algebra su(2). It is generated by the operators L_+, L_-, and L_0, obeying the commutation relations (see Ref. [4] and references therein),

$$[L_0, L_\pm] = \pm L_\pm, \tag{1}$$

$$[L_+, L_-] = [2L_0] = \frac{q^{2L_0} - q^{-2L_0}}{q - q^{-1}}, \tag{2}$$

where q numbers and q operators are defined by

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{3}$$

There are two distinct cases for the domain of the deformation parameter: (a) $q = e^\tau$, $\tau \in \mathbf{R}$, in which

$$[x] = \frac{\sinh \tau x}{\sinh \tau}, \tag{4}$$

and (b) $q = e^{i\tau}$, $\tau \in \mathbf{R}$, in which

$$[x] = \frac{\sin \tau x}{\sin \tau}. \tag{5}$$

In both cases one has

$$[x] \to x \quad \text{as} \quad q \to 1. \tag{6}$$

If the deformation parameter q is not a root of unity [q is a root of unity in case b) if one has $q^n = 1$, $n \in \mathbf{N}$] the finite-dimensional irreducible representation $D^l_{(q)}$ of $\mathrm{su}_q(2)$ is determined by the highest weight vector $|l,l\rangle_q$ with

$$L_+ |l,l\rangle_q = 0, \tag{7}$$

and the basis states $|l,m\rangle_q$ are expressed as

$$|l,m\rangle_q = \sqrt{\frac{[l+m]!}{[2l]![l-m]!}} (L_-)^{l-m} |l,l\rangle_q, \tag{8}$$

where $[n]! = [n][n-1]\dots[1]$ is the notation for the q factorial. Then the explicit form of the irreducible representation (irrep) $D^l_{(q)}$ of the $\mathrm{su}_q(2)$ algebra is determined by the equations

$$L_\pm |l,m\rangle_q = \sqrt{[l \mp m][l \pm m + 1]} |l, m \pm 1\rangle_q, \tag{9}$$

$$L_0 |l,m\rangle_q = m |l,m\rangle_q, \tag{10}$$

and the dimension of the corresponding representation is the same as in the nondeformed case, i.e., $\dim D^l_{(q)} = 2l+1$ for $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \dots$.

The second-order Casimir operator of $\mathrm{su}_q(2)$ is

$$C_2^{(q)} = \frac{1}{2}(L_+L_- + L_-L_+ + [2][L_0]^2) = L_-L_+ + [L_0][L_0+1]$$
$$= L_+L_- + [L_0][L_0-1], \tag{11}$$

while its eigenvalues in the space of the irreducible representation $D^l_{(q)}$ are $[l][l+1]$

$$C_2^{(q)} |l,m\rangle_q = [l][l+1] |l,m\rangle_q. \tag{12}$$

It has been suggested (see Refs. [4,5] and references therein) that rotational spectra of deformed nuclei and diatomic molecules can be described by a phenomenological Hamiltonian based on the symmetry of the quantum algebra $\mathrm{su}_q(2)$,

$$H = \frac{\hbar^2}{2\mathcal{J}_0} C_2^{(q)} + E_0, \tag{13}$$

where $C_2^{(q)}$ is the second order Casimir operator of Eq. (11), $\mathcal{J}_0$ is the moment of inertia for the nondeformed case $q \to 1$, and E_0 is the bandhead energy for a given band.

The eigenvalues of the Hamiltonian of Eq. (13) in the basis of Eq. (8) are then

$$E_l^{(\tau)} = A[l][l+1] + E_0, \tag{14}$$

where the definition

$$A = \frac{\hbar^2}{2\mathcal{J}_0} \tag{15}$$

has been used for brevity.

In the case with $q = e^\tau$, $\tau \in \mathbf{R}$ the spectrum of the model Hamiltonian of Eq. (13) takes the form

$$E_l^{(\tau)} = A \frac{\sinh(l\tau)\sinh((l+1)\tau)}{\sinh^2(\tau)} + E_0, \quad q = e^\tau, \tag{16}$$

while, in the case with $q = e^{i\tau}$, $\tau \in \mathbf{R}$ and $q^n \neq 1$, $n \in \mathbf{N}$, the spectrum of the model Hamiltonian of Eq. (13) takes the form

$$E_l^{(\tau)} = A \frac{\sin(l\tau)\sin((l+1)\tau)}{\sin^2(\tau)} + E_0, \quad q = e^{i\tau}. \tag{17}$$

It is known (see Refs. [4,5] and references therein) that only the spectrum of Eq. (17) exhibits behavior that is in agreement with experimentally observed rotational bands.

III. ROTATIONAL INVARIANCE OF THE $\mathbf{su}_q(2)$ HAMILTONIAN

In this section we are going to use both the usual quantum mechanical operators of angular momentum, denoted by $\hat{l}_+$,

$\hat{l}_-$, and $\hat{l}_0$, and the q deformed ones, which are related to $\mathrm{su}_q(2)$ and denoted by $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_0$, as in Sec. II. In this section we are going to use hats ($\hat{\ }$) for the operators, in order to give emphasis to the distinction between the operators and their eigenvalues. For brevity we are going to call the operators $\hat{l}_+$, $\hat{l}_-$, and $\hat{l}_0$ "*classical*," while the operators $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_0$ will be called "*quantum*." For the classical basis the symbol $|l,\mathrm{m}\rangle_c$ will be used, while the quantum basis will be denoted by $|l,m\rangle_q$, as in Sec. II. Therefore l and m are the quantum numbers related to the usual quantum mechanical angular momentum, which is characterized by the su(2) symmetry, while l and m are the quantum numbers related to the deformed angular momentum, which is characterized by the $\mathrm{su}_q(2)$ symmetry.

The "classical" operators satisfy the usual su(2) commutation relations

$$[\hat{l}_0,\hat{l}_\pm]=\pm\hat{l}_\pm, \quad [\hat{l}_+,\hat{l}_-]=2\hat{l}_0, \tag{18}$$

while the finite-dimensional irreducible representation D^l of su(2) is determined by the highest weight vector $|l,l\rangle_c$ with

$$\hat{l}_+|l,l\rangle_c=0, \tag{19}$$

and the basis states $|l,\mathrm{m}\rangle_c$ are expressed as

$$|l,\mathrm{m}\rangle_c=\sqrt{\frac{(l+\mathrm{m})!}{(2l)!(l-\mathrm{m})!}}(\hat{l}_-)^{l-\mathrm{m}}|l,l\rangle_c. \tag{20}$$

The action of the generators of su(2) on the vectors of the classical basis is described by

$$\hat{l}_\pm|l,\mathrm{m}\rangle_c=\sqrt{(l\mp\mathrm{m})(l\pm\mathrm{m}+1)}|l,\mathrm{m}\pm1\rangle_c, \tag{21}$$

$$\hat{l}_0|l,\mathrm{m}\rangle_c=\mathrm{m}|l,\mathrm{m}\rangle_c, \tag{22}$$

the dimension of the corresponding representation being $\dim D^l=2l+1$ for $l=0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2,\ldots$.

The second order Casimir operator of su(2) is

$$\hat{C}_2=\frac{1}{2}(\hat{l}_+\hat{l}_-+\hat{l}_-\hat{l}_+)+\hat{l}_0^2=\hat{l}_-\hat{l}_++\hat{l}_0(\hat{l}_0+1)$$
$$=\hat{l}_+\hat{l}_-+\hat{l}_0(\hat{l}_0-1), \tag{23}$$

where the symbol 1 is used for the unit operator, while its eigenvalues in the space of the irreducible representation D^l are $l(l+1)$:

$$\hat{C}_2|l,\mathrm{m}\rangle_c=l(l+1)|l,\mathrm{m}\rangle_c. \tag{24}$$

It is useful to introduce the operator $\hat{l}$ through the definition

$$\hat{C}_2\equiv\hat{l}(\hat{l}+1). \tag{25}$$

Insisting that $\hat{l}$ should be a positive operator one then has by solving the relevant quadratic equation and keeping only the positive sign in front of the square root [16]:

$$\hat{l}=\frac{1}{2}(-1+\sqrt{1+4\hat{C}_2}). \tag{26}$$

The action of the operator $\hat{l}$ on the vectors of the classical basis is then given by

$$\hat{l}|l,\mathrm{m}\rangle_c=\frac{1}{2}(-1+\sqrt{1+4\hat{C}_2})|l,\mathrm{m}\rangle_c$$
$$=\frac{1}{2}(-1+\sqrt{1+4l(l+1)})|l,\mathrm{m}\rangle_c$$
$$=\frac{1}{2}(-1+\sqrt{(2l+1)^2})|l,\mathrm{m}\rangle_c$$
$$=\frac{1}{2}(-1+2l+1)|l,\mathrm{m}\rangle_c=l|l,\mathrm{m}\rangle_c, \tag{27}$$

where again only the positive value of the square root has been taken into account.

In this "classical" environment one can introduce the operators [16,17]

$$\hat{\mathcal{L}}_+=\sqrt{\frac{[\hat{l}+\hat{l}_0][\hat{l}-\hat{l}_0+1]}{(\hat{l}+\hat{l}_0)(\hat{l}-\hat{l}_0+1)}}\hat{l}_+, \tag{28}$$

$$\hat{\mathcal{L}}_-=\hat{l}_-\sqrt{\frac{[\hat{l}+\hat{l}_0][\hat{l}-\hat{l}_0+1]}{(\hat{l}+\hat{l}_0)(\hat{l}-\hat{l}_0+1)}}, \tag{29}$$

$$\hat{\mathcal{L}}_0=\hat{l}_0, \tag{30}$$

where square brackets denote q-operators, as defined in Eq. (3).

The action of these operators on the vectors of the classical basis is given by

$$\hat{\mathcal{L}}_+|l,\mathrm{m}\rangle_c=\sqrt{\frac{[\hat{l}+\hat{l}_0][\hat{l}-\hat{l}_0+1]}{(\hat{l}+\hat{l}_0)(\hat{l}-\hat{l}_0+1)}}\hat{l}_+|l,\mathrm{m}\rangle_c$$
$$=\sqrt{\frac{[\hat{l}+\hat{l}_0][\hat{l}-\hat{l}_0+1]}{(\hat{l}+\hat{l}_0)(\hat{l}-\hat{l}_0+1)}}$$
$$\times\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}|l,\mathrm{m}+1\rangle_c$$
$$=\sqrt{\frac{[l+\mathrm{m}+1][l-\mathrm{m}]}{(l+\mathrm{m}+1)(l-\mathrm{m})}}$$
$$\times\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}|l,\mathrm{m}+1\rangle_c$$
$$=\sqrt{[l+\mathrm{m}+1][l-\mathrm{m}]}|l,\mathrm{m}+1\rangle_c, \tag{31}$$

$$\begin{aligned}\hat{\mathcal{L}}_-|l,\mathrm{m}\rangle_c&=\hat{l}_-\sqrt{\frac{[\hat{l}+\hat{l}_0][\hat{l}-\hat{l}_0+1]}{(\hat{l}+\hat{l}_0)(\hat{l}-\hat{l}_0+1)}}|l,\mathrm{m}\rangle_c\\&=\hat{l}_-\sqrt{\frac{[l+\mathrm{m}][l-\mathrm{m}+1]}{(l+\mathrm{m})(l-\mathrm{m}+1)}}|l,\mathrm{m}\rangle_c\\&=\sqrt{(l+\mathrm{m})(l-\mathrm{m}+1)}\\&\quad\times\sqrt{\frac{[l+\mathrm{m}][l-\mathrm{m}+1]}{(l+\mathrm{m})(l-\mathrm{m}+1)}}|l,\mathrm{m}-1\rangle_c\\&=\sqrt{[l+\mathrm{m}][l-\mathrm{m}+1]}|l,\mathrm{m}-1\rangle_c,\end{aligned}\tag{32}$$

$$\hat{\mathcal{L}}_0|l,\mathrm{m}\rangle_c=\hat{l}_0|l,\mathrm{m}\rangle_c=m|l,\mathrm{m}\rangle_c,\tag{33}$$

or, in compact form,

$$\begin{gathered}\hat{\mathcal{L}}_\pm|l,\mathrm{m}\rangle_c=\sqrt{[l\mp\mathrm{m}][l\pm\mathrm{m}+1]}|l,\mathrm{m}\pm1\rangle_c,\\ \hat{\mathcal{L}}_0|l,\mathrm{m}\rangle_c=\mathrm{m}|l,\mathrm{m}\rangle_c.\end{gathered}\tag{34}$$

It is clear that the operators $\hat{\mathcal{L}}_+$ and $\hat{l}_+$ do not commute:

$$\begin{aligned}[\hat{\mathcal{L}}_+,\hat{l}_+]|l,\mathrm{m}\rangle_c&=\hat{\mathcal{L}}_+\hat{l}_+|l,\mathrm{m}\rangle_c-\hat{l}_+\hat{\mathcal{L}}_+|l,\mathrm{m}\rangle_c\\&=\hat{\mathcal{L}}_+\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}|l,\mathrm{m}+1\rangle_c\\&\quad-\hat{l}_+\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c\\&=(\sqrt{[l-\mathrm{m}-1][l+\mathrm{m}+2]}\\&\quad\times\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}\\&\quad-\sqrt{(l-\mathrm{m}-1)(l+\mathrm{m}+2)}\\&\quad\times\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]})|l,\mathrm{m}+2\rangle_c\neq0.\end{aligned}\tag{35}$$

This result is expected if one considers Eq. (28): The operator $\hat{l}_+$ does commute with itself and with the operator $\hat{l}$, which is a function of the relevant Casimir operator, as Eq. (26) indicates, but it does not commute with the operator $\hat{l}_0$, as Eq. (18) shows. In the same way one can see that

$$[\hat{\mathcal{L}}_-,\hat{l}_-]|l,\mathrm{m}\rangle_c\neq0.\tag{36}$$

One can now prove that the "new" operators satisfy the commutation relations of Eqs. (1) and (2). Indeed, one has

$$\begin{aligned}[\hat{\mathcal{L}}_0,\hat{\mathcal{L}}_+]|l,\mathrm{m}\rangle_c&=\hat{\mathcal{L}}_0\hat{\mathcal{L}}_+|l,\mathrm{m}\rangle_c-\hat{\mathcal{L}}_+\hat{\mathcal{L}}_0|l,\mathrm{m}\rangle_c\\&=\hat{\mathcal{L}}_0\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c\\&\quad-\hat{\mathcal{L}}_+m|l,\mathrm{m}\rangle_c\\&=(\mathrm{m}+1)\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c\\&\quad-\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}\mathrm{m}|l,\mathrm{m}+1\rangle_c\\&=(\mathrm{m}+1-\mathrm{m})\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c\\&=\hat{\mathcal{L}}_+|l,\mathrm{m}\rangle_c,\end{aligned}\tag{37}$$

and, in exactly the same way,

$$[\hat{\mathcal{L}}_0,\hat{\mathcal{L}}_-]|l,\mathrm{m}\rangle_c=-\hat{\mathcal{L}}_-|l,\mathrm{m}\rangle_c,\tag{38}$$

while for the commutator of Eq. (2) one has

$$\begin{aligned}[\hat{\mathcal{L}}_+,\hat{\mathcal{L}}_-]|l,\mathrm{m}\rangle_c&=\hat{\mathcal{L}}_+\hat{\mathcal{L}}_-|l,\mathrm{m}\rangle_c-\hat{\mathcal{L}}_-\hat{\mathcal{L}}_+|l,\mathrm{m}\rangle_c\\&=\hat{\mathcal{L}}_+\sqrt{[l+\mathrm{m}][l-\mathrm{m}+1]}|l,\mathrm{m}-1\rangle_c\\&\quad-\hat{\mathcal{L}}_-\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c\\&=([l+\mathrm{m}][l-\mathrm{m}+1]\\&\quad-[l-\mathrm{m}][l+\mathrm{m}+1])|l,\mathrm{m}\rangle_c\\&=[2\mathrm{m}]|l,\mathrm{m}\rangle_c=[2\hat{\mathcal{L}}_0]|l,\mathrm{m}\rangle_c,\end{aligned}\tag{39}$$

where use of the identity

$$[l+\mathrm{m}][l-\mathrm{m}+1]-[l-\mathrm{m}][l+\mathrm{m}+1]=[2\mathrm{m}],\tag{40}$$

which can be easily proved by using Eq. (3), has been made.

We have therefore demonstrated that the operators $\hat{\mathcal{L}}_+$, $\hat{\mathcal{L}}_-$, and $\hat{\mathcal{L}}_0$ satisfy the commutation relations of the $\mathrm{su}_q(2)$ algebra. As a consequence, the quantities appearing on the right-hand side of Eqs. (28)–(30) are just the realizations of the generators of $\mathrm{su}_q(2)$ in the "classical" basis. Therefore, from now on we can use the symbols $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_0$ in the place of $\hat{\mathcal{L}}_+$, $\hat{\mathcal{L}}_-$, and $\hat{\mathcal{L}}_0$.

One can also see that the operator

$$\hat{C}=\hat{L}_-\hat{L}_++[\hat{L}_0][\hat{L}_0+1]\tag{41}$$

acts on the vectors of the classical basis as

$$\begin{aligned}\hat{C}|l,\mathrm{m}\rangle_c&=\hat{L}_-\hat{L}_+|l,\mathrm{m}\rangle_c+[\hat{L}_0][\hat{L}_0+1]|l,\mathrm{m}\rangle_c\\&=\hat{L}_-\sqrt{[l+\mathrm{m}][l-\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c\\&\quad+[\mathrm{m}][\mathrm{m}+1]|l,\mathrm{m}\rangle_c\\&=[l+\mathrm{m}][l-\mathrm{m}+1]|l,\mathrm{m}\rangle_c+[\mathrm{m}][\mathrm{m}+1]|l,\mathrm{m}\rangle_c\\&=[l][l+1]|l,\mathrm{m}\rangle_c,\end{aligned}\tag{42}$$

where in the last step the identity

$$[l+\mathrm{m}][l-\mathrm{m}+1]+[m][m+1]=[l][l+1],\tag{43}$$

which can easily be verified using Eq. (3), has been used.

Using Eqs. (34) and (42) one can now prove that the operator $\hat{C}$ commutes with the generators $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_0$ of $\mathrm{su}_q(2)$, i.e. that $\hat{C}$ is the second order Casimir operator of $\mathrm{su}_q(2)$. Indeed, one has

$$\begin{aligned}[\hat{C},\hat{L}_+]|l,\mathrm{m}\rangle_c &= \hat{C}\hat{L}_+|l,\mathrm{m}\rangle_c - \hat{L}_+\hat{C}|l,\mathrm{m}\rangle_c \\ &= \hat{C}\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c \\ &\quad -\hat{L}_+[l][l+1]|l,\mathrm{m}\rangle_c \\ &= [l][l+1]\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}|l,\mathrm{m}+1\rangle_c \\ &\quad -\sqrt{[l-\mathrm{m}][l+\mathrm{m}+1]}[l][l+1]|l,\mathrm{m}\rangle_c = 0. \end{aligned} \tag{44}$$

In exactly the same way one can prove that

$$[\hat{C},\hat{L}_-]|l,\mathrm{m}\rangle_c = 0, \tag{45}$$

while in addition one has

$$\begin{aligned}[\hat{C},\hat{L}_0]|l,\mathrm{m}\rangle_c &= \hat{C}\hat{L}_0|l,\mathrm{m}\rangle_c - \hat{L}_0\hat{C}|l,\mathrm{m}\rangle_c \\ &= [l][l+1]\mathrm{m}|l,\mathrm{m}\rangle_c - m[l][l+1]|l,\mathrm{m}\rangle_c = 0. \end{aligned} \tag{46}$$

Thus we have proved that the operator $\hat{C}$ is the second order Casimir operator of $\mathrm{su}_q(2)$. We are now going to prove that the operator $\hat{C}$ commutes also with the generators $\hat{l}_+$, $\hat{l}_-$, and $\hat{l}_0$ of the usual su(2) algebra. Indeed, one has

$$\begin{aligned}[\hat{C},\hat{l}_+]|l,\mathrm{m}\rangle_c &= \hat{C}\hat{l}_+|l,\mathrm{m}\rangle_c - \hat{l}_+\hat{C}|l,\mathrm{m}\rangle_c \\ &= \hat{C}\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}|l,\mathrm{m}+1\rangle_c \\ &\quad -\hat{l}_+[l][l+1]|l,\mathrm{m}\rangle_c \\ &= [l][l+1]\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}|l,\mathrm{m}+1\rangle_c \\ &\quad -\sqrt{(l-\mathrm{m})(l+\mathrm{m}+1)}[l][l+1]|l,\mathrm{m}+1\rangle_c \\ &= 0. \end{aligned} \tag{47}$$

In exactly the same way one can prove that

$$[\hat{C},\hat{l}_-]|l,\mathrm{m}\rangle_c = 0, \tag{48}$$

while the relation

$$[\hat{C},\hat{l}_0]|l,\mathrm{m}\rangle_c = 0 \tag{49}$$

occurs from Eq. (46), since $\hat{L}_0 = \hat{l}_0$ by definition [see Eq. (30)]. The following comments are now in place.

(a) The fact that the operator $\hat{C}$, which will be from now on denoted by $\hat{C}_2^{(q)}$, commutes with the generators of su(2) implies that this operator is a function of the second order Casimir operator of su(2), given in Eq. (23). As a consequence, it should be possible to express the eigenvalues of $\hat{C}_2^{(q)}$, which are $[l][l+1]$ [as we have seen in Eq. (42)], in terms of the eigenvalues of $\hat{C}_2$, which are $l(l+1)$ [as we have seen in Eq. (24)]. This task will be undertaken in Sec. IV.

(b) Equations (47)–(49) also tell us that the Hamiltonian of Eq. (13) commutes with the generators of the usual su(2) algebra, i.e., it is rotationally invariant. The Hamiltonian of Eq. (13) does not break rotational symmetry. It corresponds to a function of the second order Casimir operator of the usual su(2) algebra. This function, however, has been chosen in an appropriate way, in order to guarantee that the Hamiltonian of Eq. (13) is also invariant under a more complicated symmetry, namely the symmetry $\mathrm{su}_q(2)$.

(c) From the contents of the present section it is also clear that the irrep $D^l_{(q)}$ of $\mathrm{su}_q(2)$ and the irrep D^l of su(2) have the same structure, the relevant states being in a one to one correspondence to each other. The similarity between Eqs. (34) and (21) and (22) implies that the distinction between the "classical" basis of the present section and the "quantum" basis of Sec. II turns out to be unnecessary, as well as that the quantum numbers l and m can be identified with the usual angular momentum quantum numbers l and m.

(d) These conclusions are valid in the case of q being not a root of unity, as already mentioned in Sec. II.

IV. EXACT EXPANSION OF THE $\mathrm{su}_q(2)$ SPECTRUM

Let us consider the spectrum of Eq. (17), which has been found relevant to rotational nuclear and molecular spectra, assuming for simplicity $E_0=0$ and $\tau>0$. Since the Hamiltonian of Eq. (13) is invariant under su(2), as we saw in Sec. III, it should be possible in principle to express it as a function of the Casimir operator C_2 of the usual su(2) algebra. As a consequence, it should also be possible to express the eigenvalues of this Hamiltonian, given in Eq. (17), as a function of the eigenvalues of the Casimir operator of the usual su(2), i.e. as a function of $l(l+1)$. This is a nontrivial task, since in Eq. (17) two different functions of the variable l appear, while we are in need of a single function of the variable $l(l+1)$, which is related to the length of the angular momentum vector. In order to represent the expression of Eq. (17) as a power series of the variable $l(l+1)$, one can use the identity

$$\sin(l\tau)\sin((l+1)\tau) = \frac{1}{2}\{\cos(\tau)-\cos((2l+1)\tau)\}. \tag{50}$$

It turns out that the coefficients of the relevant expansion can be expressed in terms of the spherical Bessel functions of the first kind $j_n(x)$ [18], which are determined through the generating function

$$\frac{1}{x}\cos\sqrt{x^2-2xt} = \sum_{n=0}^{\infty} j_{n-1}(x)\frac{t^n}{n!}, \tag{51}$$

and are characterized by the asymptotic behavior

$$j_n(x) \approx \frac{x^n}{(2n+1)!!}, \quad x \ll 1. \tag{52}$$

Performing the substitutions

$$x = \tau, \quad t = -2\tau l(l+1), \tag{53}$$

which imply

$$x^2 - 2xt = \tau^2 (2l+1)^2, \tag{54}$$

one obtains the expression

$$\frac{1}{\tau}\cos((2l+1)\tau) = \sum_{n=0}^{\infty} \frac{(-2\tau)^n}{n!} j_{n-1}(\tau)\{l(l+1)\}^n, \tag{55}$$

which in the special case of $l=0$ reads

$$\frac{1}{\tau}\cos\tau = j_{-1}(\tau), \tag{56}$$

in agreement with the definition [18]

$$j_{-1}(x) = \frac{\cos x}{x}. \tag{57}$$

Substituting Eqs. (55) and (56) into Eq. (50), and taking into account that [18]

$$j_0(x) = \frac{\sin x}{x}, \tag{58}$$

Eq. (17) takes the form

$$E_l^{(\tau)} = \frac{A}{j_0^2(\tau)} \sum_{n=0}^{\infty} \frac{(-1)^n (2\tau)^n}{(n+1)!} j_n(\tau)\{l(l+1)\}^{n+1}, \tag{59}$$

which is indeed an expansion in terms of $l(l+1)$.

V. APPROXIMATE EXPANSION OF THE $\mathrm{su}_q(2)$ SPECTRUM

We are now going to consider an approximate form of this expansion, which will allow us to connect the present approach to the description of nuclear spectra proposed by Amal'sky [7]. For "small deformation," i.e., for $\tau \ll 1$, one can use the asymptotic expression of Eq. (52). Keeping only the terms of the lowest order one then obtains the following approximate series

$$E_l^{(\tau)} \approx A \sum_{n=0}^{\infty} \frac{(-1)^n (2\tau)^{2n}}{(n+1)(2n+1)!} \{l(l+1)\}^{n+1}, \tag{60}$$

where use of the identity

$$2^n (n+1)!(2n+1)!! = (n+1)(2n+1)! \tag{61}$$

has been made. The first few terms of this expansion are

$$E_l^{(\tau)} \approx A\left(l(l+1) - \frac{\tau^2}{3}\{l(l+1)\}^2 + \frac{2\tau^4}{45}\{l(l+1)\}^3 - \frac{\tau^6}{315}\{l(l+1)\}^4 + \cdots \right), \tag{62}$$

in agreement with the findings of Ref. [6].

One can now observe that the expansion appearing in Eq. (60) is similar to the power series of the function

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) = \sum_{k=1}^{\infty} (-1)^{k+1} 2^{2k-1} \frac{x^{2k}}{(2k)!}. \tag{63}$$

Then, performing the auxiliary substitution

$$\xi = \sqrt{l(l+1)}, \quad \eta = l(l+1) = \xi^2, \tag{64}$$

one can put the expansion of Eq. (60) in the form

$$E_l^{(\tau)} \approx A \frac{\sin^2(\tau\xi)}{\tau^2} = \frac{\hbar^2}{2\mathcal{J}_0} \frac{\sin^2(\tau\sqrt{l(l+1)})}{\tau^2}, \quad q = e^{i\tau}. \tag{65}$$

This result is similar to the expression proposed for the unified description of nuclear rotational spectra by Amal'sky [7],

$$E_l = \varepsilon_0 \sin^2\left(\frac{\pi}{N}\sqrt{l(l+1)} \right), \tag{66}$$

where ε_0 is a phenomenological constant ($\varepsilon_0 \approx 6.664$ MeV) which remains the same for all nuclei, while N is a free parameter varying from one nucleus to the other.

VI. ANALYTIC EXPRESSIONS BASED ON THE APPROXIMATE EXPANSION OF THE $\mathrm{su}_q(2)$ SPECTRUM

In this section we will consider some analytic expressions, which are based on the approximate result of Eq. (65), with the purpose of connecting the present approach to the Harris formalism [8]. In the study of high spin phenomena the rotational frequency ω and the kinematic moment of inertia $\mathcal{J}$ are defined by

$$\hbar\omega = \frac{\partial E}{\partial \xi}, \tag{67}$$

$$\frac{\hbar^2}{2\mathcal{J}} = \frac{\partial E}{\partial \eta} = \frac{1}{2\xi}\frac{\partial E}{\partial \xi}, \tag{68}$$

where ξ has been defined in Eq. (64), and

$$\eta = l(l+1) = \xi^2. \tag{69}$$

From Eqs. (67) and (68) it is clear that the two quantities are connected by the relation

$$\mathcal{J}\omega = \hbar\xi. \tag{70}$$

Applying these definitions to the analytical expression of Eq. (65), one obtains

$$\hbar\omega = A\frac{\sin(2\tau\xi)}{\tau} = \frac{\hbar^2}{2\mathcal{J}_0}\frac{\sin(2\tau\xi)}{\tau}, \tag{71}$$

$$\mathcal{J} = \mathcal{J}_0\frac{2\tau\xi}{\sin(2\tau\xi)}, \tag{72}$$

where the identity

$$\sin 2x = 2\sin x\cos x \tag{73}$$

has been used. Using the expressions for E (for which we drop the superscript and subscript) and ω given in Eqs. (65) and (71) one can easily verify that

$$\frac{\mathcal{J}_0\omega^2}{2} = E\left(1-\frac{\tau^2}{A}E\right), \tag{74}$$

where use of the identity of Eq. (73) has been made. Defining

$$\varepsilon = \frac{\tau^2}{A}E = \sin^2(\tau\xi), \tag{75}$$

$$t = \frac{\hbar\tau}{A} = \frac{2\mathcal{J}_0}{\hbar}\tau, \tag{76}$$

where t is a constant possessing dimensions of time, Eq. (74) takes the form

$$(\omega t)^2 = 4\varepsilon(1-\varepsilon) = 4\varepsilon - 4\varepsilon^2. \tag{77}$$

This expression can be considered as a quadratic equation for ε, allowing us to express ε as a function of ωt. Indeed, one finds

$$\varepsilon = \frac{1}{2}(1\pm\sqrt{1-(\omega t)^2}). \tag{78}$$

Using the Taylor expansion [18]

$$(1+x)^{-1/2} = 1-\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\cdots,$$
$$-1<x\leqslant 1 \tag{79}$$

one obtains

$$\varepsilon = \frac{1}{2}\left(1\pm\left(1-\frac{1}{2}(\omega t)^2-\frac{1}{8}(\omega t)^4-\frac{5}{16}(\omega t)^6-\cdots\right)\right). \tag{80}$$

The choice of the negative sign then leads to

$$\varepsilon = \frac{1}{4}(\omega t)^2+\frac{1}{16}(\omega t)^4+\frac{5}{32}(\omega t)^6+\cdots, \tag{81}$$

which, through Eq. (75), gives

$$E = \frac{A}{(2\tau)^2}\left((\omega t)^2+\frac{1}{4}(\omega t)^4+\frac{5}{8}(\omega t)^6+\cdots\right). \tag{82}$$

The choice of the positive sign gives, correspondingly,

$$\varepsilon = 1-\frac{1}{4}(\omega t)^2-\frac{1}{16}(\omega t)^4-\frac{5}{32}(\omega t)^6-\cdots \tag{83}$$

and

$$E = \frac{A}{\tau^2}\left(1-\frac{1}{4}(\omega t)^2-\frac{1}{16}(\omega t)^4-\frac{5}{32}(\omega t)^6-\cdots\right). \tag{84}$$

It is clear that Eq. (82) corresponds to E increasing as a function of ω, while Eq. (84) corresponds to E decreasing as a function of ω. Therefore only the first solution can be relevant to the description of nuclear rotational spectra.

We are now trying to find a similar expansion for the kinematic moment of inertia $\mathcal{J}$. Using Eq. (76) one can rewrite Eq. (71) in the form

$$\omega t = \sin(2\tau\xi). \tag{85}$$

Then Eq. (72) gives

$$\frac{\mathcal{J}}{\mathcal{J}_0} = \frac{2\tau\xi}{\sin(2\tau\xi)} = \frac{\arcsin(\omega t)}{\omega t}. \tag{86}$$

Then using the Taylor expansion [18]

$$\arcsin x = x+\frac{1}{2}\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7}+\cdots, \tag{87}$$

one obtains

$$\frac{\mathcal{J}}{\mathcal{J}_0} = 1+\frac{1}{6}(\omega t)^2+\frac{3}{40}(\omega t)^4+\frac{5}{112}(\omega t)^6+\cdots. \tag{88}$$

Using Eqs. (70) and (76) one finds from this result that

$$\xi = \sqrt{l(l+1)} = \omega\frac{\mathcal{J}}{\hbar} = \frac{1}{2\tau}\left(\omega t+\frac{1}{6}(\omega t)^3+\frac{3}{40}(\omega t)^5 + \frac{5}{112}(\omega t)^7+\cdots\right). \tag{89}$$

The expansions appearing in Eqs. (82) and (89) are of the form occurring in the Harris formalism [8]:

$$E = E_0+\frac{1}{2}(\mathcal{J}_0\omega^2+3C\omega^4+5D\omega^6+7F\omega^8+\cdots), \tag{90}$$

$$\sqrt{l(l+1)} = \mathcal{J}_0\omega+2C\omega^3+3D\omega^5+4F\omega^7+\cdots, \tag{91}$$

the main difference between the two formalisms being the fact that in the case of Harris the coefficients of the various terms in the series are independent from each other, while in the present case the coefficients in the series are interdependent, since they all contain the constant t. It should be noticed at this point that the Harris formalism is known [19] to be equivalent to the variable moment of inertia (VMI) model [20]. The similarities between the $su_q(2)$ approach and the VMI model have been directly considered in Ref. [6].

VII. IRREDUCIBLE TENSOR OPERATORS UNDER $su_q(2)$

A different path toward the construction of a Hamiltonian appropriate for the description of rotational spectra can be taken through the construction of irreducible tensor operators under $su_q(2)$ [9,10]. In this discussion we limit ourselves to real values of q, i.e., to $q=e^\tau$, with τ real, as in Refs. [9,10].

An irreducible tensor operator of rank k is the set of $2k+1$ operators $T^{(q)}_{k,\kappa}$ $(\kappa=k,k-1,k-2,\ldots,-k)$, which satisfy with the generators of the $su_q(2)$ algebra the commutation relations [9,10]

$$[L_0,T^{(q)}_{k,\kappa}]=\kappa T^{(q)}_{k,\kappa}, \tag{92}$$

$$[L_\pm,T^{(q)}_{k,\kappa}]_{q^\kappa}=\sqrt{[k\mp\kappa][k\pm\kappa+1]}T^{(q)}_{k,\kappa\pm1}q^{-L_0}, \tag{93}$$

where q commutators are defined by

$$[A,B]_{q^\alpha}=AB-q^\alpha BA. \tag{94}$$

It is clear that in the limit $q\rightarrow1$ these commutation relations reduce to the usual ones, which occur in the definition of irreducible tensor operators under su(2). It should also be noticed that the operators

$$R^{(q)}_{k,\kappa}=(-1)^\kappa q^{-\kappa}(T^{(q)}_{k,-\kappa})^\dagger, \tag{95}$$

where † denotes Hermitian conjugation, satisfy the same commutation relations (92) and (93) as the operators $T^{(q)}_{k,\kappa}$, i.e., the operators $R^{(q)}_{k,\kappa}$ also form an irreducible tensor operator of rank k under $su_q(2)$.

We can construct an irreducible tensor operator of rank 1 using as building blocks the generators of $su_q(2)$. This irreducible tensor operator will consist of the operators J_{+1}, J_{-1}, and J_0, which should satisfy the commutation relations

$$[L_0,J_m]=mJ_m, \tag{96}$$

$$[L_\pm,J_m]_{q^m}=\sqrt{[1\mp m][2\pm m]}J_{m\pm1}q^{-L_0}, \tag{97}$$

which are a special case of Eqs. (92) and (93), while the relevant Hermitian conjugate operators will be

$$(J_m)^\dagger=(-1)^m q^{-m}J_{-m}, \tag{98}$$

which is a consequence of Eq. (95). It turns out [9,10,21] that the explicit forms of the relevant operators are

$$J_{+1}=-\frac{1}{\sqrt{[2]}}q^{-L_0}L_+, \tag{99}$$

$$J_{-1}=\frac{1}{\sqrt{[2]}}q^{-L_0}L_-, \tag{100}$$

$$\begin{aligned}J_0&=\frac{1}{[2]}(qL_+L_--q^{-1}L_-L_+)\\&=\frac{1}{[2]}(qL_-L_+-q^{-1}L_+L_-)+[2L_0]\\&=\frac{1}{2}\left([2L_0]+\frac{(q-q^{-1})}{[2]}(L_-L_++L_+L_-)\right)\\&=\frac{1}{[2]}\{q[2L_0]+(q-q^{-1})L_-L_+)\}\\&=\frac{1}{[2]}\{q[2L_0]+(q-q^{-1})(C_2^{(q)}-[L_0][L_0+1])\},\end{aligned} \tag{101}$$

while the Hermitian conjugate operators are

$$(J_{+1})^\dagger=-q^{-1}J_{-1},\quad (J_{-1})^\dagger=-qJ_{+1},\quad (J_0)^\dagger=J_0. \tag{102}$$

It is clear that in the limit $q\rightarrow1$ these results reduce to the usual expressions for spherical tensors of rank 1 under su(2), formed out of the usual angular momentum operators

$$J_+=-\frac{L_+}{\sqrt{2}}=-\frac{L_x+iL_y}{\sqrt{2}},\quad J_-=\frac{L_-}{\sqrt{2}}=\frac{L_x-iL_y}{\sqrt{2}},\quad J_0=L_0, \tag{103}$$

$$(J_+)^\dagger=-J_-,\quad (J_-)^\dagger=-J_+,\quad (J_0)^\dagger=J_0. \tag{104}$$

The commutation relations among the operators J_{+1}, J_{-1}, J_0 can be obtained using Eqs. (99)–(101), (96), and (97), as well as the fact that from Eq. (1) one has

$$\begin{aligned}[L_0,L_+]=L_+\Rightarrow L_0L_+=L_+(L_0+1)\Rightarrow f(L_0)L_+\\=L_+f(L_0+1),\end{aligned} \tag{105}$$

$$\begin{aligned}[L_0,L_-]=-L_-\Rightarrow L_0L_-=L_-(L_0-1)\Rightarrow f(L_0)L_-\\=L_-f(L_0-1),\end{aligned} \tag{106}$$

where $f(x)$ is any function which can be written as a Taylor expansion in powers of x. Indeed, one has

$$\begin{aligned}[J_{+1},J_0]&=-\frac{1}{\sqrt{[2]}}(q^{-L_0}L_+J_0-J_0q^{-L_0}L_+)\\&=-\frac{1}{\sqrt{[2]}}(L_+J_0-J_0L_+)q^{-L_0-1}\\&=-\frac{1}{\sqrt{[2]}}\sqrt{[2]}J_{+1}q^{-L_0}q^{-L_0-1}=-q^{-2L_0+1}J_{+1},\end{aligned} \tag{107}$$

$$[J_{-1},J_0]=\frac{1}{\sqrt{[2]}}(q^{-L_0}L_-J_0-J_0q^{-L_0}L_-)$$
$$=\frac{1}{\sqrt{[2]}}(L_-J_0-J_0L_-)q^{-L_0+1}$$
$$=\frac{1}{\sqrt{[2]}}\sqrt{[2]}J_{-1}q^{-L_0}q^{-L_0+1}=q^{-2L_0-1}J_{-1}, \tag{108}$$

$$[J_{+1},J_{-1}]=-\frac{1}{[2]}(q^{-L_0}L_+q^{-L_0}L_- -q^{-L_0}L_-q^{-L_0}L_+)$$
$$=-\frac{1}{[2]}(q^{-2L_0+1}L_+L_- -q^{-2L_0-1}L_-L_+)$$
$$=-\frac{1}{[2]}q^{-2L_0}(qL_+L_- -q^{-1}L_-L_+)$$
$$=-q^{-2L_0}J_0, \tag{109}$$

or, in compact form,

$$[J_{+1},J_0]=-q^{-2L_0+1}J_{+1},$$
$$[J_{-1},J_0]=q^{-2L_0-1}J_{-1},$$
$$[J_{+1},J_{-1}]=-q^{-2L_0}J_0. \tag{110}$$

In the limit $q\rightarrow 1$ these results reduce to the usual commutation relations related to spherical tensor operators under su(2):

$$[J_+,J_0]=-J_+,\quad [J_-,J_0]=J_-,\quad [J_+,J_-]=-J_0. \tag{111}$$

It is clear that the commutation relations of Eq. (110) are different from these of Eqs. (1) and (2), as it is expected since the commutation relations of Eq. (111) are different from the usual commutation relations of su(2), given in Eq. (18).

One can now try to build out of these operators the scalar square of the angular momentum operator. For this purpose one needs the definition of the tensor product of two irreducible tensor operators, which has the form [9,10,21–24]

$$[A^{(q)}_{j_1}\otimes B^{(q)}_{j_2}]^{(1/q)}_{j,m}=\sum_{m_1,m_2}\langle j_1m_1j_2m_2|jm\rangle_{1/q}A^{(q)}_{j_1,m_1}B^{(q)}_{j_2,m_2}. \tag{112}$$

One should observe that the irreducible tensor operators $A^{(q)}_{j_1}$ and $B^{(q)}_{j_2}$, which correspond to the deformation parameter q, are combined into an irreducible tensor operator $[A^{(q)}_{j_1}\times B^{(q)}_{j_2}]^{(1/q)}_{j,m}$, which corresponds to the deformation parameter $1/q$, through the use of the deformed Clebsch-Gordan coefficients $\langle j_1m_1j_2m_2|jm\rangle_{1/q}$, which also correspond to the deformation parameter $1/q$.

Analytic expressions for several q-deformed Clebsch-Gordan coefficients, as well as their symmetry proporties, can be found in Refs. [9,22]. Using the general formulas of Refs. [9,22] we derive here the Clebsch-Gordan coefficients which we will immediately need:

$$\langle 1110|11\rangle_q=q\sqrt{\frac{[2]}{[4]}},\quad \langle 1011|11\rangle_q=-q^{-1}\sqrt{\frac{[2]}{[4]}}, \tag{113}$$

$$\langle 101-1|1-1\rangle_q=q\sqrt{\frac{[2]}{[4]}},$$
$$\langle 1-110|1-1\rangle_q=-q^{-1}\sqrt{\frac{[2]}{[4]}}, \tag{114}$$

$$\langle 111-1|10\rangle_q=\sqrt{\frac{[2]}{[4]}},\quad \langle 1-111|10\rangle_q=-\sqrt{\frac{[2]}{[4]}},$$
$$\langle 1010|10\rangle_q=(q-q^{-1})\sqrt{\frac{[2]}{[4]}}. \tag{115}$$

Using the definition of Eq. (112), the Clebsch-Gordan coefficients just given, as well as the commutation relations of Eq. (110), one finds the tensor products

$$[J\otimes J]^{(1/q)}_{1,+1}=\langle 1110|11\rangle_{1/q}J_{+1}J_0+\langle 1011|11\rangle_{1/q}J_0J_{+1}$$
$$=\sqrt{\frac{[2]}{[4]}}\{q^{-1}J_{+1}J_0-qJ_0J_{+1}\}$$
$$=\sqrt{\frac{[2]}{[4]}}\{q^{-1}(J_0J_{+1}-q^{-2L_0+1}J_{+1})-qJ_0J_{+1}\}$$
$$=\sqrt{\frac{[2]}{[4]}}\{(q^{-1}-q)J_0-q^{-2L_0}\}J_{+1}$$
$$=-\sqrt{\frac{[2]}{[4]}}\{q^{-2L_0}+(q-q^{-1})J_0\}J_{+1}, \tag{116}$$

$$[J\otimes J]^{(1/q)}_{1,-1}=\langle 101-1|1-1\rangle_{1/q}J_0J_{-1}$$
$$+\langle 1-110|1-1\rangle_{1/q}J_{-1}J_0$$
$$=\sqrt{\frac{[2]}{[4]}}\{q^{-1}J_0J_{-1}-qJ_{-1}J_0\}$$
$$=\sqrt{\frac{[2]}{[4]}}\{q^{-1}J_0J_{-1}-q(J_0J_{-1}+q^{-2L_0-1}J_{-1})\},$$
$$=\sqrt{\frac{[2]}{[4]}}\{(q^{-1}-q)J_0-q^{-2L_0}\}J_{-1}$$
$$=-\sqrt{\frac{[2]}{[4]}}\{q^{-2L_0}+(q-q^{-1})J_0\}J_{-1}, \tag{117}$$

$$\begin{aligned}[J\otimes J]^{(1/q)}_{1,0}=&\langle 111-1|10\rangle_{1/q}J_{+1}J_{-1}\\&+\langle 1-111|10\rangle_{1/q}J_{-1}J_{+1}\\&+\langle 1010|10\rangle_{1/q}(J_0)^2\\=&\sqrt{\frac{[2]}{[4]}}\{J_{+1}J_{-1}-J_{-1}J_{+1}+(q^{-1}-q)(J_0)^2\}\\=&\sqrt{\frac{[2]}{[4]}}\{-q^{-2L_0}J_0-(q-q^{-1})(J_0)^2\}\\=&-\sqrt{\frac{[2]}{[4]}}\{q^{-2L_0}+(q-q^{-1})J_0\}J_0.\end{aligned}\quad(118)$$

We remark that all these tensor products are of the general form

$$\begin{aligned}[J\otimes J]^{(1/q)}_{1,m}=&-\sqrt{\frac{[2]}{[4]}}\{q^{-2L_0}+(q-q^{-1})J_0\}J_m\\=&-\sqrt{\frac{[2]}{[4]}}ZJ_m,\quad m=0,\pm 1\end{aligned}\quad(119)$$

where, by definition,

$$Z=q^{-2L_0}+(q-q^{-1})J_0.\quad(120)$$

One can now prove that the operator Z is a scalar quantity, since it is a function of the second order Casimir operator of $\mathrm{su}_q(2)$, given in Eq. (11). Indeed, one has

$$\begin{aligned}Z=&q^{-2L_0}+(q-q^{-1})J_0\\=&q^{-2L_0}+\frac{(q-q^{-1})}{[2]}\{q[2L_0]+(q-q^{-1})(C_2^{(q)}-[L_0][L_0+1])\}\\=&q^{-2L_0}+\frac{1}{[2]}\{q(q^{2L_0}-q^{-2L_0})+(q-q^{-1})^2C_2^{(q)}-(q^{L_0}-q^{-L_0})(q^{L_0+1}-q^{-L_0-1})\}\\=&q^{-2L_0}+\frac{1}{[2]}\{\underline{q^{2L_0+1}}-q^{-2L_0+1}-\underline{q^{2L_0+1}}+q+q^{-1}-q^{-2L_0-1}+(q-q^{-1})^2C_2^{(q)}\}\\=&q^{-2L_0}+\frac{1}{[2]}\{-q^{-2L_0}(q+q^{-1})+(q+q^{-1})+(q-q^{-1})^2C_2^{(q)}\}\\=&q^{-2L_0}-q^{-2L_0}+1+\frac{(q-q^{-1})^2}{[2]}C_2^{(q)}=1+\frac{(q-q^{-1})^2}{[2]}C_2^{(q)},\end{aligned}\quad(121)$$

or, in more compact form,

$$Z=q^{-2L_0}+(q-q^{-1})J_0=1+\frac{(q-q^{-1})^2}{[2]}C_2^{(q)}.\quad(122)$$

Since Z is a scalar quantity, symmetric under the exchange $q\leftrightarrow q^{-1}$ (as one can see from the last expression appearing in the last equation), Eq. (119) can be written in the form

$$\left[\frac{J}{Z}\otimes\frac{J}{Z}\right]^{(1/q)}_{1,m}=-\sqrt{\frac{[2]}{[4]}}\frac{J_m}{Z}\Rightarrow[J'\otimes J']^{(1/q)}_{1,m}=-\sqrt{\frac{[2]}{[4]}}J'_m,\quad(123)$$

where, by definition,

$$J'_m=\frac{J_m}{Z},\quad m=+1,0,-1.\quad(124)$$

It is clear that the operators J'_m also form an irreducible tensor operator, since Z is a function of the second order Casimir $C_2^{(q)}$ of $\mathrm{su}_q(2)$, which commutes with the generators L_+, L_-, and L_0 of $\mathrm{su}_q(2)$, and therefore does not affect the commutation relations of Eqs. (96) and (97).

The scalar product of two irreducible tensor operators is defined as [10,24]

$$\begin{aligned}(A_j^{(q)}\cdot B_j^{(q)})^{(1/q)}=&(-1)^{-j}\sqrt{[2j+1]}[A_j^{(q)}\times B_j^{(q)}]^{(1/q)}_{0,0}\\=&\sum_m(-q)^{-m}A^{(q)}_{j,m}B^{(q)}_{j,-m}.\end{aligned}\quad(125)$$

Substituting the irreducible tensor operators J_m in this definition we obtain [10]

$$\begin{aligned}(J\cdot J)^{(1/q)}=&-\sqrt{[3]}[J\times J]^{(1/q)}_{0,0}\\=&\frac{2}{[2]}C_2^{(q)}+\frac{(q-q^{-1})^2}{[2]^2}(C_2^{(q)})^2=\frac{Z^2-1}{(q-q^{-1})^2},\end{aligned}\quad(126)$$

where in the last step the identity

$$Z^2-1=(Z-1)(Z+1)$$
$$=\frac{(q-q^{-1})^2}{[2]}C_2^{(q)}\left(2+\frac{(q-q^{-1})^2}{[2]}C_2^{(q)}\right), \quad (127)$$

has been used, obtained through use of Eq. (122). In the same way the irreducible tensor operators J'_m give the result

$$(J'\cdot J')^{(1/q)}=\frac{1-Z^{-2}}{(q-q^{-1})^2}. \quad (128)$$

We have therefore determined the scalar square of the angular momentum operator. We can assume at this point that this quantity can be used (up to an overall constant) as the Hamiltonian for the description of rotational spectra, defining

$$H=A\frac{1-Z^{-2}}{(q-q^{-1})^2}, \quad (129)$$

where A is a constant, which we also write in the form

$$A=\frac{\hbar^2}{2\mathcal{J}_0} \quad (130)$$

for future reference.

The eigenvalues $\langle Z\rangle$ of the operator Z in the basis $|l,m\rangle$ can be easily found from the last expression given in Eq. (122), using the eigenvalues of the Casimir operator $C_2^{(q)}$ in this basis, which are $[l][l+1]$, as already mentioned in Sec. II

$$\langle Z\rangle=1+\frac{(q-q^{-1})^2}{[2]}[l][l+1]=\frac{1}{[2]}(q^{2l+1}+q^{-2l-1})$$
$$=\frac{1}{[2]}([2l+2]-[2l]). \quad (131)$$

The eigenvalues $\langle (J\cdot J)^{(1/q)}\rangle$ of the scalar quantity $(J\cdot J)^{(1/q)}$ can be found in a similar manner from Eq. (126)

$$\langle (J\cdot J)^{(1/q)}\rangle=\frac{2}{[2]}[l][l+1]+\frac{(q-q^{-1})^2}{[2]^2}[l]^2[l+1]^2$$
$$=\frac{[2l][2l+2]}{[2]^2}=[l]_{q^2}[l+1]_{q^2}, \quad (132)$$

where, by definition,

$$[x]_{q^2}=\frac{q^{2x}-q^{-2x}}{q^2-q^{-2}}. \quad (133)$$

Finally, the eigenvalues $\langle H\rangle$ of the Hamiltonian can be found by substituting the eigenvalues of Z from Eq. (131) into Eq. (129),

$$E=\langle H\rangle=A\frac{1}{(q-q^{-1})^2}\left(1-\frac{[2]^2}{(q^{2l+1}+q^{-2l-1})^2}\right)$$
$$=A\frac{1}{4\sinh^2\tau}\left(1-\frac{\cosh^2\tau}{\cosh^2((2l+1)\tau)}\right), \quad q=e^\tau, \quad (134)$$

where in the last step the identities

$$q-q^{-1}=2\sinh\tau, \quad [2]=q+q^{-1}=2\cosh\tau, \quad (135)$$

$$q^{2l+1}+q^{-2l-1}=2\cosh((2l+1)\tau), \quad (136)$$

which are valid in the present case of $q=e^\tau$ with τ being real, have been used. In the same way one sees that

$$\langle Z\rangle=\frac{\cosh((2l+1)\tau)}{\cosh\tau}. \quad (137)$$

The following comments are now in place.

(a) The last expression in Eq. (132) indicates that the eigenvalues of the scalar quantity $(J\cdot J)^{(1/q)}$ are equivalent to the eigenvalues of the Casimir operator of $\mathrm{su}_q(2)$ (which are $[l][l+1]$), up to a change in the deformation parameter from q to q^2.

(b) From Eq. (131) it is clear that the eigenvalues of the scalar operator Z go to the limiting value 1 as $q\to 1$. Therefore, one can think of Z as a "unity" operator. Furthermore the last expression in Eq. (131) indicates that $\langle Z\rangle$ is behaving like a "measure" of the unit of angular momentum in the deformed case.

VIII. ROTATIONAL INVARIANCE OF THE $\mathrm{su}_q(2)$ ITO HAMILTONIAN

In this section the notation and tools of Sec. III will be used once more. We wish to prove that the Hamiltonian of Eq. (129) commutes with the generators $\hat{l}_+$, $\hat{l}_-$, and $\hat{l}_0$ of the usual su(2) algebra, i.e., with the usual angular momentum operators. Taking into account Eq. (122) we see that acting on the "classical" basis described in Sec. III we have

$$\hat{Z}|l,\mathrm{m}\rangle_c=\left(1+\frac{(q-q^{-1})^2}{[2]}\hat{C}_2^{(q)}\right)|l,\mathrm{m}\rangle_c =\left(1+\frac{(q-q^{-1})^2}{[2]}[l][l+1]\right)|l,\mathrm{m}\rangle_c. \quad (138)$$

Then, using Eq. (129), we see that

$$\hat{H}|l,\mathrm{m}\rangle_c=\frac{A}{(q-q^{-1})^2}\left(1-\frac{1}{\hat{Z}^2}\right)|l,\mathrm{m}\rangle_c =\frac{A}{(q-q^{-1})^2}\times\left(1-\frac{1}{\left(1+\frac{(q-q^{-1})^2}{[2]}[l][l+1]\right)^2}\right)|l,\mathrm{m}\rangle_c. \quad (139)$$

Using this result, as well as Eq. (21), one finds

$$\begin{aligned}[\hat{H},\hat{l}_+]|l,\mathrm{m}\rangle_c&=\hat{H}\hat{l}_+|l,\mathrm{m}\rangle_c-\hat{l}_+\hat{H}|l,\mathrm{m}\rangle_c\\ &=\hat{H}\sqrt{(l-m)(l+m+1)}|l,\mathrm{m}+1\rangle_c-\hat{l}_+\frac{A}{(q-q^{-1})^2}\left(1-\frac{1}{\left(1+\frac{(q-q^{-1})^2}{[2]}[l][l+1]\right)^2}\right)|l,\mathrm{m}\rangle_c\\ &=\frac{A}{(q-q^{-1})^2}\left(1-\frac{1}{\left(1+\frac{(q-q^{-1})^2}{[2]}[l][l+1]\right)^2}\right)\sqrt{(l-m)(l+m+1)}|l,\mathrm{m}+1\rangle_c\\ &\quad-\sqrt{(l-m)(l+m+1)}\frac{A}{(q-q^{-1})^2}\left(1-\frac{1}{\left(1+\frac{(q-q^{-1})^2}{[2]}[l][l+1]\right)^2}\right)|l,\mathrm{m}+1\rangle_c=0. \end{aligned} \quad (140)$$

In exactly the same way, using Eqs. (21) and (139), one finds that

$$[\hat{H},\hat{l}_-]|l,\mathrm{m}\rangle_c=0, \quad [\hat{H},\hat{l}_0]|l,\mathrm{m}\rangle_c=0. \quad (141)$$

We have thus proved that the Hamiltonian of Eq. (129) is invariant under usual angular momentum. This result is expected, since the Hamiltonian is a function of the operator $\hat{Z}$, which in turn [as seen from Eq. (122)] is a function of the second order Casimir operator of su$_q(2)$, $\hat{C}_2^{(q)}$, which was proved to be rotationally invariant in Sec. III.

Since the Hamiltonian of Eq. (129) is rotationally invariant, it should be possible to express it as a function of $\hat{C}_2$ [the second order Casimir operator of su(2)]. It should also be possible to express the eigenvalues of the Hamiltonian of Eq. (129) as a function of $l(l+1)$, i.e., as a function of the eigenvalues of $\hat{C}_2$. This task will be undertaken in Sec. IX.

For completeness we mention that using Eqs. (34) and (139) one can prove in an analogous way that

$$[\hat{H},\hat{L}_+]|l,\mathrm{m}\rangle_c=0, \ [\hat{H},\hat{L}_-]|l,\mathrm{m}\rangle_c=0, \ [\hat{H},\hat{L}_0]|l,\mathrm{m}\rangle_c=0, \quad (142)$$

i.e., that the Hamiltonian of Eq. (129) commutes with the generators of su$_q(2)$ as well. Then from Eq. (103) it is clear that, in addition, one has

$$[\hat{H},\hat{J}_+]|l,\mathrm{m}\rangle_c=0, \quad [\hat{H},\hat{J}_-]|l,\mathrm{m}\rangle_c=0, \quad [\hat{H},\hat{J}_0]|l,\mathrm{m}\rangle_c=0. \quad (143)$$

Then from Eqs. (124) and (138) one furthermore obtains

$$[\hat{H},\hat{J}'_+]|l,\mathrm{m}\rangle_c=0, \quad [\hat{H},\hat{J}'_-]|l,\mathrm{m}\rangle_c=0, \quad [\hat{H},\hat{J}'_0]|l,\mathrm{m}\rangle_c=0. \quad (144)$$

IX. EXACT EXPANSION OF THE su$_q(2)$ ITO SPECTRUM

Since the Hamiltonian of Eq. (129) is invariant under su(2), as we have seen in Sec. VIII, it should be possible to write its eigenvalues [given in Eq. (134)] as an expansion in terms of $l(l+1)$. This is a nontrivial task, since in Eq. (134) a function of the variable l appears, while we are in need of a function of the variable $l(l+1)$, which is related to the length of the angular momentum vector. For this purpose it turns out that one should use the Taylor expansion [18]

$$\tanh x=\sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}x^{2n-1} =\sum_{n=0}^{\infty}\frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n+2)!}x^{2n+1}, \quad |x|<\frac{\pi}{2}, \quad (145)$$

where B_n are the Bernoulli numbers [18], defined through the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \tag{146}$$

the first few of them being

$$B_0=1,\quad B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},$$

$$B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},\ \ldots,$$

$$B_{2n+1}=0 \quad \text{for} \quad n=1,2,\ldots. \tag{147}$$

From Eq. (145) the following identities, concerning the derivatives of $\tanh x$, occur:

$$(\tanh x)'=\frac{1}{\cosh^2 x}=1-\tanh^2 x$$

$$=\sum_{n=0}^{\infty} \frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n)!(2n+2)} x^{2n}, \tag{148}$$

$$(\tanh x)''=-2\frac{\tanh x}{\cosh^2 x}=-2\frac{\sinh x}{\cosh^3 x}$$

$$=\sum_{n=0}^{\infty} \frac{2^{2n+4}(2^{2n+4}-1)B_{2n+4}}{(2n+1)!(2n+4)} x^{2n+1}. \tag{149}$$

From these equations the following auxiliary identities occur:

$$\frac{\sinh x}{x\cosh^3 x}=-\frac{1}{2x}(\tanh x)''$$

$$=\sum_{n=0}^{\infty} \frac{2^{2n+3}(1-2^{2n+4})B_{2n+4}}{(2n+1)!(2n+4)} x^{2n}, \tag{150}$$

$$\tanh^2 x=1-\frac{1}{\cosh^2 x}=\sum_{n=0}^{\infty} \frac{2^{2n+4}(1-2^{2n+4})B_{2n+4}}{(2n+2)!(2n+4)} x^{2n+2}. \tag{151}$$

The expression for the energy, given in Eq. (134), can be put in the form

$$\frac{E}{A}=\left(\frac{\cosh^2\tau\cdot\tau^2}{\sinh^2\tau}\right)\frac{1}{(2\tau)^2}\left\{\frac{1}{\cosh^2\tau}-\frac{1}{\cosh^2((2l+1)\tau)}\right\}. \tag{152}$$

Denoting

$$z=(2l+1)\tau,\quad x=l(l+1), \tag{153}$$

which imply

$$z^2=(4x+1)\tau^2,\quad z^{2n}=\tau^{2n}\sum_{k=0}^{n}\binom{n}{k}2^{2k}x^k, \tag{154}$$

(the latter through use of the standard binomial formula), from Eq. (148) one obtains the expansion

$$\begin{aligned}\frac{1}{\cosh^2((2l+1)\tau)}&=\frac{1}{\cosh^2 z}\\&=\sum_{n=0}^{\infty}\frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n)!(2n+2)}z^{2n}\\&=\sum_{n=0}^{\infty}\underbrace{\frac{2^{2n+2}(2^{2n+2}-1)B_{2n+2}}{(2n)!(2n+2)}\tau^{2n}}_{a_n}\\&\quad\times\sum_{k=0}^{n}\underbrace{\binom{n}{k}2^{2k}}_{b_{n,k}}x^k\\&=\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}b_{n,k}x^k.\end{aligned} \tag{155}$$

The double sum appearing in the last expression can be rearranged using the general procedure

$$\begin{aligned}S&=\sum_{n=0}^{\infty}a_n\sum_{k=0}^{n}b_{n,k}x^k\\&=a_0b_{00}+a_1(b_{10}+b_{11}x)+a_2(b_{20}+b_{21}x+b_{22}x^2)\\&\quad+a_3(b_{30}+b_{31}x+b_{32}x^2+b_{33}x^3)+\cdots\\&=(a_0b_{00}+a_1b_{10}+a_2b_{20}+a_3b_{30}+\cdots)\\&\quad+(a_1b_{11}+a_2b_{21}+a_3b_{31}+a_4b_{41}+\cdots)x\\&\quad+(a_2b_{22}+a_3b_{32}+a_4b_{42}+a_5b_{52}+\cdots)x^2\\&\quad+(a_3b_{33}+a_4b_{43}+a_5b_{53}+a_6b_{63}+\cdots)x^3+\cdots\\&=\sum_{n=0}^{\infty}\underbrace{\left\{\sum_{k=n}^{\infty}a_k b_{k,n}\right\}}_{c_n}x^n=\sum_{n=0}^{\infty}c_n x^n,\end{aligned} \tag{156}$$

where

$$c_n=\sum_{k=n}^{\infty}a_k b_{k,n}=\sum_{k=0}^{\infty}a_{n+k}b_{n+k,n}. \tag{157}$$

Applying this general procedure in the case of Eq. (155) we obtain

$$\frac{1}{\cosh^2((2l+1)\tau)}=\frac{1}{\cosh^2 z}=\sum_{n=0}^{\infty}c_n x^n, \tag{158}$$

where

$$\begin{aligned} c_n &= \sum_{k=0}^{\infty} a_{n+k} b_{n+k,n} \\ &= \sum_{k=0}^{\infty} \frac{2^{2n+2k+2}(2^{2n+2k+2}-1)B_{2n+2k+2}}{(2n+2k)!(2n+2k+2)} \tau^{2n+2k} \\ &\quad \times \binom{n+k}{n} 2^{2n} \\ &= (2\tau)^{2n} \sum_{k=0}^{\infty} \frac{2^{2n+2k+2}(2^{2n+2k+2}-1)B_{2n+2k+2}}{(2n+2k)!(2n+2k+2)} \\ &\quad \times \binom{n+k}{n} \tau^{2k}. \end{aligned} \tag{159}$$

The first term in Eq. (158) is

$$c_0 = \sum_{k=0}^{\infty} \frac{2^{2k+2}(2^{2k+2}-1)B_{2k+2}}{(2k)!(2k+2)} \tau^{2k} = \frac{1}{\cosh^2 \tau}. \tag{160}$$

Then one has

$$\begin{aligned} &\frac{1}{(2\tau)^2} \left\{ \frac{1}{\cosh^2 \tau} - \frac{1}{\cosh^2((2l+1)\tau)} \right\} \\ &\quad = -\frac{1}{(2\tau)^2} \sum_{n=1}^{\infty} c_n x^n = -\frac{1}{(2\tau)^2} \sum_{n=0}^{\infty} c_{n+1} x^{n+1} \\ &\quad = \sum_{n=0}^{\infty} d_n x^{n+1}, \end{aligned} \tag{161}$$

where the coefficients d_n are

$$d_n = -\frac{1}{(2\tau)^2} c_{n+1} = \frac{(-1)^n (2\tau)^n}{(n+1)!} f_n(\tau), \quad n = 0,1,2,\ldots, \tag{162}$$

with

$$\begin{aligned} f_n(\tau) &= (-1)^{n+1} (2\tau)^n (n+1)! \\ &\quad \times \sum_{k=0}^{\infty} \frac{2^{2n+2k+4}(2^{2n+2k+4}-1)B_{2n+2k+4}}{(2n+2k+2)!(2n+2k+4)} \\ &\quad \times \binom{n+k+1}{n+1} \tau^{2k}. \end{aligned} \tag{163}$$

For $n=0$ one has

$$f_0(\tau) = -\sum_{k=0}^{\infty} \frac{2^{2k+4}(2^{2k+4}-1)B_{2k+4}}{(2k+2)!(2k+4)} (k+1) \tau^{2k} = \frac{\sinh \tau}{\tau \cosh^3 \tau}, \tag{164}$$

where in the last step Eq. (150) has been used. It is worth noticing that

$$f_n(\tau) = (-1)^n \tau^n \left(\frac{1}{\tau} \frac{d}{d\tau} \right)^n f_0(\tau). \tag{165}$$

With the help of Eqs. (161) and (162), the spectrum of Eq. (152) is put into the form

$$\frac{E}{A} = \left(\frac{\tau^2 \cosh^2 \tau}{\sinh^2 \tau} \right) \sum_{n=0}^{\infty} \frac{(-1)^n (2\tau)^n}{(n+1)!} f_n(\tau) (l(l+1))^{n+1}, \tag{166}$$

since $x = l(l+1)$ from Eq. (153). It is clear that Eq. (166) is an expansion in terms of $l(l+1)$, as expected.

X. APPROXIMATE EXPANSION OF THE $su_q(2)$ ITO SPECTRUM

In the limit of $|\tau| \ll 1$ one is entitled to keep in Eq. (163) only the term with $k=0$. Then the function $f_n(\tau)$ takes the form

$$f_n(\tau) \to \frac{(-1)^{n+1} 2^{2n+2}(2^{2n+4}-1)B_{2n+4}}{(2n+1)!!(n+2)} \tau^n, \tag{167}$$

where the Bernoulli numbers appear again and use of the identity

$$(2n+2)! = 2^{n+1}(n+1)!(2n+1)!! \tag{168}$$

has been made. Taking into account the Taylor expansions

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots, \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \tag{169}$$

and keeping only the lowest order terms, one easily sees that Eq. (166) is put in the form

$$\frac{E}{A} \approx \sum_{n=0}^{\infty} \frac{2^{2n+4}(1-2^{2n+4})B_{2n+4}}{(2n+2)!(2n+4)} (2\tau)^{2n} (l(l+1))^{k+1}, \tag{170}$$

where use of the identity of Eq. (168) has been made once more and use of the fact that

$$\frac{\tau^2 \cosh^2 \tau}{\sinh^2 \tau} \approx 1 \quad \text{for} \quad |\tau| \ll 1 \tag{171}$$

has been made. Comparing this result with Eq. (151) and making the identifications

$$x = 2\tau\sqrt{l(l+1)} = 2\tau\xi, \quad \xi = \sqrt{l(l+1)}, \tag{172}$$

Eq. (170) is put into the compact form

$$\begin{aligned} E &\approx \frac{A}{(2\tau)^2} \tanh^2(2\tau\sqrt{l(l+1)}) = \frac{A}{(2\tau)^2} \tanh^2(2\tau\xi), \\ q &= e^{\tau}. \end{aligned} \tag{173}$$

The extended form of the Taylor expansion of E is easily obtained from Eq. (170):

$$E \approx A\left(l(l+1) - \frac{2}{3}(2\tau)^2(l(l+1))^2 + \frac{17}{45}(2\tau)^4(l(l+1))^3 - \frac{62}{315}(2\tau)^6(l(l+1))^4 + \cdots \right). \quad (174)$$

Equation (173) will be referred to as the "hyperbolic tangent formula."

XI. ANALYTIC EXPRESSIONS BASED ON THE APPROXIMATE EXPANSION OF THE $su_q(2)$ ITO SPECTRUM

We are now going to derive analytic formulas for the rotational frequency ω and the kinematic moment of inertia $\mathcal{J}$, based on the approximate expression for the energy given in Eq. (173). From Eqs. (67) and (68) one immediately obtains

$$\hbar\omega = \frac{\partial E}{\partial \xi} = \frac{A}{\tau}\frac{\sinh(2\tau\xi)}{\cosh^3(2\tau\xi)} = \frac{A}{\tau}\tanh(2\tau\xi)(1-\tanh^2(2\tau\xi)), \quad (175)$$

$$\frac{\hbar^2}{2\mathcal{J}} = \frac{\partial E}{\partial \eta} = \frac{1}{2\xi}\frac{\partial E}{\partial \xi} = \frac{A}{2\tau\xi}\frac{\sinh(2\tau\xi)}{\cosh^3(2\tau\xi)} = \frac{A}{2\tau\xi}\tanh(2\tau\xi)(1-\tanh^2(2\tau\xi)), \quad (176)$$

where by definition $\eta = l(l+1) = \xi^2$, as in Eq. (69). Using the expressions for E and ω given in Eqs. (173) and (175) one can easily verify that

$$\frac{\mathcal{J}_0\omega^2}{2} = E\left(1 - \frac{(2\tau)^2}{A}E\right)^2, \quad (177)$$

where use of Eq. (15) and of the identities

$$\cosh^2 x - \sinh^2 x = 1, \quad \frac{1}{\cosh^2 x} = 1 - \tanh^2 x, \quad (178)$$

has also been made. Defining

$$\varepsilon = \frac{(2\tau)^2}{A}E = \tanh^2(2\tau\xi), \quad (179)$$

$$t = \frac{\hbar\tau}{A} = \frac{2\mathcal{J}_0}{\hbar}\tau, \quad (180)$$

where t is a constant having dimensions of time, Eq. (177) takes the form

$$(\omega t)^2 = \varepsilon(1-\varepsilon)^2 = \varepsilon - 2\varepsilon^2 + \varepsilon^3. \quad (181)$$

From this equation one can determine ε as a function of ωt, in the following way. One can define

$$s(x) = 1 - \varepsilon(x) \Rightarrow \varepsilon(x) = 1 - s(x), \quad x = (\omega t)^2. \quad (182)$$

Then Eq. (181) takes the form

$$\varepsilon(1-\varepsilon)^2 = (1-s)s^2 = x \Rightarrow s^2 - s^3 = x. \quad (183)$$

From Eq. (181) it is clear that

$$\varepsilon(\omega = 0) = 0, \quad (184)$$

which immediately implies

$$s(x=0) = 1. \quad (185)$$

One can now try to express $s(x)$ as a power series in x, having the form

$$s(x) = 1 + a_1x^2 + a_2x^3 + a_3x^4 + \cdots. \quad (186)$$

For a series of this form one can use the fact that $s^2(x)$ is of the form [18]

$$s^2(x) = 1 + b_1x + b_2x^2 + b_3x^3 + \cdots, \quad (187)$$

where the coefficients b_n are given by the recursion relation

$$b_n = \frac{1}{n}\sum_{k=1}^{n}(3k-n)a_kb_{n-k}, \quad n \geq 1, \quad b_0 = 1, \quad (188)$$

as well as the fact that $s^3(x)$ is of the form [18]

$$s^3(x) = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots, \quad (189)$$

where the coefficients c_n are given by the recursion relation

$$c_n = \frac{1}{n}\sum_{k=1}^{n}(4k-n)a_kc_{n-k}, \quad n \geq 1, \quad c_0 = 1, \quad (190)$$

the explicit form of the first few coefficients being

$$b_1 = 2a_1, \quad c_1 = 3a_1, \quad (191)$$

$$b_2 = a_1^2 + 2a_2, \quad c_2 = 3(a_1^2 + a_2), \quad (192)$$

$$b_3 = 2(a_1a_2 + a_3), \quad c_3 = a_1^3 + 6a_1a_2 + 3a_3, \quad (193)$$

$$b_4 = a_2^2 + 2a_1a_3 + 2a_4, \quad c_4 = 3(a_1^2a_2 + a_2^2 + 2a_1a_3 + a_4), \quad (194)$$

$$b_5 = 2(a_2a_3 + a_1a_4 + a_5),$$

$$c_5 = 3(a_1a_2^2 + a_1^2a_3 + 2a_2a_3 + 2a_1a_4 + a_5). \quad (195)$$

The coefficients in Eq. (186) can now be determined by considering Eq. (183) written in the form

$$s^2(x) - s^3(x) = d_1x + d_2x^2 + d_3x^3 + \cdots \equiv x, \quad (196)$$

which implies that

$$d_1 = 1 \quad \text{and} \quad d_0 = d_2 = d_3 = \ldots = 0. \quad (197)$$

The first few coefficients in Eq. (196) are then

$$d_1 = 2a_1 - 3a_1 = -a_1 = 1 \Rightarrow a_1 = -1, \quad (198)$$

$$d_2=(a_1^2+2a_2)-3(a_1^2+a_2)=-2-a_2=0\Rightarrow a_2=-2, \tag{199}$$

$$d_3=2(a_1a_2+a_3)-(a_1^3+6a_1a_2+3a_3) =-7-a_3=0\Rightarrow a_3=-7. \tag{200}$$

By this procedure one obtains

$$s(x)=1-x-2x^2-7x^3-30x^4-143x^5-\cdots \tag{201}$$

and

$$\varepsilon(x)=1-s(x)=x+2x^2+7x^3+30x^4+143x^5+\cdots, \tag{202}$$

which, using Eq. (182), takes the form

$$\varepsilon(\omega)=(\omega t)^2+2(\omega t)^4+7(\omega t)^6+30(\omega t)^8+143(\omega t)^{10} +\cdots. \tag{203}$$

It is clear that this expression corresponds to a real root of the cubic equation of Eq. (181), which is of the form

$$\varepsilon^3+f_2\varepsilon^2+f_1\varepsilon+f_0=0, \tag{204}$$

with

$$f_2=-2, \quad f_1=1, \quad f_0=-(\omega t)^2. \tag{205}$$

Using the standard way of solving a cubic equation [18] one has

$$g=\frac{1}{3}f_1-\frac{1}{9}f_2^2=-\frac{1}{9}, \tag{206}$$

$$h=\frac{1}{6}(f_1f_2-3f_0)-\frac{1}{27}f_2^3=\frac{1}{2}(\omega t)^2-\frac{1}{9}, \tag{207}$$

while the discriminant is

$$D=g^3+h^2=\left(\frac{1}{2}(\omega t)^2-\frac{1}{9}\right)^2-\left(\frac{1}{27}\right)^2 =\left(\frac{1}{2}(\omega t)^2-\frac{2}{27}\right)\left(\frac{1}{2}(\omega t)^2-\frac{4}{27}\right). \tag{208}$$

One obtains three real roots when $D<0$ [i.e., when $4/27 \leqslant(\omega t)^2\leqslant 8/27$], while for $D>0$ [i.e., for $(\omega t)^2>8/27$ or for $(\omega t)^2<4/27$] one has only one real root. In the case of rotational spectra it is clear that we are interested in the region including $\omega=0$, i.e., the relevant region is $0\leqslant(\omega t)^2 <4/27$, in which only one real root exists. Using the standard procedure [18] one can write in this case the explicit form of the real root, expand the square and cubic roots appearing there, and verify that the Taylor expansion of the root is of the form given in Eq. (203).

Using Eqs. (179) and (203) one finally obtains the expansion of the energy in terms of powers of ω^2

$$E=\frac{A}{(2\tau)^2}\varepsilon=\frac{A}{(2\tau)^2}((\omega t)^2+2(\omega t)^4+7(\omega t)^6+30(\omega t)^8 +143(\omega t)^{10}+\cdots). \tag{209}$$

On the other hand from Eq. (176) using Eq. (15) one obtains

$$\frac{\mathcal{J}}{\mathcal{J}_0}=\frac{2\tau\xi}{\tanh(2\tau\xi)(1-\tanh^2(2\tau\xi))}=\frac{\operatorname{arctanh}(\sqrt{\varepsilon})}{\sqrt{\varepsilon}(1-\varepsilon)}, \tag{210}$$

where, in the last step, Eq. (179) has been taken into account. In the case of $0<\varepsilon<1$ (which guarantees that the Taylor expansion of $\operatorname{arctanh}(\sqrt{\varepsilon})$ is possible) one can use the expansion [18]

$$\operatorname{arctanh}x=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \quad |x|<1. \tag{211}$$

In addition, the following expansion holds:

$$\frac{1}{1-x^2}=\sum_{n=0}^{\infty}x^{2n}, \quad |x|<1. \tag{212}$$

Using the general result [18] that the series

$$s_1(x)=1+a_1x+a_2x^2+a_3x^3+\cdots \tag{213}$$

and

$$s_2(x)=1+b_1x+b_2x^2+b_3x^3+\cdots \tag{214}$$

can be combined into

$$s_3(x)=s_1(x)s_2(x)=\sum_{n=0}^{\infty}c_nx^n, \tag{215}$$

with

$$c_n=\sum_{k=0}^{n}a_kb_{n-k} \tag{216}$$

from Eqs. (211) and (212) one obtains

$$\frac{\operatorname{arctanh}x}{x}\frac{1}{1-x^2}=\sum_{n=0}^{\infty}\underbrace{\frac{1}{2n+1}}_{a_n}x^{2n}\sum_{n=0}^{\infty}\underbrace{1}_{b_n}x^{2n} =\sum_{n=0}^{\infty}c_nx^{2n}, \quad x^2<1 \tag{217}$$

with

$$c_n=\sum_{k=0}^{n}a_kb_{n-k}=\sum_{k=0}^{n}\frac{1}{2k+1}, \tag{218}$$

TABLE I. Parameter values and quality measure σ [Eq. (226)] for models I [Eq. (17)], I′ [Eq. (65)], II [Eq. (134)], II′ [Eq. (173)], III [Eq. (224)], and IV [Eq. (225)], obtained from least square fits to experimental spectra of Th isotopes (shown in Tables II and III). Data have been taken from Refs. [25] (^{222}Th), [26] (^{224}Th), [27] (^{226}Th), [28] (^{228}Th), [29] (^{230}Th, ^{232}Th, ^{234}Th). The $R_4=E(4)/E(2)$ ratio for each isotope is also shown.

	^{222}Th	^{224}Th	^{226}Th	^{228}Th	^{230}Th	^{232}Th	^{234}Th
R_4	2.399	2.896	3.136	3.235	3.271	3.283	3.308
Model I							
A (keV)	12.577	11.855	10.047	8.873	8.149	7.437	7.845
$10^2\tau$	4.857	5.527	4.701	4.507	3.512	3.141	3.312
σ (keV)	154.213	38.135	26.404	11.601	17.074	26.700	10.839
Model I′							
A (keV)	12.582	11.861	10.052	8.876	8.150	7.438	7.847
$10^2\tau$	4.858	5.528	4.702	4.508	3.512	3.141	3.313
σ (keV)	154.210	38.134	26.403	11.600	17.074	26.700	10.839
Model II							
A (keV)	13.797	12.253	10.289	8.988	8.261	7.559	7.928
$10^2\tau$	2.156	2.229	1.858	1.728	1.351	1.218	1.260
σ (keV)	125.815	32.420	21.631	9.582	13.585	21.724	8.204
Model II′							
A (keV)	13.792	12.247	10.286	8.986	8.260	7.564	7.927
$10^2\tau$	2.155	2.228	1.858	1.727	1.351	1.220	1.260
σ (keV)	125.815	32.420	21.631	9.581	13.585	21.730	8.204
Model III							
A (keV)	11.928	11.602	9.884	8.793	8.067	7.350	7.785
10^2B (keV)	0.703	0.977	0.616	0.525	0.291	0.210	0.253
σ (keV)	173.357	42.528	30.137	13.238	19.912	30.710	13.026
Model IV							
$10^{-2}a$ (keV)	13.812	22.413	30.636	36.853	54.344	58.577	63.701
10^2b	2.909	1.211	0.720	0.505	0.316	0.270	0.256
σ (keV)	53.745	18.244	10.080	4.677	5.216	9.754	2.139

the first few coefficients being

$$c_0=1,\quad c_1=\frac{4}{3},\quad c_2=\frac{23}{15},$$

$$c_3=\frac{176}{105},\quad c_4=\frac{563}{315},\quad c_5=\frac{6508}{3465}. \tag{219}$$

Using Eq. (217) with $x=\varepsilon$ one can put Eq. (210) in the form

$$\frac{\mathcal{J}}{\mathcal{J}_0}=\sum_{n=0}^{\infty} c_n\varepsilon^n, \tag{220}$$

where the coefficients are the ones given in Eqs. (218) and (219). Equation (220) is written analytically as

$$\frac{\mathcal{J}}{\mathcal{J}_0}=1+\frac{4}{3}\varepsilon+\frac{23}{15}\varepsilon^2+\frac{176}{105}\varepsilon^3+\frac{563}{315}\varepsilon^4+\cdots, \tag{221}$$

which can be rewritten with the help of Eq. (209) as

$$\frac{\mathcal{J}}{\mathcal{J}_0}=1+\frac{4}{3}(\omega t)^2+\frac{21}{5}(\omega t)^4+\frac{120}{7}(\omega t)^6+\frac{715}{9}(\omega t)^8+\frac{4368}{11}(\omega t)^{10}+\cdots. \tag{222}$$

Using Eq. (70) one then additionally has

$$\xi=\sqrt{l(l+1)}=\frac{\mathcal{J}\omega}{\hbar}=\frac{\mathcal{J}_0}{\hbar}\omega\left(1+\frac{4}{3}(\omega t)^2+\frac{21}{5}(\omega t)^4+\frac{120}{7}(\omega t)^6+\frac{715}{9}(\omega t)^8+\frac{4368}{11}(\omega t)^{10}+\cdots\right). \tag{223}$$

Equations (209) and (223) give the energy and the quantity $\sqrt{l(l+1)}$ as series in powers of the rotational frequency ω, thus making contact between the present approach and the Harris formalism [8].

XII. NUMERICAL TESTS

The formulas developed in the previous sections will be now tested against the experimental spectra of the Th isotopes [25–29], which range from vibrational [^{222}Th with $R_4=E(4)/E(2)=2.399$] to clearly rotational (^{234}Th with $R_4=3.308$). The purpose of this study is twofold.

(a) To test the quality of the approximations used in Secs. V and X.

(b) To test the agreement between theoretical predictions and experimental data. The standard rotational expansion

$$E=Al(l+1)+B(l(l+1))^2+C(l(l+1))^3+D(l(l+1))^4 +\cdots, \tag{224}$$

from which only the first two terms will be included in order to keep the number of parameters equal to 2, as well as the Holmberg–Lipas two-parameter expression [11]

$$E=a(\sqrt{1+bl(l+1)}-1), \tag{225}$$

which is known to give the best fits to experimental rotational nuclear spectra among all two-parameter expressions [12], will be included in the test for comparison. For brevity we are going to use the following terminology: model I for Eq. (17) (original su$_q(2)$ formula), model I′ for Eq. (65) (''the sinus formula''), model II for Eq. (134) [''the su$_q(2)$ irreducible tensor operator (ITO) formula''], model II′ for Eq. (173) (''the hyperbolic tangent formula''), model III for Eq. (224) (the standand rotational formula), and model IV for Eq. (225) (the Holmberg–Lipas formula).

It should be emphasized at this point that in models I and I′ the deformation parameter is a phase factor ($q=e^{i\tau}$, τ real), while in models II and II′ the deformation parameter is a real number ($q=e^{\tau}$, τ real). A consequence of this fact is the presence of trigonometric functions in models I and I′, while in models II and II′ hyperbolic functions appear.

The parameters resulting from the relevant least square fits, together with the quality measure

$$\sigma=\sqrt{\frac{2}{l_{max}}\sum_{i=2}^{l_{max}}(E_{expt}(l)-E_{th}(l))^2}, \tag{226}$$

TABLE II. Theoretical predictions of models I [Eq. (17)], I′ [Eq. (65)], II [Eq. (134)], II′ [Eq. (173)], III [Eq. (224)], and IV [Eq. (225)], obtained from least square fits to the experimental spectrum (expt.) of ^{232}Th, taken from Ref. [29]. All energies are given in keV. The relevant model parameters and quality measure σ [Eq. (226)] are given in Table I.

				^{232}Th			
l	expt.	I	I′	II	II′	III	IV
2	49.4	44.5	44.5	45.2	45.3	44.0	47.3
4	162.2	147.8	147.8	150.0	150.1	146.2	156.3
6	333.3	308.1	308.1	312.3	312.5	305.0	323.6
8	557.1	523.0	523.0	529.1	529.4	518.3	544.8
10	826.9	789.0	789.0	796.6	797.0	783.0	814.4
12	1136.9	1102.1	1102.1	1110.0	1110.6	1095.4	1126.9
14	1482.3	1457.1	1457.1	1464.2	1464.8	1450.8	1476.7
16	1858.3	1848.6	1848.6	1853.4	1854.0	1843.7	1858.8
18	2261.7	2270.3	2270.3	2271.7	2272.3	2267.9	2268.7
20	2690.5	2715.7	2715.7	2713.1	2713.6	2716.4	2702.3
22	3142.9	3177.6	3177.6	3171.7	3171.9	3181.2	3156.3
24	3618.3	3648.9	3648.9	3641.8	3641.7	3653.7	3627.6
26	4114.9	4122.0	4122.0	4118.0	4117.9	4124.5	4113.9
28	4630.5	4589.5	4589.5	4595.6	4594.6	4583.2	4613.1

TABLE III. Theoretical predictions of models I′ [Eq. (65)], II′ [Eq. (173)], III [Eq. (224)], and IV [Eq. (225)], obtained from least square fits to the experimental spectra (expt.) of ^{222}Th [25] and ^{224}Th [26]. All energies are given in keV. The relevant model parameters and quality measure σ [Eq. (226)] are given in Table I.

			^{222}Th					^{224}Th		
l	expt.	I′	II′	III	IV	expt.	I′	II′	III	IV
2	183.3	75.1	82.1	71.3	115.7	98.1	70.7	72.9	69.3	80.0
4	439.8	247.7	269.1	235.7	356.0	284.1	232.4	238.6	228.1	256.8
6	750.0	511.2	550.4	488.6	677.7	534.7	477.2	487.1	470.1	511.8
8	1093.5	855.7	910.7	822.3	1048.6	833.9	793.2	804.1	784.7	825.5
10	1461.1	1268.3	1332.0	1227.0	1449.6	1173.8	1164.9	1172.8	1158.0	1181.8
12	1850.7	1733.4	1795.0	1689.6	1869.4	1549.8	1574.4	1575.5	1572.1	1568.8
14	2259.7	2233.6	2281.0	2194.8	2301.7	1958.9	2001.6	1995.4	2005.5	1978.1
16	2687.8	2749.9	2773.1	2934.3	2742.5	2398.0	2425.7	2417.4	2432.8	2403.8
18	3133.5	3263.0	3257.1	3257.0	3189.4	2864.0	2826.2	2829.1	2824.9	2841.7
20	3596.0	3753.6	3721.7	3769.5	3640.7					
22	4077.6	4203.2	4158.7	4235.4	4095.4					
24	4577.9	4594.9	4562.9	4625.8	4552.7					
26	5097.9	4914.0	4931.3	4908.8	5012.0					

TABLE IV. Same as Table III, but for ^{226}Th [27], and ^{228}Th [28].

	^{226}Th					^{228}Th				
l	expt.	I′	II′	III	IV	expt.	I′	II′	III	IV
2	72.2	60.0	61.4	59.1	65.5	57.8	53.0	53.7	52.6	55.4
4	226.4	198.1	202.0	195.2	213.3	186.8	175.1	176.9	173.8	181.7
6	447.3	409.3	415.9	404.2	432.9	378.2	362.3	365.1	360.0	372.2
8	721.9	686.1	694.2	679.7	711.9	622.5	608.5	611.6	605.9	618.3
10	1040.3	1018.9	1026.1	1012.6	1038.1	911.8	905.8	907.9	903.7	911.3
12	1395.2	1395.9	1399.5	1391.9	1401.2	1239.4	1244.4	1244.5	1244.0	1242.6
14	1781.5	1803.8	1802.2	1803.8	1792.9	1599.5	1613.5	1611.3	1615.1	1605.2
16	2195.8	2228.1	2221.9	2232.5	2206.9	1988.1	2001.0	1998.3	2003.4	1993.0
18	2635.1	2654.1	2647.9	2659.5	2638.3	2407.9	2394.5	2396.1	2393.4	2401.2
20	3097.1	3066.5	3070.4	3064.2	3083.4					

where l_{max} is the angular momentum of the highest level included in the fit, are listed in Table I, while in Table II the theoretical predictions of all models for ^{222}Th are listed together with the experimental spectrum. Finally in Tables III–V the theoretical predictions of models I′, II′, III, and IV for the rest of the Th isotopes are listed, together with the relevant experimental spectra. From these tables the following observations can be made.

(a) As seen in Tables I and II, models I and I′ give results which are almost identical. The same is true for models II and II′. We therefore conclude that the approximations carried out in Secs. V and X are very accurate. This is the reason that in Tables III–V the results of models I and II are omitted in favor of models I′ and II′.

(b) All models give good results for ^{226}Th-^{234}Th, which lie in the rotational region, with R_4 ratio between 3.136 and 3.308, with model IV giving the best results and model III giving the worst ones, while in all cases models II and II′ are better than models I and I′. It should be noticed, however, that all models tend to underestimate the first several levels of the spectra and the last one or two levels, while they overestimate the rest of the levels. In other words, all models "fail in the same way."

(c) A similar picture holds for the transitional nucleus ^{224}Th ($R_4=2.896$) and the near-vibrational nucleus ^{222}Th ($R_4=2.399$), i.e. still model IV gives the best results and model III the worst, while models II and II′ are better than models I and I′. However, the deviations from the data become much larger, indicating that all these models are inappropriate for describing spectra in the vibrational and transitional regions, in which the presence of a term linear in l is required, as in the u(5) and o(6) limits of the Interacting Boson model [30].

These observations lead to the following conclusions:

(a) One can freely use model I′ in the place of model I, and model II′ in the place of model II, since the relevant approximations turn out to be very accurate. Models I′ and II′ have the advantage of providing simple analytic expressions for the energy, the rotational frequency and the moment of inertia.

(b) The fact that models II and II′ are better than models I and I′ indicates that within the same symmetry [$su_q(2)$ in this case] it is possible to construct different rotational Hamiltonians characterized by different degrees of agreement with the data. However, these Hamiltonians are too "rigid," in the sense that they can describe only rotational

TABLE V. Same as Table III, but for ^{230}Th [29] and ^{234}Th [29].

	^{230}Th					^{234}Th				
l	expt.	I′	II′	III	IV	expt.	I′	II′	III	IV
2	53.2	48.8	49.4	48.3	51.3	49.6	47.0	47.4	46.6	48.7
4	174.0	161.7	163.6	160.2	169.0	164.1	155.8	157.2	154.7	161.1
6	356.5	336.4	340.0	333.7	349.3	337.5	324.5	327.1	322.5	333.8
8	593.9	569.7	574.5	565.8	586.4	565.7	550.3	553.8	547.4	562.4
10	879.6	856.7	862.1	852.2	873.9	843.5	829.0	832.9	825.7	841.5
12	1207.5	1192.0	1196.7	1187.8	1205.4	1165.8	1155.8	1159.2	1152.8	1165.6
14	1572.8	1568.8	1571.6	1565.9	1574.5	1527.6	1525.1	1527.0	1523.1	1529.1
16	1970.7	1979.8	1979.7	1979.2	1975.6	1924.4	1930.3	1930.0	1930.1	1926.8
18	2397.5	2416.8	2413.6	2419.0	2403.9	2352.0	2364.3	2361.7	2366.1	2354.1
20	2849.8	2871.3	2866.3	2875.4	2855.1	2806.1	2819.5	2815.6	2822.7	2806.8
22	3325.2	3334.3	3330.8	3337.7	3325.7	3282.4	3288.0	3285.4	3290.5	3281.4
24	3820.2	3796.6	3800.8	3793.7	3812.7	3776.1	3761.5	3765.0	3758.8	3774.8

spectra, while vibrational and transitional spectra are outide their realm.

Some additional comments on the convergence of the various expansions can be made by considering the quantities [31]

$$r_1=\frac{\frac{C}{A}}{\left(\frac{B}{A}\right)^2}=\frac{AC}{B^2},\quad r_2=\frac{\frac{D}{A}}{\left(\frac{B}{A}\right)^3}=\frac{A^2D}{B^3},\tag{227}$$

which refer to the coefficients of the expansion of Eq. (224). Keeping only the first two terms in the Harris formalism for the energy and the moment of inertia leads to the values [31]

$$r_1^{Harris}=4,\quad r_2^{Harris}=24.\tag{228}$$

From Eq. (62) we obtain, for model I′,

$$r_1^{I'}=\frac{2}{5},\quad r_2^{I'}=\frac{3}{35},\tag{229}$$

while from Eq. (174) we obtain, for model II′,

$$r_1^{II'}=\frac{17}{20},\quad r_2^{II'}=\frac{93}{140}.\tag{230}$$

The Taylor expansion of the Holmberg-Lipas formula [Eq. (225)] reads

$$\frac{E}{a}=\frac{b}{2}l(l+1)-\frac{b^2}{8}(l(l+1))^2+\frac{b^3}{16}(l(l+1))^3-\frac{5b^4}{128}(l(l+1))^4+\cdots,\tag{231}$$

from which one obtains

$$r_1^{IV}=2,\quad r_2^{IV}=5.\tag{232}$$

We observe that for models I′, II′, and IV the quality of the fits is improved as the values of the ratios r_1 and r_2 become larger.

Finally, a word of warning: One could think of fitting the experimental spectra by Eq. (224), keeping the first four terms in the expansion, and then trying to use the parameter values obtained from fitting several nuclei in order to determine ''optimal'' values for the ratios r_1 and r_2 from Eq. (227) as a function of the mass number. This procedure, however, is very unsafe, since the values of the parameters C and especially D obtained from the fits are very unstable.

XIII. DISCUSSION

The main results of the present work are the following.

(a) The rotational invariance of the original su$_q$(2) Hamiltonian [5,6] under the usual physical angular momentum has been proved explicitly and its connections to the formalisms of Amal'sky [7] (''the sinus formula'') and Harris [8] have been given.

(b) An irreducible tensor operator (ITO) of rank 1 under su$_q$(2) has been found and used, through q-deformed tensor product and q-deformed Clebsch-Gordan coefficient techniques [9,10,22,23], for the construction of a Hamiltonian appropriate for the description of rotational spectra, the su$_q$(2) ITO Hamiltonian. The rotational invariance of this new Hamiltonian under the usual physical angular momentum has been proved explicitly. Furthermore, an approximate simple closed expression (''the hyperbolic tangent formula'') for the energy spectrum of this Hamiltonian has been found and its connection to the Harris [8] formalism has been demonstrated.

From the results of the present work it is clear that the su$_q$(2) Hamiltonian, as well as the su$_q$(2) ITO Hamiltonian, are complicated functions of the Casimir operator of the usual su(2), i.e. of the square of the usual physical angular momentum. These complicated functions possess su$_q$(2) symmetry, in addition to the usual su(2) symmetry. Matrix elements of these functions can be readily calculated in the deformed basis, but also in the usual physical basis. A similar study of a q-deformed quadrupole operator is called for. This operator would allow the study of multi-band spectra, in analogy to the Elliott model [32], as well as the study of B(E2) transition probabilities. Since q deformation appears to describe well the stretching effect of rotational nuclear spectra, ir is interesting to check what its influence on the corresponding B(E2) transition probabilities will be. Work in this direction is in progress.

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