Tensor representation of the nucleon-nucleon amplitude

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(Received 2 October 2001; published 17 April 2002)

Many approaches to nucleon-nucleus elastic and inelastic scattering are based on the use of the free-space nucleon-nucleon transition amplitude. In calculations where the full spin dependence of this amplitude is needed, its use is more tractable when it is expressed in terms of irreducible tensor operators of the spins of the interacting nucleons. We present general formulas for this representation, which is particularly useful for inelastic scattering studies involving spin-flip transitions of a target nucleon.

DOI: 10.1103/PhysRevC.65.054001 PACS number(s): 24.10.Ht, 24.70.+s, 25.40.Cm

I. INTRODUCTION

The free nucleon-nucleon (NN) transition amplitude, both on and off the energy shell [1,2] is a basic dynamical input off *ab initio* multiple scattering scattering formalisms either within a many-body [3,4] or a few-body [5] framework. These formalisms have been applied with success to describe elastic and inelastic scattering from stable [6] (and references therein) and unstable nuclei [7,8]. The spin and isospin dependent amplitudes of the transition amplitude can be calculated readily from realistic NN interaction models, such as the Paris [9] and Bonn [10] interactions. It was shown in Ref. [2] that the off-shell behavior of the NN transition amplitude is very stable against the underlying NN interaction. In addition, the on-shell values must reproduce the available experimental NN data. Thus, the dynamical NN input of the multiple scattering expansions is very well defined, at least on the energy shell.

Traditionally, the NN transition amplitude has been presented using the Wolfenstein parametrization [11]. This representation involves six amplitudes $\mathcal{A},\ldots,\mathcal{F}$, being the coefficients of spin operators, which are scalar products of the Pauli spin vectors $\vec{\sigma}_i$ for the projectile and struck nucleons with a set of unit vectors defined by the scattering plane of the nucleon pair. This representation is not convenient in cases, such as in inelastic scattering, where one needs to account fully for the the spin dependence of the NN interaction [12].

Equivalently, the amplitude can be expressed in central, spin-orbit, and the usual tensor component S_{12} [13]. Alternatively, the NN transition amplitude can also be expressed in terms of irreducible tensor operators in the space of spin S $(=0,1)$ of the interacting pair, $\tau_{\kappa q}(S)$ [14]. This is, for instance, a convenient representation for analysis of deuteron scattering from spin-zero targets.

A more tractable representation for treating the spin dependence of the NN interaction in proton scattering analysis is presented here, in which the scattering amplitude is expressed in terms of spin tensor operators associated with the colliding particles.

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II. FORMALISM

Assuming that we use plane wave states, normalized such that

$$\langle \vec{r}|\vec{k}\rangle=(2\pi)^{-3/2}\exp(i\vec{k}\cdot\vec{r}), \tag{1}$$

then the free NN scattering amplitude $M(\omega,\vec{\mathcal{K}}',\vec{\mathcal{K}})$, describing scattering from two-nucleon states with relative momenta $\vec{\mathcal{K}}$ and $\vec{\mathcal{K}}'$ for relative energy ω in their center-of-mass (c.m.) frame, is related to the antisymmetrized transition matrix elements by

$$M(\omega,\vec{\mathcal{K}}',\vec{\mathcal{K}})=\langle\vec{\mathcal{K}}'|M(\omega)|\vec{\mathcal{K}}\rangle=-\frac{4\pi^2\mu}{\hbar^2}\langle\vec{\mathcal{K}}'|t^f_{01}(\omega)|\vec{\mathcal{K}}\rangle, \tag{2}$$

where μ is the NN reduced mass. These amplitudes are operators in both the NN spin and isospin spaces.

The Wolfenstein decomposition of the NN amplitude for the scattering of an incident (0) and struck (1) nucleon has been used extensively. It gives the most general form of the amplitude, consistent with time-reversal, parity, and rotational invariance, as

$$\begin{aligned}M(\omega,\vec{\mathcal{K}}',\vec{\mathcal{K}})=&\mathcal{A}+\mathcal{B}(\vec{\sigma}_0\cdot\hat{n})(\vec{\sigma}_1\cdot\hat{n})+\mathcal{C}(\vec{\sigma}_0+\vec{\sigma}_1)\cdot\hat{n}\\&+\mathcal{D}(\vec{\sigma}_0\cdot\hat{m})(\vec{\sigma}_1\cdot\hat{m})+\mathcal{E}(\vec{\sigma}_0\cdot\hat{l})(\vec{\sigma}_1\cdot\hat{l})\\&+\mathcal{F}[(\vec{\sigma}_0\cdot\hat{l})(\vec{\sigma}_1\cdot\hat{m})+(\vec{\sigma}_1\cdot\hat{m})(\vec{\sigma}_0\cdot\hat{l})],\end{aligned} \tag{3}$$

where the orthogonal set of unit vectors $\hat{n}=(\vec{\mathcal{K}}\times\vec{\mathcal{K}}')/|\vec{\mathcal{K}}\times\vec{\mathcal{K}}'|$, $\hat{l}=(\vec{\mathcal{K}}'+\vec{\mathcal{K}})/|\vec{\mathcal{K}}'+\vec{\mathcal{K}}|$, and $\hat{m}=\hat{l}\times\hat{n}$ are defined by the NN scattering plane [11]. The coefficient amplitudes $\mathcal{A},\ldots,\mathcal{F}$ can also be expressed as complex functions of ω, the momentum transfer $\vec{q}=(\vec{\mathcal{K}}'-\vec{\mathcal{K}})$, and the total momentum $\vec{\mathcal{Q}}=(\vec{\mathcal{K}}'+\vec{\mathcal{K}})/2$ of the NN pair in their c.m. frame. They remain operators in isotopic spin space, so for instance

$$\mathcal{A}(\omega,\vec{q},\vec{\mathcal{Q}})=\mathcal{A}_0+\mathcal{A}_\tau(\vec{\tau}_0\cdot\vec{\tau}_1)=\mathcal{A}^{T=0}P_0+\mathcal{A}^{T=1}P_1, \tag{4}$$

where the $\vec{\tau}_i$ are the isospin Pauli operators for the two nucleons and the P_T are projectors for the total isospin states. In this equation, $\mathcal{A}_0$ and $\mathcal{A}_\tau$ represent the isoscalar and isovector components of $\mathcal{A}$ in the isospin space.

The $\mathcal{A}, \ldots, \mathcal{F}$ amplitudes can be derived on and off the energy shell [1,2,6] from a realistic NN interaction, such as, for example, the Paris [9] and Bonn [10] interactions.

Equivalently, the NN scattering amplitude can be represented in terms of central, spin-orbit, and tensor components. The work of Franey and Love (FL) [13] makes use of this representation, where the form factors of the components are nonlocal, with a direct and an exchange term and given in terms of sums of Yukawa forms, the ranges and strenght of each term being determined by fitting the on-shell NN scattering data. The FL transition amplitude is usually referred as a pseudo-T matrix, since it is not constructed from a potential model and violates unitarity badly [15].

In the example of nucleon elastic scattering on a spin-zero target only the central $\mathcal{A}$ and spin-orbit $\mathcal{C}$ components contribute to the first order term of the multiple scattering expansion of the optical potential [3,4]. Second and higher order terms, however, involve contributions from all components [6]. Of course, for elastic scattering from a non-spin-zero nucleus, or for inelastic scattering involving spin-flip transitions of a struck nucleon in the target, a full treatment of the scattering amplitude needs to be considered. In these applications, approximate treatments need to be performed to handle the orthogonal set of unit vectors $n=(\vec{k}\times\vec{k}')/|\vec{k}\times\vec{k}'|$, $l=(\vec{k}'+\vec{k})/|\vec{k}'+\vec{k}|$, and $m=l\times n$ [6,17].

In elastic scattering processes involving a more general target, and in inelastic scattering, in which the full spin dependence of the interacting nucleons enters, the NN amplitudes are more conveniently constructed such that

$$\langle\vec{\mathcal{K}}'|M|\vec{\mathcal{K}}\rangle=\sum_{a\alpha\ b\beta} M^{(ab)}_{\alpha\beta}(\vec{\mathcal{K}}',\vec{\mathcal{K}})\tau_{a\alpha}(s_0)\tau_{b\beta}(s_1), \qquad (5)$$

where $\tau_{a\alpha}(s_0)$ is the irreducible tensor operator for the projectile particle (0) with spin $s_0(a=0,\ldots,2s_0)$; $\tau_{b\beta}(s_1)$ is the irreducible tensor operator for the struck particle (1) with spin s_1 $(b=0,\ldots,2s_1)$. Explicitly, since $s_0=s_1=\frac{1}{2}$,

$$\tau_{00}(\tfrac{1}{2})=1, \quad \tau_{1\beta}(\tfrac{1}{2})=\sigma_\beta(1), \qquad (6)$$

with $\sigma_\beta(1)$ the spherical components of $\vec{\sigma}_1$ with respect to the chosen z axis. It is understood that the $M^{(ab)}_{\alpha\beta}$ depend on the isospin of the two nucleons. The amplitudes relevant to the pp, pn, and nn cases are obtained from the isospin singlet $(T=0)$ and triplet $(T=1)$ amplitudes. The explicit dependence of the M and $M^{(ab)}_{\alpha\beta}$ on ω and T will not be shown in the following.

We first decompose the NN amplitude of Eq. (2) into spin singlet $(S=0)$ and triplet $(S=1)$ components, $M^S_{\nu'\nu}$, where ν and ν' refer to the incident and final state spin projections in state S,

$$\langle\vec{\mathcal{K}}'|M|\vec{\mathcal{K}}\rangle=\sum_{S\nu\nu'} M^S_{\nu'\nu}(\vec{\mathcal{K}}',\vec{\mathcal{K}})|S\nu'\rangle\langle S\nu|. \qquad (7)$$

These $M^S_{\nu'\nu}=\langle\vec{\mathcal{K}}'S\nu'|M|\vec{\mathcal{K}}S\nu\rangle$ are in any case calculated during the construction of the NN amplitudes from the partial wave transition amplitudes $M^{JS}_{L'L}(\mathcal{K}',\mathcal{K})$, e.g., Appendix C of [6]. We adopt the convention that

$$\langle\vec{\mathcal{K}}'|M|\vec{\mathcal{K}}\rangle=\frac{2}{\pi}\sum_{JLL'SM} i^{L-L'}\mathcal{Y}^M_{(L'S)J}(\hat{\mathcal{K}}')M^{JS}_{L'L}(\mathcal{K}',\mathcal{K}) \times\mathcal{Y}^{M\,\dagger}_{(LS)J}(\hat{\mathcal{K}}), \qquad (8)$$

where $\mathcal{Y}^M_{(LS)J}$ is a spin-angle function

$$\mathcal{Y}^M_{(LS)J}(\hat{\mathcal{K}}')=\sum_{\Lambda\nu}(L\Lambda S\nu|JM)Y_{L\Lambda}(\hat{\mathcal{K}}')\mathcal{X}_{S\nu}, \qquad (9)$$

and $Y_{L\Lambda}$ and $\mathcal{X}_{S\nu}$ are spherical harmonics [16] and total spinors of the NN pair. Combining Eq. (7) and Eq. (8) results

$$M^S_{\nu'\nu}(\vec{\mathcal{K}}',\vec{\mathcal{K}})=\frac{2}{\pi}\sum_{JMLL'\Lambda\Lambda'} i^{L-L'}(L'\Lambda'S\nu'|JM) \times(L\Lambda S\nu|JM)Y_{L'\Lambda'}(\hat{\mathcal{K}}')Y^*_{L\Lambda}(\hat{\mathcal{K}}) \times M^{JS}_{L'L}(\mathcal{K}',\mathcal{K}). \qquad (10)$$

The partial wave sums are, of course, over values that satisfy the Pauli principle requirement, i.e., $L+S+T$ be odd.

It is now convenient to reexpress the spin-space projector in terms of irreducible tensor operators in the space of spin S [14],

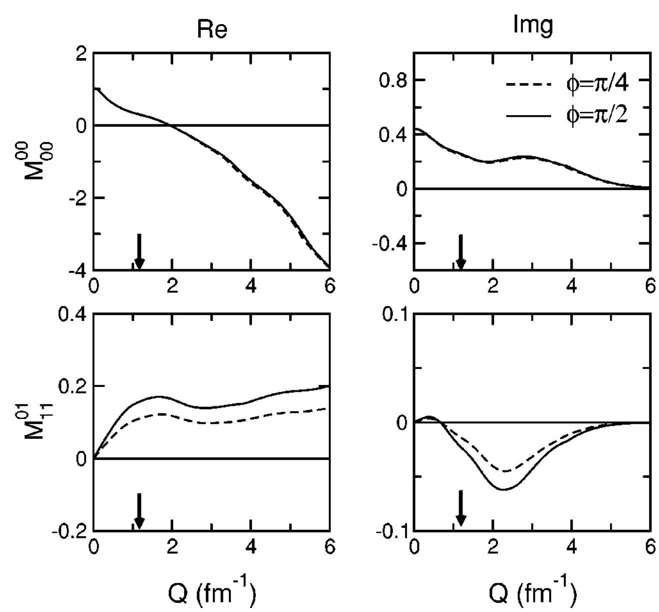

FIG. 1. Real and Imaginary parts of the Isoescalar components of M^{00}_{00} and M^{01}_{11}, at $E_{\rm Lab}=135$ MeV, as a function of the total momentum $\mathcal{Q}$, with $q=1$ fm^{-1}. The arrow indicates the on-shell value.

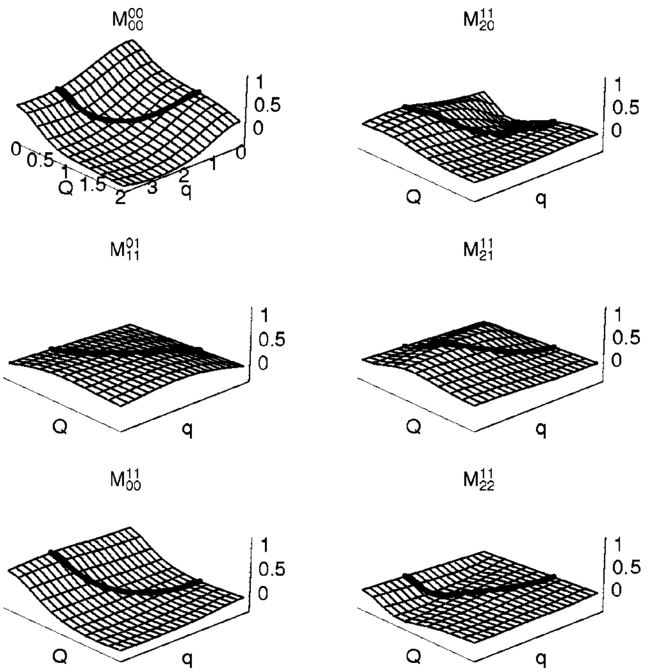

FIG. 2. Real part of the isoescalar components at $E_{\rm Lab}=135$ MeV.

$$\tau_{\kappa q}(S)=\sum_{\nu\nu'} \hat{\kappa}(S\nu\kappa q|S\nu')|S\nu'\rangle\langle S\nu|, \tag{11}$$

where $0\leqslant\kappa\leqslant 2S$ and $\hat{\kappa}=\sqrt{2\kappa+1}$ that satisfy $\tau^{\dagger}_{\kappa q}(S)=(-)^{q}\tau_{\kappa -q}(S)$, and so the spin tensor decomposition of the amplitude is

$$\langle\vec{\mathcal{K}}'|M|\vec{\mathcal{K}}\rangle=\sum_{Skq} M^{S}_{\kappa q}(\vec{\mathcal{K}}',\vec{\mathcal{K}})\tau^{\dagger}_{\kappa q}(S), \tag{12}$$

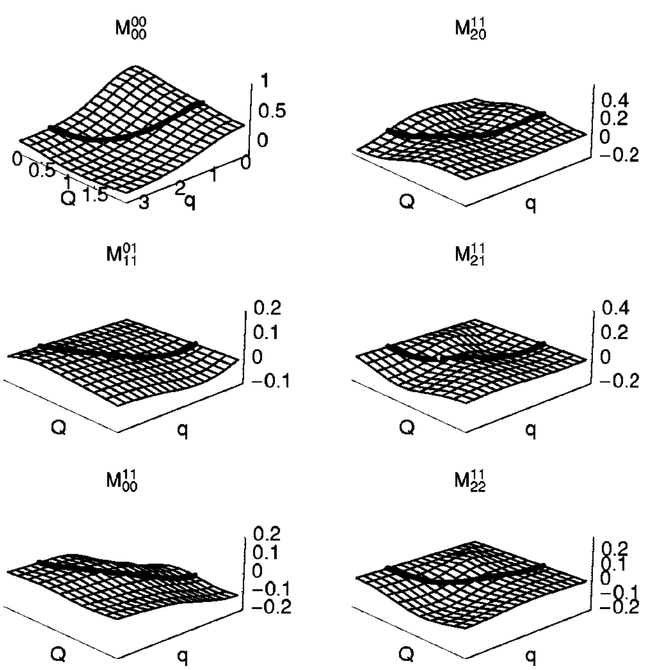

FIG. 3. Imaginary part of the isoescalar components at $E_{\rm Lab}=135$ MeV.

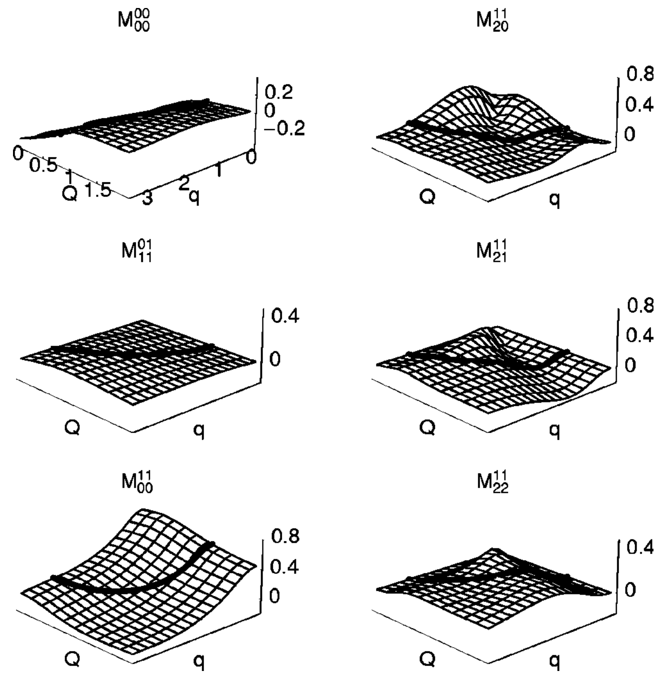

FIG. 4. Real part of the isovector components at $E_{\rm Lab}=135$ MeV.

where

$$M^{S}_{\kappa q}(\vec{\mathcal{K}}',\vec{\mathcal{K}})=\frac{\hat{\kappa}}{\hat{S}^{2}}\sum_{\nu\nu'} M^{S}_{\nu'\nu}(\vec{\mathcal{K}}',\vec{\mathcal{K}})(S\nu'\kappa q|S\nu). \tag{13}$$

It just remains to decompose the spin tensors for spin S in terms of those, $\tau_{b\beta}(s_i)$ $(i=0,1)$, of the incident (0) and struck (1) nucleons, as

$$\tau_{\kappa q}(S)=\sum_{ab}\frac{\hat{S}^{3}\hat{a}\hat{b}}{\hat{s}_0\hat{s}_1}\begin{Bmatrix} s_0 & s_1 & S\\ s_0 & s_1 & S\\ a & b & \kappa\end{Bmatrix}$$
$$\times\sum_{\alpha\beta}(a\alpha b\beta|\kappa q)\tau_{a\alpha}(s_0)\tau_{b\beta}(s_1). \tag{14}$$

This can be rewritten as

$$\tau_{\kappa q}(S)=\sum_{ab}\frac{\hat{S}^{3}\hat{a}\hat{b}}{\hat{s}_0\hat{s}_1}\begin{Bmatrix} s_0 & s_1 & S\\ s_0 & s_1 & S\\ a & b & \kappa\end{Bmatrix}\mathcal{T}_{\kappa q}(a,b), \tag{15}$$

where we have defined the new tensor operators

$$\mathcal{T}_{\kappa q}(a,b)=\sum_{\alpha\beta}(a\alpha b\beta|\kappa q)\tau_{a\alpha}(s_0)\tau_{b\beta}(s_1). \tag{16}$$

In terms of these,

$$\langle\vec{\mathcal{K}}'|M|\vec{\mathcal{K}}\rangle=\sum_{\kappa q a b} M^{(ab)}_{\kappa q}(\vec{\mathcal{K}}',\vec{\mathcal{K}})\mathcal{T}^{\dagger}_{\kappa q}(a,b), \tag{17}$$

where (a,b) refer to the ranks of the nucleon spin tensors, and

$$M_{\kappa q}^{(ab)}(\vec{\mathcal{K}}',\vec{\mathcal{K}})=\sum_{S} M_{\kappa q}^{S}(\vec{\mathcal{K}}',\vec{\mathcal{K}})\mathcal{N}_{\kappa}^{S}(ab) \tag{18}$$

with

$$\mathcal{N}_{\kappa}^{S}(ab)=\frac{\hat{S}^3\hat{a}\hat{b}}{\hat{s}_0\hat{s}_1}\begin{Bmatrix} s_0 & s_1 & S \\ s_0 & s_1 & S \\ a & b & \kappa \end{Bmatrix}. \tag{19}$$

The amplitudes $M_{\kappa q}^{(ab)}$ can be expressed in terms of the transferred momentum q, total momentum $\mathcal{Q}$, and the angle ϕ between these two vectors, that is, $M_{\kappa q}^{(ab)}(\omega,q,\mathcal{Q},\phi)$. On the energy shell, $\phi=\pi/2$ and $q^2/4+\mathcal{Q}^2=2\mu\omega/\hbar^2$.

For checking purposes, we note that $M_{00}^{(00)}=\mathcal{A}$ and $M_{11}^{(01)}=M_{11}^{(10)}=-i\mathcal{C}/\sqrt{2}$.

We need to consider the nonvanishing amplitudes $M_{00}^{(00)}$, $M_{11}^{(01)}=M_{11}^{(10)}$, $M_{00}^{(11)}$, $M_{20}^{(11)}$, $M_{21}^{(11)}$, and $M_{22}^{(11)}$. The amplitudes also satisfy $M_{\kappa-q}^{(ab)}=(-1)^{\kappa+q}M_{\kappa q}^{(ab)}$. All terms with $a=b=\kappa=1$ are seen to be zero as a result the vanishing of the 9-j coefficient. As a result of this geometric coefficient, the tensor $\mathcal{T}_{\kappa q}$ satisfies $\mathcal{T}_{\kappa q}^{+}=(-1)^q\mathcal{T}_{\kappa-q}$.

III. RESULTS

For illustrative purposes, we show in here the calculated off-shell amplitudes $M_{\kappa q}^{(ab)}$ at $E_{\mathrm{Lab}}=135$ MeV, making use of the Paris potential [9]. We use NNAMP [18], which calculates all the Wolfenstein and tensor representation amplitudes on and off the energy shell. The tensor amplitudes $M_{\kappa q}^{(ab)}$ are evaluated from the angular momentum amplitudes $M_{L'L}^{JS}$ following the procedure described in the text. These angular momentum amplitudes are obtained as Refs. [1,19]. For this particular example, we have used a maximum number of six partial waves.

We have verified that the tensor amplitudes do not vary strongly with energy and with the angle ϕ. From all the amplitudes, only M_{11}^{00} (and thus the Wolfenstein amplitude $\mathcal{C}$) represented in Fig. 1, shows a slight dependence on ϕ, the other amplitudes remaining fairly independent on this parameter. In this figure, the calculated amplitudes are evaluated at $q=1$ fm^{-1}, the solid and the dashed lines corresponding to $\phi=\pi/2$ and $\phi=\pi/4$, respectively. The arrow indicates the on-shell value.

In Figs. 2 and 3, we represent the real and imaginary parts of the isoescalar components in the isospin space. In Figs. 4 and 5 we show the corresponding isovector components. The amplitudes are represented as a function of the transferred momentum q and the total momentum $\mathcal{Q}$. The angle ϕ between these two vectors was taken to its on-shell value, $\phi=\pi/2$. The axis of quantization is chosen in the direction of the incident beam. The curve represented in each surface represents the corresponding on-shell value.

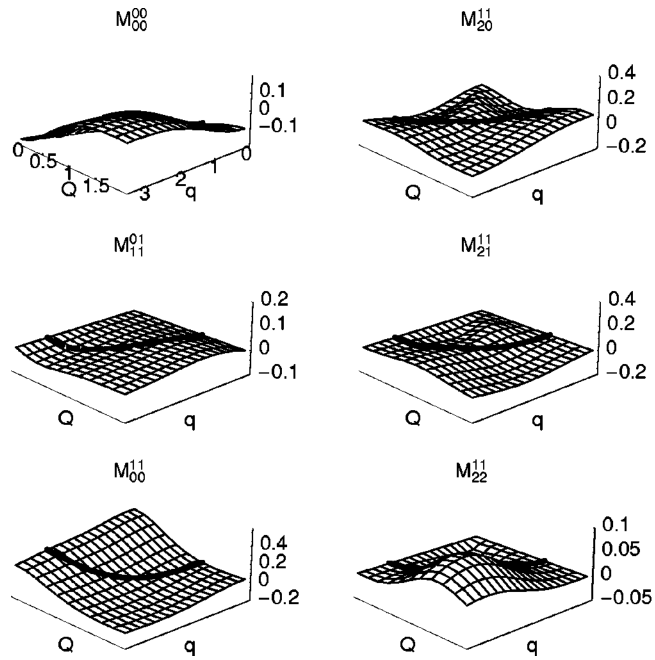

FIG. 5. Imaginary part of the Isovector components at $E_{\mathrm{Lab}}=135$ MeV.

It follows from the figures that the effect of the nonlocalities for the rank 0 components of the tensor representation, $M_{00}^{(00)}$ and $M_{(00)}^{11}$, might be significant when used in multiple scattering frameworks. The amplitudes in other reference systems can be readily obtained from these through rotation.

IV. CONCLUSION

In summary, we have described a convenient general method to express the NN transition amplitude as a linear combination for the spherical components of the spin operators of the two interacting particles. This is a more treatable representation to be used in multiple scattering formalisms that require a full treatment of the spin of the NN transition amplitude.

ACKNOWLEDGMENTS

The authors would like to thank J. A. Tostevin and R. C. Johnson for many useful discussions. This work was supported by Fundação para a Ciência e Tecnologia (Portugal) through Grant No. POCTI/1999/FIS/36282.

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