Sp(3,R) mean field theory

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The sp(3,R) mean field approximation describes collective nuclear rotation in a symplectic density matrix formalism. The densities are 6×6 matrices that are defined by the quantum mechanical expectations of the symplectic algebra generators. The 21 generators of the noncompact symplectic algebra sp(3,R) include the mass quadrupole and monopole moments, the kinetic energy, the harmonic oscillator Hamiltonian, and the angular, vibrational, and vortex momenta. The mean field approximation restricts the densities to a coadjoint orbit of the canonical transformation group Sp(3,R). The reduction of a Sp(3,R) coadjoint orbit into orbits of the dynamical symmetry group GCM(3) is proved to be consistent with the reduction of an Sp(3,R) discrete series representation into irreducible representations of GCM(3). This reduction places a strict bound on the range of the Kelvin circulation which is the Casimir of the 15-dimensional subalgebra gcm(3)$\subset$sp(3,R). The cranked anisotropic oscillator and Riemann ellipsoid model are special cases of symplectic mean field theory. The application of the Riemann model in the even-even heavy deformed region indicates that the character of low energy collective rotational modes depends only on the quadrupole deformation β. The energy of the first 2^+ state in such isotopes is a simple function of β.

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I. INTRODUCTION

The noncompact symplectic algebra sp(3,R) is the dynamical symmetry algebra of the symplectic shell model [1–9]. The symplectic algebra is generated by all one-body Hermitian operators that are quadratic in the position and momentum operators. Its irreducible unitary discrete series representations unify the geometrical collective model and the harmonic oscillator shell model. An infinite-dimensional representation space contains all np-nh core-excited basis states which are required to embed the geometrical model fully and without approximation into the shell model. Although there are no effective charges in the symplectic theory, the eigenstates of the symplectic model Hamiltonian yield $E2$ transition rates and deformations that compare favorably with experiment [10–12]. The transverse $E2$ form factors predicted by the symplectic model are sensitive to the nuclear collective current [13–20]. A conventional shell model calculation that excludes symplectic core excitations fails to provide a satisfactory theoretical explanation of transverse inelastic transitions [21].

An alternative to a dynamical symmetry model founded on irreducible representation theory is algebraic mean field theory. Recently the algebraic mean field method was applied to the su(3) algebra and the description of rotational bands [22,23]. The algebraic method is a generalization of Hartree-Fock and Hartree-Fock-Bogoliubov mean field theories that are based on the unitary and orthogonal groups, respectively [24–27]. The algebraic theory also allows for a generalization of the random phase approximation that describes time-dependent normal mode oscillations of equilibrium configurations [28].

For a given Lie algebra, the space of algebraic mean field theory is the set of densities defined as the quantum mechanical expectations of the operators of the algebra. The set of densities is contained in the dual space of the Lie algebra. The Lie group of the algebra acts as a transformation group on the dual space via the coadjoint action. For semisimple matrix Lie algebras, the dual space is naturally isomorphic to the Lie algebra and the coadjoint action is matrix conjugation. The mean field approximation restricts the model densities to a single coadjoint orbit. This orbit is a surface contained in the algebra's dual space. The energy in the mean field approximation is a real-valued function of the density; its critical points on a coadjoint orbit are the equilibrium mean field densities.

The dimension of a coadjoint orbit surface is less than the dimension of a finite-dimensional Lie algebra. Although the only nontrivial unitary representations of a noncompact Lie algebra such as sp(3,R) are infinite dimensional, the mean field approximation limits the theoretical investigation to a finite-dimensional manifold. In fact, sp(3,R) mean field theory requires only matrix computations with 6×6 matrices. Thus, algebraic mean field theory eliminates the technical difficulty of infinite-dimensional representation spaces for noncompact algebras. Even for a compact algebra, the dimension of the representation space is typically much greater than the dimension of a coadjoint orbit surface, and the mean field method is simpler than the irreducible representation method.

The shell model and the irreducible representation spaces of dynamical symmetry algebras are fundamentally inertial frame descriptions. In contrast, the geometrical model provides a simpler description of rotational and vibrational modes in the rotating body-fixed frame. Even though useful approximations to it exist [29,30], the K quantum number (the projection of the angular momentum vector onto the body's symmetry axis in the rotating frame) is not well defined in shell model theory. Symplectic mean field theory permits a transformation to the body-fixed frame that provides a clearer physical interpretation of rotational modes than shell model theory offers.

Geometric quantization is one of the methods for the con-

struction of the irreducible representations of Lie groups [31–35]. The starting point for the construction is a coadjoint orbit that satisfies a generalized Bohr-Sommerfeld quantization rule. These special coadjoint orbits are called integral. The metatheorem of geometric quantization states that, for every representation theory concept, a corresponding, and ultimately equivalent, idea can be discovered for integral coadjoint orbits [31]. For example, a coadjoint orbit of a semisimple compact Lie group is integral when it contains the density of an integral highest weight vector. A familiar problem in representation theory is the reduction of an irreducible representation of a Lie group G into a direct sum of irreducible representations of a Lie subgroup H. In geometric quantization this decomposition problem is solved by projecting an integral coadjoint orbit of G into the dual space of H and then determining the integral coadjoint orbits of H contained in the projection. All information about an irreducible representation of a Lie group is encoded in the symplectic geometry of its coadjoint orbits. Thus, the metatheorem asserts that the physical properties of a quantum system governed by a dynamical symmetry can be investigated in two equivalent ways: The direct way is through the irreducible representations of the dynamical symmetry group; the indirect method is via the group's coadjoint orbits. For some problems, the physics and the mathematics of the mean field or coadjoint orbit method is the preferred way.

In this article the mean field method is applied to the noncompact symplectic algebra. In Sec. II the symplectic algebra and its dual space are defined. There are four representations of the symplectic algebra corresponding to the deformed and isotropic phonon bases and, for both of these bases, representations associated with real $\mathrm{sp}(3,R)$ and complex $\mathrm{u}(3,3)\cap\mathrm{sp}(3,C)$ matrices. Although these four representations are mathematically equivalent, each has particular advantages for different physical applications. In this section the coadjoint action and the coadjoint orbits are also defined. The symplectic Casimirs are constant functions on each coadjoint orbit. The restriction to a coadjoint orbit is the mathematical expression of dynamical symmetry in algebraic mean field theory.

There are two important subalgebra chains for the symplectic model. The shell model is associated with the Elliott u(3) subalgebra, the symmetry algebra of the harmonic oscillator. The geometrical collective model is related to the general collective motion gcm(3) subalgebra. The intersection of the u(3) and gcm(3) algebras is the angular momentum algebra so(3) of the rotation group. This article applies symplectic mean field theory to the collective model and its associated gcm(3) subalgebra.

The general collective motion algebra gcm(3) is a 15-dimensional noncompact subalgebra of the 21-dimensional symplectic algebra [36–43]. The monopole and quadrupole mass tensors and the Lie algebra of the general linear group $\mathrm{GL}_+(3,R)$ generate gcm(3). In classical physics, gcm(3) is the dynamical symmetry of the Riemann ellipsoid theory [44–50]. The dynamical symmetry algebra of the Bohr-Mottelson collective model is gcm(3). The inequivalent irreducible representations of gcm(3) are indexed by the Kelvin circulation C which is quantized to integral multiples of $\hbar$. The nonnegative integer $C/\hbar$ labels the irreducible representations of the vorticity group SO(3). A wave function $\Psi(q)$ in an irreducible gcm(3) representation space is a function of the monopole-quadrupole tensor q with vector values in an irreducible representation space of the vorticity group. The scalar representation $C=0$ corresponds to the original Bohr-Mottelson model for an irrotational droplet. The other representations $C\neq 0$ correspond to nuclear collective currents with nonvanishing circulation.

The gcm(3) Casimir is the squared length of the vector Kelvin circulation operator [51]. This Casimir is a physical observable that measures the collective current and the character of rotational states, i.e., rigid rotation, irrotational flow, and the continuum of intermediate nuclear currents. For a classical fluid, the Kelvin circulation is a conserved quantity [52]. The gcm(3) Casimir is a five-body operator that is too complicated for shell model calculations at present. In Sec. III the values of the gcm(3) Casimir on a fixed coadjoint orbit of $\mathrm{sp}(3,R)$ are derived. It is proved that the Kelvin circulation has the same restriction on its range in mean field theory as it has in the gcm(3) reduction of a symplectic irreducible shell model representation [43]. The scope of collective currents is sharply limited by the shell model and this important property is respected without error in the mean field approximation.

The Riemann ellipsoid and Bohr-Mottelson geometrical models can be unified within the differential geometric formalism of modern gauge theory [53]. The relevant space is a principal G bundle, where the structure group $G=\mathrm{SO}(3)$ is the vorticity group and the base manifold is the space of inertia ellipsoids. A choice of a connection on this principal bundle imposes constraints on the current, e.g., rigid body, irrotational flow, or a "falling cat." The quantum Bohr-Mottelson theory, as extended to allow nonzero Kelvin circulation, is an associated G bundle where the fibers are isomorphic to an irreducible representation of the vorticity group. The integral Kelvin circulation labels this vorticity group representation, and it is a gauge invariant.

The isotropic and anisotropic harmonic oscillator Hamiltonians are elements of the symplectic Lie algebra. In Sec. IV the range of the isotropic oscillator on a symplectic coadjoint orbit is shown to be bounded from below just as it is within the corresponding shell model irreducible $\mathrm{sp}(3,R)$ representation. For the cranked anisotropic oscillator Hamiltonian, symplectic coadjoint theory is shown to be equivalent to Inglis cranking theory [54–57]. When the anisotropy of the mean field is consistent with the geometrical deformation, the nucleus rotates rigidly.

To derive the Riemann ellipsoid model in symplectic coadjoint orbit theory, a term proportional to the Kelvin circulation is added to the Routhian of the anisotropic oscillator Hamiltonian in Sec. V [58–60]. The Kelvin circulation term in the symplectic model energy modifies the nuclear current. The self-consistent moment of inertia in the Riemann theory depends on a parameter r, called the rigidity, which varies continuously from $r=0$, corresponding to irrotational flow, to $r=1$, corresponding to rigid rotation. The rigidity is the ratio of the Kelvin circulation to its rigid body value; it directly measures the character of rotational motion.

The rotational bands of deformed nuclei have approximately a constant value for the rigidity. Among the rare-earth even-even nuclei, the rigidity r is approximately a quadratic function of the nuclear quadrupole deformation β. In this heavy deformed region, experiment indicates that the character of low energy rotational states depends only on their deformation and that their excitation energies, in conformity with the Riemann approximation, are mostly kinetic in origin.

II. Sp(3,R) COADJOINT ORBIT THEORY

The mean field approximation uses a faithful matrix representation of the algebra to facilitate computations. For the symplectic Lie algebra there are two useful matrix representations. One is an algebra of 6×6 real matrices which is isomorphic to the quadratics in the Cartesian position and momentum operators. This representation relates the symplectic theory to the geometrical collective model. The other realization is a subalgebra of u(3,3), which is an algebra of 6×6 complex matrices. This second matrix representation is isomorphic to the quadratics in the oscillator phonons, and it is relevant physically to the harmonic oscillator shell model. In this section these two representations are defined. The dual space of symplectic densities also has two matrix realizations corresponding to these two faithful matrix representations. Note that some authors denote this real symplectic algebra by sp(6,R); the complexification of sp(3,R) is C_3 in the Cartan classification of simple Lie algebras.

Later, in Sec. II A, the symplectic group and its coadjoint action on the densities are defined. Subgroups of Sp(3,R), such as the rotation group SO(3), general linear group $\mathrm{GL}_+(3,R)$, and unitary group U(3), act as physically important transformation groups on each coadjoint orbit.

The three symplectic Casimirs are defined in Sec. II B. These are constant functions on each coadjoint orbit. The Casimirs are of enormous practical value for the mean field method. Without them, the determination of critical points of the energy on a coadjoint orbit would necessitate the introduction of explicit coordinates for the orbit surface. In the case of Sp(3,R), the orbits are typically 18 dimensional, and the mean field method might become intractable and would certainly become messy. However, the critical points on a level surface of the Casimirs can be found cleanly using the method of Lagrange multipliers. Only the space of densities requires coordinates—an easy task because it is a vector space.

In Sec. II C, the anisotropic oscillator representation is defined in preparation for its use in cranking theory in Sec. IV. The anisotropic and isotropic representations are related by a coadjoint transformation using the diagonal dilatation matrices in the general linear group.

Let $(x_{\alpha j},p_{\alpha j})$ denote the dimensionless Cartesian components of the position and momentum Hermitian operators of particle α in a finite system of particles. They obey the canonical commutation relation $[x_{\alpha j},p_{\beta k}]=i\delta_{\alpha\beta}\delta_{jk}$. The symplectic generators are the Hermitian operators

$$\hat{Q}_{jk}=\sum_\alpha x_{\alpha j}x_{\alpha k},$$

$$\hat{T}_{jk}=\sum_\alpha p_{\alpha j}p_{\alpha k},$$

$$\hat{N}_{jk}=\sum_\alpha \left(x_{\alpha j}p_{\alpha k}-\frac{i}{2}\delta_{jk}\right). \tag{1}$$

The observables $\hat{Q}_{jk}$ and $\hat{T}_{jk}$ are the monopole-quadrupole tensors in position and momentum space, respectively. The traces of these tensors, which are the monopole components, determine the nuclear radius and the kinetic energy, respectively. The traceless parts of these Cartesian tensors, which are the quadrupole components, determine the nuclear deformation in Euclidean space and momentum space, respectively.

The nine components of $\hat{N}_{jk}$ generate the Lie algebra gl(3,R) of the general linear group. The antisymmetric parts $\hat{L}_i=\varepsilon_{ijk}\hat{N}_{jk}$ are the vector angular momentum components which generate the algebra so(3) of the rotation group. The trace $\Sigma_j\hat{N}_{jj}=\Sigma_\alpha(\vec{r}_\alpha\cdot\vec{p}_\alpha-3i/2)$ measures the breathing mode oscillations. The traceless symmetric part $\hat{N}^{(2)}_{jk}=\hat{N}_{jk}+\hat{N}_{kj}-(2/3)\delta_{jk}\Sigma_i\hat{N}_{ii}$ is the quadrupole vibrational momentum. In the principal axis frame, the diagonal components of $\hat{N}^{(2)}_{jk}$ determine the vibrations of the principal axis lengths. The off-diagonal components of $\hat{N}^{(2)}_{jk}$ yield the Kelvin circulation. The quadrupole vibrational momentum satisfies the commutation relation

$$\hat{N}^{(2)}_{jk}=i[\hat{Q}^{(2)}_{jk},\hat{H}], \tag{2}$$

where $\hat{H}$ is the harmonic oscillator Hamiltonian. The matrix elements $\langle f|\hat{N}^{(2)}_{jk}|i\rangle$ of the quadrupole vibrational momentum are evidently identically zero for any two state vectors i and f from a single major oscillator shell. In contrast, the matrix element of $\hat{N}^{(2)}_{jk}$ between the ground state and the giant isoscalar quadrupole resonance is very large. Thus conventional shell model calculations cannot achieve a meaningful theoretical analysis of nuclear currents.

The symplectic algebra sp(3,R) of matrices consists of 6×6 real matrices S of the form

$$S=\begin{pmatrix} X & -U \\ V & -X^T \end{pmatrix}, \tag{3}$$

where X,U,V are 3×3 real matrices and U,V are symmetric. The representation σ of the algebra of matrices is given by

$$\sigma(S)=i\sum_{jk}X_{jk}\hat{N}_{jk}+\frac{i}{2}\sum_{jk}U_{jk}\hat{Q}_{jk}+\frac{i}{2}\sum_{jk}V_{jk}\hat{T}_{jk}. \tag{4}$$

When S is a matrix in the symplectic Lie algebra, the operator $\sigma(S)$ is a skew-adjoint one-body operator. The set of operators is a representation, $[\sigma(S_1),\sigma(S_2)]=\sigma([S_1,S_2])$.

The symplectic density matrix ρ corresponding to a normalized wave function Ψ is a symplectic matrix

$$\rho=\begin{pmatrix} n^T & t \\ -q & -n \end{pmatrix}, \tag{5}$$

where the 3×3 real dimensionless matrices n, q, t are defined by the expectations of the algebra generators,

$$q_{jk}=\langle\Psi|\hat{Q}_{jk}|\Psi\rangle,$$

$$t_{jk}=\langle\Psi|\hat{T}_{jk}|\Psi\rangle,$$

$$n_{jk}=\langle\Psi|\hat{N}_{jk}|\Psi\rangle. \tag{6}$$

Note that ρ is a symplectic Lie algebra matrix because q and t are symmetric. The quantum mechanical expectation of a symplectic Lie algebra representation $\sigma(S)$ equals half the trace of the product of the density matrix times the Lie algebra element,

$$\langle\rho,S\rangle\equiv\frac{1}{2}\,\mathrm{tr}\,(\rho S)=-i\langle\Psi|\sigma(S)|\Psi\rangle. \tag{7}$$

A second equivalent characterization of the symplectic Lie algebra and its space of density matrices is given by operators that are quadratics in phonon creation and destruction operators. Let $a^\dagger_{\alpha j}$ and $a_{\alpha j}$ denote oscillator creation and destruction operators for particle α in the Cartesian basis; these operators obey phonon commutation relations, $[a_{\alpha j},a^\dagger_{\beta k}]=\delta_{\alpha\beta}\delta_{jk}$. The symplectic sp(3,$R$) generators in phonon form are

$$\hat{C}_{jk}=\sum_\alpha\left(a^\dagger_{\alpha j}a_{\alpha k}+\frac{1}{2}\delta_{jk}\right),$$

$$\hat{A}_{jk}=\frac{1}{2}\sum_\alpha a^\dagger_{\alpha j}a^\dagger_{\alpha k},$$

$$\hat{B}_{jk}=\frac{1}{2}\sum_\alpha a_{\alpha j}a_{\alpha k}. \tag{8}$$

The symplectic algebra of matrices may be viewed as a subalgebra of u(3,3). Define the complex 6×6 matrices in u(3,3)$\cap$sp(3,C),

$$S'=\begin{pmatrix} Z & -W \\ -W^* & -Z^T \end{pmatrix}, \tag{9}$$

where Z,W are complex 3×3 matrices, Z is skew-Hermitian, and W is symmetric. This matrix is represented by the skew-adjoint one-body operator,

$$\sigma'(S')=\sum_{jk}Z_{jk}\hat{C}_{jk}+\sum_{jk}(W_{jk}\hat{A}_{jk}-W^*_{jk}\hat{B}_{jk}). \tag{10}$$

These operators are a representation of the symplectic Lie algebra, $[\sigma'(S'_1),\sigma'(S'_2)]=\sigma'([S'_1,S'_2])$.

The symplectic density matrix ρ' relative to the phonon basis that corresponds to a normalized wave function Ψ is a complex matrix

$$\rho'=\begin{pmatrix} z^T & -w^* \\ -w & -z \end{pmatrix}, \tag{11}$$

where the 3×3 complex matrices z, w are defined by the expectations

$$z_{jk}=-i\langle\Psi|\hat{C}_{jk}|\Psi\rangle,$$

$$w_{jk}=-2i\langle\Psi|\hat{A}_{jk}|\Psi\rangle,$$

$$w^*_{jk}=2i\langle\Psi|\hat{B}_{jk}|\Psi\rangle. \tag{12}$$

Note that ρ' is a symplectic Lie algebra matrix in u(3,3) $\cap$sp(3,C) because z is skew-Hermitian and w is symmetric. The quantum mechanical expectation of the symplectic Lie algebra representation $\sigma'(S')$ equals the trace of the product of the density matrix times the Lie algebra element,

$$\langle\rho',S'\rangle\equiv\frac{1}{2}\,\mathrm{tr}\,(\rho' S')=-i\langle\Psi|\sigma'(S')|\Psi\rangle. \tag{13}$$

When Ψ is a highest weight vector of an irreducible unitary representation of sp(3,R), the density in the phonon basis is diagonal,

$$\varrho'=\begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix},\quad z=-i\,\mathrm{diag}(N_1,N_2,N_3), \tag{14}$$

where N_i are the weights of the irreducible unitary representation, $\hat{C}_{ii}\Psi=N_i\Psi$.

The representations σ of the real Lie algebra sp(3,R) and σ' of the embedding of sp(3,R) in u(3,3) are related by a transformation from position and momentum observables to harmonic oscillator phonons. Define the unitary 6×6 complex matrix

$$K=\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -iI & iI \end{pmatrix}, \tag{15}$$

where I denotes the 3×3 identity matrix. If S is a real symplectic Lie algebra matrix, Eq. (3), then $S'=K^\dagger SK$ is its isomorphic image in u(3,3)$\cap$Sp(3,C), Eq. (9). The representations of these two matrices as skew-adjoint operators are the same, $\sigma(S)=\sigma'(S')$. When the dual space elements are related by $\rho'=K^\dagger\rho K$, the expectations also coincide, $\langle\rho,S\rangle=\langle\rho',S'\rangle$.

The complex diagonal density ϱ' of Eq. (14) corresponds to a real symplectic density matrix $\varrho=K\varrho' K^\dagger$,

$$\varrho=\begin{pmatrix} 0 & t \\ -q & 0 \end{pmatrix},\quad t=q=\mathrm{diag}(N_1,N_2,N_3). \tag{16}$$

This is the symplectic density of a wave function with N_i oscillator quanta in the ith Cartesian direction.

A. Sp(3,R) transformation group

The symplectic group Sp(3,R) consists of the real 6×6 matrices g that leave invariant an antisymmetric form J,

$$\mathrm{Sp}(3,R)=\{g\in M_6(R)\,|\,g^T J g=J\},\quad J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}. \tag{17}$$

The group is represented by unitary transformations $\sigma(g)$ which are the exponentiation of the Lie algebra representation: If S is a sp(3,R) Lie algebra matrix, $g=\exp(S)$ is a symplectic group element that is represented unitarily by $\sigma(g)=\exp[\sigma(S)]$. The set of unitary operators $\sigma(g)$ forms a ray representation of the symplectic group and a unitary representation of the metaplectic group. The metaplectic group Mp(3,R) is a twofold covering group of the symplectic group that is introduced because the spectrum of the harmonic oscillator is half-integral.

When the symplectic group acts on a wave function, $\Psi\mapsto\sigma(g)\Psi$, the corresponding symplectic density is transformed according to the coadjoint action, $\rho\mapsto \mathrm{Ad}^*_g\rho = g\rho g^{-1}$. The coadjoint orbit $\mathcal{O}_\rho$ consists of the density ρ and all transformed densities $\mathrm{Ad}^*_g\rho$ as g ranges over the entire symplectic group Sp(3,R). An orbit $\mathcal{O}_\rho$ is a smooth surface contained in the space of all symplectic density matrices. The space of all density matrices is a disjoint union of its coadjoint orbits.

The subgroups of the symplectic group are transformation groups on each coadjoint orbit. The subalgebra gl(3,R) consists of the block diagonal matrices of Eq. (3), and the subgroup $\mathrm{GL}_+(3,R)$ of Sp(3,R) is given by the matrices

$$\tau(x)=\begin{pmatrix}x & 0\\ 0 & (x^T)^{-1}\end{pmatrix}, \tag{18}$$

where x is a 3×3 real matrix with a positive determinant. The general linear group $\mathrm{GL}_+(3,R)$ is the kinematical group of invertible linear transformations of three-dimensional Euclidean space. The subgroup SL(3,R) of the general linear group consists of the matrices with unit determinant and, when acting on Euclidean space, describes incompressible fluid motion.

The rotation group SO(3) is a subgroup of the general linear group, and it is a transformation group on each coadjoint orbit. For $R\in\mathrm{SO}(3)$, the coadjoint action transforms the density ρ of Eq. (5) into the rotated density

$$\tau(R)\cdot\rho\cdot\tau(R)^{-1}=\begin{pmatrix}(RnR^T)^T & RtR^T\\ -RqR^T & -RnR^T\end{pmatrix}. \tag{19}$$

Each orbit of the rotation group contains a diagonal inertia tensor $RqR^T=\mathrm{diag}(a_1^2,a_2^2,a_3^2)$. Since a physical orbit in the dual space demands a positive definite inertia tensor, the diagonal elements a_k^2 are positive real numbers. The positive roots $a_k>0$ are proportional to the axis lengths of the inertia ellipsoid. The orbits for which the inertia tensor fails to be positive definite are unphysical; i.e., no wave function Ψ exists such that the expectation of the quadrupole-monopole tensor $\hat{Q}_{jk}$ is not a positive definite matrix. Similarly, the momentum tensor t is positive definite on the physical coadjoint orbits. Thus a physically admissible orbit must have positive-definite moments t and q.

When q is diagonal, the symplectic density represents the system in the rotating principal axis frame, and it is denoted by $\tilde{\rho}$. The space $\mathcal{M}_\rho$ of all principal axis frame densities $\tilde{\rho}$ is a submanifold of the coadjoint orbit $\mathcal{O}_\rho$.

The unitary group U(3) is a subgroup of u(3,3) $\cap$Sp(3,C) given by

$$\tau'(x)=\begin{pmatrix}x & 0\\ 0 & (x^T)^{-1}\end{pmatrix}, \tag{20}$$

where x is a 3×3 complex unitary matrix. The subgroup U(3) is the symmetry group of the harmonic oscillator Hamiltonian.

The unitary group acts on the space of complex symplectic densities ρ' by the coadjoint transformation, $\rho'\mapsto\tau'(x)\rho'\tau'(x)^{-1}$. On the space of real symplectic densities ρ, the unitary group acts equivalently according to

$$\rho=K\rho'K^\dagger\mapsto K\tau'(x)\rho'\tau'(x)^{-1}=\tau(x)\rho\tau(x)^{-1}, \tag{21}$$

where the unitary group is represented by the real matrices

$$\tau(x)=K\tau'(x)K^\dagger=\begin{pmatrix}U & -V\\ V & U\end{pmatrix} \tag{22}$$

and U,V are the real and imaginary parts of $x=U+iV\in U(3)$.

Each coadjoint orbit is a homogeneous space; i.e., it is diffeomorphic to a coset space of the group Sp(3,R) modulo a subgroup. The relevant subgroup is an isotropy subgroup which is the set of group elements g that fix an orbit point, $\mathrm{Ad}^*_g\rho=\rho$. If ϱ of Eq. (16) is the orbit point, then the isotropy subgroup H_ϱ depends on whether the eigenvalues coalesce,

$$H_\varrho=\begin{cases}\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1), & \text{when } N_i \text{ are distinct;}\\ \mathrm{U}(2)\times\mathrm{U}(1), & \text{when two } N_i \text{ are equal;}\\ \mathrm{U}(3), & \text{when all } N_i \text{ are equal.}\end{cases} \tag{23}$$

The dimension of the homogeneous space $\mathcal{O}_\varrho \simeq \mathrm{Sp}(3,R)/H_\varrho$ is $\dim\mathcal{O}_\varrho=\dim\mathrm{Sp}(3,R)-\dim H_\varrho$. Hence the dimension of the generic coadjoint orbit is 18. When two eigenvalues N_i are equal, the coadjoint orbit is 16 dimensional. If all three eigenvalues coincide, the orbit Sp(3,R)/U(3) is 12 dimensional, and it is diffeomorphic to one of the classical domains [61].

The coadjoint orbits of the symplectic group in the space of complex symplectic densities are in one-to-one correspondence with the orbits $\mathcal{O}_\rho$ in the space of real symplectic densities. When $\rho'=K^\dagger\rho K$ is a complex symplectic density (11), then the orbit through it is $\mathcal{O}_{\rho'}=K^\dagger\mathcal{O}_\rho K$.

B. Symplectic Casimirs

The symplectic Casimirs $\mathcal{C}_{2r}[\rho]$ are real-valued functions of the density

$$\mathcal{C}_{2r}[\rho]=\frac{(-1)^r}{2}\,\mathrm{tr}\,(\rho^{2r}),\quad r=1,2,3. \tag{24}$$

The Casimirs are constant on each coadjoint orbit, $\mathcal{C}_{2r}[\rho]=\mathcal{C}_{2r}[\mathrm{Ad}_g^*\rho]$ for $g\in \mathrm{Sp}(3,R)$. They are also independent of the real or complex form of the symplectic density, $\mathcal{C}_{2r}[\rho]=\mathcal{C}_{2r}[\rho']$. The trace of an odd power of the density is identically zero because of the special form of the symplectic density matrix. Only the even powers of quadratic, quartic, and sextet degrees are functionally independent.

Suppose a coadjoint orbit in the space of complex symplectic densities contains a diagonal matrix ϱ' of the form (14) where the entries N_i are real numbers. The eigenvalues N_i are unique to the coadjoint orbit and, therefore, label the coadjoint orbit. When the N_i are positive half-integers or integers, the orbit corresponds to an irreducible unitary discrete series representation of the symplectic Lie algebra, and the labels N_i are its weights. There are also orbits for which the density ρ' cannot be diagonalized by a symplectic group transformation, but these will not be considered in this paper because they do not correspond to shell model representations.

Because the coadjoint action is matrix conjugation, it does not alter the eigenvalues of the density matrix, and the coadjoint transformation $\mathrm{Ad}_g^*\rho$ is an isospectral deformation of the density ρ. The solutions to the secular equation for each density in a fixed coadjoint orbit are the constant eigenvalues N_i. Thus the coefficients to the polynomial secular equation must be likewise constant on each coadjoint orbit. These coefficients are the Casimir functions $\mathcal{C}_{2r}[\rho]$. For an orbit $\mathcal{O}_{\varrho'}$, labeled by (N_1,N_2,N_3), the values of the symplectic Casimirs are

$$\mathcal{C}_{2r}[\varrho']=\sum_i N_i^{2r}. \tag{25}$$

A level surface of the symplectic Casimir functions consists of all densities ρ, Eq. (5), that satisfy the three algebraic equations, $\mathcal{C}_{2r}[\rho]=\Sigma_i N_i^{2r}$. The corresponding coadjoint orbit $\mathcal{O}_{\varrho'}$ in the space of complex matrices consists of the densities ρ' of the form (11) that satisfy the three polynomial relations (25). In the typical case of distinct N_i, the three Casimir functions are functionally independent and, according to the implicit function theorem, the level surface is 18 dimensional, i.e., the dimension of the dual space of densities minus the number of independent functions.

Since the Casimirs are invariant with respect to the coadjoint action, each coadjoint orbit is contained in a level surface of the symplectic Casimirs. For distinct N_i, the coadjoint orbit and the level surface have the same dimension, and the coadjoint orbit is an open submanifold of the level surface. But the coadjoint orbit is not necessarily identical to the level surface; see the Appendix for a counterexample.

C. Anisotropic representation

The definition of the representation of the algebra of symplectic matrices (4) depends ultimately on the choices for the length units of the three Euclidean directions which render $(x_{\alpha j},p_{\alpha j})$ dimensionless. In the isotropic case, a single length unit b is used, and the sp(3,R) representation is denoted by $\sigma(S)$. In the anisotropic case, the length units b_k depend on the Euclidean direction $k=1,2,3$, and the symplectic representation is denoted by $\sigma_{\mathrm{def}}(S_{\mathrm{def}})$. The symplectic matrices S and S_{def} are elements of the same matrix algebra (3).

For the deformed representation the density is denoted by ρ_{def} and

$$\langle\rho_{\mathrm{def}},S_{\mathrm{def}}\rangle\equiv\frac{1}{2}\,\mathrm{tr}\,(\rho_{\mathrm{def}}S_{\mathrm{def}})=-i\langle\Psi|\sigma_{\mathrm{def}}(S_{\mathrm{def}})|\Psi\rangle. \tag{26}$$

The subgroup of diagonal matrices in the general linear group determines transformations from isotropic to deformed oscillator phonons. Let $D=\mathrm{diag}(d_1,d_2,d_3)$ be the diagonal matrix with positive elements $d_k=b/b_k=\sqrt{\omega_k/\omega_0}$. The deformed representation of sp(3,R) is related to the isotropic representation by

$$\sigma_{\mathrm{def}}(S_{\mathrm{def}})=\sigma(S),\quad S=\tau(D)S_{\mathrm{def}}\tau(D)^{-1}, \tag{27}$$

for $S,S_{\mathrm{def}}\in\mathrm{sp}(3,R)$. If the diagonal matrix is an element of SL(3,R), then the deformed frequencies satisfy $\omega_1\omega_2\omega_3=\omega_0^3$.

The symplectic density ρ_{def} of a wave function Ψ in the deformed representation is related by a coadjoint transformation to the density ρ in the isotropic representation,

$$\rho=\tau(D)\rho_{\mathrm{def}}\tau(D)^{-1}. \tag{28}$$

The oscillator phonons can be chosen to be either the isotropic harmonic oscillator phonons or the anisotropic deformed oscillator phonons. The alternate representation by deformed phonons is the same skew-adjoint operator as its representation by isotropic phonons. Thus, for every skew-adjoint operator in the symplectic Lie algebra, there are four matrices that characterize it,

$$\sigma(S)=\sigma'(S')=\sigma_{\mathrm{def}}(S_{\mathrm{def}})=\sigma'_{\mathrm{def}}(S'_{\mathrm{def}}), \tag{29}$$

where the matrices are interrelated by

$$S'_{\mathrm{def}}=K^\dagger S_{\mathrm{def}}K,\quad S'=K^\dagger SK,$$

$$S=\tau(D)S_{\mathrm{def}}\tau(D)^{-1},\quad S'=\tau'(D)S'_{\mathrm{def}}\tau'(D)^{-1}, \tag{30}$$

with $\tau'(D)=K^\dagger\tau(D)K$. Depending on the theoretical problem, one of these four equivalent matrix forms of a symplectic Lie algebra operator may optimize the mathematical computation or the physical interpretation.

III. GCM(3) DECOMPOSITION

The general collective motion algebra gcm(3) is a subalgebra of the real symplectic algebra sp(3,R),

$$\mathrm{gcm}(3)=\left\{\begin{pmatrix} X & -U \\ 0 & -X^T \end{pmatrix}\middle| X,U\in M_3(R), U^T=U\right\}. \quad (31)$$

The gcm(3) algebra is a 15-dimensional semidirect sum of two Lie subalgebras. One subalgebra is the Lie algebra gl(3,R) of the general linear group. The ideal of the semidirect sum is the six-dimensional Abelian algebra generated by the quadrupole-monopole tensor.

The general collective motion group GCM(3) is a Lie subgroup of the symplectic group Sp(3,R),

$$\mathrm{GCM}(3)=\left\{\begin{pmatrix} x & -xU \\ 0 & (x^T)^{-1} \end{pmatrix}\middle| x\in \mathrm{GL}_+(3,R), U\in M_3(R), U^T=U\right\}. \quad (32)$$

The GCM(3) group is a semidirect product of $\mathrm{GL}_+(3,R)$ and the Abelian normal subgroup

$$R^6=\left\{\begin{pmatrix} I & -U \\ 0 & I \end{pmatrix}\middle| U\in M_3(R), U^T=U\right\}. \quad (33)$$

GCM(3) is a transformation group on each coadjoint orbit of the symplectic group. The goal of this section is to determine the orbits of GCM(3) in a fixed coadjoint orbit of the symplectic group. Given a symplectic density with a positive-definite quadrupole-monopole tensor q, a general linear group element can be found that transforms q into the identity matrix I. Next a R^6 group transformation is chosen to remove the symmetric part of the n tensor. Finally a rotation group element is selected to diagonalize the positive-definite kinetic tensor t. Hence each physical orbit of the GCM(3) group in the space of real symplectic densities contains a density of the form

$$\rho=\begin{pmatrix} -n & t \\ -I & -n \end{pmatrix}, \quad (34)$$

where $n^T=-n$ is antisymmetric and $t=\mathrm{diag}(t_1,t_2,t_3)$ is a positive-definite diagonal matrix.

There is one gcm(3) Casimir function which is the squared length of the Kelvin circulation vector,

$$C^2[\rho]=\mathrm{tr}(qn^Tq^{-1}n-n^2). \quad (35)$$

The circulation is zero, $C^2[\varrho]=0$, for the symplectic orbit representative of Eq. (16). The Casimir function is invariant with respect to GCM(3) subgroup transformations, $C^2[\rho]=C^2[\mathrm{Ad}^*_g\rho]$ for $g\in$ GCM(3). Since the Kelvin circulation is constant on each GCM(3) orbit, its value may be computed at the GCM(3) representative density of Eq. (34), $C^2[\rho]=-2\,\mathrm{tr}(n^2)$. The antisymmetric matrix n determines the components of the pseudovector Kelvin circulation, $n_{ij}=\epsilon_{ijk}C_k/2$, and the Casimir function is $C^2[\rho]=C_1^2+C_2^2+C_3^2=\vec{C}\cdot\vec{C}$.

When evaluated at the GCM(3) representative density, the quadratic symplectic Casimir function implies the relation

$$\frac{1}{2}\,\vec{C}\cdot\vec{C}+\mathrm{tr}(t)=\sum N_k^2. \quad (36)$$

Since the kinetic tensor t is positive definite, the length of the circulation pseudovector is bounded from above on a coadjoint orbit of the symplectic group.

A finer upper bound is determined by identifying the maximum of the Kelvin circulation on a given coadjoint orbit. A relative maximum is a critical point of the gcm(3) Casimir function. Let ρ denote the GCM(3) orbit representative of Eq. (34) and $\rho(\epsilon)$ a smooth curve in the symplectic coadjoint orbit through ρ,

$$\rho(\epsilon)=e^{\epsilon S}\rho e^{-\epsilon S}=\rho+\epsilon[S,\rho]+\frac{\epsilon^2}{2}[S,[S,\rho]]+\cdots, \quad (37)$$

for S an element (3) of the symplectic Lie algebra. A critical point of the gcm(3) Casimir satisfies

$$\frac{d}{d\epsilon}C^2[\rho(\epsilon)]\big|_{\epsilon=0}=0, \quad (38)$$

or equivalently, after evaluating the derivative explicitly,

$$0=(t_1-t_2)C_3=(t_2-t_3)C_1=(t_3-t_1)C_2. \quad (39)$$

In addition, the symplectic density satisfies the three symplectic Casimir identities.

There are three cases to consider. First, if all components of the circulation are zero, then the GCM(3) orbit representative is

$$\rho=\begin{pmatrix} 0 & t \\ -I & 0 \end{pmatrix}, \quad t=\mathrm{diag}(N_1^2,N_2^2,N_3^2). \quad (40)$$

This density is in the GCM(3) orbit of the highest weight density ϱ since $\rho=\tau(x)\varrho\,\tau(x)^{-1}$ for $x=\mathrm{diag}(\sqrt{N_1},\sqrt{N_2},\sqrt{N_3})$ in $\mathrm{GL}_+(3,R)$.

Second, if two components of the circulation vanish and one is nonzero, say $C_1=C_2=0$, $C_3\neq 0$, then two diagonal elements of the kinetic tensor coalesce, $t_1=t_2$, according to Eq. (39). This density is on the level surface of the symplectic Casimir functions when either (1)

$$C_3=N_1-N_2, \quad t_1=t_2=(N_1+N_2)^2/4, \quad t_3=N_3^2 \quad (41)$$

or (2) $C_3=N_1+N_2$, $t_1=t_2=(N_1-N_2)^2/4$, $t_3=N_3^2$. It is shown in the Appendix that (2) is not a point on the coadjoint orbit $\mathcal{O}_\varrho$, although it lies on the level surface, and, therefore, this solution is rejected. For (1), the quanta N_k may be permuted to obtain three distinct solutions altogether.

Third, if just one component of the circulation is zero—say, $C_3=0$—then the kinetic tensor is a multiple of the identity matrix. Because of the kinetic tensor symmetry, the symplectic density may be rotated so that the circulation vector has two zero components. Thus the third case reduces to the second case. When all diagonal elements of t are equal, one of the quanta is the average of the other two—say, $N_3=(N_1+N_2)/2$.

To determine whether or not a critical point is a relative maximum, the Hessian of the gcm(3) Casimir function must be evaluated. Along a curve $\rho(\epsilon)$ through the critical point (41), the gcm(3) Casimir equals

$$C^2[\rho(\epsilon)]=(N_1-N_2)^2-\epsilon^2\{(N_1^2-N_3^2)(N_3^2-N_2^2)(V_{23}^2+V_{31}^2) +(N_1-N_2)^2N_1N_2[(V_{22}-V_{11})^2+4V_{12}^2]\}. \quad (42)$$

The GCM(3) orbit point ρ is a relative maximum of the gcm(3) Casimir function only when $N_1\geqslant N_3\geqslant N_2$ or $N_1\leqslant N_3\leqslant N_2$.

Theorem. Suppose $\mathcal{O}_\varrho$ is the symplectic coadjoint orbit containing the density ϱ of Eq. (16) and the orbit labels are ordered by $N_1\geqslant N_3\geqslant N_2$. The range of the squared length of the Kelvin circulation vector on the surface $\mathcal{O}_\varrho$ is $0\leqslant C^2[\rho]\leqslant(N_1-N_2)^2$.

In a study of the unitary sp(3,R) discrete series representations and their gcm(3) decomposition, Rowe and Repka [44] proved that the maximum value of the circulation equals the maximum value of the angular momentum in the su(3) representation $(\lambda,\mu)=(N_1-N_3,N_3-N_2)$ for $N_1\geqslant N_3\geqslant N_2$. The maximum value from su(3) representation theory is $\lambda+\mu=N_1-N_2$ which, as has been just shown, is the maximum value of the circulation on a coadjoint orbit.

IV. OSCILLATOR HAMILTONIAN

The simplest Hamiltonian of the symplectic model is the isotropic harmonic oscillator,

$$\hat{H}_0=\hbar\omega_0\sum_{k=1}^{3}\hat{C}_{kk}, \quad (43)$$

and its energy functional is

$$E[\rho]=\frac{\hbar\omega_0}{2}\,\mathrm{tr}\,(t+q), \quad (44)$$

where ω_0 is the isotropic oscillator frequency. The oscillator Hamiltonian is an element of the symplectic Lie algebra, $\hat{H}_0=-i\hbar\omega_0\sigma(h)=-i\hbar\omega_0\sigma'(h')$, where the real and complex symplectic matrices are

$$h=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}, \quad h'=\begin{pmatrix}iI & 0\\ 0 & -iI\end{pmatrix}. \quad (45)$$

A critical point ρ' of the harmonic oscillator energy functional must commute with the matrix Hamiltonian h',

$$0=[\rho',h'], \quad (46)$$

and, therefore, the symplectic density is

$$\rho'=\begin{pmatrix}z^T & 0\\ 0 & -z\end{pmatrix}, \quad (47)$$

where z is any skew-Hermitian 3×3 complex matrix. In order for ρ' to lie on the coadjoint orbit, the eigenvalues of z must be $-iN_1,-iN_2,-iN_3$. The energy equals $E=\hbar\omega_0\langle\rho',h'\rangle=i\hbar\omega_0\mathrm{tr}\,z=\hbar\omega_0\Sigma N_k$. Since the harmonic oscillator Hamiltonian is U(3) invariant, there is a corresponding U(3) degeneracy for the density,

$$\langle \mathrm{Ad}_x^*\rho',h'\rangle=\langle\rho',\mathrm{Ad}_x^{-1}h'\rangle=\langle\rho',h'\rangle=E, \quad (48)$$

for any $x\in$ U(3).

The symplectic orbit representative ϱ has energy E and it minimizes of the oscillator energy functional. The U(3)orbit $\mathrm{Ad}_x^*\varrho'$ for $x\in$U(3) is a surface of degenerate densities in the coadjoint orbit.

For the discrete series unitary representations of sp(3,R), the harmonic oscillator spectrum is bounded similarly from below at the energy E. Moreover, the eigenspace of energy E in the representation space is U(3) invariant.

A. Anisotropic oscillator

The anisotropic oscillator Hamiltonian is an element of the symplectic Lie algebra,

$$\hat{H}=\sum_{k=1}^{3}\hbar\omega_k\hat{C}_{kk}, \quad (49)$$

where ω_k are the normal mode frequencies and the operators $\hat{C}_{kk}$ are defined in terms of deformed phonons. A $\mathrm{GL}_+(3,R)$ adjoint transformation that is defined by a diagonal matrix $D=\mathrm{diag}(d_1,d_2,d_3)$, $d_k^2=\omega_k/\omega_0$, transforms the isotropic oscillator into the deformed oscillator

$$i\hat{H}/\hbar\omega_0=\sigma_{\mathrm{def}}(h_{\mathrm{def}})=\sigma'_{\mathrm{def}}(h'_{\mathrm{def}}), \quad (50)$$

where the real and complex symplectic matrices

$$h_{\mathrm{def}}=\begin{pmatrix}0 & -U\\ U & 0\end{pmatrix}, \quad U=\mathrm{diag}(\omega_1/\omega_0,\omega_2/\omega_0,\omega_3/\omega_0),$$

$$h'_{\mathrm{def}}=\begin{pmatrix}Z & 0\\ 0 & -Z\end{pmatrix}, \quad Z=iU. \quad (51)$$

The energy functional is $E[\rho'_{\mathrm{def}}]=\hbar\omega_0\langle\rho'_{\mathrm{def}},h'_{\mathrm{def}}\rangle=\langle\Psi|\hat{H}|\Psi\rangle$. The real matrix h_{def} or the equivalent complex matrix h'_{def} in sp(3,R) is the mean field Hamiltonian. A mean field solution ρ'_{def} is a complex density that is a critical point of the energy functional on the coadjoint orbit $\mathcal{O}_{\varrho'}$. A critical point commutes with the mean field Hamiltonian,

$$[\rho'_{\mathrm{def}},h'_{\mathrm{def}}]=0. \quad (52)$$

Since the mean field Hamiltonian h'_{def} is diagonal, the commuting density ρ'_{def} must also be diagonal. To be a point on the coadjoint orbit, the diagonal entries of ρ'_{def} must be some permutation of $(-iN_1,-iN_2,-iN_3)$. When the diagonal entries are not permuted, the energy of the critical point is simply $E[\rho'_{\mathrm{def}}]=\Sigma\hbar\omega_kN_k$.

B. Cranked deformed oscillator

The cranked anisotropic oscillator $\hat{H}_\omega = \hat{H} - \omega \hat{L}_1$ is the Routhian that describes the rotation of the anisotropic oscillator $\hat{H}$ about the one-axis with an angular velocity ω. The Routhian is an element of the symplectic Lie algebra because the components of the vector angular momentum operator are in the algebra. The kth angular momentum component is the representation of the real matrix S,

$$i\hat{L}_k/\hbar = \sigma(S), S = \begin{pmatrix} E_{ij} - E_{ji} & 0 \\ 0 & E_{ij} - E_{ji} \end{pmatrix}, \tag{53}$$

where E_{ij} denotes the elementary 3×3 matrix whose only nonzero entry is one at the intersection of row i with column j. With respect to the deformed phonon basis, the Routhian operator is the representation of the complex matrix h'_ω,

$$i\hat{H}_\omega/\hbar\omega_0 = \sigma'_{\mathrm{def}}(h'_\omega), \tag{54}$$

$$h'_\omega = \begin{pmatrix} Z & -W \\ -W^* & -Z^T \end{pmatrix},$$

$$Z = \begin{pmatrix} i\omega_1/\omega_0 & 0 & 0 \\ 0 & i\omega_2/\omega_0 & -u \\ 0 & u & i\omega_3/\omega_0 \end{pmatrix},$$

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -v \\ 0 & -v & 0 \end{pmatrix}, \tag{55}$$

where

$$u = \frac{\omega}{2\omega_0}(\alpha + \alpha^{-1}), \quad v = \frac{\omega}{2\omega_0}(\alpha - \alpha^{-1}), \tag{56}$$

and $\alpha = \sqrt{\omega_2/\omega_3}$. A critical point for the Routhian energy functional $E_\omega[\rho'_\omega] = \hbar\omega_0\langle \rho'_\omega, h'_\omega\rangle$ on a coadjoint orbit is a density ρ'_ω that commutes with the Routhian matrix h'_ω. To solve for the critical point, first diagonalize the Routhian matrix by an adjoint transformation

$$\mathrm{Ad}_g h'_\omega = \begin{pmatrix} Z & 0 \\ 0 & -Z \end{pmatrix}, \quad Z = i\,\mathrm{diag}(\omega_1, \Omega_+, \Omega_-)/\omega_0,$$

$$\Omega_\pm^2 = \frac{1}{2}(\omega_2^2 + \omega_3^2) + u^2 - v^2 \pm q,$$

$$q^2 = \frac{1}{4}(\omega_2^2 - \omega_3^2)^2 + u^2(\omega_2 + \omega_3)^2 - v^2(\omega_2 - \omega_3)^2. \tag{57}$$

The transformed density $\mathrm{Ad}^*_g \rho'_\omega$ commutes with the diagonalized Routhian,

$$0 = g[\rho'_\omega, h'_\omega]g^{-1} = [\mathrm{Ad}^*_g \rho'_\omega, \mathrm{Ad}_g h'_\omega]. \tag{58}$$

This forces the transformed density to be diagonal. But a diagonal density on the coadjoint orbit must have the eigenvalues iN_1, iN_2, iN_3. Hence, in the natural order of Eq. (14), the symplectic density of the cranked anisotropic oscillator is

$$\rho'_\omega = \mathrm{Ad}^*_{g^{-1}} \varrho'. \tag{59}$$

The energy of the critical point is $E_\omega[\rho'_\omega] = \hbar\omega_1 N_1 + \hbar\Omega_+ N_2 + \hbar\Omega_- N_3$.

In the usual derivation of the cranked anisotropic oscillator energy, the quantum numbers N_1, N_2, N_3 are fixed, an assumption known as the adiabatic condition [62]. In the symplectic theory, the quanta are fixed because the mean field assumption restricts the densities to a fixed coadjoint orbit. Thus the adiabatic condition and the symplectic mean field approximation are mathematically equivalent.

The density of the cranked oscillator may be determined explicitly in two ways. One method is to determine the mean field Routhian's eigenvectors, which are the columns of the symplectic group element g of Eq. (59). The other technique is to apply Feynman's lemma [63]. Suppose the mean field Hamiltonian $h(\lambda)$ depends on some parameter λ, e.g., the angular velocity ω or the frequencies ω_k. For each value of λ, let $\rho(\lambda)$ denote the equilibrium density, $[\rho(\lambda), h(\lambda)] = 0$, and let $E(\lambda) = \hbar\omega_0\langle\rho(\lambda), h(\lambda)\rangle$ be the equilibrium energy. Then Feynman's lemma, in its density matrix form, states that

$$\frac{dE}{d\lambda} = \hbar\omega_0 \left\langle \rho(\lambda), \frac{\partial}{\partial\lambda} h(\lambda) \right\rangle. \tag{60}$$

When λ is taken to be ω, the angular momentum of the equilibrium density is determined. If λ equals one of the frequencies ω_i, the squared axis lengths of the inertia ellipsoid are determined. Explicit formulas for the angular momentum and the axis lengths are given by Stamp [64].

C. Inglis cranking

For small angular velocities, the cranking term may be computed in perturbation theory. In the coadjoint orbit theory, the perturbation method is implemented by expressing the diagonalizing symplectic group element g as a power series in ω,

$$g = \exp\left(\frac{\omega}{\omega_0} M\right) = I + \frac{\omega}{\omega_0} M + \frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2 M^2 + \cdots, \tag{61}$$

where M is a complex matrix in the symplectic Lie algebra $\mathrm{sp}(3,C) \cap \mathrm{u}(3,3)$, Eq. (9). Here M is chosen so that

$$\mathrm{Ad}_g h'_\omega = h'_\omega + \frac{\omega}{\omega_0}[M, h'_\omega] + \frac{1}{2}\left(\frac{\omega}{\omega_0}\right)^2 [M,[M, h'_\omega]] + \cdots \tag{62}$$

is a diagonal matrix to quadratic order in ω. The perturbation results in the coadjoint orbit theory for the rotational energy, deformation, and angular momentum derived are identical to the well-known formulas of Inglis [54].

The advantage to the perturbation formulas is that their physical interpretation is clearer than the exact expressions of Sec. IV B and that the approximation is adequate for ap-

plications to the yrast rotational bands of deformed even-even isotopes. In particular, the angular momentum I in the Inglis approximation is a linear interpolation of the rigid body (RR) and irrotational flow (IF) values,

$$I=[r\mathcal{I}_{RR}+(1-r)\mathcal{I}_{IF}]\omega, \tag{63}$$

where the moments of inertia are

$$\mathcal{I}_{RR}=\frac{M}{5}(b^2+c^2),\quad \mathcal{I}_{IF}=\frac{M}{5}\frac{(b^2-c^2)^2}{b^2+c^2}, \tag{64}$$

when M is the nuclear mass, and b,c are the semiaxis lengths of the inertia ellipsoid perpendicular to the rotation axis, and the parameter r equals

$$r=\frac{N_3/N_2-N_2/N_3}{\omega_2/\omega_3-\omega_3/\omega_2}. \tag{65}$$

The extreme cases of rigid rotation and irrotational flow, respectively, are attained when $\omega_3 N_3=\omega_2 N_2$ or $r=1$, the self-consistency ansatz, and $N_3=N_2$ or $r=0$, respectively [56].

In the classical theory of Riemann ellipsoids, the angular momentum is also a linear combination (63) where the mixing parameter is the rigidity r, the ratio of the Kelvin circulation to its rigid body value [44,46].

Within the framework of the cranked anisotropic oscillator theory, what frequencies are most appropriate for real nuclei? The squares of the axis lengths (a,b,c) of the inertia ellipsoid are proportional to the ratios of quanta to frequency in the three Cartesian directions, $a^2\propto N_1/\omega_1$, $b^2\propto N_2/\omega_2$, $c^2\propto N_3/\omega_3$. After rearranging Eq. (65), the ratio of the frequencies depends on the rigidity and the deformation according to

$$\left(\frac{\omega_2}{\omega_3}\right)^2=\frac{r+(c/b)^2}{r+(b/c)^2}. \tag{66}$$

Rigid rotation and irrotational flow, respectively, correspond to $\omega_2/\omega_3=c/b$ and $\omega_2/\omega_3=(c/b)^2$, respectively. The rigidity of an even-even deformed nucleus is inferred from the measured deformation and moment of inertia of its yrast rotational band using Eq. (68) for a Riemann ellipsoid [46]. Equation (66) determines the frequency ratio of a nucleus modeled as a pure anisotropic oscillator using the experimental rigidity and deformation. In Fig. 1, the frequency ratio ω_2/ω_3 is plotted versus the major to minor axis ratio c/b for all deformed even-even nuclei with $c/b\geqslant 1.2$ using experimental values for the nuclear quadrupole deformation and the Riemann ellipsoid rigidity [65]. The 76 deformed isotopes in the figure range from ^{12}C to ^{252}Cf. The correlation of a straight line of slope=2 with the experimental points is 0.996. The linear relationship for real nuclei between the deformation in frequency space and the deformation in Euclidean space is remarkably simple.

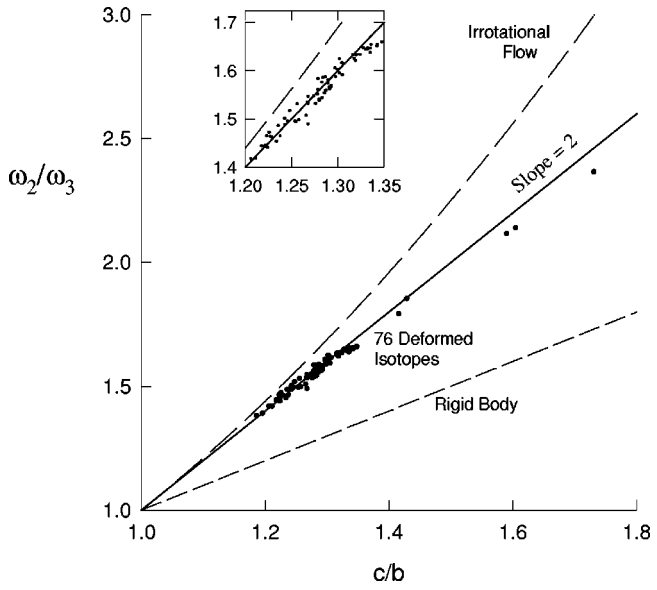

FIG. 1. The ratio of the frequencies of the anisotropic oscillator Hamiltonian is plotted versus the ratio of the axis lengths for deformed even-even nuclei using Eq. (66).

V. RIEMANN ELLIPSOIDS

The gcm(3) subalgebra of sp(3,R) is the dynamical symmetry algebra of the Riemann ellipsoid model. A Riemann ellipsoid is a classical fluid with an ellipsoidal boundary such that its velocity field $\vec{v}(\vec{r})$ is a linear function of the Cartesian position vector $\vec{r}$. When $\vec{v}(\vec{r})=\Omega\,\vec{r}$ with Ω an antisymmetric 3×3 matrix, the ellipsoid rotates rigidly. When $\vec{v}(\vec{r})=\Xi\,\vec{r}$ with Ξ a symmetric 3×3 matrix, the ellipsoid rotates irrotationally, $\vec{\nabla}\times\vec{v}=0$. A general linear velocity field is defined by a sum of antisymmetric Ω and symmetric Ξ matrices.

An S-type Riemann ellipsoid is one that is rotating about a principal axis—say, the one-axis. The S-type equilibrium ellipsoids are indexed by the rigidity parameter r. The kinetic energy of an S-type ellipsoid is an interpolation between the rigid and irrotational values,

$$E=\frac{I^2}{2\mathcal{I}_r}, \tag{67}$$

where the moment of inertia is

$$\mathcal{I}_r=\frac{[r\mathcal{I}_{RR}+(1-r)\mathcal{I}_{IF}]^2}{r^2\mathcal{I}_{RR}+(1-r^2)\mathcal{I}_{IF}}. \tag{68}$$

For a heavy deformed nucleus, the low energy yrast rotational band is described by the classical Riemann formula. Because the experimental moment of inertia is typically about one-half the rigid body value and 5 times the irrotational flow value, the rigidity is about 1/6. In Fig. 2, the energy levels of the yrast band of ^{166}Er are compared to the spectra of a rigid rotor ($r=1$), irrotational fluid ($r=0$), and S-type Riemann ellipsoid ($r=0.15$).

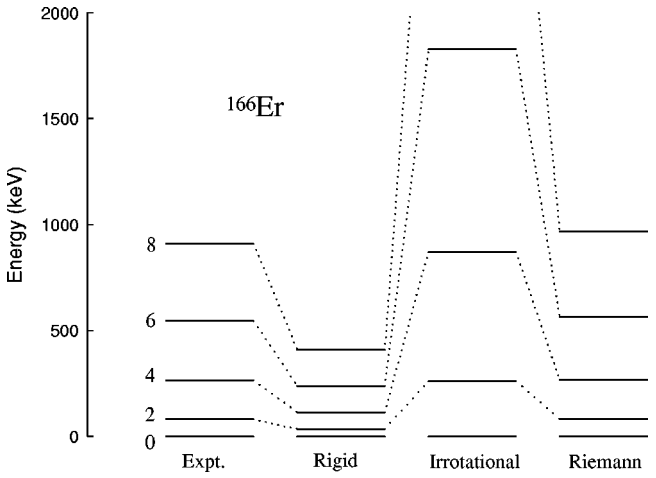

FIG. 2. The yrast band spectrum of ^{166}Er is compared with the theoretical spectrum of a rigid rotor, irrotational droplet, and a Riemann ellipsoid with $r=0.15$.

If the rigidity were only an interpolating parameter that enables a fit to the experimental moment of inertia, it would be of minor interest. But the rigidity characterizes the nuclear current and the nature of rotational motion in deformed isotopes. A direct experimental method for determining the nuclear circulation and rigidity is provided by inelastic electron scattering measurements of the transverse $E2$ form factor, as has been emphasized by Moya de Guerra [13] and Vassanji and Rowe [14]. The author has shown elsewhere that the transverse $E2$ form factor for a Riemann rotor is a weighted interpolation of the rigid rotor and irrotational flow form factors [16]. There is to date no published experimental measurement of transverse form factors in the heavy deformed region. But projected Hartree-Fock (PHF) calculations of the transverse $E2$ form factor in the heavy deformed region were done by Berdichevsky *et al.* [15]. For ^{156}Gd the PHF transverse form factor implies a rigidity $r\approx 0.12$ and this is consistent with the rigidity $r=0.15$ inferred from the moment of inertia.

The rigidity depends on the nuclear deformation. For the rare-earth even-even isotopes, Fig. 3 plots the experimental rigidity r versus the quadrupole deformation β. The optimal theoretical fit of a quadratic function of the deformation to the rigidity in this region is $r=10\beta^2-5.1\beta+0.68$. In Fig. 4, this quadratic function is used in conjunction with Eqs. (67), and (68) to predict the energy of the 2^+ excited state in the rare-earth region.

The Riemann model is an approximate solution to a symplectic model Routhian which includes a Kelvin circulation term. The Kelvin circulation is a vector operator with components $\hat{C}_i=\varepsilon_{ijk}(\hat{Q}^{-1/2}\hat{N}\hat{Q}^{1/2})_{jk}$. Despite its physical importance, the expectation of the circulation has not been calculated in shell model irreducible representations of the symplectic algebra because the Kelvin circulation operator is very complicated. However, in mean field theory, this operator may be evaluated in the principal axis frame,

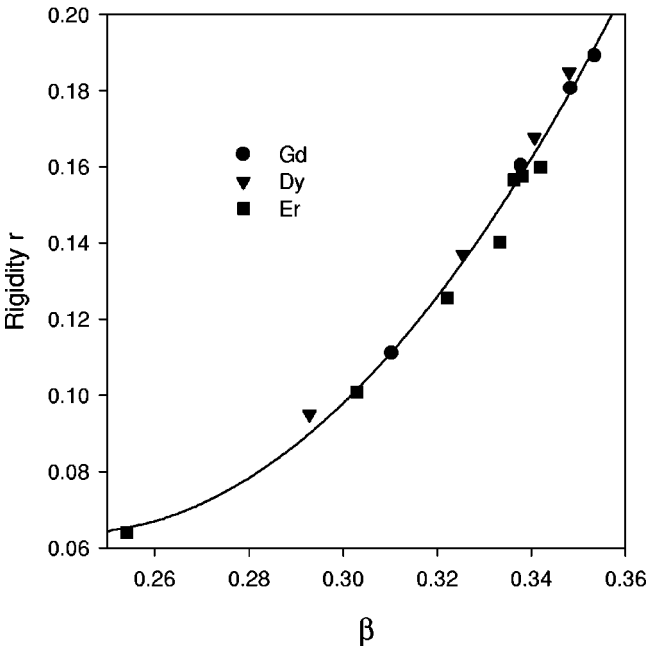

FIG. 3. The rigidity of 16 rare earth even-even isotopes is plotted versus their quadrupole deformation.

$$i\hat{C}_k/\hbar=\sigma(S[\tilde{\rho}]),\quad S[\tilde{\rho}]=\begin{pmatrix} X & 0 \\ 0 & -X^T \end{pmatrix},$$

$$X=(a_j/a_i)E_{ij}-(a_i/a_j)E_{ji}, \tag{69}$$

where the quadrupole-monopole tensor $\tilde{q}$ of the symplectic density $\tilde{\rho}$ in the rotating frame is diagonal $\tilde{q}=\mathrm{diag}(a_1^2,a_2^2,a_3^2)$. The circulation operator is the representation of a density dependent matrix $S[\tilde{\rho}]$ in the sp(3,R) Lie algebra.

The symplectic Routhian in the Riemann ellipsoid approximation is [58,60]

$$\hat{H}_{\omega\lambda}=\hat{H}-\lambda\hat{C}_1-\omega\hat{L}_1. \tag{70}$$

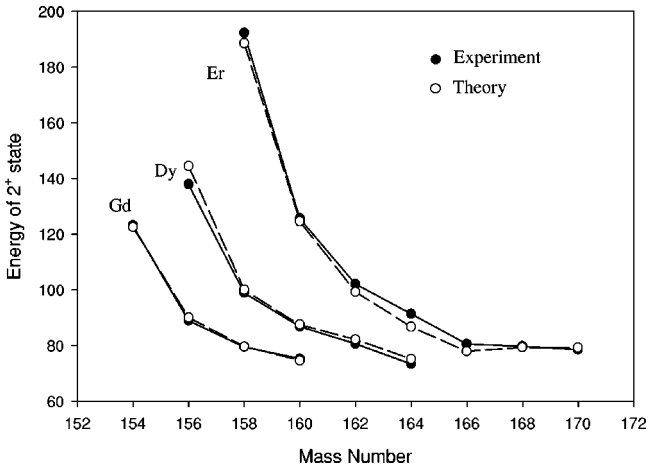

FIG. 4. The theoretical energy of the first 2^+ excited state for 16 rare-earth even-even isotopes is compared to experiment.

In the classical Riemann model, the real parameter λ is the vortex velocity. With respect to the deformed basis, the Routhian operator is the representation of the complex symplectic matrix $h'_{\omega\lambda}[\tilde{\rho}]$,

$$i\hat{H}_{\omega\lambda}/\hbar\omega_0=\sigma'_{\mathrm{def}}(h'_{\omega\lambda}), \tag{71}$$

where the matrix is given by Eq. (55) with

$$u=\frac{\omega}{2\omega_0}(\alpha+\alpha^{-1})+\frac{\lambda}{2\omega_0}(\nu+\nu^{-1}),$$

$$v=\frac{\omega}{2\omega_0}(\alpha-\alpha^{-1})+\frac{\lambda}{2\omega_0}(\nu-\nu^{-1}), \tag{72}$$

with $\nu=(a_2/a_3)\alpha$ and $\alpha^2=\omega_2/\omega_3$. A critical point for the Routhian energy functional $E_{\omega\lambda}[\rho'_{\omega\lambda}]=\hbar\omega_0\langle\rho'_{\omega\lambda},h'_{\omega\lambda}\rangle$ on a coadjoint orbit is a density that commutes with the Routhian matrix

$$[\rho'_{\omega\lambda},h'_{\omega\lambda}]=0. \tag{73}$$

For small angular and vortex velocities, a critical point for the Riemann ellipsoid Routhian can be derived using perturbation theory. When the self-consistency condition is satisfied, $\omega_1N_1=\omega_2N_2=\omega_3N_3$, the classical expressions for the angular momentum (63), and circulation of a Riemann ellipsoid are attained [58],

$$C=(M/5)[2bc\omega-(b^2+c^2)\lambda],$$

$$I=(M/5)[(b^2+c^2)\omega-2bc\lambda], \tag{74}$$

where M is the nuclear mass, and b,c are the axis lengths perpendicular to the rotation axis. The rotational kinetic energy also approximates its classical Riemann ellipsoid value (68) or

$$E=(M/10)[(b^2+c^2)(\omega^2+\lambda^2)-4bc\omega\lambda]$$

$$=\frac{1}{4}\left[\frac{(I+C)^2}{(b+c)^2}+\frac{(I-C)^2}{(b-c)^2}\right]. \tag{75}$$

When $\lambda=0$, the rigid rotation formulas of Inglis cranking are obtained [54].

When the angular and vortex velocities are not small, a critical point for the Routhian energy is discovered by an iterative scheme. The procedure is similar to the conventional technique to find Hartree-Fock solutions. First make an initial guess for the symplectic density $\rho'_{\omega\lambda}$ and calculate the symplectic Routhian matrix $h'_{\omega\lambda}[\rho'_{\omega\lambda}]$. Next, diagonalize the Routhian matrix by an adjoint transformation

$$\mathrm{Ad}_g h'_{\omega\lambda}=\begin{pmatrix} Z & 0 \\ 0 & -Z \end{pmatrix},\quad Z=i\,\mathrm{diag}(\omega_1,\Omega_+,\Omega_-)/\omega_0,$$

for $g\in \mathrm{U}(3,3)\cap \mathrm{Sp}(3,C)$. The eigenvalues of the Routhian matrix are given by Eq. (57).

The transformed density $\mathrm{Ad}^*_g\rho'_{\omega\lambda}$ commutes with the diagonalized Routhian,

$$0=g[\rho'_{\omega\lambda},h'_{\omega\lambda}]g^{-1}=[\mathrm{Ad}^*_g\rho'_{\omega\lambda},\mathrm{Ad}_g h'_{\omega\lambda}]. \tag{76}$$

This forces the transformed density to be diagonal. But a diagonal density on the coadjoint orbit must have the eigenvalues iN_1, iN_2, iN_3. Hence, in the natural order of Eq. (14), the symplectic density of the cranked anisotropic oscillator is

$$\rho'_{\omega\lambda}=\mathrm{Ad}^*_{g^{-1}}\varrho'. \tag{77}$$

The energy of the critical point is $E_\omega[\rho'_{\omega\lambda}]=\hbar\omega_1N_1+\hbar\Omega_+N_2+\hbar\Omega_-N_3$.

The symplectic density, determined in this way, is not yet a critical point for the Routhian when $\lambda\neq0$. In contrast to ordinary Inglis cranking of the anisotropic oscillator, the Routhian matrix $h'_{\omega\lambda}[\rho'_{\omega\lambda}]$ depends on the symplectic density due to the Kelvin circulation term. Using the transformed density (77) as the new starting point, one repeats the procedure a second time, thereby yielding a transformed density (77). This sequence must be iterated until convergence is achieved:

$$\rho'_{\omega\lambda}\rightarrow h'_{\omega\lambda}\rightarrow \mathrm{Ad}_g h'_{\omega\lambda}\ \text{is diagonal}\rightarrow\rho'_{\omega\lambda}=\mathrm{Ad}^*_{g^{-1}}\varrho'. \tag{78}$$

Numerical results for the deformation, angular momentum, circulation, and energy obtained by this iterative scheme were reported elsewhere [58].

The pairing force influences significantly the rigidity. In prior work the rigidity was shown to be given approximately by

$$r\approx 1-\sinh^{-1}x/(x\sqrt{1+x^2}), \tag{79}$$

where $x=(\hbar\omega_2-\hbar\omega_3)/2\Delta$ and Δ is the BCS pairing gap [66]. When the pair gap is large, x and the rigidity r are small and the rotational motion approaches irrotational flow. When the pairing gap is small, x is large, the rigidity nears unity and the system rotates almost rigidly. Using experimental values for the moment of inertia and deformation, the rigidity is calculated from Eq. (68), and the pairing field is deduced from the Belyaev formula for the moment of inertia of a self-consistent anisotropic oscillator plus BCS pairing field [67]. In Fig. 5, the BCS pairing field Δ is plotted versus the rigidity r in the rare earth region. BCS pairing and the Kelvin circulation modify the moment of inertia of a cranked anisotropic oscillator in similar ways.

VI. CONCLUSION

The symplectic shell model presents special technical challenges because its irreducible representations are infinite dimensional. Moreover, explanations of nuclear rotational motion in the shell model share a common problem: the rotating body-fixed frame is difficult or impossible to define rigorously and work with effectively. Symplectic coadjoint orbit theory eliminates these two obstructions to a tractable robust theory of geometrical collective motion.

When $\mathrm{sp}(3,R)$ is an exact dynamical symmetry of the shell model, the wave functions are vectors from a single symplectic irreducible representation space. Spin-orbit and

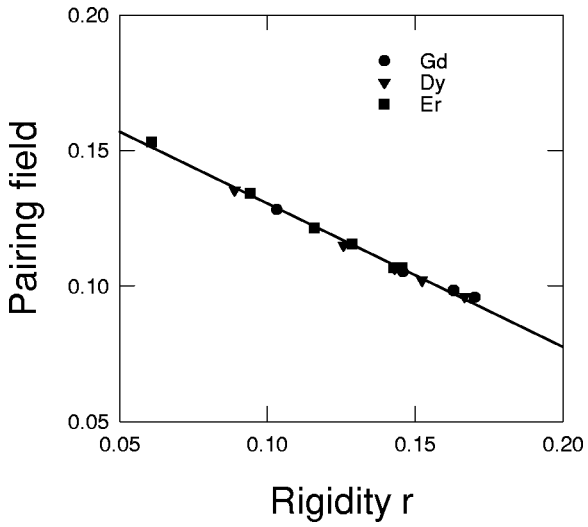

FIG. 5. The pairing field Δ depends linearly on the rigidity r in the rare earth region.

pairing forces are not operators in the symplectic enveloping algebra and, except for special cases, cause a violation of exact symplectic symmetry. The symplectic Lie algebra is a maximal finite dimensional Lie algebra of hermitian operators; i.e., any Hermitian microscopic operator adjoined to the sp(3,R) algebra generates an infinite-dimensional Lie algebra. The exclusion of monopole pairing from the sp(3,R) dynamical symmetry model is partially mitigated by the inclusion of the Kelvin circulation. The effect of the Kelvin circulation on the nuclear moment of inertia is similar to BCS pairing. The spin-orbit force usually mixes symplectic representations. One exception is when Wigner supermultiplet symmetry is good. In some cases, the representation of the symplectic algebra by pseudospin operators enables sp(3,R) to be a good symmetry [68,69]. In heavy deformed nuclei, strong coupling of the spin to rotor bands can contribute positively to well-defined rotational bands [70,71]. However, the overall landscape is that sp(3,R) symmetry is broken for most isotopes, and its application to the description of geometrical states requires careful analysis. This is a puzzle because experiment unambiguously demonstrates the existence of rotational bands, yet the fundamental theory of geometrical collective states indicates that such bands are only possible in special circumstances.

Symplectic mean field theory places less stringent demands than irreducible representation theory and provides a conceivable explanation for the data. In mean field theory, the basic object is a sp(3,R) density matrix that is defined by the expectations of algebra generators. Even when a set of wave functions that forms a rotational band are a superposition of vectors from many sp(3,R) irreducible representations, the densities of band members can share approximately common values for the three symplectic Casimirs.

The sp(3,R) densities determine nuclear geometrical properties and these may be predicted accurately in mean field theory. A symplectic density matrix makes no predictions for nongeometrical operators, while wave functions from a sp(3,R) irreducible representation predict the matrix elements of all operators. In essence, symplectic irreducible representation theory makes strong claims about nuclear wave functions that are irrelevant to the description of geometrical states. Symplectic mean field theory can explain geometrical collective densities, as observed directly in experiment, but it does not try to construct wave functions that incorporate all the degrees of freedom in phase space.

One of the oldest and most fundamental questions in nuclear structure science is, how does a deformed nucleus rotate? This article uses the Riemann rotor model to create a simple physical picture of rotating deformed isotopes. In the proposed model, the excitation energies of yrast rotational band members are essentially kinetic in nature, the moment of inertia is a function of the deformation and the rigidity, and, for even-even rare earth isotopes, the character of collective rotation depends only on the quadrupole deformation. These physical assertions need to be put to a stringent and direct experimental test in electron scattering measurements of the transverse $E2$ form factor in the heavy deformed region. Such measurements could finally answer the physical question about nuclear collective rotation in this domain.

The symplectic model is a rich theory of geometrical collective modes, but this article has explored just a small part of it, primarily topics related to the Kelvin circulation. The range of the circulation is bounded on a symplectic coadjoint orbit, as was shown in Sec. III. The character of rotational modes in atomic nuclei is circumscribed by this theorem, which parallels the reduction of gcm(3) in an irreducible sp(3,R) discrete series representation [43].

In many prior applications of the symplectic theory to the shell model, the rotationally invariant Hamiltonian is the sum of the harmonic oscillator $\hat{H}_0$ and a collective potential energy $V(\beta,\gamma)$ that is a function of the quadrupole deformation. In future work on symplectic coadjoint orbit theory, the critical points of energy functionals corresponding to such shell model Hamiltonians will be found and compared to the prior shell model calculations.

The mean field Hamiltonian is, in general, a density-dependent element of the Lie algebra [72]. A subsequent article will derive the symplectic mean field Hamiltonian from the energy functional using the symplectic geometry of a coadjoint orbit. The mean field Hamiltonian can be applied to the description of normal mode oscillations of symplectic equilibrium states. For su(3) dynamical symmetry, the mean field Hamiltonian and normal mode theory have been reported [23,28].

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APPENDIX: CASIMIR-LEVEL SURFACE

A symplectic coadjoint orbit is properly contained in a level surface of the symplectic Casimirs. Consider the following matrix in the symplectic Lie algebra sp(2,R):

$$M(u,v)=\begin{pmatrix} 0 & u/2 & v^2/4 & 0 \\ -u/2 & 0 & 0 & v^2/4 \\ -1 & 0 & 0 & u/2 \\ 0 & -1 & -u/2 & 0 \end{pmatrix}, \tag{A1}$$

where u,v are real numbers. When (I) $u=N_1-N_2$, $v=N_1+N_2$ or (II) $u=N_1+N_2$, $v=N_2-N_1$, the eigenvalues of the matrix $M(u,v)$ are $\pm iN_1$ and $\pm iN_2$. Thus both cases I and II lie on the same level surface of the sp(2,R) Casimirs.

In case I, the matrix $M(u,v)$ is in the orbit $\mathcal{O}_\varrho$ for

$$\varrho=\begin{pmatrix} 0 & 0 & N_1 & 0 \\ 0 & 0 & 0 & N_2 \\ -N_1 & 0 & 0 & 0 \\ 0 & -N_2 & 0 & 0 \end{pmatrix}, \tag{A2}$$

since $gM(u,v)g^{-1}=\varrho$ for the real symplectic group element g in Sp(2,R),

$$g=\begin{pmatrix} 0 & 1/\sqrt{v} & \sqrt{v}/2 & 0 \\ -1/\sqrt{v} & 0 & 0 & -\sqrt{v}/2 \\ -1/\sqrt{v} & 0 & 0 & \sqrt{v}/2 \\ 0 & 1/\sqrt{v} & -\sqrt{v}/2 & 0 \end{pmatrix}. \tag{A3}$$

In case II, the matrix $M(u,v)$ is *not* in the orbit $\mathcal{O}_\varrho$. In order for $gM(u,v)g^{-1}=\varrho$, the transformation g is

$$g=\begin{pmatrix} 0 & \sqrt{v}/2 & i\sqrt{v}/2 & 0 \\ -i\sqrt{v}/2 & 0 & 0 & \sqrt{v}/2 \\ i/\sqrt{v} & 0 & 0 & 1/\sqrt{v} \\ 0 & -1/\sqrt{v} & i/\sqrt{v} & 0 \end{pmatrix}. \tag{A4}$$

This g is an element of Sp(2,C); it is not a member of Sp(2,R).

The result may be extended to Sp(n,R) for $n>2$ by embedding the matrix $M(u,v)$ in an obvious way.

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