# Soft electroweak bremsstrahlung: Theorems and astrophysical relevance

R. G. E. Timmermans,* A. Yu. Korchin,† E. N. E. van Dalen, and A. E. L. Dieperink
*Theory Group, Kernfysisch Versneller Instituut, University of Groningen, Zernikelaan 25, 9747 AA Groningen, The Netherlands*
(Received 14 November 2001; published 3 June 2002)

We analyze the structure of the amplitudes for electroweak bremsstrahlung in nucleon-nucleon collisions, for the charged ($N+N\rightarrow N+N+e^-+\overline{\nu}_e$) and neutral ($N+N\rightarrow N+N+\nu_f+\overline{\nu}_f$) weak current. Theorems are derived for the matrix elements of the vector and axial-vector currents in the soft regime. A comparison is made with previous work, usually performed in the nonrelativistic limit and by using a one-pion exchange two-nucleon interaction in Born approximation. Such approaches are argued to be unrealistic. This is explicitly shown for the neutrino-pair emission process in neutron-neutron scattering. Our results are relevant for calculations of neutrino emissivities in supernovae and in cooling scenarios of neutron stars.

DOI: 10.1103/PhysRevC.65.064007 PACS number(s): 13.75.Cs, 13.10.+q, 26.60.+c, 21.30.−x

## I. INTRODUCTION

The thermal evolution of neutron stars is governed by the electroweak interaction, in particular, by neutrino emission processes. In the dense neutron-rich matter in the interior of the star, the one-body neutrino emission processes are strongly suppressed. The dominant cooling mechanisms for a neutron star are thought to be the two-body modified Urca processes [1,2] via the charged weak current,

$$n+n\rightarrow n+p+e^-+\overline{\nu}_e,$$

$$n+p\rightarrow p+p+e^-+\overline{\nu}_e$$

(and the inverse reactions, such as $n+p+e^-\rightarrow n+n+\nu_e$), referred to as the $nne\overline{\nu}$ and $npe\overline{\nu}$ processes, respectively. Also considered are the neutrino-pair bremsstrahlung processes [3] via the neutral weak current,

$$n+n\rightarrow n+n+\nu_f+\overline{\nu}_f,$$

$$n+p\rightarrow n+p+\nu_f+\overline{\nu}_f,$$

referred to as the $nn\nu\overline{\nu}$ and $np\nu\overline{\nu}$ processes, respectively. In this case, all three neutrino flavors $(e,\mu,\tau)$ contribute. We will refer to these two-nucleon processes collectively as neutrino bremsstrahlung processes, and together with photon bremsstrahlung as electroweak bremsstrahlung.

More than twenty years ago, the neutrino emissivities due to the modified Urca and neutrino-pair bremsstrahlung processes were calculated by Friman and Maxwell [4,5]. They assumed, in Born approximation, a two-nucleon ($NN$) interaction consisting of the long-range one-pion exchange (OPE) and a Landau Fermi-liquid-type of short-range interaction. The extreme nonrelativistic limit was used. The resulting emissivities were about one order of magnitude larger than previous calculations [2,3], which did not take into account the OPE interaction. The modified Urca processes were found to dominate over the neutrino-pair bremsstrahlung processes. An interesting aspect of this Friman-Maxwell (FM) work were the cancellations they observed for the various bremsstrahlung matrix elements in the "soft" limit, i.e., for the lepton-pair energy $\omega \ll m$, where $m$ is the nucleon mass. The contributions of the polar-vector part of the weak interaction vanish for the $nne\overline{\nu}$, $nn\nu\overline{\nu}$, and $np\nu\overline{\nu}$ matrix elements, both for the OPE and the Landau-type interactions. The axial-vector part of the weak interaction, on the other hand, which involves the nucleon spin operator, receives a finite contribution from the tensor force that arises from OPE.

At the typical temperatures existing in neutron stars, the neutrino bremsstrahlung can be classified as soft. For neutrino-pair emission, the energy and momentum of the $\nu_f\overline{\nu}_f$ pairs, $q=(\omega,\vec{q})$, are relatively small, up to about 10 MeV. For these processes, the leading $\mathcal{O}(1/q)$ contribution to the bremsstrahlung amplitude is expected to be dominant. For the modified Urca processes, the subleading-order $\mathcal{O}(1)$ contribution is likely to be significant, because the momentum of the emitted electron must be higher than the Fermi momentum of the electrons in the star. In this paper, we derive the neutrino bremsstrahlung amplitudes on the basis of theorems for the matrix elements of the weak vector and axial-vector currents in the soft regime. Our framework is inspired by the soft-photon approach to electromagnetic bremsstrahlung. In this case, Low's soft-photon theorem states that both the leading $\mathcal{O}(1/q)$ and subleading $\mathcal{O}(1)$ terms of the bremsstrahlung amplitude are fixed from the corresponding nonradiative process [6,7]. By using an analogous approach for the electroweak case, it is possible to include the full $NN$ transition amplitude ($T$ matrix), instead of just the Born approximation, and in its full complexity, instead of just the OPE interaction. Special care has to be taken to treat the exchange interactions correctly. In particular, we will follow Ref. [8] to take the Pauli principle into account. Based on these theorems, we can then investigate to what extent previous approximations and assumptions are actually valid.[1] A better understanding of these points in free space is

*Email address: timmermans@KVI.nl
†Permanent address: Kharkov Institute of Physics and Technology, Ukraine.

[1]Similar techniques also apply to the spacelike analogue, $NN$ correlation effects in neutrino scattering.

 

mandated before turning to the neutrino emission processes inside the nuclear medium [9–12].

Our investigation was partly motivated by more recent papers dealing with neutrino emission and scattering processes in neutron-star cooling and in supernova cores [13–17]. These works demonstrate that the framework adopted in the pioneering FM paper has become the standard one. However, for energies that are relevant in neutron stars, with nucleon momenta near the Fermi surface, it is not realistic to treat the $NN$ interaction by OPE in Born approximation. The Burnett-Kroll extension [18,19] of Low's soft-photon theorem states that the leading- and subleading-order bremsstrahlung *cross section* (summed over spins) is proportional to the corresponding unpolarized nonradiative *cross section*. This implies that the use of a model that does not describe the $NN$ phase shifts well already misses the bremsstrahlung cross section by a corresponding factor. For OPE in Born approximation this factor can be very significant, e.g., the $np$ cross section at 300 MeV calculated with OPE in Born approximation is too large by roughly a factor of 5. Although the Burnett-Kroll theorem does not apply to the axial-vector current, similar reduction factors could be expected for neutrino bremsstrahlung. Recently, Hanhart, Phillips, and Reddy [20], by using the leading term in the soft expansion, related the bremsstrahlung process via phase shifts directly to the $NN$ scattering data. They found, in fact, a significant reduction of the neutrino emissivity in the $nn\nu\bar{\nu}$ process compared to FM.

This paper is organized as follows. In Sec. II we specify the electroweak currents. Section III contains the derivation of the soft-bremsstrahlung theorems for the polar- and axial-vector currents. Section IV is devoted to a discussion of the general structure of the covariant $NN$ scattering amplitude and its nonrelativistic limit. In Sec. V we study more closely the structure of the bremsstrahlung amplitudes in the "ultrasoft" [i.e., $\mathcal{O}(1/q)$] regime, and make the connection to the nonrelativistic limit that is commonly used. For the reason mentioned above, we focus here on the neutrino-pair processes. Next, in Sec. VI we recover from our our general study the contributions from one-boson exchange and compare to the results obtained previously by FM for one-pion exchange. In Sec. VII we calculate for one case, the $nn\nu\bar{\nu}$ process, the cross section in free space and compare to calculations with only one-boson exchanges in Born approximation. We end with a summary of our conclusions and an outlook in Sec. VIII. A brief Appendix is devoted to a subtlety in the treatment of the pion pole in the axial-vector current matrix element.

## II. ELECTROWEAK CURRENTS

In the following, we need the structure of the electromagnetic and weak currents and the single-nucleon matrix elements of the currents. In this section they will be specified.[2]

[2]The space-time metric $g_{\mu\nu}$ has elements $g_{ii}=1(i=1,2,3)$ and $g_{00}=-1$; Dirac matrices are defined such that $\{\gamma_\mu,\gamma_\nu\}=2g_{\mu\nu}$, $\sigma_{\mu\nu}=[\gamma_\mu,\gamma_\nu]/2i$, and $\gamma_5=i\gamma_0\gamma_1\gamma_2\gamma_3$.

The validity of the standard model of the electroweak interactions will be assumed and the neutrinos will be taken to be massless. The standard model implies the conserved vector current (CVC) condition at the hadronic level. We will also use the partially-conserved axial-vector current hypothesis (PCAC) that is a consequence of the spontaneous breakdown of chiral symmetry in QCD. SU(2) isospin symmetry (charge independence) of the strong interaction will also be assumed.

At low momentum transfer, $W^{\pm}$ and $Z^0$ exchange reduce to four-fermion couplings. Urca processes via the charged current are described by the effective $\beta$-decay interaction,

$$\mathcal{L}_\beta=\frac{G_F}{\sqrt{2}}i[\bar{e}\gamma^\lambda(1+\gamma_5)\nu_e]\mathcal{J}^W_\lambda+\text{H.c.}; \tag{1}$$

$G_F\simeq1.0\times10^{-5}/m_p^2$, with $m_p$ the proton mass, is the Fermi coupling constant. Neutrino-pair bremsstrahlung is described by the effective four-fermion interaction

$$\mathcal{L}_\nu=\frac{G_F}{\sqrt{2}}i\sum_f\,[\bar{\nu}_f\gamma^\lambda(1+\gamma_5)\nu_f]\mathcal{J}^Z_\lambda, \tag{2}$$

where the sum is over the neutrino flavors $e,\mu,\tau$.

In Eqs. (1) and (2), the charged and neutral weak hadronic currents read

$$\mathcal{J}^W_\lambda=\cos\theta_C(\mathcal{J}^1_\lambda+i\mathcal{J}^2_\lambda),$$

$$\mathcal{J}^Z_\lambda=\mathcal{J}^3_\lambda-2\sin^2\theta_W J^{em}_\lambda-\frac{1}{2}\mathcal{J}^{(s)}_\lambda,$$

$$\mathcal{J}^{(s)}_\lambda=i\bar{s}\gamma_\lambda(1+\gamma_5)s, \tag{3}$$

where the Cabibbo angle is given by $\cos\theta_C\simeq0.975$, and the Weinberg angle by $\sin^2\theta_W\simeq0.231$. The currents are the sum of polar- and axial-vector parts, viz.

$$\mathcal{J}^a_\lambda=V^a_\lambda+A^a_\lambda, \tag{4}$$

where $a=1,2,3$; $J^{em}_\lambda$ is the electromagnetic current. More generally, with the help of the quark-triplet fields $\psi=(u,d,s)$, one defines the octets of polar- and axial-vector currents

$$V^a_\lambda=i\bar{\psi}\gamma_\lambda\lambda^a\psi/2,\quad A^a_\lambda=i\bar{\psi}\gamma_\lambda\gamma_5\lambda^a\psi/2; \tag{5}$$

where $\lambda^a$ for $a=1,\ldots,8$ are the Gell-Mann matrices. A possible contribution of the strange quarks to the neutral current of the nucleons, $\mathcal{J}^{(s)}_\lambda$, will be ignored in this paper. The electromagnetic current then reads, in terms of isovector and isoscalar parts,

$$J^{em}_\lambda=V^3_\lambda+V^8_\lambda/\sqrt{3}\equiv J^{em}_\lambda(I=1)+J^{em}_\lambda(I=0). \tag{6}$$

To derive the theorems below in Secs. III B and III C, it suffices to have the matrix elements of the currents between on-mass-shell nucleons. The matrix elements of the current $\mathcal{J}^a_\lambda$ $(a=1,2,3)$ of Eq. (4) are, at space-time point $x=0$, defined by the general Lorentz and isospin structures

$$\langle N_2(p')|V^a_\lambda|N_1(p)\rangle = i\,\overline{u}(p')[G_V\gamma_\lambda + G_T\sigma_{\lambda\varrho}(p-p')^\varrho]\times u(p)(N_2^\dagger\tau^a N_1)/2,$$

$$\langle N_2(p')|A^a_\lambda|N_1(p)\rangle = i\,\overline{u}(p')[G_A\gamma_\lambda\gamma_5 + G_P i\gamma_5(p-p')_\lambda]\times u(p)(N_2^\dagger\tau^a N_1)/2, \quad (7)$$

in terms of vector, axial-vector, weak magnetism, and induced pseudoscalar form factors, which are functions of four-momentum transfer $t=-(p'-p)^2$. $N_i$ $(i=1,2)$ are the isospin wave functions of the nucleons; $\tau_a \equiv \lambda_a$ for $a=1,2,3$ are the Pauli isospin matrices. In terms of isovector and isoscalar form factors, we write for $J^{em}_\lambda$ of Eq. (6)

$$\langle N_2(p')|J^{em}_\lambda|N_1(p)\rangle = i\,\overline{u}(p')[F^V_1\gamma_\lambda + F^V_2\sigma_{\lambda\varrho}(p-p')^\varrho]u(p)(N_2^\dagger\tau^3 N_1)/2 + i\overline{u}(p')[F^S_1\gamma_\lambda + F^S_2\sigma_{\lambda\varrho}(p-p')^\varrho]u(p)(N_2^\dagger N_1)/2. \quad (8)$$

The matrix element of the charged weak current of Eq. (3) in beta decay is defined by

$$\langle p'|\mathcal{J}^W_\lambda|n\rangle = i\cos\theta_C\overline{u}(p')[G^\beta_V\gamma_\lambda + G^\beta_T\sigma_{\lambda\varrho}(n-p')^\varrho + G^\beta_A\gamma_\lambda\gamma_5 + G^\beta_P i\gamma_5(n-p')_\lambda]u(n)(N_p^\dagger\tau^+ N_n), \quad (9)$$

where $\tau^+=(\tau_1+i\tau_2)/2$. For the proton, the matrix elements of the neutral weak and electromagnetic current read

$$\langle p'|\mathcal{J}^Z_\lambda|p\rangle = i\overline{u}(p')[G^p_V\gamma_\lambda + G^p_T\sigma_{\lambda\varrho}(p-p')^\varrho + G^p_A\gamma_\lambda\gamma_5 + G^p_P i\gamma_5(p-p')_\lambda]u(p)(N_p^\dagger N_p),$$

$$\langle p'|J^{em}_\lambda|p\rangle = i\overline{u}(p')[F^p_1\gamma_\lambda + F^p_2\sigma_{\lambda\varrho}(p-p')^\varrho]u(p)(N_p^\dagger N_p), \quad (10)$$

and analogously for the neutron, where $F^V_i = F^p_i - F^n_i$ and $F^S_i = F^p_i + F^n_i$ for $i=1,2$.

By using Eq. (3) and isospin symmetry, one finds that for the proton the form factors of the neutral current are related to the ones for beta decay and electromagnetism by

$$G^p_V = G^\beta_V/2 - 2\sin^2\theta_W F^p_1, \quad G^p_A = G^\beta_A/2,$$

$$G^p_T = G^\beta_T/2 - 2\sin^2\theta_W F^p_2, \quad G^p_P = G^\beta_P/2, \quad (11)$$

while for the neutron similarly

$$G^n_V = -G^\beta_V/2 - 2\sin^2\theta_W F^n_1, \quad G^n_A = -G^\beta_A/2,$$

$$G^n_T = -G^\beta_T/2 - 2\sin^2\theta_W F^n_2, \quad G^n_P = -G^\beta_P/2. \quad (12)$$

CVC implies additionally that

$$G^\beta_V = F^V_1, \quad G^\beta_T = F^V_2. \quad (13)$$

It remains to fix the normalization of the various form factors at $t=0$. Inspecting Eqs. (11), (12), and (13), it suffices to specify that $F^p_1(0)=1$ and $F^n_1(0)=0$ for the Dirac form factors, while $F^p_2(0)=\kappa_p/2m$ and $F^n_2(0)=\kappa_n/2m$ for the Pauli form factors; the anomalous magnetic moments of the nucleons are $\kappa_p=1.793$ and $\kappa_n=-1.913$. Moreover, from neutron beta decay we have $G^\beta_A(0)=g_A\simeq 1.257$, the Gamow-Teller coupling constant. The induced pseudoscalar form factor will be further discussed below in Sec. III C.

## III. THEOREMS FOR SOFT ELECTROWEAK BREMSSTRAHLUNG

### A. Introduction

In this section, we will derive the theorems for the matrix elements of the polar-vector current $V^a_\lambda$ and axial-vector current $A^a_\lambda$ in soft electroweak bremsstrahlung. The corresponding theorem for electromagnetic bremsstrahlung, usually called the soft-photon theorem (SPhT),[3] was first derived by Low [6] and augmented by others [7,18,19]. It states that the first two terms in an expansion of the bremsstrahlung amplitude in photon momentum $q$ are determined by the amplitude for the corresponding nonradiative process.

From a general argument it can be understood that soft electroweak bremsstrahlung theorems (SBTs) can be written down. Let $M^a_\lambda$ be the matrix element of the pertinent electroweak current and $q$ the (timelike) four-momentum of the photon or lepton pair. The divergence is given by $q^\lambda M^a_\lambda$. For instance, when the current is conserved, one has $q^\lambda M^a_\lambda=0$. We expand $M^a_\lambda$ in powers of $q$, starting with the singular term of order $\mathcal{O}(1/q)$ coming from radiation off the external legs. Consider two different bremsstrahlung amplitudes constructed from the same amplitude for the nonradiative process. The leading-order $\mathcal{O}(1/q)$ term in the expansion of the two amplitudes, proportional to the nonradiative amplitude, is identical. Therefore, the difference $\Delta M^a_\lambda$ of these two amplitudes must be analytical at $q^\lambda=0$. It follows then from $q^\lambda\Delta M^a_\lambda=0$ that this difference is of the order $\mathcal{O}(q)$. Thus, the first two terms in the amplitude, of order $\mathcal{O}(1/q)$ and $\mathcal{O}(1)$, are model independent, and given by the corresponding nonradiative process. This is the content of the SBT. This argument also makes clear that in order to derive a SBT it is not necessary that the divergence $q^\lambda M^a_\lambda$ is zero. As pointed

[3]In our case, one could call the corresponding theorem a soft-neutrino theorem. We avoid, however, the often used but confusing term "low-energy theorem." There is no restriction on the energy of the nonradiative system considered: It is the *radiated* four-momentum $q^\lambda$ that is low.

out by Adler and Dothan [7], it is sufficient that the divergence is known in terms of hadronic quantities. This is the case for the axial-vector current, for which the PCAC assumption can be made.

The SPhT was first applied to the $NN$ case by Nyman [21], and subsequently by a number of other authors [22,23]. In the real-photon case, one can derive [8] a soft-photon amplitude that provides a good description of the available experimental $pp$ bremsstrahlung cross sections up to the pion-production threshold at 280 MeV [23]. Since neither the relation $q^2=0$ nor the relation $\varepsilon^\lambda q_\lambda=0$ ($\varepsilon$ is the photon polarization vector) is needed in the derivation, it is obvious that the theorem also holds for virtual timelike photons ($q^2<0$), that is, dilepton ($e^+e^-$-pair) production. A specific amplitude for the latter case has been constructed, for instance, in Ref. [24].

We consider the electroweak bremsstrahlung process in $NN$ collisions, viz.

$$N(p_1)+N(p_2)\to N(p_1')+N(p_2')+B^\lambda(q), \tag{14}$$

where $B$ is the vector boson, *in casu* the photon or the $W^\pm$ or $Z^0$ bosons, which decays into two leptons ($e^+e^-$, $e^-\overline{\nu}_e$, $\nu_f\overline{\nu}_f$), $B(q)\to l_1(q_1)+l_2(q_2)$. Energy-momentum conservation reads $p_1+p_2=p_1'+p_2'+q$ for the four-vectors. In our case of interest, the radiated four-momentum $q$ is small, which implies that the vector bosons $W^\pm$ and $Z^0$ are far off their mass shells, and the effective four-fermion interactions specified in Sec. II can be used. We denote by $M_\lambda^a$ the matrix element of the electroweak current $\mathcal{J}_\lambda^a$ of Eq. (4). Contracted with the photon polarization or with the leptonic current, it gives the bremsstrahlung amplitude. We write the bremsstrahlung amplitude as $M_\lambda^a=M_\lambda^{vec,a}+M_\lambda^{ax,a}$, and focus first on the polar-vector part. It is implied in this section that the nucleon spinors are present, so one should always read

$$M_\lambda^a\to\overline{u}(p_1')\overline{u}(p_2')M_\lambda^a u(p_1)u(p_2). \tag{15}$$

Isospin wave functions will also not be written explicitly.

### B. Soft-bremsstrahlung theorem for the vector current

Not surprisingly, the derivation of the SBT for the electroweak vector current is entirely analogous to the standard SPhT. Complications only arise due to the flavor (isospin) structure of the currents. For completeness, we summarize here the main points as an introduction to the more complex case of the axial-vector current in the following section.

Following Low [6], we split the matrix element into the sum of external and internal contributions, $M_\lambda^{vec,a}=M_\lambda^{ext,a}+M_\lambda^{int,a}$. The expression for radiation off the external legs is given by

$$\begin{aligned}M_\lambda^{ext,a}={}&T_1 iS(p_1-q)\Gamma_\lambda^a+\Gamma_\lambda^a iS(p_1'+q)T_1'\\&+T_2 iS(p_2-q)\Gamma_\lambda^a+\Gamma_\lambda^a iS(p_2'+q)T_2',\end{aligned} \tag{16}$$

where $S(p)=(i\gamma\cdot p+m)^{-1}$, and the vector-current vertex, cf. Eq. (7), is

$$\Gamma_\lambda^a(q)=[G_V(q^2)\gamma_\lambda+G_T(q^2)\sigma_{\lambda\varrho}q^\varrho]\tau^a/2. \tag{17}$$

The covariant $NN$ scattering amplitudes ($T$ matrices) with one nucleon off its mass shell are defined as

$$\begin{aligned}T_1&=\langle p_1',p_2'|T|p_1-q,p_2\rangle, & T_1'&=\langle p_1'+q,p_2'|T|p_1,p_2\rangle,\\T_2&=\langle p_1',p_2'|T|p_1,p_2-q\rangle, & T_2'&=\langle p_1',p_2'+q|T|p_1,p_2\rangle.\end{aligned} \tag{18}$$

Equation (16) is expanded in powers of $q$ to order $\mathcal{O}(1)$. In the expansion of the form factors $G_V$ and $G_T$, only the zero-momentum terms are kept, since the corrections of order $q^2dG/dq^2$ contribute only at order $\mathcal{O}(q)$ in the bremsstrahlung amplitude. The $T$ matrices are expanded around a conveniently chosen kinematical point $T_0$ that corresponds to elastic $NN$ scattering. As argued above, differences due to the specific choice of these points show up only at order $\mathcal{O}(q)$, and are expected to be small for the neutrino emission processes under the conditions that exist in neutron stars. The Lorentz and isospin structures of the $T$ matrices will be specified later. We get

$$T_1=T_0-q\cdot\frac{\partial}{\partial p_1}T_0+\ \ldots, \quad T_1'=T_0+q\cdot\frac{\partial}{\partial p_1'}T_0+\ \ldots, \tag{19}$$

and similarly for nucleon 2. We also introduce the momenta

$$r_1=2p_1-q, \quad r_1'=2p_1'+q, \tag{20}$$

and likewise for nucleon 2.

The expansion of Eq. (16) gives three terms,

$$M_\lambda^{ext,a}=M_\lambda^{conv,a}+M_\lambda^{spin,a}+M_\lambda^{deriv,a}+\mathcal{O}(q), \tag{21}$$

where

$$\begin{aligned}M_\lambda^{conv,a}={}&-T_0\frac{r_{1\lambda}}{r_1\cdot q}\frac{\tau^a}{2}G_V(0)\\&+G_V(0)\frac{\tau^a}{2}\frac{r_{1\lambda}'}{r_1'\cdot q}T_0+(1\rightleftharpoons 2),\end{aligned} \tag{22}$$

$$\begin{aligned}M_\lambda^{spin,a}={}&-T_0G'(0)\frac{i\sigma_{\lambda\varrho}q^\varrho}{r_1\cdot q}\frac{\tau^a}{2}\\&+\frac{\tau^a}{2}\frac{i\sigma_{\lambda\varrho}q^\varrho}{r_1'\cdot q}G'(0)T_0+(1\rightleftharpoons 2),\end{aligned} \tag{23}$$

and

$$\begin{aligned}M_\lambda^{deriv,a}={}&q\cdot\frac{\partial T_0}{\partial p_1}\frac{r_{1\lambda}}{r_1\cdot q}\frac{\tau^a}{2}G_V(0)\\&+G_V(0)\frac{\tau^a}{2}\frac{r_{1\lambda}'}{r_1'\cdot q}q\cdot\frac{\partial T_0}{\partial p_1'}+(1\rightleftharpoons 2),\end{aligned} \tag{24}$$

with

$$G(0)=G_V(0)+\Lambda^+(p_1)G_T(0),$$
$$G'(0)=G_V(0)+\Lambda^+(p_1')G_T(0). \tag{25}$$

We defined here, for any momentum, the operator

$$\Lambda^+(p)=-i\gamma\cdot p+m. \tag{26}$$

The "convection current" term $M_\lambda^{conv,a}$ has vanishing divergence, since

$$q^\lambda M_\lambda^{conv,a}=-T_0\frac{\tau^a}{2}G_V(0)+G_V(0)\frac{\tau^a}{2}T_0+(1\leftrightarrow 2)$$
$$=-G_V(0)[T_0,I^a]=0, \tag{27}$$

from conservation of isospin. The "spin" term $M_\lambda^{spin,a}$ clearly has zero divergence by itself. However, the term that contains the derivatives of $T_0$, $M_\lambda^{deriv,a}$, does not. Therefore, we add a nonsingular "internal" term,

$$M_\lambda^{int,a}=-\frac{\partial T_0}{\partial p_1^\lambda}\frac{\tau^a}{2}G_V(0)-G_V(0)\frac{\tau^a}{2}\frac{\partial T_0}{\partial p_1'^\lambda}+(1\rightleftharpoons 2), \tag{28}$$

such that the sum does have zero divergence. Introducing the derivative operators

$$\mathcal{D}_\lambda^1=\frac{r_{1\lambda}}{r_1\cdot q}q\cdot\frac{\partial}{\partial p_1}-\frac{\partial}{\partial p_1^\lambda},\quad \mathcal{D}_\lambda^{1'}=\frac{r_{1\lambda}'}{r_1'\cdot q}q\cdot\frac{\partial}{\partial p_1'}-\frac{\partial}{\partial p_1'^\lambda}, \tag{29}$$

and similarly for nucleon 2, which satisfy $q^\lambda\mathcal{D}_\lambda^1=q^\lambda\mathcal{D}_\lambda^{1'}=0$, we get

$$M_\lambda^{deriv,a}+M_\lambda^{int,a}=(\mathcal{D}_\lambda^1T_0)\frac{\tau^a}{2}G_V(0)$$
$$+G_V(0)\frac{\tau^a}{2}(\mathcal{D}_\lambda^{1'}T_0)+(1\rightleftharpoons 2). \tag{30}$$

Thus, we finally have

$$M_\lambda^{vec,a}=M_\lambda^{ext,a}+M_\lambda^{int,a}$$
$$=M_\lambda^{conv,a}+M_\lambda^{spin,a}+M_\lambda^{deriv,a}+M_\lambda^{int,a}+\mathcal{O}(q), \tag{31}$$

with $q^\lambda M_\lambda^{vec,a}=0$, which is the CVC condition. In the evaluation of the derivative operators, derivatives in the off-mass-shell direction need not be considered, since they cancel in the end; only derivatives with respect to any two Mandelstam variables are needed [6,7].

### C. Soft-bremsstrahlung theorem for the axial-vector current

Theorems for the charged axial-vector current were pioneered by Adler and Dothan [7], and applied to, for instance, the process of pion production by neutrinos. For an excellent review, see the authoritative Ref. [25]. After the discovery of the neutral weak currents, this early work was extended by Adler to neutrino scattering processes [26]. In the context of the linear sigma model, the charged axial-vector current process was considered in Ref. [27].

The derivation of the SBT for the axial current requires the PCAC assumption for the matrix element of the divergence of the axial current [25], i.e.,

$$iq^\lambda M_\lambda^{ax,a}=\frac{F_\pi}{2}\frac{m_\pi^2}{q^2+m_\pi^2}M_\pi^a, \tag{32}$$

where $M_\pi^a$ is the pion emission matrix element; $F_\pi\simeq 185$ MeV is the pion decay constant. As for the vector-current case, the derivation starts by splitting the matrix element of the axial-vector current into an external and an internal part, $M_\lambda^{ax,a}=M_\lambda^{ext,a}+M_\lambda^{int,a}$. The external part reads

$$M_\lambda^{ext,a}=T_1iS(p_1-q)\Gamma_\lambda^a+\Gamma_\lambda^a iS(p_1'+q)T_1'+(1\rightleftharpoons 2), \tag{33}$$

where the axial-vector-current vertex, cf. Eq. (7), is

$$\Gamma_\lambda^a(q)=[\gamma_\lambda G_A(q^2)+iq_\lambda G_P(q^2)]\gamma_5\tau^a/2, \tag{34}$$

and the $T$ matrices are given in Eq. (18). The pion emission matrix element is likewise split into an external and internal part, $M_\pi^a=M_\pi^{ext,a}+M_\pi^{int,a}$. The external part reads

$$M_\pi^{ext,a}=T_1iS(p_1-q)\Gamma_\pi^a+\Gamma_\pi^a iS(p_1'+q)T_1'+(1\rightleftharpoons 2), \tag{35}$$

where the pion-nucleon strong-interaction vertex is given by

$$\Gamma_\pi^a(q)=\gamma_5G_{NN\pi}(q^2)\tau^a; \tag{36}$$

the pion-nucleon coupling constant is defined at the pion pole as $g_{NN\pi}\equiv G_{NN\pi}(-m_\pi^2)$. Consistent with PCAC applied to neutron decay, we assume the pion-pole dominance of the induced pseudoscalar form factor, viz.

$$G_P(q^2)=\frac{2mG_A(q^2)}{q^2+m_\pi^2}=\frac{F_\pi G_{NN\pi}(q^2)}{q^2+m_\pi^2}, \tag{37}$$

and, correspondingly, the Goldberger-Treiman relation

$$g_A/F_\pi\simeq g_{NN\pi}/2m=\sqrt{4\pi}f_{NN\pi}/m_{\pi^+}, \tag{38}$$

where $f_{NN\pi}$ is the (rationalized) pion-nucleon coupling constant in pseudovector notation. With $f_{NN\pi}^2=0.0750(9)$ [28], this relation is well satisfied.

The external part $M_\lambda^{ext,a}$ in Eq. (33) is written as the sum of two terms,

$$M_\lambda^{ext,a}=M_\lambda^{gt,a}+M_\lambda^{ps,a}, \tag{39}$$

where, with the momenta as defined in Eq. (20), and the operators $\Lambda^+$ as defined in Eq. (26), the "Gamow-Teller" part is

$$M_\lambda^{gt,a} = -T_1 \frac{\Lambda^+(p_1-q)}{r_1\cdot q} G_A i\gamma_\lambda\gamma_5 \frac{\tau^a}{2} + G_A i\gamma_\lambda\gamma_5 \frac{\tau^a}{2}\frac{\Lambda^+(p_1'+q)}{r_1'\cdot q} T_1' + (1\rightleftharpoons 2), \quad (40)$$

and the induced pseudoscalar term is

$$M_\lambda^{ps,a} = T_1 \frac{\Lambda^+(p_1-q)}{r_1\cdot q} G_P q_\lambda\gamma_5 \frac{\tau^a}{2} - G_P q_\lambda\gamma_5 \frac{\tau^a}{2}\frac{\Lambda^+(p_1'+q)}{r_1'\cdot q} T_1' + (1\rightleftharpoons 2). \quad (41)$$

With these definitions, the PCAC relation Eq. (32) reads

$$iq^\lambda M_\lambda^{gt,a} + iq^\lambda M_\lambda^{ps,a} + iq^\lambda M_\lambda^{int,a} = \frac{F_\pi}{2}\frac{m_\pi^2}{q^2+m_\pi^2}(M_\pi^{ext,a}+M_\pi^{int,a}). \quad (42)$$

This equation is expanded in powers of $q$. In the external amplitudes, we expand the $T$-matrices according to Eq. (19), as well as the form factors, setting $g_A = G_A(0)$. Also the internal amplitudes are expanded. On the right-hand side of Eq. (42), we use the identity $m_\pi^2/(q^2+m_\pi^2) = 1 - q^2/(q^2 + m_\pi^2)$ in the term multiplying $M_\pi^{int,a}$. After a little algebra, and by using the Goldberger-Treiman relation (38), we arrive at the two conditions

$$\frac{F_\pi}{2} M_\pi^{int,a}(0) = -g_A T_0 i\gamma_5 \frac{\tau^a}{2} - g_A i\gamma_5 \frac{\tau^a}{2} T_0 + (1\rightleftharpoons 2), \quad (43)$$

and

$$M_\lambda^{int,a}(0) = -g_A \frac{\partial T_0}{\partial p_1^\lambda}\gamma_5\frac{\tau^a}{2} + g_A\gamma_5\frac{\tau^a}{2}\frac{\partial T_0}{\partial p_1'^\lambda} + (1\rightleftharpoons 2) - i\frac{F_\pi}{2}\frac{\partial}{\partial q^\lambda}M_\pi^{int,a}(0) + i\frac{F_\pi}{2}\frac{q_\lambda}{q^2+m_\pi^2}M_\pi^{int,a}(0). \quad (44)$$

Equation (43) must be imposed on the internal pion emission matrix element $M_\pi^{int,a}$ in order for $M_\lambda^{int,a}$ to be analytic; it is the analog of Adler's "consistency relation" in pion-nucleon scattering [7,25]. Furthermore, in order to calculate the matrix element of the axial-vector current to order $\mathcal{O}(1)$, we need to evaluate the quantity $\partial M_\pi^{int,a}(0)/\partial q^\lambda$. This is the derivative, at $q=0$, of the internal part of the pion emission amplitude in $NN$ scattering. (In order to extrapolate the amplitude to $q=0$, a smoothness assumption typical for "soft-pion" physics is required. The limit $q\to 0$ can be performed by first setting $\vec{q}=0$ and then taking $\omega\to 0$ [25].) The last term in Eq. (44), proportional to $M_\pi^{int,a}(0)$, is formally of order $\mathcal{O}(q)$ for a finite pion mass and, therefore, beyond the SBT. However, it may vary fast due to the smallness of the pion mass, and following Ref. [7] one can keep in the total amplitude the pion-pole term in Eq. (44). As for the vector case, only the $T$ matrix for on-mass-shell nucleons enters. In particular, antiparticle ("negative-energy") states only contribute in order $\mathcal{O}(q)$ and higher, that is, beyond the SBT.

## IV. ELASTIC SCATTERING AMPLITUDE

So far, we have succinctly denoted the $NN$ $T$ matrices by $T_0$. In Secs. V B, V C, and V D, we need to specify the Lorentz and isospin structures. We also reinstate the nucleon spinors, cf. Eq. (15). Care should be taken of the Pauli principle for the interchange of the nucleon variables in the final (or the initial) state.[4] It is convenient to follow Ref. [8], which treats the $pp$ bremsstrahlung case, and start from the Goldberger-Grisaru-MacDowell-Wong (GGMW) representation of the (antisymmetrized) $NN$ scattering amplitude [29,30]. For the process $N(p_1)+N(p_2)\to N(p_1')+N(p_2')$, this amplitude can be decomposed in isospin space as

$$T = \sum_{I=0,1}\sum_{\alpha=1}^{5} F_\alpha^{(I)}(s,t,u)[\overline{u}(p_2')\,\Omega_\alpha u(p_2)\times\overline{u}(p_1')\Omega^\alpha u(p_1) + (-)^\alpha\overline{u}(p_1')\,\Omega_\alpha u(p_2)\times\overline{u}(p_2')\Omega^\alpha u(p_1)]\mathcal{B}_I, \quad (45)$$

where the projection operators on isosinglet and isotriplet states are

$$\mathcal{B}_0 = (1-\vec{\tau}_1\cdot\vec{\tau}_2)/4, \quad \mathcal{B}_1 = (3+\vec{\tau}_1\cdot\vec{\tau}_2)/4, \quad (46)$$

respectively. The five Fermi covariants (tensors) are

$$\Omega_\alpha = (\Omega_1,\Omega_2,\Omega_3,\Omega_4,\Omega_5) = (1,\sigma_{\mu\nu}/\sqrt{2}, i\gamma_5\gamma_\mu, \gamma_\mu, \gamma_5). \quad (47)$$

$F_\alpha^{(I)}(s,t,u)$ are the invariant functions of the Mandelstam variables $s=-(p_1+p_2)^2$, $t=-(p_1'-p_1)^2$, and $u=-(p_2'-p_1)^2$, which satisfy $s+t+u=4m^2$. The relation of these functions to the $NN$ helicity amplitudes and phase shifts is specified in Ref. [29]. The restriction put on the functions $F_\alpha^{(I)}$ by the Pauli principle is

$$F_\alpha^{(I)}(s,t,u) = (-)^{\alpha+I}F_\alpha^{(I)}(s,u,t). \quad (48)$$

For the $nn\nu\overline{\nu}$, $np\nu\overline{\nu}$, $nne\overline{\nu}$, and $npe\overline{\nu}$ processes we need the isospin combinations

$$F_\alpha^{(nn)}(s,t,u) = F_\alpha^{(pp)}(s,t,u) = F_\alpha^{(1)}(s,t,u)$$

$$F_\alpha^{(np)}(s,t,u) = (F_\alpha^{(0)}(s,t,u)+F_\alpha^{(1)}(s,t,u))/2, \quad (49)$$

for $\alpha=1,\ldots,5$. We also define the combination

$$F_\alpha^{(ce)}(s,t,u) = (F_\alpha^{(1)}(s,t,u)-F_\alpha^{(0)}(s,t,u))/2. \quad (50)$$

[4] See the discussion following Eq. (85).

The (charge-)exchange contribution to the $np$ scattering amplitude is then, using Eq. (48), $(-)^{\alpha}F_{\alpha}^{(np)}(s,t,u)=-F_{\alpha}^{(ce)}(s,u,t)$. When, in the following, the dependence of these $F$ functions on the Mandelstam variables is suppressed, the usual order $(s,t,u)$ is implied.

For matrix elements with the nucleons on the mass shell, the most general nonrelativistic charge-independent $NN$ interaction [31,32] contains central, spin-spin, tensor, spin-orbit, and quadratic spin-orbit terms. For each value of the isospin, it can be written in configuration space as

$$T=T_C+T_S\,\vec{\sigma}_1\cdot\vec{\sigma}_2+T_T S_{12}+T_{SO}\,\vec{L}\cdot\vec{S}+T_Q\,Q_{12}, \quad (51)$$

where $S_{12}=3\,\vec{\sigma}_1\cdot\vec{r}\,\vec{\sigma}_2\cdot\vec{r}-\vec{\sigma}_1\cdot\vec{\sigma}_2$, and $Q_{12}=(\vec{\sigma}_1\cdot\vec{L}\vec{\sigma}_2\cdot\vec{L}+\vec{\sigma}_2\cdot\vec{L}\,\vec{\sigma}_1\cdot\vec{L})/2$. The coefficients $T_i$ are functions of $r^2$ and the operators $p^2$ and $L^2$. In momentum space we define the vectors $\vec{k}=\vec{p}\,'-\vec{p}$, $\vec{k}\,'=\vec{p}\,'+\vec{p}$, and $\vec{n}=\vec{k}\,'\times\vec{k}$. In our calculations in Sec. VII we use the corresponding decomposition of the $T$ matrix,

$$T=A+B\sigma_{1n}\sigma_{2n}+C\sigma_{1k}\sigma_{2k}+D\sigma_{1k'}\sigma_{2k'}+iE(\sigma_{1n}+\sigma_{2n}), \quad (52)$$

where $\sigma_{1n}=\vec{\sigma}_1\cdot\hat{n}$, etc. The momentum-space coefficients $A,\ldots,E$ are in general functions of $\vec{k}$, $\vec{k}\,'$, $\vec{n}$, and the energy.

## V. ANALYSIS OF THE ULTRASOFT REGIME

### A. Leading-order bremsstrahlung amplitudes

The amplitudes derived in the preceding section are Lorentz covariant, consistent with the symmetries of the electroweak currents in the standard model, and take into account the terms of order $\mathcal{O}(1/q)$ and order $\mathcal{O}(1)$ in a model-independent manner. They can, therefore, provide a good starting point for calculating neutrino emissivities in neutron stars. To analyze the bremsstrahlung amplitudes further and, in particular, to make contact with the pioneering FM work [4], we will consider in more detail the leading-order (''ultrasoft'') $\mathcal{O}(1/q)$ terms, in this section, which are dominant for low lepton-pair energies $\omega$. (In the photon case, this is sometimes called external emission dominance.) These leading terms were previously studied in Ref. [20] for the $nn\nu\bar{\nu}$ process.

The leading-order $\mathcal{O}(1/q)$ part of the vector-current matrix element is obtained from Eq. (22), by neglecting $q^2$ in the denominators, compared to the terms $p\cdot q$, as

$$M_{\lambda}^{vec,a}=-g_V T_0\frac{p_{1\lambda}}{p_1\cdot q}\frac{\tau^a}{2}+g_V\frac{\tau^a}{2}\frac{p'_{1\lambda}}{p'_1\cdot q}T_0+(1\rightleftharpoons 2), \quad (53)$$

where $g_V\equiv G_V(0)$. The leading-order $\mathcal{O}(1/q)$ term of the purely isovector, axial-vector matrix element can be written, from Eq. (40), as

$$M_{\lambda}^{ax,a}=-g_A T_0\frac{\Lambda^+(p_1)}{2p_1\cdot q}i\gamma_\lambda\gamma_5\frac{\tau^a}{2}+g_A i\gamma_\lambda\gamma_5\frac{\tau^a}{2}\frac{\Lambda^+(p'_1)}{2p'_1\cdot q}T_0+(1\rightleftharpoons 2). \quad (54)$$

The pion-pole term in the matrix element of the axial-vector current is not explicitly needed in the remainder of this paper. Its role is briefly discussed in the Appendix. Formally, it is needed to obey PCAC in order $\mathcal{O}(1/q)$. However, in most cases of interest, the pion-pole contribution can be neglected. For the neutral-current processes, it drops out when contracted with the neutrino-pair current, since it is proportional to the radiated momentum $q_\lambda$, and hence

$$q_\lambda\overline{u}(q_1)i\gamma^\lambda(1+\gamma_5)v(q_2)=-2m_\nu\overline{u}(q_1)\gamma_5 v(q_2)=0, \quad (55)$$

for massless neutrinos. The reason, as is well known, is that the spin-zero pion cannot decay to two massless leptons, one left handed and the other right handed. For the charged-current (Urca) processes, the contribution is proportional to the lepton mass. For electrons, it is therefore small compared to the leading-order term [Eq. (54)]. It would contribute, however, for muons, that are present at higher densities in the neutron-star core.

In the following sections, we will consider separately in more detail the $nn\nu\bar{\nu}$, $np\nu\bar{\nu}$, $nne\bar{\nu}$, and $npe\bar{\nu}$ processes. In order to gain more insight, and especially in order to compare to FM, we will, for the neutrino-pair bremsstrahlung processes, make the nonrelativistic expansion of the amplitudes. Let us introduce the vector $\vec{V}=\vec{q}/\omega$. Neglecting the lepton masses, its magnitude,

$$|\vec{V}|=\sqrt{\vec{q}_1^{\,2}+\vec{q}_2^{\,2}+2|\vec{q}_1||\vec{q}_2|\cos\theta_{l\bar{l}}}/(\omega_1+\omega_2), \quad (56)$$

is bounded between 0 and 1; $q_1=(\omega_1,\ \vec{q}_1)$ and $q_2=(\omega_2,\ \vec{q}_2)$ are the four-momenta of the two leptons; $\theta_{l\bar{l}}$ is the angle between $\vec{q}_1$ and $\vec{q}_2$. The expansion of a typical pole term in Eqs. (53) and (54) is

$$\frac{1}{p\cdot q}=-\frac{1}{m\omega}\left[1+\frac{\vec{p}}{m}\cdot\vec{V}+\left(\frac{\vec{p}}{m}\cdot\vec{V}\right)^2-\frac{\vec{p}^{\,2}}{2m^2}+\cdots\right]. \quad (57)$$

Two parameters are important in the expansions of the amplitudes: (i) $|\vec{p}|/m$, related to the nonrelativistic reduction of the covariant expressions and (ii) $\omega/m_\pi$, where $m_\pi$ is the range of the two-nucleon interaction, related to the soft expansion. The nucleon momentum $|\vec{p}|$ is of the order of the Fermi momentum, which for neutron matter is typically around 300 MeV/$c$. For neutrino-pair bremsstrahlung the energy of the neutrinos is of the order of the temperature of the star, i.e., several MeV. It is then sufficient to consider only the leading term in the soft expansion. For the modified Urca processes the lepton-pair energy can be significantly higher. In that case, not only the leading $\mathcal{O}(1/q)$ terms, but also the subleading $\mathcal{O}(1)$ terms need to be considered in a calculation based on the SBT.

### B. The $nn\nu\bar{\nu}$ process

We consider first the process $n+n\rightarrow n+n+\nu_f+\bar{\nu}_f$. The $pp\nu\bar{\nu}$ process, which is much less relevant in neutron stars, can be treated in an analogous manner. It differs only in the coupling of the proton to the neutral current as compared to the neutron, and because $T^{(nn)}$ differs from $T^{(pp)}$ due to electromagnetic and small charge-symmetry breaking effects.

The leading-order $\mathcal{O}(1/q)$ vector-current amplitude for the $nn\nu\bar{\nu}$ process takes the form, combining Eqs. (45) and (53),

$$M_\lambda^{vec}=g_V^n\left(-\frac{p_{1\lambda}}{p_1\cdot q}+\frac{p'_{1\lambda}}{p'_1\cdot q}-\frac{p_{2\lambda}}{p_2\cdot q}+\frac{p'_{2\lambda}}{p'_2\cdot q}\right)T^{(nn)}, \tag{58}$$

where, cf. Eq. (12), $g_V^n\equiv G_V^n(0)=-1/2$, and

$$T^{(nn)}\equiv\sum_{\alpha=1}^{5}F_\alpha^{(nn)}[\bar{u}(p'_2)\Omega_\alpha u(p_2)\bar{u}(p'_1)\Omega^\alpha u(p_1)+(-)^\alpha\bar{u}(p'_1)\Omega_\alpha u(p_2)\bar{u}(p'_2)\Omega^\alpha u(p_1)] \tag{59}$$

is the $nn$ scattering amplitude in the GGMW notation of Eq. (45). To the extent that we can make the replacement $p\cdot q=\vec{p}\cdot\vec{q}-m\omega\simeq-m\omega$, that is, ignore the recoil terms of order $\mathcal{O}(|\vec{p}|/m)$ and higher in Eq. (57), we obtain from Eq. (58) $M_\lambda^{vec}=0$, independent of any approximation made for the $nn$ $T$ matrix. Thus, the vector-current contribution vanishes in leading-order $\mathcal{O}(1/q)$ in the extreme nonrelativistic limit. This generalizes the FM result, where this cancellation was observed for Landau-type interactions and OPE, to the complete $nn$ $T$ matrix. The CVC condition was required to obtain this result.

While the vector-current contribution vanishes in the approximation of FM, the corrections are easily obtained from Eq. (58). When we include the recoil terms of Eq. (57), we get for the space and time components of the matrix element $M_\lambda^{vec}=(M_0^{vec},\vec{M}^{vec})$,

$$\vec{M}^{vec}=\frac{2g_V^n}{\omega}\frac{\vec{p}(\vec{p}\cdot\vec{V})-\vec{p}'(\vec{p}'\cdot\vec{V})}{m^2}T^{(nn)},$$

$$M_0^{vec}=-\vec{V}\cdot\vec{M}^{vec}, \tag{60}$$

with $\vec{p}=\vec{p}_1=-\vec{p}_2$ and $\vec{p}'=\vec{p}'_1=-\vec{p}'_2$ in the center-of-mass (c.m.) frame of the nucleons, and $T^{(nn)}$ is given by the nonrelativistic reduction of Eq. (59), as in Eq. (52). (The minus sign in the second line is due to the metric, i.e., $M_0^{vec}=-M^{vec,0}$.) Due to the fact that identical particles are involved, the terms of order $\mathcal{O}(|\vec{p}|/m)$ cancel. This is analogous to the absence of electric-dipole radiation in photon bremsstrahlung processes when the center of mass coincides with the center of charge of the radiating system, e.g., in $pp$ bremsstrahlung, where at low energies electric-quadrupole radiation dominates.

The leading-order axial-current matrix element for the $nn\nu\bar{\nu}$ process reads, combining Eqs. (45) and (54),

$$\begin{aligned}M_\lambda^{ax}=-\frac{g_A}{2}\sum_{\alpha=1}^{5}F_\alpha^{(nn)}\Bigg[&\bar{u}(p'_2)\Omega_\alpha u(p_2)\bar{u}(p'_1)\\&\times\left(-\Omega^\alpha\frac{\Lambda^+(p_1)}{2p_1\cdot q}i\gamma_\lambda\gamma_5+i\gamma_\lambda\gamma_5\frac{\Lambda^+(p'_1)}{2p'_1\cdot q}\Omega^\alpha\right)u(p_1)\\&+(-)^\alpha\bar{u}(p'_1)\Omega_\alpha u(p_2)\bar{u}(p'_2)\\&\times\left(-\Omega^\alpha\frac{\Lambda^+(p_1)}{2p_1\cdot q}i\gamma_\lambda\gamma_5+i\gamma_\lambda\gamma_5\frac{\Lambda^+(p'_2)}{2p'_2\cdot q}\Omega^\alpha\right)u(p_1)\Bigg]\\&+(1\rightleftharpoons2),\end{aligned} \tag{61}$$

where we used $G_A^n(0)=-g_A/2$, cf. Eq. (12). In contrast to the vector-current case in Eq. (58), the axial-current matrix element does not factorize into bremsstrahlung pole terms and the $T$ matrix, and the exchange terms have to be included explicitly.

In the nonrelativistic limit, the expression (61) simplifies considerably. By using $p\cdot q\simeq-m\omega$, we can write

$$\vec{M}^{ax}=\frac{g_A}{\omega}[T^{(nn)},\vec{S}],$$

$$M_0^{ax}=0, \tag{62}$$

where $\vec{S}=(\vec{\sigma}_1+\vec{\sigma}_2)/2$ is the total spin of the $nn$ system. Pauli spinors are not written explicitly in Eq. (62). We conclude that for the $nn\nu\bar{\nu}$ process of the general $T$ matrix, only the tensor, spin-orbit, and quadratic spin-orbit terms contribute, and only the spin-triplet $nn$ partial waves contribute. This result generalizes FM and is in agreement with Ref. [20].

### C. The $np\nu\bar{\nu}$ process

Next, we consider the process $n+p\rightarrow n+p+\nu_f+\bar{\nu}_f$. To distinguish neutrons from protons, the momenta will be denoted in this section by $n$ and $n'$ ($p$ and $p'$), for the neutron (proton) in the initial and final states, respectively. Hence, energy-momentum conservation reads $n+p=n'+p'+q$.

The vector-current matrix element, including four direct ($n\rightarrow n',p\rightarrow p'$) and four exchange ($n\rightarrow p',p\rightarrow n'$) diagrams takes the form

$$M_\lambda^{vec}=\left[g_V^n\left(-\frac{n_\lambda}{n\cdot q}+\frac{n'_\lambda}{n'\cdot q}\right)+g_V^p\left(-\frac{p_\lambda}{p\cdot q}+\frac{p'_\lambda}{p'\cdot q}\right)\right]T^{(np)}, \tag{63}$$

where as before $g_V^n=-1/2$ and, cf. Eq. (11), $g_V^p\equiv G_V^p(0)=1/2-2\sin^2\theta_W$, and

$$T^{(np)} \equiv \sum_{\alpha=1}^{5} F_{\alpha}^{(np)} [\overline{u}(p') \Omega_{\alpha} u(p) \overline{u}(n') \Omega^{\alpha} u(n) + (-)^{\alpha} \overline{u}(n') \Omega_{\alpha} u(p) \overline{u}(p') \Omega^{\alpha} u(n)] \quad (64)$$

is the $np$ scattering amplitude in the GGMW notation of Eq. (45). It is easily checked from Eq. (63) that the vector current is conserved, i.e., $q^{\lambda} M_{\lambda}^{vec} = 0$.

In the nonrelativistic limit we find for the vector-current matrix element,

$$\vec{M}^{vec} = -\frac{g_V^n - g_V^p}{\omega} \frac{\vec{k}}{m} T^{(np)},$$

$$M_0^{vec} = -\vec{V} \cdot \vec{M}^{vec}, \quad (65)$$

where $\vec{k} = \vec{n}' - \vec{n} = \vec{p} - \vec{p}'$. We conclude that for the $np\nu\overline{\nu}$ case the vector-current contribution is of order $\mathcal{O}(|\vec{p}|/m)$, whereas for the $nn\nu\overline{\nu}$ case it is of order $\mathcal{O}(\vec{p}^2/m^2)$. As mentioned, this is analogous to the case of photon bremsstrahlung in $NN$ scattering, where electric-dipole radiation is dominant for the $np$ case, but suppressed for the $pp$ case. Moreover, since the momentum transfer $|\vec{k}|$ runs up to $2|\vec{p}|$ the factor $\vec{k}/m$ is not small for nucleon momenta at the Fermi surface. It is, therefore, questionable to drop the vector-current terms for the $np\nu\overline{\nu}$ process, as was done in FM for OPE in Born approximation.

The axial-current matrix element for the $np\nu\overline{\nu}$ process follows from Eqs. (45) and (54). We write in this case

$$M_{\lambda}^{ax} = M_{\lambda}^{(dir)} + M_{\lambda}^{(ex)}, \quad (66)$$

where

$$M_{\lambda}^{(dir)} = \frac{g_A}{2} \sum_{\alpha=1}^{5} F_{\alpha}^{(np)} \Bigg[ \overline{u}(p') \Omega_{\alpha} u(p) \overline{u}(n') \times \left( \Omega^{\alpha} \frac{\Lambda^{+}(n)}{2n \cdot q} i\gamma_{\lambda}\gamma_5 - i\gamma_{\lambda}\gamma_5 \frac{\Lambda^{+}(n')}{2n' \cdot q} \Omega^{\alpha} \right) u(n) + \overline{u}(p') \left( -\Omega^{\alpha} \frac{\Lambda^{+}(p)}{2p \cdot q} i\gamma_{\lambda}\gamma_5 + i\gamma_{\lambda}\gamma_5 \frac{\Lambda^{+}(p')}{2p' \cdot q} \Omega^{\alpha} \right) \times u(p) \overline{u}(n') \Omega_{\alpha} u(n) \Bigg], \quad (67)$$

and

$$M_{\lambda}^{(ex)} = \frac{g_A}{2} \sum_{\alpha=1}^{5} (-)^{\alpha} F_{\alpha}^{(np)} \Bigg[ \overline{u}(n') \Omega_{\alpha} u(p) \overline{u}(p') \times \left( \Omega^{\alpha} \frac{\Lambda^{+}(n)}{2n \cdot q} i\gamma_{\lambda}\gamma_5 + i\gamma_{\lambda}\gamma_5 \frac{\Lambda^{+}(p')}{2p' \cdot q} \Omega^{\alpha} \right) u(n) + \overline{u}(n') \left( -\Omega^{\alpha} \frac{\Lambda^{+}(p)}{2p \cdot q} i\gamma_{\lambda}\gamma_5 - i\gamma_{\lambda}\gamma_5 \frac{\Lambda^{+}(n')}{2n' \cdot q} \Omega^{\alpha} \right) u(p) \overline{u}(p') \Omega_{\alpha} u(n) \Bigg]. \quad (68)$$

Note that corresponding neutron and proton legs (i.e., both incoming or both outgoing) differ by a minus sign, since $G_A^p(0) = -G_A^n(0) = g_A/2$, cf. Eqs. (11) and (12).

In the nonrelativistic limit we find for this axial-current matrix element, with $p \cdot q \simeq -m\omega$,

$$\vec{M}^{ax} = \frac{g_A}{\omega} ([T^{(dir)}, \vec{D}] + \hat{P}_{\sigma} \{T^{(ex)}, \vec{D}\}),$$

$$M_0^{ax} \simeq 0, \quad (69)$$

where $\vec{D} = (\vec{\sigma}_1 - \vec{\sigma}_2)/2$, and $T^{(dir)}$ and $T^{(ex)}$ are operators in spin space given by the nonrelativistic reduction of the direct and exchange parts of the $np$ $T$ matrix of Eq. (64). Pauli spinors are not written explicitly. $\hat{P}_{\sigma} = (1 + \vec{\sigma}_1 \cdot \vec{\sigma}_2)/2$ is the spin-exchange operator. Equation (69) is a generalization of the FM result for the $np\nu\overline{\nu}$ process in several ways. First, it includes the complete $np$ $T$ matrix. Second, as a consequence, the commutator term receives contributions not only from the spin-spin and the tensor force, as in FM, but also from the spin-orbit interaction, not present in FM. The anticommutator term represents the corresponding exchange diagrams. This term receives contributions from the central and spin-spin forces, taken into account in FM by Landau interactions, and also from tensor and spin-orbit forces.

### D. The modified Urca processes $nne\overline{\nu}$ and $npe\overline{\nu}$

Consider first the process $n + n \to n + p + e^- + \overline{\nu}_e$. We introduce the notations $n_1$ and $n_2$ for the momenta of the initial neutrons, and $n'$ and $p'$ for the final neutron and proton, so energy-momentum conservation reads $n_1 + n_2 = n' + p' + q$; the Mandelstam variables are defined by $t = -(n' - n_1)^2$ and $u = -(p' - n_1)^2$. The matrix element includes three direct and three exchange diagrams, and takes the following form for the vector current:

$$M_{\lambda}^{vec} = \cos\theta_C \times \sum_{\alpha=1}^{5} \left( -\frac{n_{1\lambda}}{n_1 \cdot q} F_{\alpha}^{(ce)} - \frac{n_{2\lambda}}{n_2 \cdot q} F_{\alpha}^{(np)} + \frac{p'_{\lambda}}{p' \cdot q} F_{\alpha}^{(nn)} \right) \times [\overline{u}(p') \Omega_{\alpha} u(n_2) \overline{u}(n') \Omega^{\alpha} u(n_1) + (-)^{\alpha} \overline{u}(n') \Omega_{\alpha} u(n_2) \overline{u}(p') \Omega^{\alpha} u(n_1)]. \quad (70)$$

It is seen that for the modified Urca processes the isospin decomposition of the $T$ matrix is more involved than in the case of the neutral current. By using Eqs. (49) and (50), it is easily checked that CVC is obeyed.

The axial-vector matrix element for the $nne\overline{\nu}$ process reads

$$M_{\lambda}^{ax} = M_{\lambda}^{(dir)} + M_{\lambda}^{(ex)}, \quad (71)$$

where

$$M_\lambda^{(dir)}=g_A\cos\theta_C \times\sum_{\alpha=1}^{5}\Bigg(-F_\alpha^{(ce)}\overline{u}(p')\Omega^\alpha u(n_2)\overline{u}(n')\Omega_\alpha \times\frac{\Lambda^+(n_1)}{2n_1\cdot q}i\gamma_\lambda\gamma_5 u(n_1)-F_\alpha^{(np)}\overline{u}(p')\Omega^\alpha \times\frac{\Lambda^+(n_2)}{2n_2\cdot q}i\gamma_\lambda\gamma_5 u(n_2)\overline{u}(n')\Omega_\alpha u(n_1)+F_\alpha^{(nn)}\overline{u}(p') \times i\gamma_\lambda\gamma_5\frac{\Lambda^+(p')}{2p'\cdot q}\Omega_\alpha u(n_2)\overline{u}(n')\Omega^\alpha u(n_1)\Bigg) \tag{72}$$

and

$$M_\lambda^{(ex)}=g_A\cos\theta_C \times\sum_{\alpha=1}^{5}(-)^\alpha\Bigg(-F_\alpha^{(ce)}\overline{u}(n')\Omega^\alpha u(n_2) \times\overline{u}(p')\Omega_\alpha\frac{\Lambda^+(n_1)}{2n_1\cdot q}i\gamma_\lambda\gamma_5 u(n_1) -F_\alpha^{(np)}\overline{u}(n')\Omega^\alpha\frac{\Lambda^+(n_2)}{2n_2\cdot q}i\gamma_\lambda\gamma_5 u(n_2)\overline{u}(p')\Omega_\alpha u(n_1) +F_\alpha^{(nn)}\overline{u}(n')\Omega^\alpha u(n_2)\overline{u}(p') \times i\gamma_\lambda\gamma_5\frac{\Lambda^+(p')}{2p'\cdot q}\Omega_\alpha u(n_1)\Bigg). \tag{73}$$

The vector and axial-vector matrix elements, $M_\lambda^{vec}$ and $M_\lambda^{ax}$, are antisymmetric under the interchange of the initial neutrons, as can be verified using Eqs. (48)–(50).

As was stressed in Sec. V A, the $\mathcal{O}(1/q)$ approximation for the modified Urca processes can serve only as an order-of-magnitude estimate, since the contribution of the terms of order $\mathcal{O}(1)$ is likely significant. The process $n+p\rightarrow p+p+e^-+\overline{\nu}_e$ is very similar to the $nne\overline{\nu}$ case. In fact, it is easily seen that the same isospin combinations of the $T$ matrix enter as in the $nne\overline{\nu}$ process.

## VI. ONE-BOSON EXCHANGE CONTRIBUTIONS

### A. Motivation

In this section, we illustrate the main points of the previous sections by replacing the two-nucleon $T$ matrix by one-meson exchange in Born approximation. This discussion is for illustrative purposes only, since, as argued before, it clearly is not realistic. Our aim in this section is twofold.

First, we wish to reexamine the contribution from one-pion exchange, since it was used in the pioneering FM work, and also practically in all the more recent calculations of neutrino emissivities. In that (FM) approach, the following simplifications were made: (i) Both the weak-interaction operators and the nucleon propagators were treated in the (extreme) nonrelativistic limit. (ii) All lepton momenta were neglected. (iii) The $T$ matrix was replaced by an $NN$ interaction that is assumed to be velocity independent; in particular, all momentum-dependent terms, such as spin-orbit forces, were discarded.

In this situation, depending on the time ordering of the weak- and the strong-interaction vertices in the bremsstrahlung amplitude, the off-mass-shell nucleon propagators behave as $\pm 1/\omega$. As a result, one finds that there is no contribution to the vector-current matrix element, and that the only contribution to the axial-current matrix element comes from the nonvanishing commutator of the weak axial spin (Gamow-Teller) operator with the $\vec{\sigma}\cdot\vec{k}$ operator of the strong pion-nucleon vertex (or anticommutator in case of the $np$ system). This has led to the widely accepted point of view in the literature that only spin correlations induced by the tensor force play a role for soft bremsstrahlung. In addition, it is often assumed that the strength of these correlations is given by the OPE potential in free space.

Second, in order to get an idea about the relevance of the contributions from other mesons, we will consider the isovector-vector $\varrho(770)$ and the isoscalar-scalar $\sigma$ [or $\varepsilon(760)$] mesons. The reason for considering one-rho exchange (ORE) is that, as is well known, its tensor force cuts down the OPE tensor force at short distance. We will take into account only the part of rho-exchange that gives rise to this tensor force. The reason for considering one-sigma exchange (OSE) is that it contributes to the spin-orbit interaction (we will ignore the quadratic spin-orbit term). Such a term has also a nonvanishing (anti-)commutator with the axial spin operator, and hence could contribute to the neutrino emissivities at the same order as OPE. A spin-orbit interaction is allowed by Galilean covariance and, therefore, occurs in the general nonrelativistic $T$ matrix [31], cf. Eqs. (51) and (52). One-boson exchange models for the $NN$ interaction contain strong spin-orbit forces from vector- and scalar-meson exchange. Also the more fundamental chiral two-pion exchange interaction [28] has a strong spin-orbit component. For simplicity, we will take only the tensor force of ORE into account. We stress that these one-boson exchanges (OBEs) in Born approximation are used for illustrative purposes only.

To obtain the nonrelativistic amplitudes, we neglect the lepton momenta and work in the c.m. frame of the nucleons. We define $\vec{k}=\vec{p}\,'-\vec{p}$ and $\vec{k}'=\vec{p}\,'+\vec{p}$, with $\vec{k}\cdot\vec{k}'=0$, and $-t=\vec{k}^2$ and $-u=\vec{k}'^2$. We also introduce the total spin operator $\vec{S}=(\vec{\sigma}_1+\vec{\sigma}_2)/2$, and the vector operators $\vec{D}=(\vec{\sigma}_1-\vec{\sigma}_2)/2$ and $\vec{Q}=(\vec{\sigma}_1\times\vec{\sigma}_2)/2$. The OPE, ORE (spin-dependent part), and OSE potentials read

$$V_\pi=-\vec{\tau}_1\cdot\vec{\tau}_2\frac{\vec{\sigma}_1\cdot\vec{k}\,\vec{\sigma}_2\cdot\vec{k}}{4m^2}v_\pi(k^2),$$

$$V_\varrho^{(spin)}=-\vec{\tau}_1\cdot\vec{\tau}_2\Bigg(\frac{\vec{\sigma}_1\times\vec{k}\,\vec{\sigma}_2\times\vec{k}}{4m^2}(1+\kappa_\varrho)^2 -i\frac{\vec{S}\cdot(\vec{k}\times\vec{k}')}{4m^2}(3+4\kappa_\varrho)\Bigg)v_\varrho(k^2), \tag{74}$$

$$V_\sigma = -\left(1+\frac{\vec{k}^2}{8m^2}-\frac{\vec{k}'^2}{8m^2}-i\frac{\vec{S}\cdot(\vec{k}\times\vec{k}')}{4m^2}\right)v_\sigma(k^2),$$

with $v_\pi(k^2)=g^2_{NN\pi}/(\vec{k}^2+m_\pi^2)$, $v_\varrho(k^2)=g^2_{NN\varrho}/(\vec{k}^2+m_\varrho^2)$, $\kappa_\varrho = f_{NN\varrho}/g_{NN\varrho}$, and $v_\sigma(k^2)=g^2_{NN\sigma}/(\vec{k}^2+m_\sigma^2)$.

### B. The $nn\nu\bar{\nu}$ process

This corresponds to the isospin $I=1$ channel, so $\vec{\tau}_1\cdot\vec{\tau}_2=1$. Up to order $\mathcal{O}(\vec{p}^2/m^2)$ OPE does not contribute to the vector-current amplitude. The contribution of OSE to the space components of this amplitude is

$$\vec{M}^{vec}_\sigma = \frac{g_V^n}{\omega m^2}(\vec{k}'(\vec{k}\cdot\vec{V})+\vec{k}(\vec{k}'\cdot\vec{V}))\times[v_\sigma(k^2)-v_\sigma(k'^2)\hat{P}_\sigma], \quad (75)$$

and from CVC, $M^{vec,0}_\sigma = -\vec{V}\cdot\vec{M}^{vec}_\sigma$. The OPE contribution to the space components of the axial-vector current matrix element is obtained by using the (anti)commutation relations of the Pauli spin matrices. We get

$$\vec{M}^{ax}_\pi = i\frac{g_A}{4\omega m^2}(v_\pi(k^2)[(\vec{\sigma}_1\cdot\vec{k})(\vec{k}\times\vec{\sigma}_2)+(\vec{\sigma}_2\cdot\vec{k})(\vec{k}\times\vec{\sigma}_1)] - v_\pi(k'^2)[(\vec{\sigma}_1\cdot\vec{k'})(\vec{k'}\times\vec{\sigma}_2)+(\vec{\sigma}_2\cdot\vec{k'})(\vec{k'}\times\vec{\sigma}_1)]); \quad (76)$$

in this order the contribution to the time component vanishes. OSE gives two contributions to the axial-vector current matrix element. The first one, of order $\mathcal{O}(\vec{k}^2/m^2)$ comes from the commutator of the spin-orbit interaction of OSE with the axial spin operator of the weak vertex. The second one, of order $\mathcal{O}(|\vec{k}|/m)$, is due to the expansion of the nucleon propagator. We have

$$\vec{M}^{ax}_\sigma = -\frac{g_A}{2\omega m}\Bigg(-2v_\sigma(k^2)(\vec{k}\cdot\vec{V})\vec{D}-2iv_\sigma(k'^2)(\vec{k}'\cdot\vec{V})\vec{Q} + \frac{1}{2m}v^{(+)}_\sigma(k^2,k'^2)[\vec{k}(\vec{S}\cdot\vec{k}')-\vec{k}'(\vec{S}\cdot\vec{k})] - \frac{2}{m}v^{(-)}_\sigma(k^2,k'^2)(\vec{k}\cdot\vec{V})(\vec{k}'\cdot\vec{V})\vec{S}\Bigg), \quad (77)$$

with $v^{(\pm)}_\sigma(k^2,k'^2)=v_\sigma(k^2)\pm v_\sigma(k'^2)$. The time component coming from OSE is given by

$$M^{ax,0}_\sigma = \frac{g_A}{2\omega m}\Bigg(2v_\sigma(k^2)(\vec{D}\cdot\vec{k})+2iv_\sigma(k'^2)(\vec{Q}\cdot\vec{k}') + \frac{1}{m}v^{(-)}_\sigma(k^2,k'^2)[(\vec{S}\cdot\vec{k})(\vec{k'}\cdot\vec{V})+(\vec{S}\cdot\vec{k}')(\vec{k}\cdot\vec{V})]\Bigg). \quad (78)$$

The matrix element between the Pauli spinors is implied in the above expressions. The Pauli principle is satisfied by explicit antisymmetrization of the matrix element.

The formulas for ORE can easily be written down analogously by using $(\vec{\sigma}_1\times\vec{k})(\vec{\sigma}_2\times\vec{k})=\vec{k}^2(\vec{\sigma}_1\cdot\vec{\sigma}_2)-(\vec{\sigma}_1\cdot\vec{k})\times(\vec{\sigma}_2\cdot\vec{k})$.

### C. The $np\nu\bar{\nu}$ process

This process differs in two ways from the previous one: both isospins $I=0$ and 1 contribute, and the weak vertices for neutron and proton have opposite signs. This results in the following expressions that include both direct $(\pi^0,\sigma)$ and exchange $(\pi^\pm)$ contributions. We take the incoming neutron as particle 1, and the incoming proton as particle 2, so that the operators below are sandwiched between the spinors $\chi^+_{s'_n}(1)\chi^+_{s'_p}(2)$ and $\chi_{s_n}(1)\chi_{s_p}(2)$. The OPE contribution is given by

$$\vec{M}^{ax}_\pi = i\frac{g_A}{4\omega m^2}(v_\pi(k^2)[(\vec{\sigma}_1\cdot\vec{k})(\vec{k}\times\vec{\sigma}_2) - (\vec{\sigma}_2\cdot\vec{k})(\vec{k}\times\vec{\sigma}_1)]-4v_\pi(k'^2)\vec{k}'(\vec{Q}\cdot\vec{k}')). \quad (79)$$

The OSE contribution to the space part of the matrix element is

$$\vec{M}^{ax}_\sigma = -\frac{g_A}{2\omega m}v_\sigma(k^2)\Bigg(-2(\vec{k}\cdot\vec{V})\vec{S}+\frac{1}{2m}[\vec{k}(\vec{D}\cdot\vec{k}') - \vec{k}'(\vec{D}\cdot\vec{k})]-\frac{2}{m}(\vec{k}\cdot\vec{V})(\vec{k}'\cdot\vec{V})\vec{D}\Bigg), \quad (80)$$

and the corresponding contribution to the time component is

$$M^{ax,0}_\sigma = \frac{g_A}{2\omega m}v_\sigma(k^2)\Bigg(2\vec{S}\cdot\vec{k}+\frac{1}{m}[(\vec{D}\cdot\vec{k})(\vec{k}'\cdot\vec{V}) + (\vec{D}\cdot\vec{k}')(\vec{k}\cdot\vec{V})]\Bigg). \quad (81)$$

In this order the vector-current matrix element is

$$\vec{M}^{vec}_\sigma = \frac{v_\sigma(k^2)}{m\omega}\times\left[(g_V^n-g_V^p)\vec{k}+\frac{g_V^n+g_V^p}{2m}(\vec{k}(\vec{k'}\cdot\vec{V})+\vec{k}'(\vec{k}\cdot\vec{V}))\right]. \quad (82)$$

### D. Comparison with previous results

To compare in detail with previous results [4,5] for OPE, we contract the bremsstrahlung matrix element $M_\lambda$ with the leptonic current (see the following section), square the resulting amplitudes, and sum over the nucleon and lepton spins. Keeping close to the notation used by Maxwell [5], the final results, including the exchange contributions, can be expressed in the form

$$\frac{G_F^2}{2}\sum_{spins}|M_\pi|^2 = 128\gamma^2\frac{\omega_1\omega_2}{\omega^2}\left[C_1\left(\frac{k^2}{k^2+m_\pi^2}\right)^2+C_2\left(\frac{k'^2}{k'^2+m_\pi^2}\right)^2 + \frac{C_3k^2k'^2+C_4(\vec{k}\cdot\vec{k'})^2}{(k^2+m_\pi^2)(k'^2+m_\pi^2)}\right], \quad (83)$$

with $\omega=\omega_1+\omega_2$ and, cf. Eq. (38),

$$\gamma\equiv\frac{G_F}{\sqrt{2}}\frac{f_{NN\pi}^2}{m_\pi^2}\delta, \quad (84)$$

where $\delta=g_A$ for the neutral-current processes, and $\delta=g_A\cos\theta_C$ for the charged-current (Urca) processes. Equation (83) holds for one neutrino flavor. The coefficients $C_i$ $(i=1,\ldots,4)$ are given by

| | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
|---|---|---|---|---|
| $nn\nu\bar{\nu}$ | 1 | 1 | 1 | $-3$ |
| $np\nu\bar{\nu}$ | 1 | 2 | $-2$ | 2 |
| $nne\bar{\nu}$ | 4 | 4 | $-3$ | 1 |
| $npe\bar{\nu}$ | 4 | 4 | $-3$ | 1 |

(85)

These results agree with Refs. [4,5,9] only for the $nn\nu\bar{\nu}$ process and the direct contribution, $C_1$, for the other processes. The differences can be traced back to the exchange contribution. Our formulas agree with Ref. [5]. However, our $C_i$s for $i=2,3,4$ disagree with the Tables of Ref. [5] for several entries. They agree with Ref. [33].

## VII. NEUTRINO EMISSION CROSS SECTION

In this section, we will use the leading-order $\mathcal{O}(1/q)$ amplitudes to calculate the neutrino emission cross section in free space. We will compare these results with the $NN$ $T$ matrix to those that are obtained when OPE in Born approximation is used.

The cross section for the electroweak bremsstrahlung process $N(p_1)+N(p_2)\rightarrow N(p_1')+N(p_2')+l(q_1)+l(q_2)$, with $q=q_1+q_2=(\omega,\vec{q})$, integrated over the kinematics of the two leptons, is given by

$$d\sigma=\frac{m^4}{4j(2\pi)^8}\int\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}d^4q\,\delta^4(P_f-P_i) \times\int\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\delta^4(q-q_1-q_2)\frac{G_F^2}{2}\sum_{spins}|M|^2, \quad (86)$$

where the flux factor is $j=\sqrt{s(s-4m^2)}/2$, with $s=4m^2+2T_Lm$; $T_L$ is the kinetic energy of the bombarding nucleon in the laboratory frame; $P_i=p_1+p_2$ and $P_f=p_1'+p_2'+q$. The sum runs over the spins of all particles in the final state, for unpolarized initial nucleons the usual average over spins is implied. The squared matrix element has been defined as

$$|M|^2=M_\lambda M_\varrho^* L^{\lambda\varrho}(q_1,q_2), \quad (87)$$

with the symmetric lepton tensor given by

$$L^{\lambda\varrho}(q_1,q_2)=\mathrm{Tr}[(-i\gamma\cdot q_1+m_1)i\gamma^\lambda(1+\gamma_5) \times(-i\gamma\cdot q_2+m_2)i\gamma^\varrho(1+\gamma_5)] =8(q_1^\lambda q_2^\varrho+q_1^\varrho q_2^\lambda-q_1\cdot q_2 g^{\lambda\varrho}+i\varepsilon^{\lambda\alpha\varrho\beta}q_{1\alpha}q_{2\beta}). \quad (88)$$

We first isolate the leptonic part given by the tensor

$$I^{\lambda\varrho}(q)=\int\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\delta^4(q-q_1-q_2)L^{\lambda\varrho}(q_1,q_2) =\frac{8}{3}(q^\lambda q^\varrho-q^2g^{\lambda\varrho})\int\frac{d^3q_1}{\omega_1}\frac{d^3q_2}{\omega_2}\delta^4(q-q_1-q_2) =\frac{16\pi}{3}(q^\lambda q^\varrho-q^2g^{\lambda\varrho}). \quad (89)$$

The cross section Eq. (86) then reduces to

$$d\sigma=\frac{m^4}{4j(2\pi)^8}\frac{16\pi}{3}\frac{G_F^2}{2}\int\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}d^4q \times\delta^4(P_f-P_i)M_\lambda(q^\lambda q^\varrho-q^2g^{\lambda\varrho})M_\varrho^*. \quad (90)$$

This result is exact. Since we have all along assumed that the momentum $q$ is soft, we drop $\vec{q}$ in

$$\delta^4(P_f-P_i) =\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_1'-\vec{p}_2'-\vec{q})\,\delta(E_1+E_2-E_1'-E_2'-\omega).$$

Then we find, in the c.m. frame of the nucleons,

$$\int\frac{d^3p_1'}{E_1'}\frac{d^3p_2'}{E_2'}\delta^4(p_1+p_2-p_1'-p_2'-q) =\int\frac{d^3p'}{E'^2}\delta(\sqrt{s}-\omega-2E')=\frac{|\vec{p}'|}{\sqrt{s}-\omega}d\Omega(\hat{p}'), \quad (91)$$

where $|\vec{p}'|=[(\sqrt{s}-\omega)^2/4-m^2]^{1/2}$ is the momentum of the outgoing nucleon. We get

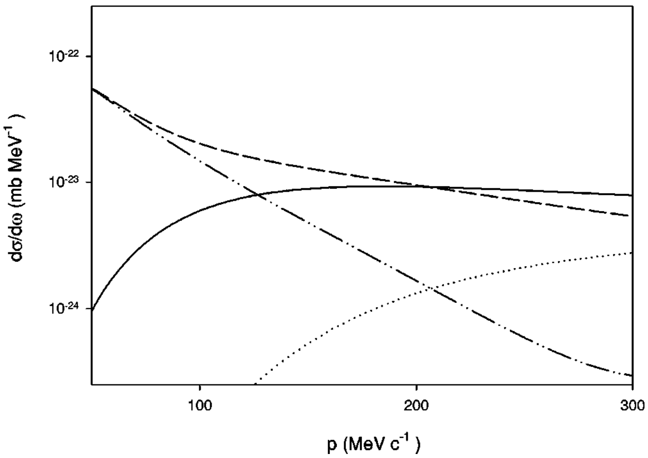


FIG. 1. Cross section $d\sigma/d\omega$ for the process $n+n\rightarrow n+n+\nu_f+\bar{\nu}_f$ as a function of neutron c.m. momentum, for $\omega=1$ MeV (summed over neutrino flavors). Shown are the results for the complete $T$ matrix (full curve), as well as the separate contributions from the tensor (dashed curve), quadratic spin-orbit (dashed-dotted curve), and spin-orbit (dotted curve) terms.

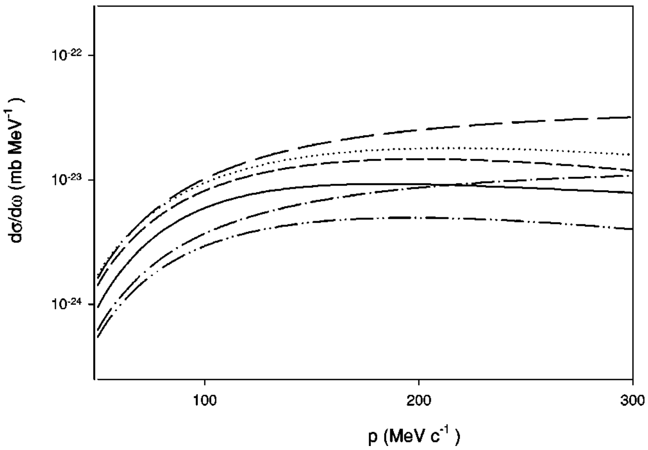


FIG. 2. Various OBE contributions to the cross section $d\sigma/d\omega$ for the process $n+n\rightarrow n+n+\nu_f+\bar{\nu}_f$ as a function of neutron c.m. momentum, for $\omega=1$ MeV (summed over neutrino flavors). Shown are the results for the complete $T$ matrix (full curve), OPE (long-dashed curve), OPE plus the tensor term of ORE (short-dashed curve), OPE plus the tensor term of ORE plus OSE (dotted curve), OPE without exchange terms (dashed-dotted curve), and OPE plus tensor term of ORE without exchange terms (dashed-double-dotted curve).

$$d\sigma=\frac{m^4}{4j(2\pi)^8}\frac{16\pi}{3}\frac{G_F^2}{2}\frac{|\vec{p}\,'|}{\sqrt{s}-\omega}\times\int d\Omega(\hat{p}\,')d^4q(M_\lambda q^\lambda M_\varrho^* q^\varrho-q^2M^\lambda M_\lambda^*). \tag{92}$$

For OPE this can be simplified further using the independence of the matrix elements of $\vec{q}$; see Eqs. (76) and (79). This allows one to perform integration over $\vec{q}$ in Eq. (92), making use of

$$\int_{|\vec{q}|\leqslant\omega}d^3q(q^\lambda q^\varrho-q^2g^{\lambda\varrho})=\frac{4\pi}{5}\omega^5\delta^{\lambda\varrho}, \tag{93}$$

where $\delta^{\lambda\varrho}=1$ if $\lambda=\varrho$, and 0 otherwise. In this way one obtains

$$\frac{d\sigma}{d\omega}=\frac{m^4}{4j(2\pi)^8}\frac{64\pi^2\omega^5}{15}\frac{G_F^2}{2}\frac{|\vec{p}\,'|}{\sqrt{s}-\omega}\times 2\pi\int(|M_0|^2+|\vec{M}|^2)\sin\theta_{p'}d\theta_{p'}. \tag{94}$$

This result can be shown to be equivalent to that of FM if in addition the time component of the matrix element is discarded as being of higher order in $k/m$.

We calculate the cross section Eq. (92) for the $nn\nu\bar{\nu}$ process (with unpolarized initial neutrons), for OPE and with the full $T$ matrix. Equation (92) is multiplied by a factor of 3 to take into account the three neutrino flavors. The $nn$ phase shifts are, for simplicity, assumed to be equal to the $pp$ phase shifts, which are taken from Ref. [34]. The results are presented in Figs. 1 and 2. We show the cross section Eq. (92) as a function of nucleon c.m. momenta in a range relevant for neutron matter at saturation density. We take a typical neutrino-pair energy of $\omega=1$ MeV.

In Fig. 1 we plot the result for the full $T$ matrix as well as the contributions from different spin interactions in Eq. (52). It is seen that the contribution of the tensor terms [the $C$ and $D$ terms in Eq. (52)] is fairly independent of nucleon momentum and that it dominates over the spin-orbit (the $E$ term) and the quadratic spin-orbit (the $B$ term) contributions above 200 MeV/$c$. The amplitudes for the tensor terms ($C$ and $D$) and the quadratic spin-orbit term ($B$) have opposite signs at low momenta and thus cancel in the cross section. As a comparison, we show in Fig. 2 several results for the OBE contributions, for the same $nn\nu\bar{\nu}$ process.[5] The OPE result strongly overpredicts the result obtained with the full $T$ matrix. In particular, at 300 MeV/$c$ OPE is larger by about a factor of 5. This is in qualitative agreement with the recent result of Ref. [20].

As mentioned, the tensor force from OPE gets cancelled at short distance by the tensor force from ORE, which has an opposite sign. Indeed, it is seen in Fig. 2 that when the ORE tensor force is added to OPE, the result is significantly closer to that obtained with the full $T$ matrix, although the difference varies with momentum. We also show the result for OPE without the "exchange" (antisymmetrization) contribution, that, apparently, is used in "standard cooling scenarios," and which is lower than the full OPE by about a factor of 5. The rationale for not taking this OPE exchange contribution into account in FM was the observation that it is largely cancelled by the tensor-force part of ORE. Finally, to

[5]Numerical values are taken from the OBE model Nijm93 [32]; specifically, we use $m_\sigma=488$ MeV, $g_{NN\sigma}^2/4\pi=4.8$ for OSE, and $(g_{NN\varrho}+f_{NN\varrho})^2/4\pi=22.0$ for the tensor term of ORE.

give an estimate of the effect of other mesons, we also plot the contribution from OSE, which gives rise to a spin-orbit force.

## VIII. CONCLUSIONS AND SUMMARY

In this paper, we studied the structure of the electroweak $NN$ bremsstrahlung amplitude in the soft regime, based on theorems for the matrix elements of the weak polar- and axial-vector currents, both charged and neutral. The resulting amplitudes are considerably more complicated than for the photon bremsstrahlung case. Especially the exchange contributions require careful treatment. A main conclusion is that it appears possible to calculate reliably and accurately soft-neutrino bremsstrahlung by using the symmetries of the electroweak currents in the standard model and the available knowledge of the $NN$ interaction.

We compared our amplitudes to the pioneering FM work based on OPE in Born approximation. It was shown that for the most general $NN$ $T$ matrix the lowest-order contribution to the vector-current matrix element cancels. The finite subleading-order terms were also obtained in our approach. Leading- and subleading-order terms for the axial-vector current matrix elements were derived. Specifically, we pointed out that spin-orbit forces contribute to the $nn\nu\bar{\nu}$ process, and not just tensor forces.

Our numerical estimate of the cross section for the $nn\nu\bar{\nu}$ process indicates large differences in a calculation based on $NN$ phase shifts, compared to treatments restricted to OPE in Born approximation. Similar differences may be expected for the $np\nu\bar{\nu}$ and modified Urca processes as well. This has implications for neutrino emissivities in neutron-star cooling. In the simplest approach, as used in the work of FM and in many later studies, one adopts the quasiparticle (or mean-field) picture. This basically leads to a convolution of the free-space bremsstrahlung amplitudes squared with the Fermi-Dirac functions at finite temperature and density. In that case the reduction factor from OPE in Born approximation to realistic $T$ matrix discussed above is not affected. In more consistent approaches additional "medium effects" must be taken into account, such as the replacement of the $T$ matrix by the $G$ matrix and the use of effective masses and effective couplings. This is under investigation [35].

## ACKNOWLEDGMENTS

We thank A. D. Sedrakian, J. A. Tjon, and D. Van Neck for helpful discussions. We also acknowledge a clarifying communication with B. L. Friman and O. V. Maxwell. The research of R.G.E.T. was made possible by the support of the Royal Netherlands Academy of Arts and Sciences (KNAW); that of A.Yu.K. by the "Nederlandse Organisatie voor Wetenschappelijk Onderzoek" (NWO); that of A.E.L.D. and E.N.E.v.D. was supported by the Stichting voor Fundamenteel Onderzoek der Materie, with financial support from the Nederlandse Organisatie voor Wetenschappelijk Onderzoek.

## APPENDIX: PION POLE IN THE AXIAL-VECTOR MATRIX ELEMENT

The axial-vector amplitude in order $\mathcal{O}(1/q)$, see Eq. (54), obeys the relation

$$iq^{\lambda}M_{\lambda}^{ax,a}=F_{\pi}M_{\pi}^{a}(0)/2. \tag{A1}$$

In principle, one could add the pion-pole term $(F_{\pi}/2)M_{\pi}^{a}(0)q_{\lambda}/(q^2+m_{\pi}^2)$ (see Sec. III C). This term is needed to obey PCAC in order $\mathcal{O}(1/q)$, as can be seen as follows. When the pion pole is included in Eq. (54), the divergence is found to be

$$iq^{\lambda}M_{\lambda}^{ax,a}=\frac{F_{\pi}}{2}\frac{m_{\pi}^2}{q^2+m_{\pi}^2}M_{\pi}^{a}(0). \tag{A2}$$

This is the PCAC condition [Eq. (32)], where the limit $q\to 0$ is taken in $M_{\pi}^{a}$.

However, for the reasons mentioned in Sec. V A the pion contribution can be neglected. Moreover, for $q^2 \ll m_{\pi}^2$ the variation of this term due to the pion propagator is also negligible and Eq. (A2) reduces to Eq. (A1).

[1] H.Y. Chiu and E.E. Salpeter, Phys. Rev. Lett. **12**, 413 (1964).
[2] J.N. Bahcall and R.A. Wolf, Phys. Rev. **140**, B1452 (1965).
[3] E.G. Flowers, P.G. Sutherland, and J.R. Bond, Phys. Rev. D **12**, 315 (1975).
[4] B.L. Friman and O.V. Maxwell, Astrophys. J. **232**, 541 (1979).
[5] O.V. Maxwell, Astrophys. J. **316**, 691 (1987).
[6] F.E. Low, Phys. Rev. **110**, 974 (1958).
[7] S.L. Adler and Y. Dothan, Phys. Rev. **151**, 1267 (1966).
[8] M.K. Liou, R.G.E. Timmermans, and B.F. Gibson, Phys. Rev. C **54**, 1574 (1996).
[9] D.N. Voskresenskii and A.V. Senatorov, Sov. Phys. JETP **63**, 885 (1986); Zh. Eksp. Teor. Fiz. **90**, 1505 (1986); Sov. J. Nucl. Phys. **45**, 411 (1987); Yad. Fiz. **45**, 657 (1987).
[10] D. Blaschke, G. Röpke, H. Schulz, A.D. Sedrakian, and D.N. Voskresensky, Mon. Not. R. Astron. Soc. **273**, 596 (1995).
[11] Ch. Schaab, D. Voskresensky, A.D. Sedrakian, F. Weber, and M.K. Weigel, Astron. Astrophys. **321**, 591 (1997).
[12] A. Sedrakian and A.E.L. Dieperink, Phys. Lett. B **463**, 145 (1999); Phys. Rev. D **62**, 083002 (2000).
[13] N. Iwamoto and C.J. Pethick, Phys. Rev. D **25**, 313 (1982).
[14] R.F. Sawyer, Phys. Rev. Lett. **75**, 2260 (1995).
[15] G. Raffelt and D. Seckel, Phys. Rev. D **52**, 1780 (1995).
[16] G. Sigl, Phys. Rev. Lett. **76**, 2625 (1996); Phys. Rev. D **56**, 3179 (1997).
[17] S. Hannestad and G. Raffelt, Astrophys. J. **507**, 339 (1998).
[18] T.H. Burnett and N.M. Kroll, Phys. Rev. Lett. **20**, 86 (1968).
[19] J.S. Bell and R. Van Royen, Jr., Nuovo Cimento A **60**, 62 (1969).
[20] Ch. Hanhart, D.R. Phillips, and S. Reddy, Phys. Lett. B **499**, 5 (2001).
[21] E.M. Nyman, Phys. Rev. **170**, 1628 (1968).

[22] H.W. Fearing, Phys. Rev. C **6**, 1136 (1972).
[23] M.K. Liou, R.G.E. Timmermans, and B.F. Gibson, Phys. Lett. B **345**, 372 (1995).
[24] A.Yu. Korchin and O. Scholten, Nucl. Phys. **A581**, 493 (1995).
[25] V. De Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973).
[26] S.L. Adler, Phys. Rev. Lett. **33**, 1511 (1974); Phys. Rev. D **12**, 2644 (1975); **12**, 3501 (1975).
[27] W. Bentz, Nucl. Phys. **A446**, 678 (1985).
[28] M.C.M. Rentmeester, R.G.E. Timmermans, J.L. Friar, and J.J. de Swart, Phys. Rev. Lett. **82**, 4992 (1999).
[29] M.L. Goldberger, M.T. Grisaru, S.W. MacDowell, and D.Y. Wong, Phys. Rev. **120**, 2250 (1960).
[30] J.A. Tjon and S.J. Wallace, Phys. Rev. C **32**, 267 (1985).
[31] S. Okubo and R.E. Marshak, Ann. Phys. (N.Y.) **4**, 166 (1958).
[32] V.G.J. Stoks, R.A.M. Klomp, C.P.F. Terheggen, and J.J. de Swart, Phys. Rev. C **49**, 2950 (1994).
[33] D.G. Yakovlev, A.D. Kaminker, O.Y. Gnedin, and P. Haensel, Phys. Rep. **354**, 1 (2001).
[34] V.G.J. Stoks, R.A.M. Klomp, M.C.M. Rentmeester, and J.J. de Swart, Phys. Rev. C **48**, 792 (1993).
[35] E.N.E. van Dalen *et al.* (in preparation).