Algebraic solutions of mean-field plus $T=1$ pairing interaction

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A general procedure, based on the Bethe ansatz, is proposed for finding algebraic solutions for low-lying $J=0$ states of $2k$ nucleons interacting with one another through a $T=1$ charge-independent pairing interaction. The results provided by Richardson are shown to be valid for up to two pairs, $k\leqslant 2$; expressions are given here for up to three pairs, $k\leqslant 3$. The results show that a set of highly nonlinear equations must be solved for $k\geqslant 3$.

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I. INTRODUCTION

Pairing has long been considered to be an important interaction in nuclei. The concept was first introduced by Racah within the context of a seniority coupling scheme [1]. Various applications to realistic nuclear systems have been carried out [2] following suggestions from Bohr, Mottelson, and Pines [3]. A lot of effort has been dedicated to the pure neutron or pure proton pairing interactions using various techniques. Extensions to neutron-neutron, neutron-proton, and proton-proton pairing interactions have been formulated [4–7]. It is well known that the $T=1$ charge-independent pairing Hamiltonian can be built by using generators of the quasispin group $\mathrm{Sp}_j(4)$, where j labels the orbits considered in the model space, and from this it also follows that the pairing Hamiltonian can be diagonalized within a given irreducible representation (irrep) of the direct product group $\mathrm{Sp}_1(4)\times\cdots\times\mathrm{Sp}_p(4)$, where p is the number of orbits. In this case exact solutions—even if only generated numerically—can be given [8]. It is also well known that approximate numerical solutions can be obtained by using the BCS formalism [9–11].

A lot of effort has been devoted to finding exact analytic solutions of the nuclear pairing Hamiltonian [12–15]. Extensions to a consideration of generalized and orbit-dependent pairing interactions have been the focus of recent work based on the algebraic Bethe ansatz and infinite-dimensional Lie algebraic methods [16–19]. A method for finding roots of the Bethe ansatz equations for the equal strength pairing model that was solved earlier by Richardson has also been proposed [20]. However, these exact solutions are for proton-proton or neutron-neutron pairing interactions only. In this paper, exact solutions for the mean field plus $T=1$ charge-independent equal strength pairing interaction are revisited using the Bethe ansatz method. We find that the solutions offered by Richardson [21] and Chen and Richardson [22,23] are only valid when the number of pairs is less than or equal to two.

In this paper we introduce a new formulism for solving the problem. Numerical routines for calculations and follow-on applications will be introduced elsewhere. In Sec. II, the mean field plus $T=1$ pairing Hamiltonian and its Sp(4) quasispin structure are reviewed. In Sec. III, a general procedure for solving the $T=1$ charge-independent pairing Hamiltonian is outlined and detailed results for seniority-zero states for up to six nucleon are provided in Sec. IV. The results show that a set of highly nonlinear equations will enter whenever the number of nucleons is greater than or equal to six. Section V is reserved for a short discussion regarding implications of our findings.

II. THE $T=1$ PAIRING HAMILTONIAN AND THE Sp(4) QUASISPIN STRUCTURE

It is well known [8] that the mean field plus $T=1$ charge independent pairing Hamiltonian can be expressed in terms of generators of quasispin groups $\mathrm{Sp}_j(4)$, where j labels the total spin of the corresponding orbits. Generators of $\mathrm{Sp}_j(4)$ are the pair creation, $A_j^\dagger(\mu)$, and the pair annihilation, $A_j(\mu)$, operators with $\mu=+,-,0$; the total nucleon number operator N_j for orbit j; and the isospin operators $T_\mu(j)$:

$$A_j^\dagger(\mu)=\sum_{m>0}(-1)^{j-m}a_{jm,\mu}^\dagger a_{j-m,\mu}^\dagger \quad \text{for} \quad \mu=+,-, \tag{2.1a}$$

$$A_j^\dagger(0)=\sqrt{\frac{1}{2}}\left\{\sum_{m>0}(-1)^{j-m}a_{jm,+}^\dagger a_{j-m,-}^\dagger + \sum_{m>0}(-1)^{j-m}a_{jm,-}^\dagger a_{j-m,+}^\dagger\right\}, \tag{2.1b}$$

$$A_j(\mu)=\sum_{m>0}(-1)^{j-m}a_{j-m,\mu}a_{jm,\mu} \quad \text{for} \quad \mu=+,-, \tag{2.2a}$$

$$A_j(0)=\sqrt{\frac{1}{2}}\left\{\sum_{m>0}(-1)^{j-m}a_{j-m,-}a_{jm,+} + \sum_{m>0}(-1)^{j-m}a_{j-m,+}a_{jm,-}\right\}, \tag{2.2b}$$

$$N_j=\sum_{mm_t}a_{jm,m_t}^\dagger a_{jm,m_t}, \quad T_+(j)=\sum_m a_{jm,+}^\dagger a_{jm,-},$$

$$T_-(j)=\sum_m a_{jm,-}^\dagger a_{jm,+},$$

$$T_0(j)=\frac{1}{2}\sum_m (a^\dagger_{jm,+}a_{jm,+}-a^\dagger_{jm,-}a_{jm,-}), \quad (2.3)$$

where $a^\dagger_{jmm_t}$ (a_{jmm_t}) is the creation (annihilation) operator for a nucleon in the state with angular momentum j, angular momentum projection m, and isospin projection m_t with $m_t=+\frac{1}{2}, -\frac{1}{2}$. These operators satisfy the following commutation relations:

$$[T_-(j),A^\dagger_{j'}(-)]=0, \quad [T_+(j),A^\dagger_{j'}(+)]=0,$$

$$[T_-(j),A^\dagger_{j'}(0)]=\delta_{jj'}\sqrt{2}A^\dagger_j(-),$$

$$[T_+(j),A^\dagger_{j'}(0)]=-\delta_{jj'}\sqrt{2}A^\dagger_j(+),$$

$$[T_-(j),A^\dagger_{j'}(+)]=-\delta_{jj'}\sqrt{2}A^\dagger_j(0),$$

$$[T_+(j),A^\dagger_{j'}(-)]=\delta_{jj'}\sqrt{2}A^\dagger_j(0),$$

$$[\Omega_j-\tfrac{1}{2}N_j-T_0(j),A^\dagger_{j'}(-)]=0,$$

$$[A_j(0),A^\dagger_{j'}(-)]=-\delta_{jj'}\frac{1}{\sqrt{2}}T_-(j),$$

$$[A_j(+),A^\dagger_{j'}(+)]=\delta_{jj'}\left(\Omega_j-\frac{1}{2}N_j-T_0(j)\right),$$

$$[A_j(-),A^\dagger_{j'}(+)]=0,$$

$$[A_j(0),A^\dagger_{j'}(+)]=\delta_{jj'}\frac{1}{\sqrt{2}}T_+(j),$$

$$[A_j(+),A^\dagger_{j'}(0)]=\delta_{jj'}\frac{1}{\sqrt{2}}T_-(j),$$

$$[A_j(-),A^\dagger_{j'}(-)]=\delta_{jj'}\left(T_0(j)+\Omega_j-\frac{1}{2}N_j\right),$$

$$[A_j(0),A^\dagger_{j'}(0)]=\delta_{jj'}\left(\Omega_j-\frac{1}{2}N_j\right), \quad [A_j(+),A^\dagger_{j'}(-)]=0,$$

$$[A_j(-),A^\dagger_{j'}(0)]=-\delta_{jj'}\frac{1}{\sqrt{2}}T_+(j),$$

$$[\Omega_j-\tfrac{1}{2}N_j,A^\dagger_{j'}(+)]=-\delta_{jj'}A^\dagger_j(+),$$

$$[\Omega_j-\tfrac{1}{2}N_j,A^\dagger_{j'}(0)]=-\delta_{jj'}A^\dagger_j(0),$$

$$[\Omega_j-\tfrac{1}{2}N_j,A^\dagger_{j'}(-)]=-\delta_{jj'}A^\dagger_j(-),$$

$$[\Omega_j-\tfrac{1}{2}N_j+T_0(j),A^\dagger_{j'}(+)]=0,$$

$$[\Omega_j-\tfrac{1}{2}N_j+T_0(j),A^\dagger_{j'}(0)]=-\delta_{jj'}A^\dagger_j(0),$$

$$[\Omega_j-\tfrac{1}{2}N_j+T_0(j),A^\dagger_{j'}(-)]=-\delta_{jj'}2A^\dagger_j(-),$$

$$[\Omega_j-\tfrac{1}{2}N_j-T_0(j),A^\dagger_{j'}(+)]=-\delta_{jj'}2A^\dagger_j(+),$$

$$[\Omega_j-\tfrac{1}{2}N_j-T_0(j),A^\dagger_{j'}(0)]=-\delta_{jj'}A^\dagger_j(0), \quad (2.4)$$

where $\Omega_j\equiv j+\frac{1}{2}$ is the pair degeneracy of orbit j. According to the Wigner-Eckart theorem, the pair creation operators $\mathcal{A}^\dagger_j(\mu)$ with $\{\mathcal{A}^\dagger_j(+)=-A^\dagger_j(+),\mathcal{A}^\dagger_j(0)=A^\dagger_j(0),\mathcal{A}^\dagger_j(-)=A^\dagger_j(-)\}$ and the pair annihilation operators $\mathcal{A}_j(\mu)$ with $\{\mathcal{A}_j(+)=A_j(-),\mathcal{A}_j(0)=-A_j(0),\mathcal{A}_j(-)=-A_j(+)\}$ are $T=1$ irreducible tensor operators, that satisfy the following conjugation relation:

$$\mathcal{A}_j(\mu)=(-1)^{1-\mu}[\mathcal{A}^+_j(-\mu)]^\dagger. \quad (2.5)$$

The mean field, with single-particle energies ε_j from the spherical shell model, plus $T=1$ charge-independent pairing interaction Hamiltonian can be expressed as

$$\hat{H}=\sum_j \varepsilon_j N_j-G\sum_{jj'\mu}A^\dagger_j(\mu)A_{j'}(\mu), \quad (2.6)$$

where $G>0$ is the overall pairing interaction strength. Since $\hat{H}$ is invariant under isospin rotation, both the isospin quantum number T and its third component T_0 with eigenvalue M_T are good quantum numbers of the system.

III. THE ALGEBRAIC BETHE ANSATZ METHOD

Let $\{g_i(\alpha)\}\in G_\alpha$, where G_α is a semisimple Lie algebra, satisfying

$$[g_i(\alpha),g_j(\beta)]=\delta_{\alpha\beta}\sum_k c^k_{ij}g_k(\beta) \quad (3.1)$$

for $i,j=1,2,\ldots,\dim(G)$, and $\alpha,\beta=1,2,\ldots,p$, where p is the total number of the orbits considered in the problem, and the c^k_{ij} are the structure constants of G. Suppose G can be decomposed into three gradings with

$$G_\alpha=h^\alpha\oplus n^\alpha_+\oplus n^\alpha_-, \quad (3.2)$$

where h^α is the subalgebra of G_α containing the Cartan subalgebra of G_α, and elements in n^α_+ and those in n^α_- satisfy the following relations:

$$[h_i(\alpha),h_j(\beta)]=\delta_{\alpha\beta}\sum_k c^k_{ij}h_k(\alpha)\forall h_i(\alpha)\in h^\alpha, \quad h_j(\beta)\in h^\beta, \quad (3.3a)$$

$$[\mathbf{h}(\alpha),\mathbf{A}(\beta)]\propto\delta_{\alpha\beta}\mathbf{A}(\alpha)\forall\mathbf{h}(\alpha)\in h^\alpha, \quad \mathbf{A}(\beta)\in n^\beta_+, \quad (3.3b)$$

$$[\mathbf{h}(\alpha),\mathbf{B}(\beta)]\propto\delta_{\alpha\beta}\mathbf{B}(\alpha)\forall\mathbf{h}(\alpha)\in h^\alpha, \quad \mathbf{B}(\beta)\in n^\beta_-, \quad (3.3c)$$

$$[\mathbf{A}(\alpha),\mathbf{B}(\beta)]\propto\delta_{\alpha\beta}\mathbf{h}(\alpha)\forall\mathbf{A}(\beta)\in n^\alpha_+, \quad \mathbf{B}(\beta)\in n^\beta_-. \quad (3.3d)$$

A class of Hamiltonians with up to two-body interactions and p orbits may be written in terms of these $G_1\otimes G_2\otimes\cdots\otimes G_p$ generators as

$$\hat{H}=\sum_{i\alpha} \epsilon_\alpha h_i^0(\alpha)-G\sum_{\alpha\beta} (\mathbf{A}(\alpha)\times\mathbf{B}(\beta))_0^{(0)}, \quad (3.4)$$

where $h_i^0(\alpha)$ are elements of the Cartan subalgebra of G_α, and the term $(\mathbf{A}(\alpha)\times\mathbf{B}(\beta))_0^{(0)}$, which should be an invariant of the Cartan subalgebra, is a scalar with respect to the subalgebra h. Then, one can express a k-particle excitation state as

$$|\zeta;[\lambda]_k M,\nu\eta\rangle=\sum_{P\in S_k} Q^{[\lambda]}(x_{P(1)},x_{P(2)},\cdots,x_{P(k)})(\mathbf{A}(x_{P(1)})\times\mathbf{A}(x_{P(2)})\times\cdots\times\mathbf{A}(x_{P(k)})\times|\mathrm{lw}\rangle)_\eta^{(\nu)}, \quad (3.5)$$

where ζ represents additional quantum numbers that are required for a unique labeling of the states, ν is an irrep of the subalgebra h, η denotes a complete set of labels for the irrep ν, $[\lambda]_k$ is an irrep of the permutation group S_k containing k-boxes in the corresponding Young diagram with M labeling the basis of $[\lambda]$, the x_i $(i=1,2,\ldots,k)$ on the right-hand-side are spectral parameters, the $Q^{[\lambda]}(x_1,x_2,\ldots,x_k)$ are expansion coefficients that may in general be spectral parameter dependent, $|\mathrm{lw}\rangle$ is the lowest-weight state satisfying

$$\mathbf{B}(\alpha)|\mathrm{lw}\rangle=\mathbf{0}\forall\alpha, \quad (3.6)$$

and $\mathbf{A}(x)$ is an operator functional of $\mathbf{A}(\alpha)$ that is dependent on the spectral parameter x.

To determine the operator functional $\mathbf{A}(x)$, one can construct the following affine Lie algebra $\hat{G}$ without central extension [16–19]:

$$g_n^i=\sum_\alpha \epsilon_\alpha^n g_i(\alpha), \quad g_i(\alpha)\in G_\alpha, \quad (3.7)$$

with $n=0,1,2,\ldots,$ which satisfy

$$[g_m^i,g_n^j]=\sum_k c_{ij}^k g_{m+n}^k. \quad (3.8)$$

Then, one can rewrite Eq. (3.4) in terms of generators of $\hat{G}$ as

$$\hat{H}=\sum_i h_1^{(0)i}-G(\mathbf{A}_0\times\mathbf{B}_0)_0^0. \quad (3.9)$$

The k-particle wave functions can be expanded in terms of these operators as

$$|\zeta;[\lambda]_k M,\nu\eta\rangle=\sum_{P\in S_k} Q^{[\lambda]}(x_{P(1)},x_{P(2)},\cdots,x_{P(k)})\times\sum_{n_1\cdots n_k} a_{n_1}a_{n_2}\cdots a_{n_k}x_{P(1)}^{n_1}x_{P(2)}^{n_2}\cdots x_{P(k)}^{n_k}\times(\mathbf{A}_{n_1}\times\mathbf{A}_{n_2}\times\cdots\times\mathbf{A}_{n_k}\times|\mathrm{lw}\rangle)_\eta^{(\nu)}, \quad (3.10)$$

where

$$a_{n_i}\mathbf{A}_{n_i}=\frac{1}{2\pi i}\oint_0 x^{n_i}\mathbf{A}(x)dx \quad (3.11)$$

is the Fourier-Laurent expansion of $\mathbf{A}(x)$ around $|x|=0$ on a complex plane. The spectral parameters x_i and the expansion coefficients $Q^{[\lambda]}(x_1,x_2,\ldots,x_k)$ should then be determined by the corresponding eigenvalue equation.

It can be seen that the $T=1$ pairing Hamiltonian (2.6) can be regarded as a special example of Eq. (3.4), in which the Lie algebra G is Sp(4) and the subalgebra of Sp(4) is $\mathrm{U}(2)\sim\mathrm{SU}_T(2)\oplus\mathrm{U}(1)$, where $\mathrm{SU}_T(2)$ is the isospin algebra, and U(1) is generated by the total number of nucleons N. In this paper, we only consider seniority-zero states. Hence, the lowest-weight state is an isospin scalar. Similar to Eq. (3.5), a k pair excitation eigenstate can be written as

$$|\zeta;[\lambda]_k M,TT_0\rangle=\sum_{P\in S_k} Q^{[\lambda]}(x_{P(1)},x_{P(2)},\cdots,x_{P(k)})\times(\mathcal{A}^\dagger(x_{P(1)})\times\mathcal{A}^\dagger(x_{P(2)})\times\cdots\times\mathcal{A}^\dagger(x_{P(k)})_{T_0}^T)|0\rangle, \quad (3.12)$$

where $|0\rangle$ is the seniority-zero and isospin scalar state satisfying

$$A_j(\mu)|0\rangle=0 \quad \text{for} \quad \mu=+,-,0, \quad (3.13)$$

and $[\lambda]$ is an irrep of S_k that can be constructed from the $\mathcal{A}_j^\dagger$ operators of Eq. (3.12).

It has been confirmed in exact solutions of the equal strength pairing problem with only neutron-neutron or proton-proton pairing interaction [12–19] that the building blocks $A_\mu^\dagger(x)$ can be expressed as elements of the nonlinear Gaudin algebra $\mathcal{G}(\mathrm{SU}(2))$ with

$$A_\mu^\dagger(x)=\sum_j \frac{A_j^\dagger(\mu)}{1-\varepsilon_j x} \quad \text{for} \quad \mu=+,-. \quad (3.14)$$

It suffices to use the nonlinear Gaudin algebra $\mathcal{G}(\mathrm{Sp}(4))$ to construct the eigenstates (3.12), which is generated by

$$g_\mu(x)=\sum_{j=1}^p \frac{1}{1-\varepsilon_j x}g_j(\mu), \quad (3.15)$$

where p is the total number of orbits, $g_j(\mu)$ are the $\mathrm{Sp}_j(4)$ generators, and ε_j is the single-particle energy of the jth orbit.

It should be noted that the possible irreps $[\lambda]$ occuring in Eq. (3.12) should be determined by properties of the $A^\dagger(\mu)$ operators. Because μ can only take on three different values, $\mu=+,-,0$, Young diagrams constructed from those $A_\mu^\dagger$ operators can have at most three rows. Furthermore, because the Schur-Weyl duality relation between the permutation group S_k and the unitary group $\mathrm{U}(N)$, the irrep $[\lambda]$ with exact k boxes of S_k can be regarded as the same irrep of $\mathrm{U}(N)$. Since the irreps $[\lambda]$ contain at most three rows, in this case they can be considered to be equivalent to the same irreps of U(3). Therefore, the possible isospin quantum number T for a given irrep $[\lambda]$ of S_k can be obtained by the reduction [23]

$$U(3)\supset SO(3),$$
$$[\lambda]\downarrow T. \tag{3.16}$$

The remaining problem is to find the expansion coefficients $Q^{[\lambda]}(x_1,x_2,\ldots,x_k)$ and to establish the Bethe ansatz equations based on the corresponding eigenvalue equation, which will be addressed in the next section.

IV. EXACT SOLUTIONS FOR UP TO SIX NUCLEONS

In this section, we use the algebraic Bethe ansatz method established in the preceding section to derive exact solutions for the mean field plus $T=1$ charge-independent equal strength pairing Hamiltonian problem. The following elements of the Gaudin algebra $\mathcal{G}(\mathrm{Sp}(4))$ will be useful:

$$A^\dagger_\mu(x)=\sum_j \frac{A^\dagger_j(\mu)}{1-\varepsilon_j x},\quad A_\mu(x)=\sum_j \frac{A_j(\mu)}{1-\varepsilon_j x},$$

$$T_\mu(x)=\sum_j \frac{T_\mu(j)}{1-\varepsilon_j x} \tag{4.1a}$$

for $\mu=+,-,0$, and

$$N(x)=\sum_j \frac{N_j}{1-\varepsilon_j x}. \tag{4.1b}$$

Then, the Hamiltonian (2.6) can be rewritten as

$$\hat{H}=\frac{\partial N(x)}{\partial x}\Big|_{x=0}+G\mathcal{A}^\dagger(0)\cdot\mathcal{A}(0). \tag{4.2}$$

Solving the eigenvalue equation

$$\hat{H}|\zeta;[\lambda]_k M,TT_0\rangle=E_\zeta^{[\lambda]_k T}|\zeta;[\lambda]_k M,TT_0\rangle \tag{4.3}$$

with the Bethe ansatz wave function (3.12) implies that one simultaneously determines the expansion coefficients $Q^{[\lambda]}(x_1,x_2,\ldots,x_k)$ and the Bethe ansatz equations that the spectral parameters $x_1,x_2,\ldots,x_k$, should satisfy. In the following, we will list exact solutions for $k\leqslant 3$.

The $k=1$ *case*. Since the $k=0$ case is trivial, corresponding to a zero eigenvalue with the seniority zero eigenstate $|0\rangle$, our derivation starts with $k=1$. In this case, the eigenstate can be written, up to a constant, as

$$|\zeta;[1],T=1,\ M_T=\mu\rangle=\mathcal{A}^\dagger_\mu(x^{(\zeta)})|0\rangle. \tag{4.4}$$

It follows that the expansion coefficient $Q^{[1]}(x)$ can be taken simply as a constant. Using the commutation relations

$$\left[\frac{\partial N(x)}{\partial x}\Big|_{x=0},\mathcal{A}^\dagger_\mu(x)\right]=\frac{2}{x}\mathcal{A}^\dagger_\mu(x)-\frac{2}{x}\mathcal{A}^\dagger_\mu(0) \tag{4.5}$$

and

$$[\hat{H},\mathcal{A}^\dagger_\mu(x)]=\frac{2}{x}\mathcal{A}^\dagger_\mu(x)-\mathcal{A}^\dagger_\mu(0)\left(\frac{2}{x}+G\left(\Omega(x)-\frac{1}{2}N(x)\right)\right)+G(\mathcal{A}^\dagger(0)\times\mathcal{T}(x))^1_\mu, \tag{4.6}$$

where $(\mathcal{A}^\dagger(0)\times\mathcal{T}(x))^1_\mu$ are defined as follows:

$$(\mathcal{A}^\dagger(0)\times\mathcal{T}(x))^1_\mu=\sqrt{2}\sum_{\mu_1\mu_2}\mathcal{A}^\dagger_{\mu_1}(0)\mathcal{T}_{\mu_2}(x)\langle 1\mu_1,1\mu_2|1\mu\rangle, \tag{4.7}$$

$\langle 1\mu_1,1\mu_2|1\mu\rangle$ is the SU(2) CG coefficient, the $\mathrm{SU}_T(2)$ rank-1 tensor operator $\mathcal{T}$ with $\mathcal{T}_+=-\sqrt{\tfrac{1}{2}}\,T_+$, $\mathcal{T}_0=T_0$, $\mathcal{T}_-=\sqrt{\tfrac{1}{2}}T_-$, and

$$\Omega(x)=\sum_j\frac{\Omega_j}{1-\varepsilon_j x}. \tag{4.8}$$

One can easily derive that the eigenvalue is given by

$$E_\zeta^{[1]T=1}=\frac{2}{x^{(\zeta)}}, \tag{4.9}$$

where the spectral parameter $x^{(\zeta)}$ satisfies

$$\frac{2}{x^{(\zeta)}}+G\Omega(x^{(\zeta)})=0. \tag{4.10}$$

The additional quantum number ζ in this case indicates the solution $x^{(\zeta)}$ is the ζth root of Eq. (4.10).

The $k=2$ *case*. In this case, there are two irreps of S_2 with [2,0,0] and [1,1,0]. The allowed values of T are $T=2$ and $T=0$ for [2,0,0], and $T=1$ for [1,1,0]. Using the operators $\mathcal{A}^\dagger_\mu(x)$, one can construct symmetric and antisymmetric operators with respect to the spectral parameter permutation $x_1\leftrightarrow x_2$. The symmetric ones are

$$B^{T=2}_{M_T=2}(x_1,x_2)=A^\dagger_+(x_1)A^\dagger_+(x_2), \tag{4.11a}$$

and

$$B^{T=0}(x_1,x_2)=-\mathcal{A}^\dagger(x_1)\cdot\mathcal{A}^\dagger(x_2)=A^\dagger_+(x_1)A^\dagger_-(x_2)+A^\dagger_-(x_1)A^\dagger_+(x_2)+A^\dagger_0(x_1)A^\dagger_0(x_2). \tag{4.11b}$$

The antisymmetric one is

$$P^{T=1}_\mu(x_1,x_2)=\sqrt{2}(\mathcal{A}^\dagger(x_1)\times\mathcal{A}^\dagger(x_2))^1_\mu, \tag{4.12}$$

which is similar to the definition given in Eq. (4.7). Using these operators, one can easily obtain the corresponding wave function for the $k=2$ cases. It can be proven that the eigenenergies for both symmetric and antisymmetric case are all given by

$$E_\zeta^{[\lambda]_2 T}=\frac{2}{x_1^{(\zeta)}}+\frac{2}{x_2^{(\zeta)}}. \tag{4.13}$$

For [2,0,0] and $T=2$, spectral parameters x_1 and x_2 should satisfy

$$\frac{2}{x_i^{(\zeta)}}+G\Omega(x_i^{(\zeta)})+G\frac{2x_j^{(\zeta)}}{x_i^{(\zeta)}-x_j^{(\zeta)}}=0, \tag{4.14}$$

where $i=1,2$ with $j=2,1$. Up to a constant, the corresponding wavefunction can be expressed as

$$|\zeta;[2],\ T=2,\ M_T=2\rangle=A_+^\dagger(x_1^{(\zeta)})A_+^\dagger(x_2^{(\zeta)})|0\rangle. \tag{4.15}$$

Wave functions with $M_T\neq T$ can be obtained by applying $T_-=\Sigma_j T_-(j)$ consecutively on Eq. (4.15) because the total spin T is conserved. For [2,0,0] and $T=0$, by using the commutation relation

$$\begin{aligned}[\hat{H},B^0(x_1,x_2)]=&\sum_i \frac{2}{x_i}B^0(x_1,x_2)-\sum_{i\neq j}B^0(0,x_j)\\ &\times\left(\frac{2}{x_i}+G\left(\Omega(x_i)-N(x_i)/2-\frac{x_j}{x_i-x_j}\right)\right)\\ &+G(\mathcal{A}^\dagger(0)\cdot[\mathcal{A}^\dagger(x_1)\times\mathcal{T}(x_2)]\\ &+\mathcal{A}^\dagger(0)\cdot[\mathcal{A}^\dagger(x_2)\times\mathcal{T}(x_1)]),\end{aligned} \tag{4.16}$$

it can be proven that the spectral parameters $x_1^{(\zeta)}$ and $x_2^{(\zeta)}$ should satisfy

$$\begin{aligned}&\frac{2}{x_1^{(\zeta)}}+G\left(\Omega(x_1^{(\zeta)})-\frac{x_2^{(\zeta)}}{x_1^{(\zeta)}-x_2^{(\zeta)}}\right)=0,\\ &\frac{2}{x_2^{(\zeta)}}+G\left(\Omega(x_2^{(\zeta)})-\frac{x_1^{(\zeta)}}{x_2^{(\zeta)}-x_1^{(\zeta)}}\right)=0.\end{aligned} \tag{4.17}$$

Up to a constant the corresponding wave function is

$$|\zeta;[2,0,0],\ \ T=0,\ \ M_T=0\rangle=B^0(x_1,x_2)|0\rangle. \tag{4.18}$$

For the antisymmetric case with [1,1,0] and $T=1$, using the commutation relation

$$\begin{aligned}[\hat{H},P_\mu^1(x_1,x_2)]=&\left(\frac{2}{x_1}+\frac{2}{x_2}\right)P_\mu^1(x_1,x_2)-P_\mu^1(x_1,0)\\ &\times\left(\frac{2}{x_2}+G(\Omega(x_2)-\frac{1}{2}N(x_2))\right)-P_\mu^1(0,x_2)\\ &\times\left(\frac{2}{x_1}+G(\Omega(x_1)-\frac{1}{2}N(x_1))\right)+G\mathcal{A}_\mu^\dagger(0)\\ &\times[\mathcal{A}^\dagger(x_1)\cdot\mathcal{T}(x_2)-\mathcal{A}^\dagger(x_2)\cdot\mathcal{T}(x_1)]\\ &+G(B^0(x_1,0)\mathcal{T}_\mu(x_2)-B^0(0,x_2)\mathcal{T}_\mu(x_1)),\end{aligned} \tag{4.19}$$

one can easily prove that the parameters $x_1^{(\zeta)}$ and $x_2^{(\zeta)}$ should satisfy

$$\frac{2}{x_1^{(\zeta)}}+G\Omega(x_1^{(\zeta)})=0,\quad \frac{2}{x_2^{(\zeta)}}+G\Omega(x_2^{(\zeta)})=0. \tag{4.20}$$

The solutions should be valid only for $x_1^{(\zeta)}\neq x_2^{(\zeta)}$ because the wave fucntion is totally antisymmetric with respect to a spectral parameter permutation, which, up to a constant, can be written as

$$|\zeta;[1,1,0],T=1,M_T=1\rangle=P_1^1(x_1^{(\zeta)},x_2^{(\zeta)})|0\rangle. \tag{4.21}$$

The $k=3$ case. For $k=3$, we need to consider all terms occurring in Eq. (3.12) due to the permutations involved. Using the building blocks $\mathcal{A}^\dagger(x)$, one can construct the Bethe ansatz wave function (3.12) for different irreps of the permutation group S_3 by using the induced representation method [24,25]. Therefore, the wave functions for the symmetric $k=3$ and $T=1$ case should be written as

$$\begin{aligned}|\zeta;[3,0,0],\ \ T=1,M_T=1\rangle=&(1+g_2+g_1g_2)Q^{[3]}\\ &\times(x_1x_2;x_3)B^0(x_1x_2)A_+^\dagger(x_3)|0\rangle,\end{aligned} \tag{4.22}$$

where g_i ($i=1,2$) are generators of S_3, which are nothing but nearest-neighbor permutations defined by $g_i=(i,i+1)$ for $i=1,2,\ldots,k-1$. It is obvious that x_1 and x_2 in the primitive vector $B^0(x_1x_2)$ are symmetric with respect to the $x_1\leftrightarrow x_2$ permutation. Up to a constant, the coefficients in $(1+g_2+g_1g_2)$ are taken from the induction coefficients [24,25] (IDCs) of $S_2\times S_1\downarrow S_3$ for the coupling $[2]\otimes[1]\downarrow[3]$. It should be emphasized that, generally, $Q^{[3]}(x_1x_2;x_3)\neq Q^{[3]}(x_1x_3;x_2)\neq Q^{[3]}(x_2x_3;x_1)$, where

$$\begin{aligned}Q^{[3]}(x_1x_3;x_2)&=g_2Q^{[3]}(x_1x_2;x_3),\\ Q^{[3]}(x_2x_3;x_1)&=g_1g_2Q^{[3]}(x_1x_2;x_3).\end{aligned} \tag{4.23}$$

For the symmetric $k=3$ case with $T=3$ rather than $T=1$, the wave function can be written as

$$|\zeta;[3,0,0],\ T=3,M_T=3\rangle=A_+^\dagger(x_1)A_+^\dagger(x_2)A_+^\dagger(x_3)|0\rangle \tag{4.24}$$

because it has to be symmetric with respect to any permutaton of the spectral parameters.

Solutions for the spectral parameters in this stretched $T=3$ case are the same as for the neutron-neutron or proton-proton pairing problem derived previously [12–15], which are given by

$$\frac{2}{x_i^{(\zeta)}}+G\Omega(x_i^{(\zeta)})+G\sum_{j\neq i}\frac{2x_j^{(\zeta)}}{x_i^{(\zeta)}-x_j^{(\zeta)}}=0 \tag{4.25}$$

for $i=1,2,3$. The corresponding eigenenergy is given by

$$E_\zeta^{[30]\ T=3}=\sum_{i=1}^3\frac{2}{x_i^{(\zeta)}}. \tag{4.26}$$

For the [3,0,0] and $T=1$ case, we need the following commutation relation:

$$[[H,B^0(x_1,x_2)],A_1^\dagger(x_3)]|0\rangle=\Big\{-GA_1^\dagger(x_1)B^0(0,x_2)\left(\frac{x_2}{x_2-x_3}-\frac{x_1}{x_1-x_3}\right)-GA_1^\dagger(x_2)B^0(x_1,0)\left(\frac{x_1}{x_1-x_3}-\frac{x_2}{x_2-x_3}\right)$$
$$+GA_1^\dagger(0)B^0(x_1,x_2)\left(\frac{x_1}{x_1-x_3}+\frac{x_2}{x_2-x_3}\right)+GA_1^\dagger(0)B^0(x_1,x_3)\frac{x_3}{x_3-x_2}$$
$$+GA_1^\dagger(0)B^0(x_2,x_3)\frac{x_3}{x_3-x_1}-GA_1^\dagger(x_1)B^0(0,x_3)\frac{x_3}{x_3-x_2}-GA_1^\dagger(x_2)B^0(0,x_3)\frac{x_3}{x_3-x_1}$$
$$+GA_1^\dagger(x_3)B^0(0,x_1)\frac{x_3}{x_3-x_2}+GA_1^\dagger(x_3)B^0(0,x_2)\frac{x_3}{x_3-x_1}\Big\}|0\rangle. \tag{4.27}$$

Using Eq. (4.27) with Eq. (4.22), one can prove that the eigenenergies are given by

$$E_\zeta^{[30]T=1}=\sum_{i=1}^{3}\frac{2}{x_i^{(\zeta)}}. \tag{4.28}$$

However, in this case, there are nine independent basis vectors in the final expression. Except for the original eigenstate, Eq. (4.22), all other coefficients in front of these basis vectors should vanish. Therefore, $2/x_i+\Omega(x_i)$ should be chosen to satisfy the same condition,

$$\frac{2}{x_i^{(\zeta)}}+G\Omega(x_i^{(\zeta)})=GF_i^{[3]}(x_1^{(\zeta)},x_2^{(\zeta)},x_3^{(\zeta)};\alpha^{[3]},\beta^{[3]},\gamma^{[3]}) \tag{4.29}$$

for $i=1,2,3$, where

$$\alpha^{[3]}=Q^{[3]}(x_1,x_2;x_3),\quad \beta^{[3]}=Q^{[3]}(x_1,x_3;x_2),$$
$$\gamma^{[3]}=Q^{[3]}(x_2,x_3;x_1) \tag{4.30}$$

are functions of x_i $(i=1,2,3)$ satisfying conditions (4.23), and $F_i^{[3]}(x_1,x_2,x_3;\alpha^{[3]},\beta^{[3]},\gamma^{[3]})$ for $i=1,2,3$, is a function of x_i. After symmetrization, we get

$$F_1^{[3]}=\frac{x_2}{x_2-x_1}\,\frac{\beta^{[3]}+\gamma^{[3]}-\alpha^{[3]}}{\alpha^{[3]}+\beta^{[3]}+\gamma^{[3]}}+\frac{x_3}{x_3-x_1}\,\frac{\alpha^{[3]}+\gamma^{[3]}-\beta^{[3]}}{\alpha^{[3]}+\beta^{[3]}+\gamma^{[3]}},$$

$$F_2^{[3]}=\frac{x_1}{x_1-x_2}\,\frac{\beta^{[3]}+\gamma^{[3]}-\alpha^{[3]}}{\alpha^{[3]}+\beta^{[3]}+\gamma^{[3]}}+\frac{x_3}{x_3-x_2}\,\frac{\alpha^{[3]}+\beta^{[3]}-\gamma^{[3]}}{\alpha^{[3]}+\beta^{[3]}+\gamma^{[3]}},$$

$$F_3^{[3]}=\frac{x_2}{x_2-x_3}\,\frac{\alpha^{[3]}+\beta^{[3]}-\gamma^{[3]}}{\alpha^{[3]}+\beta^{[3]}+\gamma^{[3]}}+\frac{x_1}{x_1-x_3}\,\frac{\alpha^{[3]}+\gamma^{[3]}-\beta^{[3]}}{\alpha^{[3]}+\beta^{[3]}+\gamma^{[3]}}. \tag{4.31}$$

The cancellation of unwanted terms requires that $\alpha^{[3]}$, $\beta^{[3]}$, and $\gamma^{[3]}$ satisfy the following equations:

$$\alpha^{[3]}F_1^{[3]}+\frac{x_2}{x_2-x_1}(\gamma^{[3]}+\alpha^{[3]}+\beta^{[3]})-\frac{x_3}{x_3-x_1}(\gamma^{[3]}+\alpha^{[3]})=0,$$

$$\alpha^{[3]}F_2^{[3]}+\frac{x_1}{x_1-x_2}(\gamma^{[3]}+\beta^{[3]}+\alpha^{[3]})-\frac{x_3}{x_3-x_2}(\beta^{[3]}+\alpha^{[3]})=0, \tag{4.32a}$$

$$\alpha^{[3]}F_3^{[3]}-\frac{x_2}{x_2-x_3}(\alpha^{[3]}+\beta^{[3]})-\frac{x_1}{x_1-x_3}(\gamma^{[3]}+\alpha^{[3]})=0,$$

$$\beta^{[3]}F_1^{[3]}+\frac{x_3}{x_3-x_1}(\gamma^{[3]}+\beta^{[3]}+\alpha^{[3]})-\frac{x_2}{x_2-x_1}(\beta^{[3]}+\gamma^{[3]})=0,$$

$$\beta^{[3]}F_3^{[3]}+\frac{x_1}{x_1-x_3}(\gamma^{[3]}+\alpha^{[3]}+\beta^{[3]})-\frac{x_2}{x_2-x_3}(\beta^{[3]}+\alpha^{[3]})=0, \tag{4.32b}$$

$$\beta^{[3]}F_2^{[3]}-\frac{x_3}{x_3-x_2}(\alpha^{[3]}+\beta^{[3]})-\frac{x_1}{x_1-x_2}(\gamma^{[3]}+\beta^{[3]})=0,$$

$$\gamma^{[3]}F_2^{[3]}+\frac{x_3}{x_3-x_2}(\beta^{[3]}+\gamma^{[3]}+\alpha^{[3]})-\frac{x_1}{x_1-x_2}(\beta^{[3]}+\gamma^{[3]})=0,$$

$$\gamma^{[3]}F_3^{[3]}+\frac{x_2}{x_2-x_3}(\beta^{[3]}+\alpha^{[3]}+\gamma^{[3]})-\frac{x_1}{x_1-x_3}(\gamma^{[3]}+\alpha^{[3]})=0, \tag{4.32c}$$

$$\gamma^{[3]}F_1^{[3]}-\frac{x_3}{x_3-x_1}(\alpha^{[3]}+\gamma^{[3]})-\frac{x_2}{x_2-x_1}(\beta^{[3]}+\gamma^{[3]})=0.$$

Due to relations (4.23), the three sets of Eqs. (4.32a), (4.32b), and (4.32c) can be changed into one another through the permutation g_2 and g_1g_2. Therefore, they are not independent. Substituting Eq. (4.31) into Eq. (4.32), one can get relations among $\alpha^{[3]}$, $\beta^{[3]}$, and $\gamma^{[3]}$. There are three sets of solutions with

$$\alpha^{[3]}=-\frac{(\beta^{[3]}+\gamma^{[3]})(\beta^{[3]}x_2(x_1-x_3)+\gamma^{[3]}x_1(x_2-x_3))}{\gamma^{[3]}(3x_1x_2-2x_2x_3-x_1x_3)+\beta^{[3]}(3x_1x_2-x_2x_3-2x_1x_3)}, \tag{4.33}$$

$$\beta_1^{[3]}=\frac{1}{9x_1x_2(x_1-x_3)(x_3-x_2)}[x_1^2(3x_2^2+2x_2x_3-3x_3^2)+4x_1x_2x_3(x_3-2x_2)+2x_2^2x_3^2+h_3/(h_1-h_2)^{1/3}+(h_1-h_2)^{1/3}]\gamma^{[3]}, \tag{4.34a}$$

$$\beta_2^{[3]}=\frac{1}{36(x_1x_2(x_1-x_3)(x_2-x_3)}[h_4-2(\sqrt{3}i+1)h_5/(h_1+h_2)^{1/3}+2(-1+\sqrt{3}i)(h_1+h_2)^{1/3}]\gamma^{[3]}, \tag{4.34b}$$

$$\beta_3^{[3]}=\frac{1}{36(x_1x_2(x_1-x_3)(x_2-x_3)}[h_4+2(\sqrt{3}i-1)h_5/(h_1+h_2)^{1/3}-2(1+\sqrt{3}i)(h_1+h_2)^{1/3}]\gamma^{[3]}, \tag{4.34c}$$

where

$$h_1=9x_1x_2x_3(x_1-x_2)(x_1-x_3)(x_2-x_3)\sqrt{3D}, \tag{4.35a}$$

$$\begin{aligned}D=&x_1^6(-9x_2^6+27x_2^5x_3-79x_2^4x_3^2+113x_2^3x_3^3-79x_2^2x_3^4+27x_2x_3^5-9x_3^6)+x_1^5x_2x_3(x_2+x_3)(27x_2^4-4x_2^3x_3-19x_2^2x_3^2-4x_2x_3^3\\&+27x_3^4)-x_1^4x_2^2x_3^2(79x_2^4+23x_2^3x_3-69x_2^2x_3^2+23x_2x_3^3+79x_3^4)+x_1^3x_2^3x_3^3(x_2+x_3)(113x_2^2-136x_2x_3+113x_3^2)\\&-x_1^2x_2^4x_3^4(79x_2^2-23x_2x_3+79x_3^2)+27x_1x_2^5x_3^5(x_2+x_3)-9x_2^6x_3^6,\end{aligned} \tag{4.35b}$$

$$\begin{aligned}h_2=&x_1^6(216x_2^6-783x_2^5x_3+1152x_2^4x_3^2-899x_2^3x_3^3+414x_2^2x_3^4-135x_2x_3^5+27x_3^6)-3x_1^5x_2x_3(171x_2^5-537x_2^4x_3+637x_2^3x_3^3\\&-347x_2^2x_3^3+51x_2x_3^4+9x_3^5)+3x_1^4x_2^2x_3^2(159x_2^4-437x_2^3x_3+435x_2x_3^3+48x_3^4)+x_1^3x_2^3x_3^3(-199x_2^3+441x_2^2x_3\\&-135x_2x_3^2+53x_3^3)+3x_1^2x_2^4x_3^4(13x_2^2-51x_2x_3-2x_3^2)+3x_1x_2^5x_3^5(5x_2+11x_3)-8x_2^6x_3^6,\end{aligned} \tag{4.35c}$$

$$\begin{aligned}h_3=&36x_1^4x_2^4-3x_1^3x_2^3x_3(29x_1+19x_2)+x_1^2x_2^2x_3^2(76x_1^2+109x_1x_2+31x_2^2)-x_1x_2x_3^3(30x_1^3+62x_1^2x_2+47x_1x_2^2+5x_2^3)\\&+x_3^4(9x_1^4-6x_1^3x_2+40x_1^2x_2^2-11x_1x_2^3+4x_2^4),\end{aligned} \tag{4.35d}$$

$$h_4=-4(x_1^2(3x_2^2+2x_2x_3-3x_3^2)+4x_1x_2x_3(x_3-2x_2)+2x_2^2x_3^2), \tag{4.35e}$$

$$\begin{aligned}h_5=&x_1^4(36x_2^4-87x_2^3x_3+76x_2^2x_3^2-30x_2x_3^3+9x_3^4)-x_2x_3(57x_2^3-109x_2^2x_3+62x_2x_3^2+6x_3^3)\\&+x_1^2x_2^2x_3^2(31x_2^2-47x_2x_3+40x_3^2)-x_1x_2^3x_3^3(5x_2+11x_3)+4x_2^4x_3^4,\end{aligned} \tag{4.35f}$$

and $\beta_i^{[3]}$ $(i=1,2,3)$ are three different solutions in terms of $\gamma^{[3]}$. By substituting each solutions $\alpha^{[3]}(\beta_i^{[3]}),\beta_i^{[3]}$ into Eq. (4.31), the final expressions for $F_i^{[3]}$ will be $\gamma^{[3]}$ independent; and the corresponding Eq. (4.29) provides solutions for the spectral parameters x_1, x_2, and x_3 of the problem.

For the S_3 irrep $[21]$ and $T=1$, the wave function can be written as

$$|\zeta;[2,1,0]_1,\ T=1,M_T=1\rangle=(2-g_2-g_1g_2)Q^{[21]}(x_1x_2;x_3)B^{[0]}(x_1,x_2)A_+^\dagger(x_3)|0\rangle. \tag{4.36}$$

Up to a constant, the coefficients in $(2-g_2-g_1g_2)$ are taken from the IDCs [24,25] of $S_2\times S_1\downarrow S_3$ for the coupling $[2]\otimes[1]\downarrow[21]$. The excitation energy is given by

$$E_\zeta^{[21]T=1}=\sum_{i=1}^{3}\frac{2}{x_i^{(\zeta)}}. \tag{4.37}$$

In this case the Bethe ansatz equations can be written as

$$\frac{2}{x_i^{(\zeta)}}+G\Omega(x_i^{(\zeta)})=GF_i^{[21]}(x_1^{(\zeta)},x_2^{(\zeta)},x_3^{(\zeta)};\alpha^{[21]},\beta^{[21]},\gamma^{[21]}) \tag{4.38}$$

for $i=1,2,3$, which determines the spectral parameters x_i, where the parameters $\alpha^{[21]}$, $\beta^{[21]}$, and $\gamma^{[21]}$ in Eq. (4.38) are

$$\alpha^{[21]}=Q^{[21]}(x_1,x_2;x_3),\quad \beta^{[21]}=Q^{[21]}(x_1,x_3;x_2),$$
$$\gamma^{[21]}=Q^{[21]}(x_2,x_3;x_1) \tag{4.39}$$

and are functions of x_i $(i=1,2,3)$ satisfying the conditions

$$Q^{[21]}(x_1x_3;x_2)=g_2Q^{[21]}(x_1x_2;x_3),$$
$$Q^{[21]}(x_2x_3;x_1)=g_1g_2Q^{[21]}(x_1x_2;x_3). \tag{4.40}$$

After symmetrization, we get

$$F_1^{[21]}=-\frac{x_2}{x_2-x_1}\frac{\beta^{[21]}+\gamma^{[21]}+2\alpha^{[21]}}{2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]}}+\frac{x_3}{x_3-x_1}\frac{2\alpha^{[21]}-\gamma^{[21]}+\beta^{[21]}}{2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]}},$$

$$F_2^{[21]}=-\frac{x_1}{x_1-x_2}\frac{\beta^{[21]}+\gamma^{[21]}+2\alpha^{[21]}}{2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]}}+\frac{x_3}{x_3-x_2}\frac{2\alpha^{[21]}-\beta^{[21]}+\gamma^{[21]}}{2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]}},$$

$$F_3^{[21]}=\frac{x_2}{x_2-x_3}\frac{2\alpha^{[21]}-\beta^{[21]}+\gamma^{[21]}}{2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]}}+\frac{x_1}{x_1-x_3}\frac{2\alpha^{[21]}-\gamma^{[21]}+\beta^{[21]}}{2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]}}. \quad (4.41)$$

Hence, $\alpha^{[21]}$, $\beta^{[21]}$, and $\gamma^{[21]}$ should satisfy the following equations:

$$2\alpha^{[21]}F_1^{[21]}+\frac{x_2}{x_2-x_1}(2\alpha^{[21]}-\gamma^{[21]}-\beta^{[21]})-\frac{x_3}{x_3-x_1}(2\alpha^{[21]}-\gamma^{[21]})=0,$$

$$2\alpha^{[21]}F_2^{[21]}+\frac{x_1}{x_1-x_2}(2\alpha^{[21]}-\gamma^{[21]}-\beta^{[21]})-\frac{x_3}{x_3-x_2}(2\alpha^{[21]}-\beta^{[21]})=0, \quad (4.42a)$$

$$2\alpha^{[21]}F_3^{[21]}-\frac{x_2}{x_2-x_3}(2\alpha^{[21]}-\beta^{[21]})-\frac{x_1}{x_1-x_3}(2\alpha^{[21]}-\gamma^{[21]})=0,$$

$$-\beta^{[21]}F_1^{[21]}+\frac{x_3}{x_3-x_1}(2\alpha^{[21]}-\gamma^{[21]}-\beta^{[21]})+\frac{x_2}{x_2-x_1}(\beta^{[21]}+\gamma^{[21]})=0,$$

$$-\beta F_3^{[21]}+\frac{x_1}{x_1-x_3}(2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]})-\frac{x_2}{x_2-x_3}(2\alpha^{[21]}-\beta^{[21]})=0, \quad (4.42b)$$

$$-\beta F_2^{[21]}-\frac{x_3}{x_3-x_2}(2\alpha^{[21]}-\beta^{[21]})+\frac{x_1}{x_1-x_2}(\gamma^{[21]}+\beta^{[21]})=0,$$

$$-\gamma^{[21]}F_2^{[21]}+\frac{x_3}{x_3-x_2}(2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]})+\frac{x_1}{x_1-x_2}(\beta^{[21]}+\gamma^{[21]})=0,$$

$$-\gamma^{[21]}F_3^{[21]}+\frac{x_2}{x_2-x_3}(2\alpha^{[21]}-\beta^{[21]}-\gamma^{[21]})-\frac{x_1}{x_1-x_3}(2\alpha^{[21]}-\gamma^{[21]})=0, \quad (4.42c)$$

$$-\gamma^{[21]}F_1^{[21]}-\frac{x_3}{x_3-x_1}(2\alpha^{[21]}-\gamma^{[21]})+\frac{x_2}{x_2-x_1}(\beta^{[21]}+\gamma^{[21]})=0,$$

In comparison of Eq. (4.42) with Eq. (4.32), it is clear that the functions

$$\alpha^{[21]}=\frac{1}{2}\alpha^{[3]},\quad \beta^{[21]}=-\beta^{[3]},\quad \gamma^{[21]}=-\gamma^{[3]}. \quad (4.43)$$

Therefore, substituting Eqs. (4.33) and (4.34) into Eq. (4.43), and then substituting the resultants of Eq. (4.43) into Eq. (4.41), one gets the final expression for $F_i^{[21]}$ in terms the spectral parameters x_1, x_2, and x_3. Finally, substituting the functions $F_i^{[21]}$ into Eq. (4.38), one obtains the Bethe ansatz equations that determine the possible spectral parameters x_i.

For the S_3 irrep $[21]$ and $T=2$, the wave function can be written as

$$|\zeta;[2,1,0]_2,T=2,M_T=2\rangle=(2+g_2-g_1g_2)Q_A^{[21]}(x_1x_2;x_3)P_1^1(x_1,x_2)A_+^\dagger(x_3)|0\rangle, \quad (4.44)$$

where $Q_A^{[21]}(x_1x_2;x_3)$ is antisymmetric with respect to $x_1\leftrightarrow x_2$ permutation because $P_1^1(x_1,x_2)$ is antisymmetric with $x_1\leftrightarrow x_2$ permutation. Again, up to a constant, the coefficients in $(2+g_2-g_1g_2)$ are taken from the IDCs [24,25] of $S_2\times S_1\downarrow S_3$ for the coupling $[1^2]\otimes[1]\downarrow[21]$. The excitation energy is given by

$$E_\zeta^{[21]T=2}=\sum_{i=1}^{3}\frac{2}{x_i^{(\zeta)}}. \quad (4.45)$$

In this case the Bethe ansatz equations can be written as

$$\frac{2}{x_i^{(\zeta)}}+G\Omega(x_i^{(\zeta)})=GF_i^{[21](A)}(x_1^{(\zeta)},x_2^{(\zeta)},x_3^{(\zeta)};\alpha^{[21](A)},\beta^{[21](A)},\gamma^{[21](A)}) \quad (4.46)$$

for $i=1,2,3$, which determines the spectral parameters x_i. After symmetrization, we get

$$F_1^{[21](A)}=\frac{x_3}{x_3-x_1}\frac{2(2\alpha^{[21](A)}+\gamma^{[21](A)})}{2\alpha^{[21](A)}+\beta^{[21](A)}+\gamma^{[21](A)}},$$

$$F_2^{[21](A)}=\frac{x_3}{x_3-x_2}\frac{2(2\alpha^{[21](A)}+\beta^{[21](A)})}{2\alpha^{[21](A)}+\beta^{[21](A)}+\gamma^{[21](A)}},$$

$$F_3^{[21](A)}=\frac{x_1}{x_1-x_3}\frac{2(2\alpha^{[21](A)}+\gamma^{[21](A)})}{2\alpha^{[21](A)}+\beta^{[21](A)}+\gamma^{[21](A)}}+\frac{x_2}{x_2-x_3}\frac{2(2\alpha^{[21](A)}+\beta^{[21](A)})}{2\alpha^{[21](A)}+\beta^{[21](A)}+\gamma^{[21](A)}}. \quad (4.47)$$

Hence, $\alpha^{[21](A)}$, $\beta^{[21](A)}$, and $\gamma^{[21](A)}$ should satisfy the following equations:

$$2\alpha^{[21](A)}F_1^{[21](A)}-\frac{x_3}{x_3-x_1}(2\alpha^{[21](A)}+\gamma^{[21](A)})=0,$$

$$2\alpha^{[21](A)}F_2^{[21](A)}-\frac{x_3}{x_3-x_2}(2\alpha^{[21](A)}+\beta^{[21](A)})=0, \quad (4.48a)$$

$$2\alpha^{[21](A)}F_3^{[21](A)}-\frac{x_1}{x_1-x_3}(2\alpha^{[21](A)}+\gamma^{[21](A)})-\frac{x_2}{x_2-x_3}(2\alpha^{[21](A)}+\beta^{[21](A)})=0,$$

$$\beta^{[21](A)}F_1^{[21](A)}+\frac{x_2}{x_2-x_1}(\beta^{[21](A)}-\gamma^{[21](A)})=0,$$

$$\beta^{[21](A)}F_3-\frac{x_2}{x_2-x_3}(2\alpha^{[21](A)}+\beta^{[21](A)})=0, \quad (4.48b)$$

$$\beta^{[21](A)}F_2^{[21](A)}-\frac{x_3}{x_3-x_2}(2\alpha^{[21](A)}+\beta^{[21](A)})-\frac{x_1}{x_1-x_2}(\beta^{[21](A)}-\gamma^{[21](A)})=0,$$

$$\gamma^{[21](A)}F_2^{[21](A)}-\frac{x_1}{x_1-x_2}(\gamma^{[21](A)}-\beta^{[21](A)})=0,$$

$$\gamma^{[21](A)}F_3^{[21](A)}-\frac{x_1}{x_1-x_3}(2\alpha^{[21](A)}+\gamma^{[21](A)})=0 \quad (4.48c)$$

$$\gamma^{[21](A)}F_1^{[21](A)}+\frac{x_2}{x_2-x_1}(\beta^{[21](A)}-\gamma^{[21](A)})-\frac{x_3}{x_3-x_1}(2\alpha^{[21](A)}+\gamma^{[21](A)})=0,$$

There are two sets of solutions:

$$\beta_1^{[21](A)}=\gamma^{[21](A)}\frac{(x_2-x_1)x_3-\sqrt{x_3^2(x_1-x_2)^2-(x_3-x_2)(x_1-x_3)x_1x_2}}{x_1(x_3-x_2)}, \quad \alpha^{[21](A)}=\frac{1}{2}(\beta_1^{[21](A)}+\gamma^{[21](A)}). \quad (4.49)$$

The corresponding expressions for $F_i^{[21](A)}$ are

$$F_1^{[21](A)}=\frac{2x_2x_3-x_1(x_2+x_3)+\sqrt{x_3^2(x_1-x_2)^2+(x_2-x_3)(x_1-x_3)x_1x_2}}{(x_1-x_2)(x_1-x_3)}, \quad (4.50a)$$

$$F_2^{[21](A)}=\frac{x_1(x_2-2x_3)+x_2x_3-\sqrt{x_3^2(x_1-x_2)^2+(x_2-x_3)(x_1-x_3)x_1x_2}}{(x_1-x_2)(x_2-x_3)}, \quad (4.50b)$$

$$F_3^{[21](A)}=\frac{x_3(x_1+x_2)-2x_1x_2-\sqrt{x_3^2(x_1-x_2)^2+(x_2-x_3)(x_1-x_3)x_1x_2}}{(x_1-x_3)(x_3-x_2)}. \quad (4.50c)$$

$$\beta_2^{[21](A)}=\gamma^{[21](A)}\frac{(x_2-x_1)x_3+\sqrt{x_3^2(x_1-x_2)^2-(x_3-x_2)(x_1-x_3)x_1x_2}}{x_1(x_3-x_2)}, \quad \alpha^{[21](A)}=\frac{1}{2}(\beta_2^{[21](A)}+\gamma^{[21](A)}). \quad (4.51)$$

The corresponding expressions for $F_i^{[21](A)}$ are

$$F_1^{[21](A)}=\frac{x_1(x_2+x_3)-2x_2x_3+\sqrt{x_3^2(x_1-x_2)^2+(x_2-x_3)(x_1-x_3)x_1x_2}}{(x_1-x_2)(x_1-x_3)}, \quad (4.52a)$$

$$F_2^{[21](A)}=\frac{x_1(x_2-2x_3)+x_2x_3+\sqrt{x_3^2(x_1-x_2)^2+(x_2-x_3)(x_1-x_3)x_1x_2}}{(x_1-x_2)(x_2-x_3)}, \quad (4.52b)$$

$$F_3^{[21](A)}=\frac{x_3(x_1+x_2)-2x_1x_2+\sqrt{x_3^2(x_1-x_2)^2+(x_2-x_3)(x_1-x_3)x_1x_2}}{(x_1-x_3)(x_3-x_2)}. \quad (4.52c)$$

Substituting Eq. (4.50) or (4.52) into Eq. (4.46), one obtains the corresponding Bethe ansatz equations for determining the spectral parameters x_i.

Finally, for the S_3 irrep $[1^3]$ with $T=0$ the wave function can be written as

$$|\zeta;[1^3],T=0,M_T=0\rangle=(1-g_2+g_1g_2)Q_A^{[1^3]}(x_1,x_2;x_3)\times(P^1(x_1,x_2)\mathcal{A}^\dagger(x_3))^0_0|0\rangle, \tag{4.53}$$

where $Q_A^{[1^3]}(x_1x_2;x_3)$ is also antisymmetric with respect to a $x_1\leftrightarrow x_2$ permutation. Up to a constant, the coefficients in $(1-g_2+g_1g_2)$ are taken from the IDCs [24,25] of $S_2\times S_1\downarrow S_3$ for the coupling $[1^2]\otimes[1]\downarrow[1^3]$. The excitation energy is given by

$$E_\zeta^{[1^3]T=2}=\sum_{i=1}^{3}\frac{2}{x_i^{(\zeta)}}. \tag{4.54}$$

It can be proven that the three expansion coefficients $Q_A^{[1^3]}$ can be taken to be the same in this case. Hence, Eq. (4.53) can be simplified to

$$|\zeta;[1^3],T=0,M_T=0\rangle =(1-g_2+g_1g_2)(P^1(x_1^{(\zeta)},x_2^{(\zeta)})\times\mathcal{A}^+(x_3^{(\zeta)}))^0_0|0\rangle. \tag{4.55}$$

In this case the Bethe ansatz equations are simply

$$\frac{2}{x_i^{(\zeta)}}+G\Omega(x_i^{(\zeta)})=0 \tag{4.56}$$

for $i=1,2,3$. Because the wave function (4.54) is antisymmetric with any permutation among $x_1^{(\zeta)}$, $x_2^{(\zeta)}$, and $x_3^{(\zeta)}$, the solutions of x_i should be those satisfying Eq. (4.56) with $x_1^{(\zeta)}\neq x_2^{(\zeta)}\neq x_3^{(\zeta)}$.

V. DISCUSSION

A general procedure, based on the Bethe ansatz, for building algebraic solutions for seniority-zero $J=0$ states of $2k$ nucleons interacting through a $T=1$ charge-independent pairing interaction has been introduced. The method should also work for finding solutions of a quantum many-body problem with a Lie algebra symmetry G other than Sp(4). We used the procedure to generate explicit results for seniority-zero $J=0$ states for up to six nucleons.

The results derived in this paper for $k\leqslant 2$ as well as for $2k$ nucleons for symmetric irreps of S_k with $T=k$ agree with those given by Richardson [21] and by Chen and Richardson [22,23]. However, in Sec. IV, we showed that the results given in Refs. [21–23] are not valid for six or more nucleons in nonsymmetric irreps of the permutation group. The main difference lies in the fact that in the present work the expansion coefficients $Q^{[\lambda]}$ are considered to be functions of the spectral parameter x_i and different from one another for nonsymmetric irreps of the permutation groups, while the expansion coefficients in the work of Richardson and of Chen and Richardson were assumed to be independent of the spectral parameter. In fact, for $2k$ nucleon configuratons, the present calculation shows that the expansion coefficients $Q^{[\lambda]_k}$ can be taken to be the same only for totally symmetric irreps $[k]$ of the permutation groups S_k with $T=k$ or totally antisymmetric irreps $[1^k]$ with $k=1,2,3$. One can verify that the results given by Eqs. (4.29), (4.38), and (4.46) reduce to those given in Refs. [21–23] if one takes $\alpha^{[3]}=\beta^{[3]}=\gamma^{[3]}$ in Eq. (4.31), $\alpha^{[21]}=-\beta^{[21]}=-\gamma^{[21]}$ in Eq. (4.41), and $\alpha^{[21](A)}=\beta^{[21](A)}=\gamma^{[21]}$ in Eq. (4.47). But for other cases, general solutions of the type introduced here are required; those offered in Refs. [21–23] as solutions for general irreps are not possible.

The nonsymmetric [2,1] irrep of S_k is two-dimensional. The results derived in Sec. IV are for only one of the two, but a similar procedure can be applied to determine the another one. In this regard, note that the eigenenergies for a given S_k irrep are degenerate because the Hamiltonian is invariant with respect to any permutations of S_k. The most important outcome of the present analysis is the algebraic expression of the Bethe ansatz equations given in Sec. IV, which is a set of highly nonlinear equations. For example, there are three sets of solutions for the β function for the irreps ([3], $T=1$) and ([2,1], $T=1$), and there are two sets of solutions for the β function for the [2,1] irrep with $T=2$. Numerical calculations and analyses that are part of another study will be presented elsewhere.

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