Extended-soft-core NN potentials in momentum space. I. Pseudoscalar-pseudoscalar exchange potentials

Th. A. Rijken and H. Polinder
Institute for Theoretical Physics Nijmegen, University of Nijmegen, Nijmegen, The Netherlands

J. Nagata*
Venture Business Laboratory, Hiroshima University, Kagamiyama 2-313, Higashi-Hiroshima, Japan
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A momentum-space representation is derived for the Nijmegen extended-soft-core interactions. The partial wave projection of this representation is carried through, in principle for two-meson exchange in general. Explicit results for the momentum-space partial wave NN potentials from PS-PS exchange are given.

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I. INTRODUCTION

For nucleon-nucleon [1–4] and hyperon-nucleon [7] scattering we have shown that the extended-soft-core (ESC) models for baryon-baryon interactions give an excellent description of the NN and YN data. So far, applications of the ESC model have been in configuration space. In two papers we give a momentum space representation of the ESC potentials. Also, we describe a fit to the pp and np data for $0 \leqslant T_{lab} \leqslant 350$ MeV, having $\chi^2/N_{data} = 1.15$. Here, we used 20 physical parameters, being coupling constants and cutoff masses.

In synopsis, an account of a modern theoretical basis for the soft-core interactions has been given in [1]. Starting from the so-called standard model and integrating out the heavy quarks one arrives at an effective QCD for the u,d,s quarks. The generally accepted scenario is now that the QCD vacuum becomes unstable for momentum transfers for which $Q^2 \leqslant \Lambda^2_{\chi \mathrm{SB}} \approx 1\ \mathrm{GeV}^2$ [5], causing spontaneous chiral-symmetry breaking (χSB). The vacuum goes through a phase transition, generating constituent quark masses via $\langle 0|\bar{\psi}\psi|0\rangle \neq 0$ and reducing the gluon coupling α_s. Viewing the pseudoscalar mesons π, etc., as the Nambu-Goldstone bosons originating from spontaneous χSB, makes it natural to assume a quark dressing by pseudoscalar mesons and also other types of mesons in general. In this context, baryon-baryon interactions are described naturally by meson exchange using form factors at the meson-baryon vertices. As a working hypothesis, we restrict ourselves for low energy scattering to a treatment of the hadronic phase. Integrating out the heavy mesons and baryons using a renormalization procedure in the manner of Wilson [6], we restrict ourselves to mesons with $M \leqslant 1\ \mathrm{GeV}/c^2$, arriving at a so-called *effective field theory* as the proper arena to describe low energy hadron scattering. This is the general physical basis for the Nijmegen soft-core models.

In this paper, hereafter referred to as I, a representation of the ESC interactions in momentum space is presented, which is very elegant and useful for applications in momentum-space computations.

We solve basically the problem of finding a suitable representation of the two-meson-exchange (TME) potentials in momentum space in general and, in particular, with Gaussian form factors. As is apparent from the configuration-space representation, given in [2,3,12], we have to evaluate forms like

$$\frac{2}{\pi}\int_0^\infty d\lambda\, \tilde{F}(\mathbf{\Delta}^2,\lambda)\tilde{G}((\Delta-\mathbf{k})^2,\lambda),$$

which at first sight means an extra numerical integral, besides the usual convolutive integration over the $\mathbf{\Delta}$ *momentum. Here,* $\tilde{F}$ *and* $\tilde{G}$ *contain the couplings, Gaussian form factors, and momentum space Yukawa functions for the two exchanged mesons. Fortunately, the integrations over the* λ *parameter can be carried through analytically, as we shall show in the sequel. The representations obtained in this paper contain only two parameters, henceforth called* t *and* u*, to be integrated over numerically. Moreover, the* (t,u) *integration region is effectively over a finite domain, this in view of the exponentials* $\exp(-m_1^2 t)$ *and* $\exp(-m_2^2 u)$ *in the integrands. Also, the strong Gaussian-like falloff in the off-shell region, typical for soft-core interactions, is easily realized. To realize the latter property of the momentum-space potentials is problematic when these potentials have to be generated numerically from configuration space via Fourier transformation*.

The partial wave projection formulas on the LSJ partial wave basis are developed for the TME potentials in momentum space. As for the one-boson-exchange (OBE) potentials in [8], we expand the momentum-space potentials in the Pauli-spinor invariants. The partial wave basis is chosen according to the convention of [9]. Explicit formulas are worked out for general pseudoscalar-pseudoscalar (PS-PS) exchange.

In paper II [10], the momentum-space representation for the ESC meson-pair-exchange (MPE) potentials is described. Here, we moreover present a fit to the NN data for $0 \leqslant T_{lab} \leqslant 350$ MeV. Included in this ESC model are the contributions from OBE, PS-PS exchange, and MPE.

*Present address: Kyushu International University, Fukuoka 805-8512, Japan.

The contents of this paper are as follows. In Sec. II we review the definition of the ESC potentials in the context of the relativistic two-body, Thompson, and Lippmann-Schwinger equations. Here, we exploit the Macke-Klein [11] framework in field theory. For the Lippmann-Schwinger equation we introduce the usual potential forms in Pauli spinor space. We include here the central (C), the spin-spin (σ), the tensor (T), the spin-orbit (SO), the quadratic spin-orbit (Q_{12}), and the antisymmetric spin-orbit (ASO) potentials. For TME, in the approximations made in [2,3] only the central, spin-spin, tensor, and spin-orbit potentials occur. In Sec. III the special representation for the ESC interactions in momentum space is developed and illustrated using some basic examples. In Sec. IV the general PS-PS exchange is presented. In Sec. V the projection on the plane wave spinor invariants is carried through for general PS-PS exchange. The adiabatic terms, nonadiabatic, pseudovector vertex, and off-shell $1/M$ corrections are given explicitly. In Sec. VI the partial wave analysis is performed. Finally in Sec. VII the partial wave contributions from the adiabatic terms and the mentioned $1/M$ corrections are given.

Appendix A contains some basic integrals which are employed in the paper. In Appendix B a dictionary is given containing the results of the convolutive integrations for the operators that occur in PS-PS exchange. Appendix C contains the LSJ partial wave matrix elements of several important operators. In Appendix D the Fourier connection between coordinate and momentum space is illustrated, in particular for the PS-PS exchange tensor potential.

II. TWO-BODY INTEGRAL EQUATIONS IN MOMENTUM SPACE

A. Relativistic two-body equations

We consider the nucleon-nucleon reactions

$$N(p_a,s_a)+N(p_b,s_b)\to N(p_{a'},s_{a'})+N(p_{b'},s_{b'}), \tag{2.1}$$

with the total and relative four-momenta for the initial and final states,

$$P=p_a+p_b,\quad P'=p_{a'}+p_{b'},$$
$$p=\tfrac{1}{2}(p_a-p_b),\quad p'=\tfrac{1}{2}(p_{a'}-p_{b'}), \tag{2.2}$$

which become in the center-of-mass system (c.m.s.) for a and b on mass shell

$$P=(W,\mathbf{0}),\quad p=(0,\mathbf{p}),\quad p'=(0,\mathbf{p}'). \tag{2.3}$$

In general, the particles are off mass shell in the Green functions. Further on in this section, the on-mass-shell momenta for the initial and final states are denoted, respectively, by p_i and p_f. So $p_a^0=E_a(\mathbf{p}_i)=\sqrt{\mathbf{p}_i^2+M_a^2}$ and $p_{a'}^0=E_{a'}(\mathbf{p}_f)=\sqrt{\mathbf{p}_f^2+M_{a'}^2}$, and similarly for b and b'. Because of translation invariance, $P=P'$ and $W=W'=E_a(\mathbf{p}_i)+E_b(\mathbf{p}_i)=E_{a'}(\mathbf{p}_f)+E_{b'}(\mathbf{p}_f)$. The two-particle states we normalize in the following way:

$$\langle \mathbf{p}_1',\mathbf{p}_2'|\mathbf{p}_1,\mathbf{p}_2\rangle=(2\pi)^3 2E(\mathbf{p}_1)\,\delta^3(\mathbf{p}_1'-\mathbf{p}_1)\times(2\pi)^3 2E(\mathbf{p}_2)\,\delta^3(\mathbf{p}_2'-\mathbf{p}_2). \tag{2.4}$$

The relativistic two-body scattering equation reads [13–15]

$$\psi(p,P)=\psi^0(p,P)+G(p;P)\int d^4p'\,I(p,p')\,\psi(p',P), \tag{2.5}$$

where $\psi(p,P)$ is a 4×4 matrix in Dirac space. The contributions to the kernel $I(p,p')$ come from the two-nucleon irreducible Feynman diagrams. In writing Eq. (2.5) we have taken out an overall $\delta^4(P'-P)$ function and the total four-momentum conservation is implicitly understood henceforth.

The two-particle Green function $G(p;P)$ in Eq. (2.5) is simply the product of the free propagators for the baryons of lines (a) and (b). For the nucleon and more general for any baryon the Feynman propagators are given by the well-known formula

$$G^{(s)}_{\{\mu\},\{\nu\}}(p)=\int d^4x\langle 0|T(\psi^{(s)}_{\{\mu\}}(x)\,\overline{\psi}^{(s)}_{\{\nu\}}(0))|0\rangle e^{ip\cdot x}=\frac{\Pi^s(p)}{p^2-M^2+i\delta}, \tag{2.6}$$

where $\psi^{(s)}_{\{\mu\}}$ is the free Rarita-Schwinger field which describes the nucleon $(s=\frac{1}{2})$, the Δ_{33} resonance $(s=\frac{3}{2})$, etc. (see, for example, [16]). For the nucleon, the only case considered in this paper, $\{\mu\}=\varnothing$ and for, e.g., the Δ resonance, $\{\mu\}=\mu$. For the rest of this paper we deal only with nucleons.

In terms of these one-particle Green functions the two-particle Green function in Eq. (2.5) is

$$G(p;P)=\frac{i}{(2\pi)^4}\left[\frac{\Pi^{(s_a)}\left(\frac{1}{2}P+p\right)}{\left(\frac{1}{2}P+p\right)^2-M_a^2+i\delta}\right]^{(a)}\times\left[\frac{\Pi^{(s_b)}\left(\frac{1}{2}P-p\right)}{\left(\frac{1}{2}P-p\right)^2-M_b^2+i\delta}\right]^{(b)}. \tag{2.7}$$

Using now a complete set of on-mass-shell spin-s states in the first line of Eq. (2.6) one finds that the Feynman propagator of a spin-s baryon off mass shell can be written as [17]

$$\frac{\Pi^{(s)}(p)}{p^2-M^2+i\delta}=\frac{M}{E(\mathbf{p})}\left[\frac{\Lambda_+^{(s)}(\mathbf{p})}{p_0-E(\mathbf{p})+i\delta}-\frac{\Lambda_-^{(s)}(-\mathbf{p})}{p_0+E(\mathbf{p})-i\delta}\right], \tag{2.8}$$

for $s=\frac{1}{2},\frac{3}{2},\ldots$. Here, $\Lambda_+^{(s)}(\mathbf{p})$ and $\Lambda_-^{(s)}(\mathbf{p})$ are the on-mass-shell projection operators on the positive- and negative-energy states. For the nucleon they are

$$\Lambda_+(\mathbf{p})=\sum_{\sigma=-1/2}^{+1/2} u(\mathbf{p},\sigma)\otimes\overline{u}(\mathbf{p},\sigma),$$

$$\Lambda_-(\mathbf{p})=-\sum_{\sigma=-1/2}^{+1/2} v(\mathbf{p},\sigma)\otimes\overline{v}(\mathbf{p},\sigma), \tag{2.9}$$

where $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$ are the Dirac spinors for spin-$\frac{1}{2}$ particles, and $E(\mathbf{p})=\sqrt{\mathbf{p}^2+M^2}$ with M the nucleon mass. Then, in the c.m.s. where $\mathbf{P}=0$ and $P_0=W$, the Green function can be written as

$$G(p;W)=\frac{i}{(2\pi)^4}\left(\frac{M_a}{E_a(\mathbf{p})}\right)\left[\frac{\Lambda_+^{(s_a)}(\mathbf{p})}{\frac{1}{2}W+p_0-E_a(\mathbf{p})+i\delta}-\frac{\Lambda_-^{(s_a)}(-\mathbf{p})}{\frac{1}{2}W+p_0+E_a(\mathbf{p})-i\delta}\right]\left(\frac{M_b}{E_b(\mathbf{p})}\right)\times\left[\frac{\Lambda_+^{(s_b)}(-\mathbf{p})}{\frac{1}{2}W-p_0-E_b(\mathbf{p})+i\delta}-\frac{\Lambda_-^{(s_b)}(\mathbf{p})}{\frac{1}{2}W-p_0+E_b(\mathbf{p})-i\delta}\right]. \tag{2.10}$$

Multiplying out Eq. (2.10) we write the ensuing terms in shorthand notation

$$G(p;W)=G_{++}(p;W)+G_{+-}(p;W)+G_{-+}(p;W)+G_{--}(p;W), \tag{2.11}$$

where G_{++}, etc., corresponds to the term with $\Lambda_+^{s_a}\Lambda_+^{s_b}$, etc. Introducing the wave functions (see [18])

$$\psi_{rs}(p')=\Lambda_r^{s_a}\Lambda_s^{s_b}\psi(p') \quad (r,s=+,-), \tag{2.12}$$

the two-body equation (2.5) can be written for, e.g., ψ_{++} as

$$\psi_{++}(p)=\psi^0_{++}(p)+G_{++}(p;W)\int d^4p'\sum_{r,s} I(p,p')_{++,rs}\psi_{rs}(p'), \tag{2.13}$$

and similar equations for ψ_{+-}, ψ_{-+}, and ψ_{--}.

Invoking "dynamical pair suppression," as discussed in [12], we reduce Eq. (2.13) to a four-dimensional equation for ψ_{++}, i.e.,

$$\psi_{++}(p')=\psi^0_{++}(p')+G_{++}(p';W)\int d^4p\, I(p',p)_{++,++}\psi_{++}(p), \tag{2.14}$$

with the Green function

$$G_{++}(p;W)=\frac{i}{(2\pi)^4}\left[\frac{M_aM_b}{E_a(\mathbf{p})E_b(\mathbf{p})}\right]\Lambda_+^{(s_a)}(\mathbf{p})\Lambda_+^{(s_b)}(-\mathbf{p})\times\left[\frac{1}{2}W+p_0-E_a(\mathbf{p})+i\delta\right]^{-1}\times\left[\frac{1}{2}W-p_0-E_b(\mathbf{p})+i\delta\right]^{-1}. \tag{2.15}$$

B. Three-dimensional equation

Following the same procedures as in [12] we introduce the three-dimensional wave function according to Salpeter [18] by

$$\phi(\mathbf{p})=\sqrt{\frac{E_a(\mathbf{p})E_b(\mathbf{p})}{M_aM_b}}\int_{-\infty}^{\infty}\psi(p_\mu)dp_0. \tag{2.16}$$

Next, we use for the right inverse of the $\int dp_0$ operation the ansatz proposed by Klein [11],

$$\psi(p'_\mu)=\sqrt{\frac{M_aM_b}{E_a(\mathbf{p}')E_b(\mathbf{p}')}}A_W(p'_\mu)\phi(\mathbf{p}'), \tag{2.17}$$

where the right inverse is given by

$$A_W(p'_\mu)=-\frac{1}{2\pi i}\frac{W-\mathcal{W}(\mathbf{p}')}{F_W^{(a)}(\mathbf{p}',p'_0)F_W^{(b)}(-\mathbf{p}',-p'_0)} \tag{2.18}$$

with the frequently used notation

$$F_W(\mathbf{p},p_0)=p_0-E(\mathbf{p})+\frac{1}{2}W+i\delta,$$

$$\mathcal{W}(\mathbf{p})=E_a(\mathbf{p})+E_b(\mathbf{p}). \tag{2.19}$$

Applying now the p_0 integration to Eq. (2.14) and performing the p'_0 integration, made possible by the ansatz above, one arrives at the Thompson equation [19]

$$\phi_{++}(\mathbf{p}')=\phi^{(0)}_{++}(\mathbf{p}')+E_2^{(+)}(\mathbf{p}';W)\int\frac{d^3p}{(2\pi)^3}K^{irr}(\mathbf{p}',\mathbf{p}|W)\phi_{++}(\mathbf{p}), \tag{2.20}$$

where the Green function is

$$E_2^{(+)}(\mathbf{p}';W)=\frac{1}{(2\pi)^3}\frac{\Lambda_+^a(\mathbf{p}')\Lambda_+^b(-\mathbf{p}')}{(W-\mathcal{W}(\mathbf{p}')+i\delta)} \tag{2.21}$$

and the kernel is given by

$$K^{irr}(\mathbf{p}',\mathbf{p}|W)$$
$$=-\frac{1}{(2\pi)^2}\sqrt{\frac{M_aM_b}{E_a(\mathbf{p}')E_b(\mathbf{p}')}}\sqrt{\frac{M_aM_b}{E_a(\mathbf{p})E_b(\mathbf{p})}}$$
$$\times[W-\mathcal{W}(\mathbf{p}')][W-\mathcal{W}(\mathbf{p})]$$
$$\times\int_{-\infty}^{+\infty}dp_0'\int_{-\infty}^{+\infty}dp_0[\{F_W^{(a)}(\mathbf{p}',p_0')$$
$$\times F_W^{(b)}(-\mathbf{p}',-p_0')\}^{-1}[I(p_0',\mathbf{p}';p_0,\mathbf{p})]_{++,++}$$
$$\times\{F_W^{(a)}(\mathbf{p},p_0)F_W^{(b)}(-\mathbf{p},-p_0)\}^{-1}]. \tag{2.22}$$

The M/E factors in Eq. (2.22) are due to the difference between the relativistic and nonrelativistic normalizations of the two-particle states. In the following we simply put $M/E(\mathbf{p})=1$ in the kernel K^{irr}, Eq. (2.22). The corrections to this approximation would give $(1/M)^2$ corrections to the potentials, which we neglect in this paper. In the same approximation there is no difference between the Thompson equation (2.20) and the Lippmann-Schwinger equation when the connection between these equations is made using multiplication factors. Henceforth, we will not distinguish between the two.

The contributions to the two-particle irreducible kernel K^{irr} up to second order in the meson exchange are given in detail in [2]. For the definition of the TME potential in the Lippmann-Schwinger equation we shall need the complete fourth-order kernel for the Thompson equation (2.20). In operator notation, we have, from Eq. (2.20),

$$\phi_{++}=\phi_{++}^{(0)}+E_2^{(+)}K^{irr}\phi_{++}$$
$$=\phi_{++}^{(0)}+E_2^{(+)}(K^{irr}+K^{irr}E_2^{(+)}K^{irr}+\cdots)\phi_{++}^{(0)}$$
$$\equiv(1+E_2^{(+)}M)\phi_{++}^{(0)}, \tag{2.23}$$

which implies for the complete kernel M the integral equation

$$M(\mathbf{p}',\mathbf{p}|W)=K^{irr}(\mathbf{p}',\mathbf{p}|W)+\int\frac{d^3p''}{(2\pi)^3}K^{irr}$$
$$\times(\mathbf{p}',\mathbf{p}''|W).E_2^{(+)}(\mathbf{p}'';W)M(\mathbf{p}'',\mathbf{p}|W). \tag{2.24}$$

C. Lippmann-Schwinger Equation

The transformation of Eq. (2.24) to the Lippmann-Schwinger equation can be effected by defining

$$T(\mathbf{p}',\mathbf{p})=N(\mathbf{p}')M(\mathbf{p}',\mathbf{p};W)N(\mathbf{p}),$$
$$V(\mathbf{p}',\mathbf{p})=N(\mathbf{p}')K^{irr}(\mathbf{p}',\mathbf{p};W)N(\mathbf{p}), \tag{2.25}$$

where the transformation function is

$$N(\mathbf{p})=\sqrt{\frac{\mathbf{p}_i^2-\mathbf{p}^2}{2M_N[E(\mathbf{p}_i)-E(\mathbf{p})]}}. \tag{2.26}$$

Application of this transformation, yields the Lippmann-Schwinger equation

$$T(\mathbf{p}',\mathbf{p})=V(\mathbf{p}',\mathbf{p})+\int\frac{d^3p''}{(2\pi)^3}V(\mathbf{p}',\mathbf{p}'')g(\mathbf{p}'';W)T(\mathbf{p}'',\mathbf{p}), \tag{2.27}$$

with the standard Green function

$$g(\mathbf{p};W)=\frac{1}{(2\pi)^3}\Lambda_+^a(\mathbf{p})\Lambda_+^b(-\mathbf{p})\frac{M_N}{\mathbf{p}_i^2-\mathbf{p}^2+i\delta}. \tag{2.28}$$

The corrections to the approximation $E_2^{(+)}\approx g(\mathbf{p};W)$ are of order $1/M^2$, which we neglect henceforth.

The transition from Dirac spinors to Pauli spinors is given in Appendix C of [12], where we write, for the the Lippmann-Schwinger equation in the four-dimensional Pauli-spinor space,

$$\mathcal{T}(\mathbf{p}',\mathbf{p})=\mathcal{V}(\mathbf{p}',\mathbf{p})+\int\frac{d^3p''}{(2\pi)^3}\mathcal{V}(\mathbf{p}',\mathbf{p}'')\tilde{g}(\mathbf{p}'';W)\mathcal{T}(\mathbf{p}'',\mathbf{p}). \tag{2.29}$$

The $\mathcal{T}$ operator in Pauli-spinor space is defined by

$$\chi_{\sigma_a}^{(a)\dagger}\chi_{\sigma_b}^{(b)\dagger}\mathcal{T}(\mathbf{p}',\mathbf{p})\chi_{\sigma_a'}^{(a)}\chi_{\sigma_b'}^{(b)}$$
$$=\bar{u}_a(\mathbf{p},\sigma_a)\bar{u}_b(-\mathbf{p},\sigma_b)T(\mathbf{p},\mathbf{p}'')u_a(\mathbf{p}'',\sigma_a')u_b(-\mathbf{p}'',\sigma_b') \tag{2.30}$$

and similarly for the $\mathcal{V}$ operator. Like in the derivation of the OBE potentials [20,21] we make off shell and on shell the approximation $E(\mathbf{p})=M+\mathbf{p}^2/2M$ and $W=2\sqrt{\mathbf{p}_i^2+M^2}=2M+\mathbf{p}_i^2/M$ everywhere in the interaction kernels, which, of course, is fully justified for low energies only. In contrast to these kinds of approximations, of course the full $\mathbf{k}^2$ dependence of the form factors is kept throughout the derivation of the TME. Notice that the Gaussian form factors suppress the high momentum transfers strongly. This means that the contribution to the potentials from intermediate states which are far off energy shell cannot be very large.

Because of rotational invariance and parity conservation, the T matrix, which is a 4×4 matrix in Pauli-spinor space, can be expanded into the following set of in general eight spinor invariants; see, for example, [22,23]. Introducing [24]

$$\mathbf{q}=\frac{1}{2}(\mathbf{p}_f+\mathbf{p}_i),\quad \mathbf{k}=\mathbf{p}_f-\mathbf{p}_i,\quad \mathbf{n}=\mathbf{p}_i\times\mathbf{p}_f, \tag{2.31}$$

with, of course, $\mathbf{n}=\mathbf{q}\times\mathbf{k}$, we choose, for the operators P_j in spin space,

$$P_1=1,$$
$$P_2=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2,$$
$$P_3=(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})-\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2,$$

$$P_4=\frac{i}{2}(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n},$$

$$P_5=(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n}),$$

$$P_6=\frac{i}{2}(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{n},$$

$$P_7=(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{k})+(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{q}),$$

$$P_8=(\boldsymbol{\sigma}_1\cdot\mathbf{q})(\boldsymbol{\sigma}_2\cdot\mathbf{k})-(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{q}). \tag{2.32}$$

Here we follow [23], where, in contrast to [20], we have chosen P_3 to be a purely "tensor-force" operator. The expansion in spinor invariants reads

$$\mathcal{T}(\mathbf{p}_f,\mathbf{p}_i)=\sum_{j=1}^{8}\widetilde{T}_j(\mathbf{p}_f^2,\mathbf{p}_i^2,\mathbf{p}_f\cdot\mathbf{p}_i)P_j(\mathbf{p}_f,\mathbf{p}_i). \tag{2.33}$$

Similarly to Eq. (2.33) we expand the potentials V. Again following [23], we neglect the potential forms P_7 and P_8, and also the dependence of the potentials on $\mathbf{k}\cdot\mathbf{q}$. Then, the expansion (2.33) reads for the potentials as follows:

$$\mathcal{V}=\sum_{j=1}^{4}\widetilde{V}_j(\mathbf{k}^2,\mathbf{q}^2)P_j(\mathbf{k},\mathbf{q}). \tag{2.34}$$

We develop in the following subsections a new representation of the TME potentials. Included are the BW graphs shown in Fig. 1 and the TMO graphs shown in Fig. 2. Also the notation employed for the momenta is indicated in the figures.

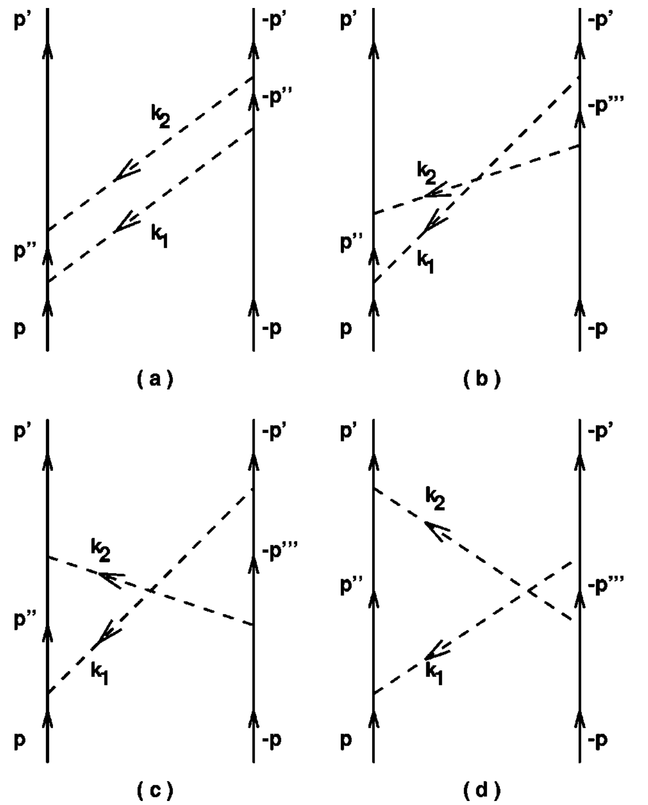

FIG. 1. BW two-meson-exchange graphs: (a) planar and (b)–(d) crossed box. The dashed line with momentum $\mathbf{k}_1$ refers to the pion and the dashed line with momentum $\mathbf{k}_2$ refers to one of the other (vector, scalar, or pseudoscalar) mesons. To these we have to add the "mirror" graphs and the graphs where we interchange the two meson lines.

III. MOMENTUM-SPACE REPRESENTATION TME POTENTIALS

In this section we give an outline of the procedures in making the partial wave analysis of TME potentials in momentum space.

A. TPS central potentials

Consider the basic integral

$$\widetilde{V}(\mathbf{k})=\int\int\frac{d^3k_1d^3k_2}{(2\pi)^3}\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\widetilde{F}(\mathbf{k}_1^2)\widetilde{G}(\mathbf{k}_2^2)$$

$$=\int\frac{d^3\Delta}{(2\pi)^3}\widetilde{F}(\boldsymbol{\Delta}^2)\cdot\widetilde{G}((\mathbf{k}-\boldsymbol{\Delta})^2), \tag{3.1}$$

with $\boldsymbol{\Delta}=\mathbf{k}_1$ and where $\widetilde{F}(\mathbf{k}_1)$ and $\widetilde{G}(\mathbf{k}_2)$ have the generic soft-core forms

$$\widetilde{F}(\mathbf{k}^2)=\frac{\exp[-\mathbf{k}^2/\Lambda_1^2]}{\mathbf{k}^2+m_1^2},\quad \widetilde{G}(\mathbf{k}^2)=\frac{\exp[-\mathbf{k}^2/\Lambda_2^2]}{\mathbf{k}^2+m_2^2}. \tag{3.2}$$

Exploiting the identity

$$\frac{\exp[-\mathbf{k}^2/\Lambda^2]}{\mathbf{k}^2+m^2}=e^{m^2/\Lambda^2}\int_1^\infty\frac{dt}{\Lambda^2}\exp\left[-\left(\frac{\mathbf{k}^2+m^2}{\Lambda^2}\right)t\right], \tag{3.3}$$

one can write Eq. (3.1) as

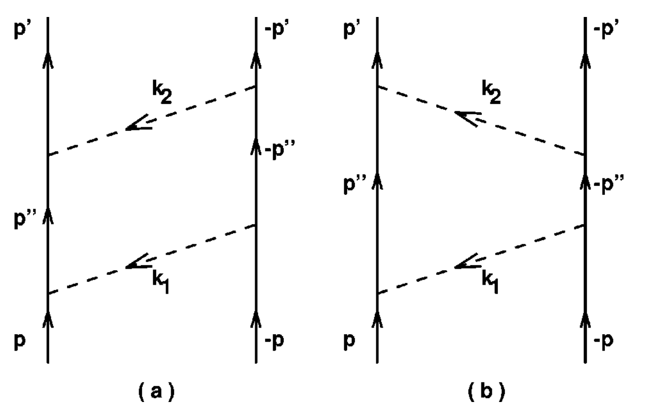

FIG. 2. Planar-box TMO two-meson-exchange graphs. Same notation as in Fig. 1. To these we have to add the "mirror" graphs and the graphs where we interchange the two meson lines.

$$\tilde{V}(\mathbf{k}) = e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \int_1^\infty \frac{dt}{\Lambda_1^2} \int_1^\infty \frac{du}{\Lambda_2^2} e^{-(m_1^2/\Lambda_1^2)t} e^{-(m_2^2/\Lambda_2^2)u} \times \int \frac{d^3\Delta}{(2\pi)^3} \exp\left[-\left(\frac{t}{\Lambda_1^2} + \frac{u}{\Lambda_2^2} \right) \mathbf{\Delta}^2 + (2\mathbf{k}\cdot\mathbf{\Delta} - \mathbf{k}^2) \frac{u}{\Lambda_2^2} \right]. \tag{3.4}$$

The Δ integral in Eq. (3.4) has been evaluated above [see Eq. (D12)] and is

$$(4\pi a)^{-3/2} \exp\left[-\frac{tu/\Lambda_1^2\Lambda_2^2}{(t/\Lambda_1^2 + u/\Lambda_2^2)} \mathbf{k}^2 \right],$$

with

$$a = \frac{t}{\Lambda_1^2} + \frac{u}{\Lambda_2^2}.$$

Redefining the variables $t \to t/\Lambda_1^2$ and $u \to u/\Lambda_2^2$, we rewrite Eq. (3.4) in the form

$$\tilde{V}(\mathbf{k}) = (4\pi)^{-3/2} e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \times \int_{t_0}^\infty dt \int_{u_0}^\infty du \frac{\exp[-(m_1^2 t + m_2^2 u)]}{(t+u)^{3/2}} \times \exp\left[-\left(\frac{tu}{t+u} \right) \mathbf{k}^2 \right] \quad (t_0 = 1/\Lambda_1^2, u_0 = 1/\Lambda_2^2). \tag{3.5}$$

This form we consider as the basic representation of the soft-core TME potentials in momentum space.

B. TPS tensor and spin-spin potentials

In this subsection we apply the same technique as used in the previous one to the more complicated case involving the tensor interaction. For PS-PS in momentum space we have

$$\tilde{V}(\mathbf{k}) = \int\int \frac{d^3k_1 d^3k_2}{(2\pi)^3} \delta(\mathbf{k} - \mathbf{k_1} - \mathbf{k_2}) \tilde{F}(\mathbf{k}_1^2) \times \tilde{G}(\mathbf{k}_2^2) \cdot [\boldsymbol{\sigma}_1 \cdot \mathbf{k}_1 \times \mathbf{k}_2][\boldsymbol{\sigma}_2 \cdot \mathbf{k}_1 \times \mathbf{k}_2] = \int \frac{d^3\Delta}{(2\pi)^3} \times [\boldsymbol{\sigma}_1 \cdot \mathbf{\Delta} \times \mathbf{k}][\boldsymbol{\sigma}_2 \cdot \mathbf{\Delta} \times \mathbf{k}] \tilde{F}(\mathbf{\Delta}^2) \cdot \tilde{G}((\mathbf{k} - \mathbf{\Delta})^2), \tag{3.6}$$

which leads to the basic integral to be evaluated:

$$\tilde{V}_{mn}(\mathbf{k}) = \int \frac{d^3\Delta}{(2\pi)^3} \Delta_m \Delta_n \tilde{F}(\mathbf{\Delta}^2) \cdot \tilde{G}((\mathbf{k} - \mathbf{\Delta})^2), \tag{3.7}$$

where $\Delta_m = (\mathbf{\Delta})_m$ and $\Delta_n = (\mathbf{\Delta})_n$. Repeating the steps taken in the previous subsection, one arrives at the analog of Eq. (3.4):

$$\tilde{V}_{mn}(\mathbf{k}) = e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \int_1^\infty \frac{dt}{\Lambda_1^2} \times \int_1^\infty \frac{du}{\Lambda_2^2} e^{-(m_1^2/\Lambda_1^2)t} e^{-(m_2^2/\Lambda_2^2)u} \int \frac{d^3\Delta}{(2\pi)^3} \Delta_m \Delta_n \times \exp\left[-\left(\frac{t}{\Lambda_1^2} + \frac{u}{\Lambda_2^2} \right) \mathbf{\Delta}^2 + (2\mathbf{k}\cdot\mathbf{\Delta} - \mathbf{k}^2) \frac{u}{\Lambda_2^2} \right]. \tag{3.8}$$

Using the integral

$$\int \frac{d^3\Delta}{(2\pi)^3} \Delta_m \Delta_n \exp[-a\mathbf{\Delta}^2 + 2b\mathbf{k}\cdot\mathbf{\Delta}] = (4\pi a)^{-3/2} \frac{1}{a} \left[\frac{1}{2}\delta_{mn} + \frac{b^2}{a} k_m k_n \right] \exp\left[+ \frac{b^2}{a} \mathbf{k}^2 \right],$$

one obtains, again after the redefinitions $t \to t/\Lambda_1^2$ and $u \to u/\Lambda_2^2$, for $\tilde{V}_{mn}$ the expression

$$\tilde{V}_{mn}(\mathbf{k}) = (4\pi)^{-3/2} e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \times \int_{t_0}^\infty dt \int_{u_0}^\infty du \frac{\exp[-(m_1^2 t + m_2^2 u)]}{(t+u)^{5/2}} \times \left[\frac{1}{2}\delta_{mn} + \frac{u^2}{(t+u)} k_m k_n \right] \exp\left[-\left(\frac{tu}{t+u} \right) \mathbf{k}^2 \right]. \tag{3.9}$$

It is clear that in Eq. (3.6) only the δ_{mn} term contributes. The result is

$$\tilde{V}(\mathbf{k}) = -\left[(\boldsymbol{\sigma}_1 \cdot \mathbf{k})(\boldsymbol{\sigma}_2 \cdot \mathbf{k}) - \frac{1}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\mathbf{k}^2 \right] \tilde{H}(\mathbf{k}^2) + \frac{2}{3}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\mathbf{k}^2 \tilde{H}(\mathbf{k}^2), \tag{3.10}$$

with

$$\tilde{H}(\mathbf{k}^2) = \frac{1}{2}(4\pi)^{-3/2} e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \int_{t_0}^\infty dt \times \int_{u_0}^\infty du \frac{\exp[-(m_1^2 t + m_2^2 u)]}{(t+u)^{5/2}} \exp\left[-\left(\frac{tu}{t+u} \right) \mathbf{k}^2 \right]. \tag{3.11}$$

From Eqs. (3.10)–(3.11) one can read off immediately the projection of Eq. (3.6) onto the spinor invariants (2.33). In Appendix D the same result is derived in an alternative way starting from the configuration-space potentials.

IV. GENERAL PS-PS EXCHANGE POTENTIALS

In [2] the derivation of the PS-PS exchange potentials in both momentum and configuration space is given. In that reference the configuration-space potentials are worked out fully. The topic of this paper is to do the same for the momentum-space description. In particular, the partial wave analysis is performed leading to a representation which is very suitable for numerical evaluation.

From [2] it appears that the momentum-space TME potential can be represented in general in the form

$$\tilde{V}_{\alpha\beta}(\mathbf{k}) = \frac{2}{\pi}\int_0^\infty d\lambda \sum_{i=1}^N w_i(\lambda) \int \frac{d^3\Delta}{(2\pi)^3} \times \tilde{O}_{\alpha\beta}(\mathbf{k},\mathbf{\Delta})\tilde{F}_\alpha^{(i)}(\mathbf{\Delta}^2,\lambda)\tilde{F}_\beta^{(i)}((\mathbf{k}-\mathbf{\Delta})^2,\lambda). \tag{4.1}$$

Here, Σ_i stands for the number of different types of λ functions, and $w_i(\lambda)$ are the corresponding weights. The operators $\tilde{O}_{\alpha\beta}$ are, for example, in the case of PS-PS exchange given by (see [2], Table II)

$$\tilde{O}^{\parallel}_{\alpha\beta}(\mathbf{k},\mathbf{\Delta}) = 2[\mathbf{\Delta}\cdot(\mathbf{k}-\mathbf{\Delta})]^2 - 2[\boldsymbol{\sigma}_1\cdot\mathbf{\Delta}\times\mathbf{k}][\boldsymbol{\sigma}_2\cdot\mathbf{\Delta}\times\mathbf{k}],$$

$$\tilde{O}^{X}_{\alpha\beta}(\mathbf{k},\mathbf{\Delta}) = 2[\mathbf{\Delta}\cdot(\mathbf{k}-\mathbf{\Delta}]^2 + 2[\boldsymbol{\sigma}_1\cdot\mathbf{\Delta}\times\mathbf{k}][\boldsymbol{\sigma}_2\cdot\mathbf{\Delta}\times\mathbf{k}]. \tag{4.2}$$

In [25] we remind the reader of the precise contents of these operators. As an example for the Σ_i we give explicitly the case of $\pi\eta$ exchange, the planar BW and TMO graphs, and the BW-crossed graphs. These can be written as [2]

$$\tilde{V}^{\parallel,\times}_{\pi\eta}(\mathbf{k}) = \pm\frac{2}{\pi}\int_0^\infty \frac{d\lambda}{\lambda^2}\int \frac{d^3\Delta}{(2\pi)^3}\tilde{O}^{\parallel,\times}_{\pi\eta}(\mathbf{k},\mathbf{\Delta})[\tilde{F}_\pi(\mathbf{\Delta}^2,m_\pi) \times\tilde{F}_\eta((\mathbf{k}-\mathbf{\Delta})^2,m_\eta) - \tilde{F}_\pi(\mathbf{\Delta}^2,\sqrt{m_\pi^2+\lambda^2}) \times\tilde{F}_\eta((\mathbf{k}-\mathbf{\Delta})^2,\sqrt{m_\eta^2+\lambda^2})e^{-\lambda^2/\Lambda_\pi^2}e^{-\lambda^2/\Lambda_\eta^2}]. \tag{4.3}$$

Here the $(+)$ sign refers to the parallel graphs ($\parallel$) and the $(-)$ sign to the crossed $(\times)$ graphs.

Integration over the λ parameter

Since due to the use of Eq. (3.3) and Gaussian form factors all integrations over λ become Gaussian, they can be performed analytically. Consider, for example, the type of integral appearing in Eq. (4.3). We write

$$\tilde{V}_{\alpha\beta}(\mathbf{k}) = \frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda^2}\int\frac{d^3\Delta}{(2\pi)^3}[\tilde{F}_\alpha(\mathbf{\Delta}^2,m_1)\tilde{F}_\beta((\mathbf{k}-\mathbf{\Delta})^2,m_2) - \tilde{F}_\alpha(\mathbf{\Delta}^2,\sqrt{m_1^2+\lambda^2}) \times\tilde{F}_\beta((\mathbf{k}-\mathbf{\Delta})^2,\sqrt{m_2^2+\lambda^2})e^{-\lambda^2/\Lambda_1^2}e^{-\lambda^2/\Lambda_2^2}]. \tag{4.4}$$

Going through the same steps as between Eqs. (3.3) and (3.5), we get for the second part between the square brackets, an extra λ-dependent factor

$$e^{-(\lambda^2/\Lambda_1^2)t}e^{-(\lambda^2/\Lambda_2^2)u}\rightarrow e^{-\lambda^2(t+u)},$$

where we applied again the same variable redefinition as before, i.e., $t\rightarrow t/\Lambda_1^2$ and $u\rightarrow u/\Lambda_2^2$. Then, the needed λ integral in Eq. (4.4) is

$$\frac{2}{\pi}\int_0^\infty\frac{d\lambda}{\lambda^2}[1-e^{-(t+u)\lambda^2}] = \frac{2}{\sqrt{\pi}}\sqrt{t+u}. \tag{4.5}$$

Therefore Eq. (4.4) becomes [compare with Eqs. (3.1)–(3.5)]

$$\tilde{V}(\mathbf{k}) = (2\pi)^{-2}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int_{t_0}^\infty dt\int_{u_0}^\infty du \times\frac{\exp[-(m_1^2t+m_2^2u)]}{(t+u)}\exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right] \quad (t_0=1/\Lambda_1^2, u_0=1/\Lambda_2^2). \tag{4.6}$$

Similarly, all λ integrations can be performed explicitly.

V. PROJECTION PS-PS EXCHANGE ON SPINOR INVARIANTS

The different contributions, such as adiabatic, nonadiabatic, pseudovector-vertex corrections, and off-shell potentials, are in direct correspondence with their configuration space analogs, given in [2].

A. Adiabatic PS-PS exchange potentials

We now have prepared all the necessary ingredients for putting things together and to list the projection of PS-PS exchange onto the potentials V_j. We write

$$\tilde{V}_j(\mathbf{k}) = (4\pi)^{-3/2}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int_{t_0}^\infty dt\int_{u_0}^\infty du \times\frac{\exp[-(m_1^2t+m_2^2u)]}{(t+u)^{3/2}} \times\exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right]\Omega_j(\mathbf{k}^2;t,u), \tag{5.1}$$

where

$$\Omega_1^{(\parallel)}(\mathbf{k}^2;t,u) = \frac{2}{\sqrt{\pi}}C^{(I)}_{NN,\parallel}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2 \times\left\{\frac{15}{4}+\frac{1}{2}\left(\frac{t^2-8ut+u^2}{t+u}\right)\mathbf{k}^2 + \left(\frac{t^2u^2}{(t+u)^2}\right)\mathbf{k}^4\right\}\frac{\sqrt{t+u}}{(t+u)^2},$$

TABLE I. Coefficients $\Upsilon_{j,k}$ for the planar ($\|$) adiabatic PS-PS contributions.

	$\Upsilon_0(\|)(t,u)$	$\Upsilon_1(\|)(t,u)$	$\Upsilon_2(\|)(t,u)$
$\Omega_1^{(\|)}$	$\frac{15}{5}\frac{\sqrt{t+u}}{(t+u)^2}$	$\frac{1}{2}\left(\frac{t^2-8ut+u^2}{t+u}\right)\frac{\sqrt{t+u}}{(t+u)^2}$	$\frac{t^2u^2}{(t+u)^2}\frac{\sqrt{t+u}}{(t+u)^2}$
$\Omega_2^{(\|)}$	—	$-\frac{1}{3}\frac{\sqrt{t+u}}{(t+u)}$	—
$\Omega_3^{(\|)}$	$+\frac{1}{2}\frac{\sqrt{t+u}}{(t+u)}$	—	—

$$\Omega_2^{(\|)}(\mathbf{k}^2;t,u)=-\frac{2}{\sqrt{\pi}}C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\frac{1}{3}\mathbf{k}^2\frac{\sqrt{t+u}}{(t+u)},$$

$$\Omega_3^{(\|)}(\mathbf{k}^2;t,u)=\frac{2}{\sqrt{\pi}}C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\frac{1}{2}\frac{\sqrt{t+u}}{(t+u)}, \tag{5.2}$$

where $C_{NN,\|}(I)$ denotes the isospin factor.

Expressions similar to Eqs. (5.2) hold for the contribution of the crossed BW graphs ($\times$) and are given below. Note the difference in the $\tilde{O}$ operators in Eq. (4.2) and $D_\times(\omega_1,\omega_2)=-D_\|(\omega_1,\omega_2)$.

$$\Omega_1^{(\times)}(\mathbf{k}^2;t,u)=-\frac{2}{\sqrt{\pi}}C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2 \times\left\{\frac{15}{4}+\frac{1}{2}\left(\frac{t^2-8ut+u^2}{t+u}\right)\mathbf{k}^2 +\left(\frac{t^2u^2}{(t+u)^2}\right)\mathbf{k}^4\right\}\frac{\sqrt{t+u}}{(t+u)^2},$$

TABLE II. Coefficients $\Upsilon_{j,k}$ for the crossed ($\times$) adiabatic PS-PS contributions.

	$\Upsilon_0(\times)(t,u)$	$\Upsilon_1(\times)(t,u)$	$\Upsilon_2(\times)(t,u)$
$\Omega_1^{(\times)}$	$-\frac{15}{4}\frac{\sqrt{t+u}}{(t+u)^2}$	$-\frac{1}{2}\left(\frac{t^2-8ut+u^2}{t+u}\right)\frac{\sqrt{t+u}}{(t+u)^2}$	$-\frac{t^2u^2}{(t+u)^2}\frac{\sqrt{t+u}}{(t+u)^2}$
$\Omega_2^{(\times)}$	—	$-\frac{1}{3}\frac{\sqrt{t+u}}{(t+u)}$	—
$\Omega_3^{(\times)}$	$+\frac{1}{2}\frac{\sqrt{t+u}}{(t+u)}$	—	—

$$\Omega_2^{(\times)}(\mathbf{k}^2;t,u)=-\frac{2}{\sqrt{\pi}}C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\frac{1}{3}\mathbf{k}^2\frac{\sqrt{t+u}}{(t+u)},$$

$$\Omega_3^{(\times)}(\mathbf{k}^2;t,u)=\frac{2}{\sqrt{\pi}}C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\frac{1}{2}\frac{\sqrt{t+u}}{(t+u)}. \tag{5.3}$$

It is convenient to introduce the expansions

$$\Omega_j^{(\|,\times)}(\mathbf{k}^2;t,u)=\frac{2}{\sqrt{\pi}}C_{NN}^{(I)}(\|,\times)\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2 \times\sum_{k=0}^{K}\Upsilon_{j,k}^{1a,1b,1c}(t,u)(\mathbf{k}^2)^k. \tag{5.4}$$

The functions $\Upsilon_{j,k}(t,u)$ are given in Tables I and II.

B. Nonadiabatic corrections

The denominators $D^{(1)}(\omega_1,\omega_2)$ for the nonadiabatic corrections are given in Ref. [2], Table III. In appendix B of [12] it is explained how $1/\omega^4$ has to be treated. It follows that

TABLE III. Coefficients $\Upsilon_{j,k}^{(1a)}$ for the planar ($\|$) nonadiabatic PS-PS contributions.

	$\Upsilon_0^{(1a)}(\|)(t,u)$	$\Upsilon_1^{(1a)}(\|)(t,u)$	$\Upsilon_2^{(1a)}(\|)(t,u)$	$\Upsilon_3^{(1a)}(\|)(t,u)$
$\Omega_1^{(1a,\|)}$	$-\frac{105}{8}\frac{1}{(t+u)^3}$	$-\frac{15}{4}\left(\frac{t^2-5ut+u^2}{(t+u)^4}\right)$	$\frac{3}{2}\left(\frac{t^2-5ut+u^2}{(t+u)^5}\right)$	$\frac{t^3u^3}{(t+u)^6}$
$\Omega_2^{(1a,\|}$	—	$\frac{5}{6}\frac{1}{(t+u)^2}$	$-\frac{1}{3}\frac{tu}{(t+u)^3}$	—
$\Omega_3^{(1a,\|)}$	$-\frac{5}{4}\frac{1}{(t+u)^2}$	$\frac{1}{2}\frac{tu}{(t+u)^3}$	—	—
$\Omega_4^{(1a,\|)}$	$-\frac{1}{(t+u)^2}$	$\frac{2tu}{(t+u)^3}$	—	—

$$D_{\|}^{(1)}(\omega_1,\omega_2)\Rightarrow -\frac{1}{2}\left(\frac{\partial}{\partial m_1^2}-\frac{1}{\Lambda_1^2}+\frac{\partial}{\partial m_2^2}-\frac{1}{\Lambda_2^2}\right)\frac{1}{\omega_1^2\omega_2^2}, \tag{5.5}$$

and one finds from Eqs. (3.4) and (5.5) that the contributions to the $\tilde{V}_j^{(1)}$ can be written in the form

$$\tilde{V}_j^{(1a)}(\mathbf{k})=(4\pi)^{-3/2}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int_{t_0}^{\infty}dt\int_{u_0}^{\infty}du \times\frac{\exp[-(m_1^2t+m_2^2u)]}{(t+u)^{3/2}}\frac{1}{2}(t+u) \times\exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right]\Omega_j^{(1a)}(\mathbf{k}^2;t,u). \tag{5.6}$$

The $\Omega_j^{(1a)}$ are given by

$$\Omega_1^{(1a,\|)}(\mathbf{k}^2;t,u)=C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \times\left\{-\frac{105}{8}+-\frac{15}{4}\left(\frac{t^2-5ut+u^2}{t+u}\right)\mathbf{k}^2 +\frac{3}{2}tu\left(\frac{t^2-5ut+u^2}{(t+u)^2}\right)\mathbf{k}^4 +\left(\frac{t^3u^3}{(t+u)^3}\right)\mathbf{k}^6\right\}\frac{1}{(t+u)^3},$$

$$\Omega_2^{(1a,\|)}(\mathbf{k}^2;t,u)=-C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right)\frac{2}{3}\mathbf{k}^2 \times\left\{-\frac{5}{4}+\frac{1}{2}\frac{tu}{t+u}\mathbf{k}^2\right\}\frac{1}{(t+u)^2},$$

$$\Omega_3^{(1a,\|)}(\mathbf{k}^2;t,u)=-C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \left\{\frac{5}{4}-\frac{1}{2}\frac{tu}{t+u}\mathbf{k}^2\right\}\frac{1}{(t+u)^2},$$

$$\Omega_4^{(1a,\|)}(\mathbf{k}^2;t,u)=-C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \left\{5-2\frac{tu}{t+u}\mathbf{k}^2\right\}\frac{1}{(t+u)^2}. \tag{5.7}$$

Again, similar expressions hold for the contribution of the crossed BW graphs ($\times$) and are given below [see for the differences in the $\tilde{O}^{(1)}$ operators [2] Eqs. (5.3) and (5.4), and $D_X^{(1)}(\omega_1,\omega_2)=-2D_{\|}^{(1)}(\omega_1,\omega_2)$, conform, Table III of [2]]:

$$\Omega_1^{(1a,\times)}(\mathbf{k}^2;t,u)=-2C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \times\left\{-\frac{105}{8}+-\frac{15}{4}\left(\frac{t^2-5ut+u^2}{t+u}\right)\mathbf{k}^2 +\frac{3}{2}tu\left(\frac{t^2-5ut+u^2}{(t+u)^2}\right)\mathbf{k}^4 +\left(\frac{t^3u^3}{(t+u)^3}\right)\mathbf{k}^6\right\}\frac{1}{(t+u)^3},$$

$$\Omega_2^{(1a,\times)}(\mathbf{k}^2;t,u)=-2C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right)\frac{2}{3} \times\mathbf{k}^2\left\{-\frac{5}{4}+\frac{1}{2}\frac{tu}{t+u}\mathbf{k}^2\right\}\frac{1}{(t+u)^2},$$

$$\Omega_3^{(1a,\times)}(\mathbf{k}^2;t,u)=-2C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \left\{\frac{5}{4}-\frac{1}{2}\frac{tu}{t+u}\mathbf{k}^2\right\}\frac{1}{(t+u)^2}. \tag{5.8}$$

C. Pseudovector-vertex corrections

Here, only the crossed BW graphs contribute [2]. From the denominators $D^{(1)}(\omega_1,\omega_2)=1/\omega_1^2\omega_2^2$ for the pseudovector-vertex corrections [2] one finds that the contributions to the $\tilde{V}_j^{(1)}$ can be written in the form

$$\tilde{V}_j^{(1b)}(\mathbf{k})=(4\pi)^{-3/2}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int_{t_0}^{\infty}dt\int_{u_0}^{\infty}du \times\frac{\exp[-(m_1^2t+m_2^2u)]}{(t+u)^{3/2}} \times\exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right]\Omega_j^{(1b)}(\mathbf{k}^2;t,u), \tag{5.9}$$

where

$$\Omega_1^{(1b,\times)}(\mathbf{k}^2;t,u)=C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \times\left\{-\frac{15}{2}+-\frac{5}{2}\left(\frac{t^2-2tu+u^2}{t+u}\right)\mathbf{k}^2 +tu\left(\frac{t^2+u^2}{(t+u)^2}\right)\mathbf{k}^4\right\}\frac{1}{(t+u)^2},$$

$$\Omega_4^{(1b,\times)}(\mathbf{k}^2;t,u)=C_{NN,\times}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{1}{M}\right) \times\left\{-1+2\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right\}\frac{1}{t+u}. \tag{5.10}$$

TABLE IV. Coefficients $\Upsilon_{j,k}^{(1a)}$ for the crossed ($\times$) nonadiabatic PS-PS contributions.

	$\Upsilon_0^{(1a)}(\times)(t,u)$	$\Upsilon_1^{(1a)}(\times)(t,u)$	$\Upsilon_2^{(1a)}(\times)(t,u)$	$\Upsilon_3^{(1a)}(\times)(t,u)$
$\Omega_1^{(1a,\times)}$	$+\frac{105}{4}\frac{1}{(t+u)^3}$	$+\frac{15}{2}\left(\frac{t^2-5ut+u^2}{(t+u)^4}\right)$	$-3\left(\frac{t^2-5ut+u^2}{(t+u)^5}\right)$	$-2\frac{t^3u^3}{(t+u)^6}$
$\Omega_2^{(1a,\times}$	—	$\frac{5}{3}\frac{1}{(t+u)^2}$	$-\frac{2}{3}\frac{tu}{(t+u)^3}$	—
$\Omega_3^{(1a,\|)}$	$-\frac{5}{2}\frac{1}{(t+u)^2}$	$\frac{tu}{(t+u)^3}$	—	—

D. Off-shell corrections in TMO diagrams

Also in this case the denominators are $D^{(1)}(\omega_1,\omega_2) = 1/\omega_1^2\omega_2^2$ [2], and one finds that the contributions to the $\tilde{V}_j^{(1)}$ can be written as

$$\tilde{V}_j^{(1c)}(\mathbf{k}) = (4\pi)^{-3/2} e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \int_{t_0}^{\infty} dt \int_{u_0}^{\infty} du \times \frac{\exp[-(m_1^2 t + m_2^2 u)]}{(t+u)^{3/2}} \times \exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right] \Omega_j^{(1c)}(\mathbf{k}^2;t,u). \quad (5.11)$$

The potentials are very similar to the pseudovector-vertex corrections given above; see [2], Sec. V C. We have

$$\Omega_1^{(1c,\|)}(\mathbf{k}^2;t,u) = C_{NN,\|}^{(I)} \left(\frac{f_{NN\pi}}{m_\pi}\right)^2 \left(\frac{f_{NN\eta}}{m_\pi}\right)^2 \left(\frac{1}{2M}\right) \times \left\{ -\frac{15}{2} + -\frac{5}{2}\left(\frac{t^2-2tu+u^2}{t+u}\right)\mathbf{k}^2 + tu\left(\frac{t^2+u^2}{(t+u)^2}\right)\mathbf{k}^4 \right\} \frac{1}{(t+u)^2},$$

$$\Omega_4^{(1c,\|)}(\mathbf{k}^2;t,u) = C_{NN,//}^{(I)} \left(\frac{f_{NN\pi}}{m_\pi}\right)^2 \left(\frac{f_{NN\eta}}{m_\pi}\right)^2 \left(\frac{1}{2M}\right) \times \left\{ -5 + 2\left(\frac{tu}{t+u}\right)\mathbf{k}^2 \right\} \frac{1}{t+u}. \quad (5.12)$$

Similarly to Eq. (5.4) it is convenient to introduce the expansion

$$\Omega_j^{(1a,1b,1c)}(\mathbf{k}^2;t,u) = C_{NN,1a,1b,1c}^{(I)} \left(\frac{f_{NN\pi}}{m_\pi}\right)^2 \left(\frac{f_{NN\eta}}{m_\pi}\right)^2 \left(\frac{1}{M}\right) \times \sum_{k=0}^{K} \Upsilon_{j,k}^{1a,1b,1c}(t,u)(\mathbf{k}^2)^k. \quad (5.13)$$

The functions $\Upsilon_{j,k}(t,u)$ are given in Tables III–VI.

VI. PARTIAL WAVE ANALYSIS

A. General structure

We write the potentials listed in the previous section in the general form

$$\tilde{V}_j(\mathbf{k}) = \int_{t_0}^{\infty} dt \int_{u_0}^{\infty} du\, w(t,u) \exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right] \times \Omega_j^{(type)}(\mathbf{k}^2;t,u), \quad (6.1a)$$

where for the adiabatic, $1/M$ PV, and off-shell corrections, the weight function $w(t,u) = w_0(t,u)$ with

$$w_0(t,u) = (4\pi)^{-3/2} e^{m_1^2/\Lambda_1^2} e^{m_2^2/\Lambda_2^2} \frac{\exp[-(m_1^2 t + m_2^2 u)]}{(t+u)^{3/2}}, \quad (6.1b)$$

and for the nonadiabatic corrections [see Eq. (5.6)],

TABLE V. Coefficients $\Upsilon_{j,k}^{(1b)}$ for the pseudovector-vertex correction PS-PS contributions.

	$\Upsilon_0^{(1b)}(t,u)$	$\Upsilon_1^{(1b)}(t,u)$	$\Upsilon_2^{(1b)}(t,u)$
$\Omega_1^{(1b,\times)}$	$-\frac{15}{2}\frac{1}{(t+u)^2}$	$-\frac{5}{2}\left(\frac{t^2-2ut+u^2}{t+u}\right)\frac{1}{(t+u)^2}$	$\frac{t^2+u^2}{(t+u)^2}\frac{tu}{(t+u)^2}$
$\Omega_4^{(1b,\times)}$	$-\frac{1}{t+u}$	$\frac{2tu}{(t+u)}$	—

TABLE VI. Coefficients $\Upsilon^{(1c)}_{j,k}$ for the off-shell correction TMO graphs PS-PS contributions.

	$\Upsilon^{(1c)}_0(t,u)$	$\Upsilon^{(1c)}_1(t,u)$	$\Upsilon^{(1c)}_2(t,u)$
$\Omega_1^{(1c,TMO)}$	$-\frac{15}{4}\frac{1}{(t+u)^2}$	$-\frac{5}{4}\left(\frac{t^2-2ut+u^2}{t+u}\right)\frac{1}{(t+u)^2}$	$\frac{1}{2}\frac{t^2+u^2}{(t+u)^2}\frac{tu}{(t+u)^2}$
$\Omega_4^{(1c,TMO)}$	$-\frac{5}{2}\frac{1}{t+u}$	$\frac{tu}{(t+u)}$	—

$$w_{na}(t,u)=\left(\frac{\Lambda_1^2+\Lambda_2^2}{\Lambda_1^2\Lambda_2^2}-(t+u)\right)w_0(t,u). \tag{6.1c}$$

From $\mathbf{k}^2=p_f^2+p_i^2-2p_fp_iz$, where $z=\cos\theta$, and the Ω_j's one readily sees that the potential contributions can be written in the form

$$\widetilde{V}_j(\mathbf{k})=\int_{t_0}^{\infty}dt\int_{u_0}^{\infty}du\, w(t,u)[X_j(t,u)+zY_j(t,u)+z^2Z_j(t,u)]\exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right], \tag{6.2}$$

where

$$X_j(t,u)=\Gamma_j[\Upsilon_{j,0}(t,u)+(p_f^2+p_i^2)\Upsilon_{j,1}(t,u)+(p_f^2+p_i^2)^2\Upsilon_{j,2}], \tag{6.3a}$$

$$Y_j(t,u)=-\Gamma_j[2p_fp_i\Upsilon_{j,1}(t,u)+4p_fp_i(p_f^2+p_i^2)\Upsilon_{j,2}], \tag{6.3b}$$

$$Z_j(t,u)=\Gamma_j[4p_f^2p_i^2\Upsilon_{j,2}]. \tag{6.3c}$$

Here Γ_j is a combination of isospin and coupling constant factors.

B. Partial wave projection

The partial wave projection of Eq. (D13) can be done easily. Using formula 10.2.36 of Ref. [26] one derives that

$$e^{-\gamma\mathbf{k}^2}=\sum_{L=0}^{\infty}(2L+1)f_L(2\gamma p_fp_i)P_L(z)e^{-\gamma(p_f-p_i)^2},$$

$$\gamma\equiv\frac{tu}{t+u}. \tag{6.4}$$

Here the function $f_L(z)$ is defined as

$$f_L(2\gamma p_fp_i)\equiv\sqrt{\frac{\pi}{4\gamma p_fp_i}}I_{L+1/2}(2\gamma p_fp_i)e^{-2\gamma p_fp_i}. \tag{6.5}$$

We note that in the form (6.4), the Gaussian damping in the off-shell momentum region is manifest.

From Eq. (6.4) and the recurrence relations for Legendre polynomials, one readily obtains, e.g.,

$$ze^{-\gamma\mathbf{k}^2}=\sum_{L=0}^{\infty}(2L+1)\left(\frac{L}{2L+1}f_{L-1}+\frac{L+1}{2L+1}f_{L+1}\right)\times(2\gamma p_fp_i)P_L(z)e^{-\gamma(p_f-p_i)^2},$$

$$z^2e^{-\gamma\mathbf{k}^2}=\sum_{L=0}^{\infty}(2L+1)\left(\frac{L(L-1)}{(2L+1)(2L-1)}f_{L-2}+\frac{4L^3+6L^2-1}{(2L-1)(2L+1)(2L+3)}f_L+\frac{(L+1)(L+2)}{(2L+1)(2L+3)}f_{L+2}\right)\times(2\gamma p_fp_i)P_L(z)e^{-\gamma(p_f-p_i)^2}. \tag{6.6}$$

Using these results the partial wave contributions can be worked out in a straightforward manner. The basic partial wave projections needed are

$$U_L(t,u)=\frac{1}{2}\int_{-1}^{+1}dz\,P_L(z)\exp(-\gamma\mathbf{k}^2),$$

$$R_L(t,u)=\frac{1}{2}\int_{-1}^{+1}dz\,z\,P_L(z)\exp(-\gamma\mathbf{k}^2),$$

$$S_L(t,u)=\frac{1}{2}\int_{-1}^{+1}dz\,z^2P_L(z)\exp(-\gamma\mathbf{k}^2), \tag{6.7}$$

where $\gamma=tu/(t+u)$, and the functions U_L,R_L,S_L can be read off from Eqs. (6.4)–(6.6). Writing

$$V(\mathbf{p}_f,\mathbf{p}_i)=\sum_{j=1}^{4}\widetilde{V}_j(\mathbf{p}_f,\mathbf{p}_i)(\mathbf{p}_f|P_j|\mathbf{p}_i), \tag{6.8}$$

the partial wave expansion of the V_j functions reads

$$V_j(\mathbf{p}_f,\mathbf{p}_i)=\sum_{L=0}^{\infty}(2L+1)\widetilde{V}_L^{(j)}(x)P_L(\cos\theta). \tag{6.9}$$

Using Eqs. (6.4)–(6.6) the partial waves $V_L^{(j)}(x)$ for the TME potentials can be written as

$$\tilde{V}_L^{(j)}(p_f,p_i)=\int_{t_0}^{\infty}dt\int_{u_0}^{\infty}du\, w(t,u)[X_j(t,u)U_L(t,u)+Y_j(t,u)R_L(t,u)+Z_j(t,u)S_L(t,u)]\equiv\mathcal{S}_L[X_j\cdot U_L+Y_j\cdot R_L+Z_j\cdot S_L]. \quad (6.10)$$

C. Partial wave projection spinor invariants

Distinguishing between the partial waves with parity $P=(-)^J$ and $P=-(-)^J$, we write the potential matrix elements on the LSJ basis in the following way (see, e.g., [22], Sec. VII):

(i) $P=(-)^J$:

$$(p_f;L'S'J'M'|V|p_i;LSJM)=4\pi V^{J,+}(S',S)\,\delta_{J'J}\delta_{M'M}\delta_{L'L}. \quad (6.11)$$

(ii) $P=-(-)^J$:

$$(p_f;L'S'J'M'|V|p_i;LSJM)=4\pi\,\delta_{J'J}\delta_{M'M}\delta_{S'S}V^{J,-}(L',L). \quad (6.12)$$

For notational convenience we will use as an index the parity factor η, which is defined by writing $P=\eta(-)^J$. The $P=(-)^J$ states contain the spin-singlet- and -triplet-uncoupled states $(\eta=+)$, and the $P=-(-)^J$ states contain the spin-triplet-coupled states $(\eta=-)$.

In nucleon-nucleon, since the proton neutron mass difference M_p-M_n is small, except for very special studies, the spin-singlet-triplet transitions can be neglected. Actually, in the TME potentials we take $M_p=M_n=M_N$. As a consequence there are no antisymmetric spin-orbit potentials, so $V_6=0$. Also, since we restrict ourselves to terms of order up to and including $1/M_N$, there are no contributions to the quadratic-spin-orbit potentials, i.e., $V_5=0$. Also, in nucleon-nucleon transitions one can neglect terms with P_7 and P_8 [22]. Therefore, we can restrict the partial wave projection of the spinor invariants to the cases P_1, P_2, P_3, and P_4.

Below we list the partial wave matrix elements for $\eta=\pm$ for the different V_jP_j $(j=1,2,3)$. Here we restrict ourselves to the matrix elements $\neq 0$.

(i) *Central* $P_1=1$:

$$(p_f;L'S'J'M'|V^{(1)}P_1|p_i;LSJM)=4\pi\,\delta_{J'J}\delta_{M'M}F_1^{J,\eta}(L'S',LS), \quad (6.13)$$

with

$$F_1^{J,\eta}(L'S',LS)=\delta_{L'L}\delta_{S'S}V_L^{(1)}(x).$$

(ii) *Spin-spin* $P_2=\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2$:

$$(p_f;L'S'J'M'|V^{(2)}P_2|p_i;LSJM)=4\pi\,\delta_{J'J}\delta_{M'M}F_2^{J,\eta}(L'S',LS), \quad (6.14)$$

with

$$F_2^{J,\eta}(L'S',LS)=\delta_{L'L}\delta_{S'S}[2S(S+1)-3]V_L^{(2)}(x).$$

(iii) *Tensor* $P_3=(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})-\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2$:

$$(p_f;L'S'J'M'|V^{(3)}P_3|p_i;LSJM)=\frac{8\pi}{3}(p_f^2+p_i^2)\,\delta_{J'J}\delta_{M'M}F_3^{J,\eta}(i,j), \quad (6.15)$$

where $i=S'$ and $j=S$ for $\eta=+$, and $i=L'$ and $j=L$ for $\eta=-$.

(a) Triplet uncoupled: $L=L'=J$, $S=S'=1$,

$$F_3^{J,+}(1,1)=\left[V_J^{(3)}-\frac{1}{2}\sin 2\psi\left(\frac{2J+3}{2J+1}V_{J-1}^{(3)}+\frac{2J-1}{2J+1}V_{J+1}^{(3)}\right)\right]. \quad (6.16)$$

(b) Triplet coupled: $L=J\pm1$, $L'=J\pm1$, $S=S'=1$,

$$F_3^{J,-}(J-1,J-1)=\frac{J-1}{2J+1}\left[-V_{J-1}^{(3)}+\frac{1}{2}\sin 2\psi\times\left\{\frac{2J-3}{2J-1}V_J^{(3)}+\frac{2J+1}{2J-1}V_{J-2}^{(3)}\right\}\right], \quad (6.17a)$$

$$F_3^{J,-}(J-1,J+1)=-3\frac{\sqrt{J(J+1)}}{2J+1}[-\sin 2\psi V_J^{(3)}+(\cos^2\psi V_{J-1}^{(3)}+\sin^2\psi V_{J+1}^{(3)})], \quad (6.17b)$$

$$F_3^{J,-}(J+1,J-1)=-3\frac{\sqrt{J(J+1)}}{2J+1}[-\sin 2\psi V_J^{(3)}+(\sin^2\psi V_{J-1}^{(3)}+\cos^2\psi V_{J+1}^{(3)})], \quad (6.17c)$$

$$F_3^{J,-}(J+1,J+1)=\frac{J+2}{2J+1}\left[-V_{J+1}^{(3)}+\frac{1}{2}\sin 2\psi\times\left\{\frac{2J+5}{2J+3}V_J^{(3)}+\frac{2J+1}{2J+3}V_{J+2}^{(3)}\right\}\right], \quad (6.17d)$$

where we introduced

$$\cos\psi=\frac{p_i}{\sqrt{p_f^2+p_i^2}},\quad \sin\psi=\frac{p_f}{\sqrt{p_f^2+p_i^2}}. \quad (6.18)$$

(iv) *Spin-orbit* $P_4=i/2(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}$:

$$(p_f;L'S'J'M'|V^{(4)}P_4|p_i;LSJM)=4\pi p_f p_i \delta_{J'J}\delta_{M'M}F_4^{J,\eta}(i,j). \tag{6.19}$$

(a) Triplet uncoupled: $L=L'=J,\ S=S'=1$,

$$F_4^{J,+}(1,1)=-(V_{J-1}^{(4)}-V_{J+1}^{(4)})/(2J+1). \tag{6.20}$$

(b) Triplet coupled: $L=J\pm1,\ L'=J\pm1,\ S=S'=1$,

$$F_4^{J,-}(J-1,J-1)=\frac{(J-1)}{(2J-1)}(V_{J-2}^{(4)}-V_J^{(4)}),$$

$$F_4^{J,-}(J+1,J+1)=-\frac{(J+2)}{(2J+3)}(V_J^{(4)}-V_{J+2}^{(4)}). \tag{6.21}$$

With the matrix elements of this section, the partial waves for the potentials can be readily derived. Henceforth, we will use the following shorthand notation [27] for the potentials:

(i) $P=(-)^J$:

$$V_{0,0}^J=V^{J,+}(0,0),\quad V_{0,2}^J=V^{J,+}(0,1),$$

$$V_{2,0}^J=V^{J,+}(1,0),\quad V_{2,2}^J=V^{J,+}(1,1). \tag{6.22}$$

(ii) $P=-(-)^J$:

$$V_{1,1}^J=V^{J,-}(J-1,J-1),\quad V_{1,3}^J=V^{J,-}(J-1,J+1),$$

$$V_{3,1}^J=V^{J,-}(J+1,J-1),\quad V_{3,3}^J=V^{J,-}(J+1,J+1), \tag{6.23}$$

where it is always understood that the final and initial state momenta are, respectively, p_f and p_i. So $V_{0,0}^J=V_{0,0}^J(p_f,p_i)$, etc. Since

$$V_{2,0}^J(p_f,p_i)=V_{0,2}^J(p_i,p_f),\quad V_{3,1}^J(p_f,p_i)=V_{1,3}^J(p_i,p_f), \tag{6.24}$$

we will give in the case of the off-diagonal terms only the explicit expressions for $V_{0,2}^J(p_f,p_i)$ and $V_{1,3}^J(p_f,p_i)$.

D. Partial wave projection potentials

The momentum-space partial wave central (C), spin-spin (σ), tensor (T), and spin-orbit (SO) potentials $V_L^{(C)}=V_L^{(1)}(p_f,p_i)$, etc., lead to the following partial wave potentials $\neq 0$:

$$V_{0,0}^J=4\pi(V_J^{(C)}-3V_J^{(\sigma)}), \tag{6.25a}$$

$$V_{2,2}^J=4\pi\Bigg[(V_J^{(C)}+V_J^{(\sigma)})+\frac{2}{3}(p_f^2+p_i^2)\times\left\{V_J^{(T)}-\frac{1}{2}\sin 2\psi\left(\frac{2J+3}{2J+1}V_{J-1}^{(T)}+\frac{2J-1}{2J+1}V_{J+1}^{(T)}\right)\right\}-p_f p_i(V_{J-1}^{(SO)}-V_{J+1}^{(SO)})/(2J+1)\Bigg], \tag{6.25b}$$

$$V_{1,1}^J=4\pi\Bigg[(V_{J-1}^{(C)}+V_{J-1}^{(\sigma)})+\frac{2}{3}(p_f^2+p_i^2)\frac{J-1}{2J+1}\times\left\{-V_{J-1}^{(T)}+\frac{1}{2}\sin 2\psi\left(\frac{2J-3}{2J-1}V_J^{(T)}+\frac{2J+1}{2J-1}V_{J-2}^{(T)}\right)\right\}+p_f p_i(J-1)(V_{J-2}^{(SO)}-V_J^{(SO)})/(2J-1)\Bigg], \tag{6.25c}$$

$$V_{1,3}^J=4\pi\Bigg[2(p_f^2+p_i^2)\frac{\sqrt{J(J+1)}}{2J+1}\{\sin 2\psi V_J^{(T)}-(\cos^2\psi V_{J-1}^{(T)}+\sin^2\psi V_{J+1}^{(T)})\}\Bigg], \tag{6.25d}$$

$$V_{3,3}^J=4\pi\Bigg[(V_{J+1}^{(C)}+V_{J+1}^{(\sigma)})+\frac{2}{3}(p_f^2+p_i^2)\frac{J+2}{2J+1}\times\left\{-V_{J+1}^{(T)}+\frac{1}{2}\sin 2\psi\left(\frac{2J+5}{2J+3}V_J^{(T)}+\frac{2J+1}{2J+3}V_{J+2}^{(T)}\right)\right\}-p_f p_i(J+2)(V_J^{(SO)}-V_{J+2}^{(SO)})/(2J+3)\Bigg]. \tag{6.25e}$$

Notice that $V_{0,2}^J=V_{2,0}^J=0$ because $V_L^{(ASO)}=0$. Furthermore, $V_{3,1}^J=V_{1,3}^J$.

VII. ADIABATIC PS-PS POTENTIALS

A. Adiabatic PS-PS coefficients

Defining the shorthand $A(p_f,p_i)$ and $B(p_f,p_i)$ by

$$\mathbf{k}^2=(p_f^2+p_i^2)-2p_f p_i z\equiv A-Bz,\quad \mathbf{k}^4\equiv A^2-2ABz+B^2z^2, \tag{7.1}$$

one obtains the following from Eqs. (5.2) for the coefficients $X,Y,Z\neq 0$:

(i) Central:

$$X_1^{(\parallel)}(t,u)=+\frac{2}{\sqrt{\pi}}C_{NN,\parallel}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\times\frac{1}{4}\left\{15+2\left(\frac{t^2-8ut+u^2}{t+u}\right)A(p_f,p_i)+4\left(\frac{t^2u^2}{(t+u)^2}\right)A^2(p_f,p_i)\right\}\frac{\sqrt{t+u}}{(t+u)^2},$$

$$Y_1^{(\parallel)}(t,u)=-\frac{1}{\sqrt{\pi}}C_{NN,\parallel}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\times\left\{\left(\frac{t^2-8ut+u^2}{t+u}\right)B(p_f,p_i)+4\left(\frac{t^2u^2}{(t+u)^2}\right)A(p_f,p_i)B(p_f,p_i)\right\}\frac{\sqrt{t+u}}{(t+u)^2},$$

$$Z_1^{(\|)}(t,u)=+\frac{2}{\sqrt{\pi}}C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\left(\frac{t^2u^2}{(t+u)^2}\right)\times B^2(p_f,p_i)\frac{\sqrt{t+u}}{(t+u)^2}. \tag{7.2}$$

(ii) Spin-spin:

$$X_2^{(\|)}(t,u)=-\frac{2}{3\sqrt{\pi}}C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2 A(p_f,p_i)\frac{\sqrt{t+u}}{(t+u)},$$

$$Y_2^{(\|)}(t,u)=+\frac{2}{3\sqrt{\pi}}C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2 B(p_f,p_i)\frac{\sqrt{t+u}}{(t+u)}. \tag{7.3}$$

(iii) Tensor:

$$X_3^{(\|)}(t,u)=+\frac{1}{\sqrt{\pi}}C_{NN,\|}^{(I)}\left(\frac{f_{NN\pi}}{m_\pi}\right)^2\left(\frac{f_{NN\eta}}{m_\pi}\right)^2\frac{\sqrt{t+u}}{(t+u)}. \tag{7.4}$$

For the $\times$ diagram contributions one has

$$X_1^{(\times)}=-X_1^{(\|)},\quad Y_1^{(\times)}=-Y_1^{(\|)},$$
$$Z_1^{(\times)}=-Z_1^{(\|)},$$
$$X_2^{(\times)}=+X_2^{(\|)},\quad Y_2^{(\times)}=+Y_2^{(\|)},$$
$$X_3^{(\times)}=+X_3^{(\|)}. \tag{7.5}$$

B. Adiabatic PS-PS partial wave potentials

The central, spin-spin, and tensor partial wave contributions are now

$$V_L^{(C)}(p_f,p_i)=\mathcal{S}_L[X_1\cdot U_L+Y_1\cdot R_L+Z_1\cdot S_L]\equiv\mathcal{S}_L[\mathbf{X}_C\cdot\mathbf{U}_L],$$
$$V_L^{(\sigma)}(p_f,p_i)=\mathcal{S}_L[X_2\cdot U_L+Y_2\cdot R_L+Z_2\cdot S_L]\equiv\mathcal{S}_L[\mathbf{X}_\sigma\cdot\mathbf{U}_L],$$
$$V_L^{(T)}(p_f,p_i)=\mathcal{S}_L[X_3\cdot U_L], \tag{7.6}$$

where

$$X_C(t,u)=X_1^{(\|)}+X_1^{(\times)},$$
$$X_\sigma(t,u)=X_2^{(\|)}+X_2^{(\times)},$$
$$X_T(t,u)=X_3^{(\|)}+X_3^{(\times)}, \tag{7.7}$$

and similar formulas for $Y_{C,\sigma,T}$ and $Z_{C,\sigma,T}$.

The momentum space partial wave central, spin-spin, and tensor potentials $V_L^{(C)}(p_f,p_i)$, etc., lead to the adiabatic (ad) contributions:

$$V_{0,0}^J(ad)=4\pi(V_J^{(C)}-3V_J^{(\sigma)}), \tag{7.8a}$$

$$V_{2,2}^J(ad)=4\pi\left[(V_J^{(C)}+V_J^{(\sigma)})+\frac{2}{3}(p_f^2+p_i^2)\left\{V_J^{(T)}-\frac{1}{2}\sin 2\psi\left(\frac{2J+3}{2J+1}V_{J-1}^{(T)}+\frac{2J-1}{2J+1}V_{J+1}^{(T)}\right)\right\}\right], \tag{7.8b}$$

$$V_{1,1}^J(ad)=4\pi\left[(V_{J-1}^{(C)}+V_{J-1}^{(\sigma)})+\frac{2}{3}(p_f^2+p_i^2)\frac{J-1}{2J+1}\left\{-V_{J-1}^{(T)}+\frac{1}{2}\sin 2\psi\left(\frac{2J-3}{2J-1}V_J^{(T)}+\frac{2J+1}{2J-1}V_{J-2}^{(T)}\right)\right\}\right], \tag{7.8c}$$

$$V_{1,3}^J(ad)=4\pi\left[2(p_f^2+p_i^2)\frac{\sqrt{J(J+1)}}{2J+1}\{\sin 2\psi V_J^{(T)}-(\cos^2\psi V_{J-1}^{(T)}+\sin^2\psi V_{J+1}^{(T)})\}\right], \tag{7.8d}$$

$$V_{3,3}^J(ad)=4\pi\left[(V_{J+1}^{(C)}+V_{J+1}^{(\sigma)})+\frac{2}{3}(p_f^2+p_i^2)\frac{J+2}{2J+1}\left\{-V_{J+1}^{(T)}+\frac{1}{2}\sin 2\psi\left(\frac{2J+5}{2J+3}V_J^{(T)}+\frac{2J+1}{2J+3}V_{J+2}^{(T)}\right)\right\}\right]. \tag{7.8e}$$

APPENDIX A: MISCELLANEOUS INTEGRALS

In this appendix we list a number of useful integrals.

(i) Consider the integrals with p components of the $\mathbf{\Delta}$ vector in the integrand: Gaussian integral

$$\mathcal{I}_{m,\ldots,n}^{(p)}(\mathbf{k})=\int\frac{d^3\Delta}{(2\pi)^3}\Delta_m\cdots\Delta_n\exp[-a\mathbf{\Delta}^2+2b\mathbf{k}\cdot\mathbf{\Delta}]$$
$$=(4\pi a)^{-3/2}(2b)^{-p}\tilde{\nabla}_m\cdots\tilde{\nabla}_l\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right]$$
$$\equiv(4\pi a)^{-3/2}\Pi_{m\cdots n}^{(p)}(\mathbf{k}^2)\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right]. \tag{A1}$$

The first tensors $\Pi_{m\cdots n}^{(p)}$ are found to be

$$\Pi^{(0)}=1,\quad \Pi_m^{(1)}=\frac{b}{a}k_m,\quad \Pi_{mn}^{(2)}=\frac{1}{a}\left\{\frac{1}{2}\delta_{mn}+\frac{b^2}{a}k_mk_n\right\},$$

$$\Pi_{mnk}^{(3)}=\frac{b}{a^2}\left\{\frac{1}{2}[\delta_{mn}k_k+\delta_{mk}k_n+\delta_{nk}k_m]+\frac{b^2}{a}k_mk_nk_k\right\},$$

$$\Pi^{(4)}_{mnkl}=\frac{1}{a^2}\left\{\frac{1}{4}[\delta_{mn}\delta_{kl}+\delta_{mk}\delta_{nl}+\delta_{nk}\delta_{ml}]+\frac{1}{2}\frac{b^2}{a}[\delta_{mn}k_k k_l+\delta_{mk}k_n k_l+\delta_{nk}k_m k_l+\delta_{kl}k_m k_n+\delta_{ml}k_k k_n+\delta_{nl}k_k k_m]+\left(\frac{b^2}{a}\right)^2 k_m k_n k_k k_l\right\}. \tag{A2}$$

(ii) Consider the integrals with p factors $\mathbf{\Delta}^2$ in the integrand:

$$\mathcal{J}^{(p)}(\mathbf{k})=\int \frac{d^3\Delta}{(2\pi)^3}(\mathbf{\Delta}^2)^p\exp[-a\mathbf{\Delta}^2+2b\mathbf{k}\cdot\mathbf{\Delta}] =\left(-\frac{\partial}{\partial a}\right)^p\cdot(4\pi a)^{-3/2}\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right] \equiv(4\pi a)^{-3/2}\Gamma^{(p)}(\mathbf{k}^2)\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right]. \tag{A3}$$

The first tensors $\Gamma^{(p)}_{m\cdots n}$ are

$$\Gamma^{(0)}=1,\quad \Gamma^{(1)}=\frac{1}{2a}\left[3+2a\left(\frac{b}{a}\right)^2\mathbf{k}^2\right],$$

$$\Gamma^{(2)}=\frac{1}{4a^2}\left[15+20a\left(\frac{b}{a}\right)^2\mathbf{k}^2+4a^2\left(\frac{b}{a}\right)^4\mathbf{k}^4\right],$$

$$\Gamma^{(3)}=\frac{1}{8a^3}\left[105+210a\left(\frac{b}{a}\right)^2\mathbf{k}^2+84a^2\left(\frac{b}{a}\right)^4\mathbf{k}^4+8a^3\left(\frac{b}{a}\right)^6\mathbf{k}^6\right]. \tag{A4}$$

(iii) Consider the integrals with p factors $\mathbf{\Delta}\cdot\mathbf{k}$ in the integrand:

$$\mathcal{K}^{(p)}(\mathbf{k})=\int \frac{d^3\Delta}{(2\pi)^3}(\mathbf{\Delta}\cdot\mathbf{k})^p\exp[-a\mathbf{\Delta}^2+2b\mathbf{k}\cdot\mathbf{\Delta}] =(4\pi a)^{-3/2}\left(\frac{1}{2}\frac{\partial}{\partial b}\right)^p\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right] \equiv(4\pi a)^{-3/2}\Sigma^{(p)}(\mathbf{k}^2)\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right]. \tag{A5}$$

The first coefficients $\Sigma^{(p)}$ are

$$\Sigma^{(0)}=1,\quad \Sigma^{(1)}=\frac{b}{a}\mathbf{k}^2,$$

$$\Sigma^{(2)}=\frac{1}{2a}\left[1+2a\left(\frac{b}{a}\right)^2\mathbf{k}^2\right]\mathbf{k}^2,$$

$$\Sigma^{(3)}=\frac{b}{2a^2}\left[3+2a\left(\frac{b}{a}\right)^2\mathbf{k}^2\right]\mathbf{k}^4. \tag{A6}$$

(iv) Consider finally the more general integrals with p factors $\mathbf{\Delta}^2$ and q factors $\mathbf{\Delta}\cdot\mathbf{k}$ in the integrand:

$$\mathcal{J}^{(p,q)}(\mathbf{k})=\int \frac{d^3\Delta}{(2\pi)^3}(\mathbf{\Delta}^2)^p(\mathbf{\Delta}\cdot\mathbf{k})^q\exp[-a\mathbf{\Delta}^2+2b\mathbf{k}\cdot\mathbf{\Delta}] =\left(-\frac{\partial}{\partial a}\right)^p\left(\frac{1}{2}\frac{\partial}{\partial b}\right)^q\cdot(4\pi a)^{-3/2}\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right] \equiv(4\pi a)^{-3/2}\Gamma^{(p,q)}(\mathbf{k}^2)\exp\left[+\frac{b^2}{a}\mathbf{k}^2\right]. \tag{A7}$$

The first tensors $\Gamma^{(p,q)}_{m\cdots n}$, with $p,q\neq 0$, are

$$\Gamma^{(1,1)}=\frac{1}{2a}\left(\frac{b}{a}\right)\left[5+2a\left(\frac{b}{a}\right)^2\mathbf{k}^2\right]\mathbf{k}^2,$$

$$\Gamma^{(1,2)}=\frac{1}{4a^2}\left[5+16a\left(\frac{b}{a}\right)^2\mathbf{k}^2+4a^2\left(\frac{b}{a}\right)^4\mathbf{k}^4\right]\mathbf{k}^2,$$

$$\Gamma^{(2,1)}=\frac{1}{4a^2}\left(\frac{b}{a}\right)\left[35+28a\left(\frac{b}{a}\right)^2\mathbf{k}^2+4a^2\left(\frac{b}{a}\right)^4\mathbf{k}^4\right]\mathbf{k}^2. \tag{A8}$$

APPENDIX B: INTEGRATION DICTIONARY

In this appendix we give a dictionary for the evaluation of the momentum integrals that occur in the matrix elements of the TME potentials. The results of the $d^3\Delta$ integration are given apart from a factor $(4\pi a)^{-3/2}$, $(a=t+u)$ common to all integrals. Using the results given in Appendix A one obtains

(a) $(\mathbf{k}_1\cdot\mathbf{k}_2)^2=(\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2)^2$

$$\Rightarrow\frac{1}{4}\left\{15+2\left(\frac{t^2-8ut+u^2}{t+u}\right)\mathbf{k}^2+4\left(\frac{t^2u^2}{(t+u)^2}\right)\mathbf{k}^4\frac{1}{(t+u)^2},\right. \tag{B1}$$

(b) $[\boldsymbol{\sigma}_1\cdot\mathbf{k}_1\times\mathbf{k}_2][\boldsymbol{\sigma}_2\cdot\mathbf{k}_1\times\mathbf{k}_2]$

$$=[\boldsymbol{\sigma}_1\cdot\mathbf{\Delta}\times\mathbf{k}][\boldsymbol{\sigma}_2\cdot\mathbf{\Delta}\times\mathbf{k}] \Rightarrow\frac{1}{2}\left\{\frac{2}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2-\left[(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})-\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2\right]\right\}\frac{1}{t+u}, \tag{B2}$$

(c) $[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}_1\times\mathbf{k}_2]\mathbf{q}(\mathbf{k}_1-\mathbf{k}_2)$

$$=[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\mathbf{\Delta}\times\mathbf{k}]\mathbf{q}\cdot(2\mathbf{\Delta}-\mathbf{k}) \Rightarrow[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\mathbf{q}\times\mathbf{k}]\frac{1}{t+u}, \tag{B3}$$

(d) $(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_1^2+\mathbf{k}_2^2)$

$$=-2(\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2)^2+(\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2)\mathbf{k}^2$$

$$\Rightarrow -\frac{1}{2}\left\{15+5\left(\frac{t^2-2tu+u^2}{t+u}\right)\mathbf{k}^2 -2tu\left(\frac{t^2+u^2}{(t+u)^2}\right)\mathbf{k}^4\right\}\frac{1}{(t+u)^2}, \tag{B4}$$

(e) $$(\mathbf{k}_1\cdot\mathbf{k}_2)=\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2\Rightarrow\frac{1}{2}\left\{-3+2\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right\}\frac{1}{t+u}. \tag{B5}$$

(f) $(\mathbf{k}_1\cdot\mathbf{k}_2)^3=(\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2)^3$

$$\Rightarrow -\frac{1}{8}\left\{105+30\left(\frac{t^2-5ut+u^2}{t+u}\right)\mathbf{k}^2 -12tu\left(\frac{t^2-5ut+u^2}{(t+u)^2}\right)\mathbf{k}^4 -8\left(\frac{t^3u^3}{(t+u)^3}\right)\mathbf{k}^6\right\}\frac{1}{(t+u)^3}, \tag{B6}$$

(g) $(\mathbf{k}_1\cdot\mathbf{k}_2)[\boldsymbol{\sigma}_1\cdot\mathbf{k}_1\times\mathbf{k}_2][\boldsymbol{\sigma}_2\cdot\mathbf{k}_1\times\mathbf{k}_2]$

$$=(\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2)[\boldsymbol{\sigma}_1\cdot\mathbf{\Delta}\times\mathbf{k}][\boldsymbol{\sigma}_2\cdot\mathbf{\Delta}\times\mathbf{k}]$$

$$\Rightarrow -\frac{1}{2}\{(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2-(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})\}\cdot\left[\frac{5}{2} -\frac{tu}{t+u}\mathbf{k}^2\right]\frac{1}{(t+u)^2}, \tag{B7}$$

(h) $(\mathbf{k}_1\cdot\mathbf{k}_2)[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{k}_1\times\mathbf{k}_2]\mathbf{q}\cdot(\mathbf{k}_1-\mathbf{k}_2)$

$$=(\mathbf{\Delta}\cdot\mathbf{k}-\mathbf{\Delta}^2)[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{\Delta}\times\mathbf{k}]\mathbf{q}\cdot(2\mathbf{\Delta}-\mathbf{k})$$

$$\Rightarrow -\frac{1}{2}[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{q}\times\mathbf{k}]\cdot\left[5-2\frac{tu}{t+u}\mathbf{k}^2\right] \times\frac{1}{(t+u)^2}. \tag{B8}$$

APPENDIX C: THE *LSJ* REPRESENTATION OPERATORS

The spherical wave functions in momentum space with quantum numbers J, L, and S, are, in the Stapp-Ypsilantis-Metropolis (SYM) convention [9],

$$\mathcal{Y}^M_{JLS}(\hat{\mathbf{p}})=i^L\ C^{JLS}_{Mm\mu}Y^L_m(\hat{\mathbf{p}})\chi^S_\mu, \tag{C1}$$

where χ is the two-nucleon spin wave function [28]. Then

$$(\mathbf{S}\cdot\hat{\mathbf{p}})\mathcal{Y}^M_{JLS}(\hat{\mathbf{p}})=-\sqrt{6}i(-)^L\left\{\sqrt{\frac{L}{2L-1}}\begin{bmatrix} L & S & J\\ 1 & 1 & 0\\ L-1 & S & J\end{bmatrix}\times\mathcal{Y}^M_{JL-1S}(\hat{\mathbf{p}}) +\sqrt{\frac{L+1}{2L+3}}\begin{bmatrix} L & S & J\\ 1 & 1 & 0\\ L+1 & S & J\end{bmatrix}\mathcal{Y}^M_{JL+1S}(\hat{\mathbf{p}})\right\}, \tag{C2}$$

where the $9j$ symbols differ from [29], formula (6.4.4), in the replacement of the $3j$ symbols by the Clebsch-Gordan coefficients and by leaving out the m_{33} summation (see [30]). Working this out explicitly, we find

$$(\mathbf{S}\cdot\hat{\mathbf{p}})\mathcal{Y}^M_{J,J-1,1}(\hat{\mathbf{p}})=-ia_J\mathcal{Y}^M_{JJ1}(\hat{\mathbf{p}}),$$

$$(\mathbf{S}\cdot\hat{\mathbf{p}})\mathcal{Y}^M_{J,J+1,1}(\hat{\mathbf{p}})=ib_J\mathcal{Y}^M_{JJ1}(\hat{\mathbf{p}}),$$

$$(\mathbf{S}\cdot\hat{\mathbf{p}})\mathcal{Y}^M_{JJ1}(\hat{\mathbf{p}})=ia_J\mathcal{Y}^M_{J,J-1,1}(\hat{\mathbf{p}})-ib_J\mathcal{Y}^M_{J,J+1,1}(\hat{\mathbf{p}}), \tag{C3}$$

where

$$a_J=-\sqrt{\frac{J+1}{2J+1}},\quad b_J=-\sqrt{\frac{J}{2J+1}}. \tag{C4}$$

Ordering the states according to $L=J-1, L=J, L=J+1$, we can write in matrix form

$$\begin{pmatrix} L=J-1 & & L=J-1\\ J & \|\mathbf{S}\cdot\hat{\mathbf{p}}\| & J\\ J+1 & & J+1\end{pmatrix} =\begin{pmatrix} 0 & ia_J & 0\\ -ia_J & 0 & ib_J\\ 0 & -ib_J & 0\end{pmatrix}. \tag{C5}$$

Similarly, using for $-i(\hat{\mathbf{p}}_f\times\hat{\mathbf{p}}_i)\cdot\mathbf{S}$ for spherical components the formula

$$-i(\hat{\mathbf{p}}_f\times\hat{\mathbf{p}}_i)_n=-\frac{4\pi}{3}\sqrt{2}C^{111}_{kln}Y^1_k(\hat{\mathbf{p}}_f)Y^1_l(\hat{\mathbf{p}}_i), \tag{C6}$$

one can work out the partial wave matrix elements involving this operator.

From the results above one can derive the following useful partial wave projections for the spin triplet states:

$$(L'1J|V(\mathbf{k}^2)(\mathbf{S}\cdot\hat{\mathbf{p}}_i)^2|L1J) =4\pi\begin{pmatrix} a_J^2V_{J-1} & 0 & -a_Jb_JV_{J-1}\\ 0 & V_J & 0\\ -a_Jb_JV_{J+1} & 0 & b_J^2V_{J+1}\end{pmatrix}, \tag{C7a}$$

$$(L'1J|(\mathbf{S}\cdot\hat{\mathbf{p}}_f)^2V(\mathbf{k}^2)|L1J)$$
$$=4\pi\begin{pmatrix} a_J^2V_{J-1} & 0 & -a_Jb_JV_{J+1} \\ 0 & V_J & 0 \\ -a_Jb_JV_{J-1} & 0 & b_J^2V_{J+1}\end{pmatrix}, \quad \text{(C7b)}$$

$$(L'1J|(\mathbf{S}\cdot\hat{\mathbf{p}}_f)V(\mathbf{k}^2)(\mathbf{S}\cdot\hat{\mathbf{p}}_i)|L1J)$$
$$=4\pi\begin{pmatrix} a_J^2V_J & 0 & -a_Jb_JV_J \\ 0 & a_J^2V_{J-1}+b_J^2V_{J+1} & 0 \\ -a_Jb_JV_J & 0 & b_J^2V_J\end{pmatrix}, \quad \text{(C7c)}$$

and

$$(L'1J|-i(\hat{\mathbf{p}}_f\times\hat{\mathbf{p}}_i)\cdot\mathbf{S}\,V(\mathbf{k}^2)|L1J)$$
$$=\frac{4\pi}{2J+1}\begin{cases}(J-1)(V_{J-2}-V_J), & L=L'=J-1,\\ -(V_{J-1}-V_{J+1}), & L=L'=J,\\ -(J+2)(V_J-V_{J+2}), & L=L'=J+1.\end{cases} \quad \text{(C8)}$$

Using the identity

$$(\boldsymbol{\sigma}_1\cdot\mathbf{a})(\boldsymbol{\sigma}_2\cdot\mathbf{a})=2(\mathbf{S}\cdot\mathbf{a})^2-\mathbf{a}^2, \quad \text{(C9)}$$

the tensor operator can be written as

$$P_3=(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})-\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2$$
$$=\frac{1}{3}[p_i^2S_{12}(\hat{\mathbf{p}}_i)+p_f^2S_{12}(\hat{\mathbf{p}}_f)]-4(\mathbf{S}\cdot\mathbf{p}_f)(\mathbf{S}\cdot\mathbf{p}_i)$$
$$+2i(\mathbf{p}_f\times\mathbf{p}_i)\cdot\mathbf{S}+\frac{4}{3}(\mathbf{p}_f\cdot\mathbf{p}_i)\mathbf{S}^2, \quad \text{(C10)}$$

where the momentum space tensor operator S_{12} is defined as

$$S_{12}(\hat{\mathbf{p}})=3(\boldsymbol{\sigma}_1\cdot\hat{\mathbf{p}})(\boldsymbol{\sigma}_2\cdot\hat{\mathbf{p}})-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2). \quad \text{(C11)}$$

From the formulas given in this appendix the partial wave projections of the several potential forms, as given in Secs. VI A and VII, can be derived in a straightforward manner. In the case of an extra factor $(\mathbf{p}_f\cdot\mathbf{p}_i)$, as occurs, for example, in the second line of Eq. (C10), we simply use the expansion

$$(\mathbf{p}_f\cdot\mathbf{p}_i)V(\mathbf{k}^2)=p_fp_i\sum_{L=0}^{\infty}(2L+1)\tilde{V}_L(x)P_L(\cos\theta), \quad \text{(C12)}$$

where

$$\tilde{V}_L=\frac{1}{2L+1}[(L+1)V_{L+1}+LV_{L-1}]. \quad \text{(C13)}$$

APPENDIX D: FOURIER TRANSFORMATION COORDINATE TO MOMENTUM SPACE

In this appendix we give an outline of how the potentials in the coordinate representation can be translated to their momentum-space counterparts in a direct way. Of course, we utilize the same techniques as described in this paper. We treat the more complicated case of the tensor potential. In this case the coordinate-space potentials are complicated. Nevertheless, we show explicitly how they are connected with our momentum-space representation.

1. TPS tensor and spin-spin potentials I

To appreciate this method in the case of TME potentials, we consider as a typical example the potential

$$V(r)=\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}e^{i(\mathbf{k_1+k_2})\cdot\mathbf{r}}[\boldsymbol{\sigma}_1\cdot(\mathbf{k}_1\times\mathbf{k}_2)]$$
$$\times[\boldsymbol{\sigma}_2\cdot(\mathbf{k}_1\times\mathbf{k}_2)]\tilde{F}(\mathbf{k}_1^2)\tilde{G}(\mathbf{k}_2^2). \quad \text{(D1)}$$

This potential, in terms of the Fourier transforms $F(r)$ and $G(r)$, has been given in [12] and reads

$$V(r)=\frac{2}{3}\left[\frac{1}{r^2}F'(r)G'(r)+\frac{1}{r}F'(r)G''(r)+\frac{1}{r}F''(r)G'(r)\right]$$
$$\times(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+\frac{1}{3}\left[\frac{2}{r^2}F'(r)G'(r)-\frac{1}{r}F'(r)G''(r)\right.$$
$$\left.-\frac{1}{r}F''(r)G'(r)\right]S_{12}, \quad \text{(D2)}$$

where $F'(r)\equiv dF(r)/dr$, etc.

In seeking the projection on the spinor invariants, we wish to write for the momentum space counterpart of Eq. (D1) as

$$\tilde{V}(\mathbf{k})=\int\int\frac{d^3k_1d^3k_2}{(2\pi)^3}\delta(\mathbf{k-k_1-k_2})\tilde{F}(\mathbf{k}_1^2)\tilde{G}(\mathbf{k}_2^2)$$
$$\times[\boldsymbol{\sigma}_1\cdot(\mathbf{k}_1\times\mathbf{k}_2)][\boldsymbol{\sigma}_2\cdot(\mathbf{k}_1\times\mathbf{k}_2)]$$
$$\equiv\tilde{V}_\sigma(\mathbf{k})(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)+\tilde{V}_T\mathbf{k})$$
$$\times\left[(\boldsymbol{\sigma}_1\cdot\mathbf{k})(\boldsymbol{\sigma}_2\cdot\mathbf{k})-\frac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\mathbf{k}^2\right]. \quad \text{(D3)}$$

It is now our task to find $\tilde{V}_{\sigma,T}(\mathbf{k})$. To proceed, we notice that one can easily see that Eq. (D2) can be written in the form

$$V(r)\equiv\frac{2}{3}H_\sigma(r)(\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_2)+\frac{1}{3}H_T(r)S_{12},$$

with

$$H_\sigma(r)=\left(\frac{2}{r}+\frac{d}{dr}\right)H'(r), \quad H_T(r)=\left(\frac{1}{r}-\frac{d}{dr}\right)H'(r), \quad \text{(D4)}$$

where the function $H(r)$ satisfies the equation

$$\frac{1}{r}\frac{d}{dr}H(r)=\frac{1}{r}\frac{d}{dr}F(r)\frac{1}{r}\frac{d}{dr}G(r). \tag{D5}$$

To solve the problem posed in this section, it is necessary to find the Fourier transform $\widetilde{H}(\mathbf{k}^2)$. To solve this problem we exploit the following lemma.

Lemma 1. If two functions $h(r)$ and $H(r)$ are related by $h(r)=(1/r)dH/dr$, *then their Fourier transforms* $\widetilde{h}(\mathbf{k}^2)$ *and* $\widetilde{H}(\mathbf{k}^2)$ satisfy

$$h(r)=\frac{1}{r}\frac{d}{dr}H(r)\leftrightarrow\frac{d\widetilde{h}(\mathbf{k}^2)}{d\mathbf{k}^2}=\frac{1}{2}\widetilde{H}(\mathbf{k}^2). \tag{D6}$$

Introducing the "little" functions $h(r),f(r),g(r)$ by

$$h(r)\equiv\frac{1}{r}\frac{d}{dr}H(r),\quad f(r)\equiv\frac{1}{r}\frac{d}{dr}F(r),\quad g(r)\equiv\frac{1}{r}\frac{d}{dr}G(r),$$

and from Eq. (D5) these satisfy the relation $h(r)=f(r)\cdot g(r)$.

Next we assume the following generic soft-core forms for $\widetilde{F}$ and $\widetilde{G}$:

$$\widetilde{F}(\mathbf{k}^2)=\frac{\exp[-\mathbf{k}^2/\Lambda_1^2]}{\mathbf{k}^2+m_1^2},\quad \widetilde{G}(\mathbf{k}^2)=\frac{\exp[-\mathbf{k}^2/\Lambda_2^2]}{\mathbf{k}^2+m_2^2}. \tag{D7}$$

The solution for $\widetilde{f}(\mathbf{k}^2)$ and $\widetilde{g}(\mathbf{k}^2)$, using the differential equations for the Fourier transforms as given in Eq. (D6), is discussed in [8], Appendix C, and reads

$$\widetilde{f}(\mathbf{k}^2)=-\frac{1}{2}\exp(m_1^2/\Lambda_1^2)E_1[(\mathbf{k}^2+m_1^2)/\Lambda_1^2],$$

$$\widetilde{g}(\mathbf{k}^2)=-\frac{1}{2}\exp(m_2^2/\Lambda_2^2)E_1[(\mathbf{k}^2+m_2^2)/\Lambda_2^2], \tag{D8}$$

where E_1 is the exponential integral [26]. Then, the Fourier transform of $h(r)=f(r)g(r)$ as follows from Eq. (D5) is given by the convolution

$$\widetilde{h}(\mathbf{k})=\widetilde{f}\star\widetilde{g}(\mathbf{k})=\int\frac{d^3\Delta}{(2\pi)^3}\widetilde{f}(\mathbf{\Delta}^2)\widetilde{g}((\mathbf{k}-\mathbf{\Delta})^2). \tag{D9}$$

With Eq. (D8) we obtain

$$\widetilde{h}(\mathbf{k})=\frac{1}{4}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int\frac{d^3\Delta}{(2\pi)^3}E_1[(\mathbf{\Delta}^2+m_1^2)/\Lambda_1^2]\times E_1\{[(\mathbf{k}-\mathbf{\Delta})^2+m_2^2]/\Lambda_2^2\}. \tag{D10}$$

Now, the Δ-integral can be performed as follows:

$$\begin{aligned}\int\frac{d^3\Delta}{(2\pi)^3}E_1\left(\frac{(\mathbf{\Delta}^2+m_1^2)}{\Lambda_1^2}\right)E_1\left(\frac{[(\mathbf{k}-\mathbf{\Delta})^2+m_2^2]}{\Lambda_2^2}\right)&=\int_1^\infty\frac{dt}{t}\int_1^\infty\frac{du}{u}\int\frac{d^3\Delta}{(2\pi)^3}\exp\left[-\frac{(\mathbf{\Delta}^2+m_1^2)}{\Lambda_1^2}t\right]\\&\quad\times\exp\left[-\frac{((\mathbf{k}-\mathbf{\Delta})^2+m_2^2)}{\Lambda_2^2}u\right]\\&=\int_1^\infty\frac{dt}{t}\int_1^\infty\frac{du}{u}\exp\left(-\frac{m_1^2}{\Lambda_1^2}t\right)\exp\left(-\frac{m_2^2}{\Lambda_2^2}u\right)\int\frac{d^3\Delta}{(2\pi)^3}\\&\quad\times\exp\left[-\left(\frac{t}{\Lambda_1^2}+\frac{u}{\Lambda_2^2}\right)\mathbf{\Delta}^2+(2\mathbf{k}\cdot\mathbf{\Delta}-\mathbf{k}^2)\frac{u}{\Lambda_2^2}\right].\end{aligned} \tag{D11}$$

The Δ integral in the last line of Eq. (D11) has a standard Gaussian form and is given by

$$\int\frac{d^3\Delta}{(2\pi)^3}\exp[\cdots]=(4\pi a)^{-3/2}\exp\left[-\frac{tu/\Lambda_1^2\Lambda_2^2}{(t/\Lambda_1^2+u/\Lambda_2^2)}\mathbf{k}^2\right], \tag{D12}$$

with

$$a=\frac{t}{\Lambda_1^2}+\frac{u}{\Lambda_2^2}.$$

In this form, the derivative with respect to $\mathbf{k}^2$ can be taken easily, and we finally obtain for $\widetilde{H}$ the solution

$$\begin{aligned}\widetilde{H}(\mathbf{k}^2)&=-\frac{1}{2}(4\pi)^{-3/2}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int_1^\infty\frac{dt}{\Lambda_1^2}\int_1^\infty\frac{du}{\Lambda_2^2}\\&\quad\times\left(\frac{t}{\Lambda_1}+\frac{u}{\Lambda_2^2}\right)^{-5/2}\exp\left(-\frac{m_1^2}{\Lambda_1^2}t\right)\exp\left(-\frac{m_2^2}{\Lambda_2^2}u\right)\\&\quad\times\exp\left[-\left(\frac{tu/\Lambda_1^2\Lambda_2^2}{t/\Lambda_1^2+u/\Lambda_2^2}\right)\mathbf{k}^2\right].\end{aligned} \tag{D13}$$

Redefining the variables $t\to t/\Lambda_1^2$ and $u\to u/\Lambda_2^2$, one can rewrite Eq. (D13) in the form

$$\begin{aligned}\widetilde{H}(\mathbf{k}^2)&=-\frac{1}{2}(4\pi)^{-3/2}e^{m_1^2/\Lambda_1^2}e^{m_2^2/\Lambda_2^2}\int_{t_0}^\infty dt\int_{u_0}^\infty du(t\\&\quad+u)^{-5/2}e^{-(m_1^2t+m_2^2u)}\exp\left[-\left(\frac{tu}{t+u}\right)\mathbf{k}^2\right]\\&\quad(t_0=1/\Lambda_1^2,u_0=1/\Lambda_2^2).\end{aligned} \tag{D14}$$

Notice that this is the same result as obtained in Eq. (3.11), as it should.

[1] Th.A. Rijken, in *Proceedings of the XIVth European Conference on Few-Body Problems in Physics*, Amsterdam, 1993, edited by B. Bakker and R. von Dantzig [Few-Body Syst., Suppl. **7**, 1 (1994)].

[2] Th.A. Rijken and V.G.J. Stoks, Phys. Rev. C **54**, 2851 (1996).

[3] Th.A. Rijken and V.G.J. Stoks, Phys. Rev. C **54**, 2869 (1996).

[4] V.G.J. Stoks and Th.A. Rijken, Nucl. Phys. **A613**, 311 (1997).

[5] S. Weinberg, Physica A **96**, 327 (1979); A. Manohar and H. Georgi, Nucl. Phys. **B234**, 189 (1984); H. Georgi, Annu. Rev. Nucl. Sci. **43**, 209 (1993).

[6] J. Polchinsky, Nucl. Phys. **B231**, 269 (1984).

[7] Th.A. Rijken, in *Proceedings of the 1st Asian-Pacific Conference on Few-Body Problems in Physics*, Tokyo, 1999 (Springer-Verlag, Berlin, 2000).

[8] T.A. Rijken, R.A.M. Klomp, and J.J. de Swart, ‘‘Soft-Core OBE-Potentials in Momentum Space,’’ report, Institute for Theoretical Physics, Nijmegen, The Netherlands, 1991.

[9] H.P. Stapp, T. Ypsilantis, and M. Metropolis, Phys. Rev. **105**, 311 (1957).

[10] Th.A. Rijken, H. Polinder, and J. Nagata, Phys. Rev. C **66,** 044009 (2002), paper II, following paper.

[11] A. Klein, Phys. Rev. **90**, 1101 (1953).

[12] Th.A. Rijken, Ann. Phys. (N.Y.) **208**, 253 (1991).

[13] R.P. Feynman, Phys. Rev. **76**, 769 (1949).

[14] J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. **37**, 452 (1951).

[15] E. Salpeter and H.A. Bethe, Phys. Rev. **84**, 1232 (1951).

[16] P.A. Carruthers, *Spin and Isospin in Particle Physics* (Gordon and Breach Science, New York, 1971).

[17] We follow the conventions of J.D. Bjorken and S.D. Drell, *Relativistic Quantum Mechanics* and *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[18] E. Salpeter, Phys. Rev. **87**, 328 (1952).

[19] R.H. Thompson, Phys. Rev. D **1**, 110 (1970).

[20] M.M. Nagels, T.A. Rijken, and J.J. de Swart, Phys. Rev. D **17**, 768 (1978).

[21] M.M. Nagels, T.A. Rijken, and J.J. de Swart, Phys. Rev. D **15**, 2547 (1977).

[22] J.J. de Swart, M.M. Nagels, T.A. Rijken, and P.A. Verhoeven, *Hyperon-Nucleon Interaction*, Springer Tracts in Modern Physics, Vol. 60 (Springer, New York, 1971).

[23] P.M.M. Maessen, T.A. Rijken, and J.J. de Swart, Phys. Rev. C **40**, 2226 (1989).

[24] At this point it is suitable to change the notation of the initial and final momenta. We use from now on the notation $\mathbf{p}_i \equiv \mathbf{p}$, $\mathbf{p}_f \equiv \mathbf{p}'$ for both on-shell and off-shell momenta.

[25] As pointed out in [2], the operators $\tilde{O}^{(\parallel,\times)}_{\alpha\beta}$ contain the contributions from all graphs, including the graphs with meson 1 and meson 2 interchanged. To avoid overcounting, in the case of identical mesons, like $\pi\pi$ or $\eta\eta$, these operators must be divided by 2. The results in the tables of this paper are given for nonidentical mesons.

[26] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I.A. Stegun (Dover, New York, 1970).

[27] T.A. Rijken, Ann. Phys. (N.Y.) **164**, 1 (1985); **164**, 23 (1985).

[28] In the SYM convention [9] the configuration space basic JLS states are

$$\mathcal{Y}^M_{JLS}(\hat{\mathbf{r}}) = C^{JLS}_{Mm\mu} Y^L_m(\hat{\mathbf{r}}) \chi^S_\mu .$$

Transformation to momentum space gives Eq. (C1). See, e.g., J.R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions* (Wiley, New York, 1972).

[29] A.R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).

[30] The explicit relation between our $9j$ symbols and those of [29], Eq. (6.4.4), is

$$\begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} = [(2j_{13}+1)(2j_{31}+1)(2j_{23}+1)(2j_{32}+1)]^{1/2} \times (-)^{j_{31}+j_{13}-j_{32}-j_{23}} \cdot \begin{Bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{Bmatrix} .$$