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# Microscopic framework for dynamical supersymmetry in nuclei

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We discuss and explore new aspects of the generalized Dyson mapping of nuclear collective superalgebras composed of an arbitrary fermion-pair algebra and a set of single-fermion creation/annihilation operators. It is shown that a direct consequence of the particular mapping procedure is the conservation of the total number of ideal particles in the resulting boson-fermion system. This provides a microscopic framework for the phenomenological supersymmetric models based on the $U(6/2\Omega)$ dynamical superalgebras. Attention is paid to the mapping of single-fermion creation and annihilation operators whose detailed form cannot be determined on the phenomenological level. We derive the general expansion of the single-fermion images that result from the similarity transformation employed to ensure nonredundant bosonization in the ideal space. The method is then illustrated in an application to the SO(4) collective algebra, a natural extension of the SU(2) seniority model.

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## I. INTRODUCTION

It is well documented that low-energy collective states in even-even nuclei can be successfully described using the interacting *s*-, *d*-boson models, the so-called IBM [1]. The dynamical groups of these models are bosonic unitary groups, either $U^{B}(6)$, or its extension $U^{B\pi}(6)\otimes U^{B\nu}(6)$ if proton and neutron collective degrees of freedom are to be treated separately. Similarly, the odd-*A* and odd-odd nuclei are described by the interacting boson-fermion models (IBFM) [2] with dynamical groups of the type $U^{B}(6)\otimes U^{F}(2\Omega)$ and $U^{B\pi}(6)\otimes U^{F\pi}(2\Omega_{\pi})\otimes U^{B\nu}(6)\otimes U^{F\nu}(2\Omega_{\nu})$, where $2\Omega$ is the capacity of the valence proton or neutron shell and $U^{F}(2\Omega)$ the corresponding fermionic unitary group. As a natural generalization of these approaches, the above product boson-fermionic groups can be embedded into the $U(6/2\Omega)$ or $U^{\pi}(6/2\Omega_{\pi})\otimes U^{\nu}(6/2\Omega_{\nu})$ supergroups [3–5]. The immediate consequence of this step is the possibility of a simultaneous description of a given even-even nucleus with its odd-*A* and odd-odd neighbors—a consequence that was recently verified experimentally [6–8] in the quartet of $^{194,195}$Pt and $^{195,196}$Au nuclei.

A crucially important question imposed by the wide-ranging success of all these phenomenological algebraic approaches [9] concerns their microscopic foundation. A variety of methods [10] have been developed with the aim to map the original fermionic problem of an even number of particles into the bosonic language. To understand the nature of the supersymmetric boson-fermion models, however, the mapping must be extended to cover also the odd-fermion degrees of freedom. In spite of numerous technical difficulties it seems now that a basic understanding arises about the link between the underlying fermionic interactions and those appearing on the boson-fermion level (see, e.g., Refs. [11–15]), as well as the reason why the $U(6/2\Omega)$-based dynamical supersymmetry is relevant in atomic nuclei [16–19]. A particularly promising approach, based on a superalgebraic extension of the so-called Dyson mapping of fermion algebras [10], was pioneered by Dobaczewski and co-workers see Refs. [17–19]. Among the main advantages of the technique proposed there belongs the direct relation of the resulting bosons to real fermionic pairs and the conservation of two-body character of the model Hamiltonian. The method achieves its nonredundant bosonization as a two-step process—no simpler construction is so far known—which requires the utilization of a particular similarity transformation and leads to typical Dyson-like non-Hermitian structures.

The aim of the present paper is to review and extend the main methodological aspects of the generalized Dyson mapping [17–19] of nuclear collective superalgebras, emphasizing those features that are directly related to phenomenological supersymmetric models. In particular, we show that the conservation of the total number of bosons plus fermions in phenomenological models is a direct consequence of the mapping procedure. We also discuss the general structure of single-fermion transfer operators that cannot be deduced in detail on the phenomenological level. Concrete new results are derived for the mapping of the SO(4) collective algebra that had been used in the first attempt [16] to investigate a possible microscopic justification for the phenomenological supersymmetry in nuclei.

The plan of the paper is as follows: Notation and the general collective superalgebra of fermion operators are introduced in Sec. II. In Secs. III and IV we sketch the method of the fermion-boson mapping and subsequent similarity transformations. The structure of the mapped Hamiltonian and properties of the single-fermion images are then discussed in Secs. V and VI. In Sec. VII we finally turn to some examples, based on a simplified single-*j* shell model, in particular, to the mapping of SU(2) and SO(4) collective superalgebras. We show how matrix elements for the single-particle transfer (relevant to the experimental identification of supersymmetry in nuclei [6–8]) can be calculated in the ideal boson-fermion space.

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## II. COLLECTIVE SUPERALGEBRA

Let us consider a fermionic system (atomic nucleus) with the Hilbert space generated by a successive application of a finite number of single-fermion creation operators $a^{\mu}$ to the physical vacuum $|0\rangle$. We assume $\mu=1\cdots 2\Omega$, enumerating the available single-particle states (their number is even due to the Kramers degeneracy) and denote the corresponding single-fermion annihilation operators as $a_{\mu}\equiv(a^{\mu})^{+}$. Any physical observable can then be obtained in terms of operators of the following form:

$$A^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}\equiv a^{\mu_1}\cdots a^{\mu_m}a_{\nu_1}\cdots a_{\nu_n}, \tag{1}$$

where $m,n=0,1,2\ldots$ As fermionic operators satisfy the familiar anticommutation relations,

$$\{a^{\mu},a_{\nu}\}=\delta^{\mu}_{\nu}, \quad \{a^{\mu},a^{\nu}\}=\{a_{\mu},a_{\nu}\}=0, \tag{2}$$

the algebraic structure formed by the operators (1) is clearly *not* an ordinary dynamical algebra [20]. However, by dividing the set of these operators to two subsets, *even* and *odd*, according to whether the difference $m-n$ of the number of creation and annihilation operators in $A^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n}$ is even or odd, respectively, we get the following schematic relations for commuting and anticommuting operators from the two sectors:

$$[\text{even},\ \text{even}]=\text{even},\quad [\text{even},\ \text{odd}]=\text{odd},$$

$$\{\text{odd},\ \text{odd}\}=\text{even}. \tag{3}$$

This means that the operators in Eq. (1) define a *superalgebra* [21,22]. Indeed, the mathematics of supergroups must be naturally involved in any fermionic many-body problem if treated in the algebraic framework [20].

The superalgebra of operators in Eq. (1) can be reduced to an ordinary dynamical algebra if one deals only with *even* numbers ($N$) of particles. Any initial state is then represented by an appropriate superposition of terms $A^{\mu_1\cdots\mu_N}|0\rangle$ and transition operators may only contain terms with $m-n=0,2,4\ldots$ (the many-body self Hamiltonian is composed of terms with $m-n=0$). Only the even sector of the set in Eq. (1) is thus invoked. Yet further crucial simplifications can be achieved if stronger restrictions are imposed on the dynamics of the system. In nuclear physics, the most important terms of the Hamiltonian and transition operators are often expressed via a certain limited set of fermion pairs, briefly *bifermions*. We assume these pairs being represented by the following most general creation operators:

$$A^{i}=\frac{1}{2}\chi^{i}_{\mu\nu}a^{\mu}a^{\nu} \tag{4}$$

($i=1\cdots M$; Greek indices occurring twice are subject to summation over the whole range from 1 to $2\Omega$), where the coefficients satisfy natural conditions $\chi^{i}_{\mu\nu}=-\chi^{i}_{\nu\mu}=(\chi^{\mu\nu}_{i})^{*}$. [The bifermion annihilation operators then read as $A_i\equiv(A^{i})^{+}=\frac{1}{2}\chi^{\mu\nu}_{i}a_{\nu}a_{\mu}$.] According to the concrete set of pairs we choose, the operators $A^{i}$ and $A_i$ belong to a particular *collective algebra* of the specific nuclear model. Note that although we usually assume that the dynamics selects a limited set of relevant pairs only, we can, in principle, consider all possible pairs, $M=\Omega(2\Omega-1)$, with $i\mapsto(\mu,\nu)$, $A^{\mu\nu}=a^{\mu}a^{\nu}$. The bifermion algebra is then identified with SO($2\Omega$) [17,18].

The bifermion states $A^{i}|0\rangle$ and $A^{j}|0\rangle$ are assumed to be orthogonal and normalized to a common factor,

$$\frac{1}{2}\chi^{\mu\nu}_{i}\chi^{j}_{\mu\nu}=g\,\delta^{j}_{i}, \tag{5}$$

as typically follows from a diagonalization of the two-particle problem. This condition ensures the closure relations for the collective algebra formed by the set of bifermion creation and annihilation operators, $A^{j}$ and $A_i$, and by their commutators,

$$[A_i,A^{j}]=g\,\delta^{j}_{i}-\chi^{j}_{\sigma\mu}\chi^{\sigma\nu}_{i}a^{\mu}a_{\nu}, \tag{6}$$

with $i,j=1\cdots M$. (Note that $[A^{i},A^{j}]=[A_i,A_j]=0$.) The closure relations read as follows (the summation convention is used for Latin indices):

$$[[A_i,A^{j}],A_k]=[[A^{i},A_j],A^{k}]^{+}=c^{jl}_{ik}A_l, \tag{7}$$

$$[[A_i,A^{j}],[A_k,A^{l}]]=c^{lj}_{km}[A_i,A^{m}]-c^{lm}_{ki}[A_m,A^{j}], \tag{8}$$

where the structure constants

$$c^{jl}_{ik}=\frac{1}{g}\chi^{\alpha\beta}_{i}\chi^{j}_{\beta\gamma}\chi^{\gamma\delta}_{k}\chi^{l}_{\delta\alpha} \tag{9}$$

satisfy symmetry relations

$$c^{jl}_{ik}=c^{jl}_{ki}=c^{lj}_{ik}=(c^{ik}_{jl})^{*}. \tag{10}$$

Note that according to Eq. (8) the commutators (6) form a core subalgebra of the collective algebra.

Of course, the use of the above collective algebra as the approximate nuclear dynamical algebra can only be possible for even nuclei. In the general case one has to consider also some odd operators. In the following, we keep the collective algebra of the above bifermion operators and extend it by considering the single-fermion creation and annihilation operators that give rise to single-fermion transfer operators between even and odd systems. The algebra of collective operators forms the even sector of the resulting superalgebra while the single-fermion creation and annihilation operators belong to the odd sector. Indeed, in agreement with the general superalgebraic rules (3), we have

$$[A^{i},a_{\mu}]=[a^{\mu},A_i]^{+}=\chi^{i}_{\nu\mu}a^{\nu}, \tag{11}$$

$$[A^{i},a^{\mu}]=[a_{\mu},A_i]^{+}=0, \tag{12}$$

$$[[A_i,A^{j}],a^{\mu}]=[a_{\mu},[A_j,A^{i}]]^{+}=\chi^{\mu\sigma}_{i}\chi^{j}_{\sigma\nu}a^{\nu}. \tag{13}$$

Equations (2), (7), (8), and (11)–(13) define the superalgebra subject to study in this paper. In fact, it is a combination of the above collective algebra with the Heisenberg-Weyl superalgebra [21]. We will call it the *collective superalgebra*.

### III. FERMION-BOSON MAPPING

The superalgebraic nature of a general fermionic many-body problem can be made explicit in terms of the usual bosonic and fermionic degrees of freedom by using fermion-boson mapping techniques [10]. In this way, the actual (real) fermionic Hilbert space is mapped onto an ''ideal'' space that describes a system of a certain number of *bosons* and so-called *ideal fermions*. To accomplish this task, a variety of different approaches have been employed in the literature. In this paper, we use a mapping technique that utilizes the so-called Usui operator [17,23,24], which is closely related to the use of coherent and supercoherent states [17].

The Usui operator $T$ acts on the *product space* $\mathbf{H}=\mathbf{H}_{\rm r}\otimes\mathbf{H}_{\rm i}$ of the real and ideal Hilbert spaces. It transforms any real state vector $|\psi\rangle\otimes|0)$ (containing the ideal vacuum) into a corresponding ideal state vector $|0\rangle\otimes|\psi)$ (with the real vacuum). If $P_0=|0\rangle\langle 0|\otimes 1$ and $\mathcal{P}_0=1\otimes|0)(0|$ are projectors onto the real and ideal vacua, respectively, one can define the *real* and *ideal subspaces* of the product Hilbert space as $\mathcal{P}_0\mathbf{H}$ and $P_0\mathbf{H}$ (they are isomorphic with the original spaces $\mathbf{H}_{\rm r}$ and $\mathbf{H}_{\rm i}$). (The rest of $\mathbf{H}$ is of no interest.) In the general case, the Usui operator does not have to map the real subspace onto the entire ideal subspace. The image of the real subspace, $T\mathcal{P}_0\mathbf{H}\subset P_0\mathbf{H}$, forms the *physical subspace* while the rest of the ideal subspace contains *spurious states*.

It seems reasonable to expect that any physically plausible mapping should conserve scalar products, i.e., must be unitary within the real and physical subspaces. We will see, however, that this condition can be relaxed without really loosing physical meaning of the mapping [10,25,26]. Let us consider the mapping of physical operators, $O\mapsto\overline{O}$, defined through the requirement $\overline{O}T=TO$, or equivalently

$$\overline{O}=TOT^{-1}, \tag{14}$$

where $\overline{O}\equiv 1\otimes\overline{O}$ is the ideal image of the real operator $O\equiv O\otimes 1$ and $T^{-1}$ is the inverse Usui operator in the physical subspace. It is clear that any set of operators within the real subspace is transformed into a set of images acting in the ideal subspace, all the algebraic relations (such as $AB=C$, $A+B=C$, $[A,B]=C$, $\{A,B\}=C\cdots$) being preserved in the physical subspace (or in the overlap of definition ranges of the operators involved with the physical subspace). If $T$ is nonunitary, the mapping does not preserve properties related to the Hermitian conjugation. In particular, ideal images of general physical operators will be non-Hermitian. Nevertheless, because $T^{-1}$ must exist within the physical subspace (this condition cannot be relaxed), all operator images remain isospectral with the respective real operators and the eigenvectors are related by $T$. Let us briefly recall that non-Hermitian operators have two sets of generally different eigenvectors, left and right: $\overline{O}|\psi_i^{\rm R})=o_i|\psi_i^{\rm R})$ and $(\psi_j^{\rm L}|\overline{O}=(\psi_j^{\rm L}|o_j$ and different eigenspaces are not orthogonal but *biorthogonal*: $(\psi_j^{\rm L}|\psi_i^{\rm R})=0$ for $o_j\neq o_i$. The nonunitarity of the mapping thus only induces the need to treat separately right and left images of the physical states according to the prescription

$$|\psi^{\rm R})=T|\psi\rangle,\quad (\psi^{\rm L}|=\langle\psi|T^{-1},$$
$$(\psi^{\rm R}|=\langle\psi|T^{+},\quad |\psi^{\rm L})=(T^{-1})^{+}|\psi\rangle. \tag{15}$$

Note the Hermitian conjugate of the Usui operator $T^{+}$ maps the physical subspace back to the real space, but it is not identical with $T^{-1}$ and, similarly, $(T^{-1})^{+}$ goes from the real to physical space but does not coincide with $T$.

It should be stressed that to keep the mapping procedure meaningful under these conditions, the proper distinction between the physical and spurious subspaces is essential. In fact, any operator that keeps the physical subspace invariant has an inverse image in the real subspace while there may be no real counterpart of operators acting within the entire ideal subspace.

Consider as the most trivial example a mapping that does nothing but renames particles. We start with a set of real fermions (created by $\{a^{\mu}\}_{\mu=1}^{2\Omega}$) and bosons (created by $\{b^{i}\}_{i=1}^{M}$) and wish to end with a set of ideal fermions ($\{\alpha^{\mu}\}_{\mu=1}^{2\Omega}$) and ideal bosons ($\{B^{i}\}_{i=1}^{M}$). The Usui operator then reads

$$T=P_0\exp(B^{i}b_i+\alpha^{\mu}a_{\mu})\mathcal{P}_0. \tag{16}$$

It is important to realize that the formal independence of physical and ideal particles translates into the fact that any boson operator commutes with all the other boson and fermion operators while the real- and ideal-fermion operators *anticommute* with each other. It is not difficult to see that under the operator in Eq. (16) any vector describing a state with fixed numbers of real particles of the given types transforms into a vector with the same numbers of the corresponding ideal particles. It means that the real subspace is mapped onto the entire ideal subspace, keeping all scalar products conserved and leaving no spurious states. The mapping (16) is thus unitary within the real and ideal subspaces while vectors orthogonal to the real subspace are annihilated by $T$. The operator mapping corresponding to Eq. (16) is trivial: $b^{i}\mapsto B^{i}, b_i\mapsto B_i, a^{\mu}\mapsto\alpha^{\mu}, a_{\mu}\mapsto\alpha_{\mu}$. This enables one to construct the ideal image of any real observable, conservation of the hermicity being guaranteed.

It is clear that the fermion-boson mapping we intend to perform is not as trivial as the mapping in the previous example. First, the role of physical bosons is not be played by some actual bosons but by fermion pairs from Eq. (4), whose annihilation and creation operators do not really commute in the bosonic way, see Eq. (6). Second, as bifermions are not independent of single fermions, their operators do not commute with fermion operators, see Eqs. (11) and (13). In spite of these difficulties, one still can keep the form of the Usui operator from the previous example,

$$T=P_0\exp(B^{i}A_i+\alpha^{\mu}a_{\mu})\mathcal{P}_0, \tag{17}$$

although some of its key properties differ from those discussed above. In particular, the spurious sector of the ideal subspace can no longer be avoided and $T$ ceases to be unitary even within the real and physical subspaces. In fact, the actual justification of Eq. (17) comes from the use of so-called supercoherent states [27] in both the real and ideal subspaces, $|C,\phi\rangle \equiv \exp(C_i A^i + \phi_\mu a^\mu)|0\rangle$ and $|C,\phi) \equiv \exp(C_i B^i + \phi_\mu \alpha^\mu)|0)$ (where $C_i$ and $\phi_\mu$ are complex and Grassman variables, respectively). Any state $|\psi\rangle$ in the real space can be represented by a function $f_\psi(C,\phi) \equiv \langle C,\phi|\psi\rangle$ and similarly any $|\psi)$ in the ideal space yields $g_\psi(C,\phi) \equiv (C,\phi|\psi)$. It can be shown that the Usui operator from Eq. (17) conserves functional representations of the associated real and ideal states [17]. [It should be stressed that not every function $f(C,\phi)$ represents a real state $|\psi\rangle$ and those functional representations $g(C,\phi)$ in the ideal space that have no counterpart in the real space constitute the spurious sector. As the real and ideal supercoherent states span the whole real and ideal spaces, respectively, $T$ does not map the real and ideal supercoherent states to each other.]

Using the Baker-Campbell-Hausdorf formulas for commuting the physical operators through the exponential in Eq. (17), one can derive the following operator mapping [17]:

$$A^i \mapsto \mathcal{A}^i + gB^i - \frac{1}{2} c^{ik}_{jl} B^j B^l B_k - \chi^i_{\mu\sigma} \chi^{\nu\sigma}_j B^j \alpha^\mu \alpha_\nu$$

$$= \mathcal{A}^i + [\mathcal{A}_j, \mathcal{A}^i] B^j - \frac{1}{2} c^{ik}_{jl} B^j B^l B_k, \tag{18}$$

$$A_i \mapsto B_i, \tag{19}$$

$$a^\mu \mapsto \alpha^\mu + \chi^{\mu\nu}_j B^j \alpha_\nu = \alpha^\mu + [\mathcal{A}_j, \alpha^\mu] B^j, \tag{20}$$

$$a_\mu \mapsto \alpha_\mu \tag{21}$$

($i=1,\ldots,M$ and $\mu=1,\ldots,2\Omega$). Here we introduced ideal-bifermion operators $\mathcal{A}^i = \frac{1}{2}\chi^i_{\mu\nu}\alpha^\mu\alpha^\nu$. Note that—in agreement with the above discussion—the ideal images of real creation and annihilation operators are not Hermitian conjugated. The mapping is nonunitary. In fact, the ideal-fermion and boson creation operators, $\alpha^\mu$ and $B^i$, have no inverse image in the real subspace (the existence of these inverse images, $X$, would require the fulfillment of the contradictory relations $\overline{X}\mathcal{P}_0 = \mathcal{P}_0 X$ with $\overline{X} = B^i$ or $\alpha^\mu$). Formulas (18) and (19) can be compared to those derived by mapping only the collective algebra without the odd sector [28]. It turns out that the reduced Usui operator $T = P_0 \exp(B^i A_i)\mathcal{P}_0$ leads to exactly the same images of the bifermion operators $A^i$ and $A_i$ except that terms associated with the ideal fermions are missing.

To complete the mapping of the whole fermionic superalgebra, we need also images of the commutators of the bifermion operators. These are given by the following formula:

$$[A_i, A^j] \mapsto g\delta^j_i - c^{jl}_{ik} B^k B_l - \chi^j_{\mu\sigma}\chi^{\nu\sigma}_i \alpha^\mu \alpha_\nu$$

$$= [\mathcal{A}_i, \mathcal{A}^j] - c^{jl}_{ik} B^k B_l, \tag{22}$$

which can be obtained either by mapping the right-hand side (rhs) of Eq. (6), or—in a simpler way—by commuting the images of bifermion operators in Eqs. (18) and (19). Also the anticommutation relations of real and mapped single-fermion operators are identical. This means that Eqs. (18)–(22) indeed define an equivalent ideal boson-fermion realization of the real-fermion superalgebra from Sec. II.

## IV. SIMILARITY TRANSFORMATIONS

### A. Hermitization

It was stressed above that the mapping in Eqs. (18)–(22) is not unitary so that the ideal images of real observables are generally non-Hermitian. On the other hand, we know that within the physical subspace the spectra of these images are real valued, identical with the spectra of physical operators. For any particular physical ideal-image operator $\overline{O}$ it should, therefore, be possible to find a similarity transformation $\overline{O} \mapsto \overline{O}' = S_{\rm H}\overline{O}S_{\rm H}^{-1}$ such that $\overline{O}'$ is Hermitian. If $|\psi^{\rm R}_i)$ and $(\psi^{\rm L}_j|$ are sets of right and left eigenvectors of $\overline{O}$, the operator $S_{\rm H}$ must satisfy $(\psi^{\rm L}_j|S^+_{\rm H}S_{\rm H}|\psi^{\rm R}_i) = \delta_{ji}$, or, equivalently, $S^+_{\rm H}S_{\rm H}\overline{O} = \overline{O}^+ S^+_{\rm H}S_{\rm H}$. Indeed, one can take, for example, $S_{\rm H} = (TT^+)^{-1/2}$, where $T^+ = \mathcal{P}_0 \exp(A^i B_i + a^\mu \alpha_\mu) P_0$ [cf. Eq. (17)]. However, as shown by Kim and Vincent [29], it is often favorable to exploit the ambiguity of the hermitization transformation—it is determined up to an arbitrary unitary transformation—to set constraints upon the image of one of the observables, e.g., the Hamiltonian $\overline{H}$. Namely, if $S_{\rm H0}$ hermitizes the Hamiltonian, then

$$S_{\rm H} = (S_{\rm H0} T T^+ S^+_{\rm H0})^{-1/2} S_{\rm H0} \tag{23}$$

hermitizes, within the physical subspace, all physical observables, while retaining the prescribed form of the Hamiltonian, $S_{\rm H}\overline{H}S_{\rm H}^{-1} = S_{\rm H0}\overline{H}S_{\rm H0}^{-1}$. This is very important since we naturally require that the hermitization does not spoil some important features of the mapped Hamiltonian, for instance, its one- plus two-body character.

Hermitization transformations preserving the two-body character of the Hamiltonian were indeed described in some particular cases [29], but no general algorithm is known. One direct approach is simply to guess the desired Hermitian operator $\overline{H}'$ isospectral with $\overline{H}$ and to construct a consistent similarity transformation. This is possible, under some specific conditions, using the following expression:

$$S_{\rm H0}^{-1} = \sum_{k=0}^{\infty} \left( \frac{1}{\hat{C} - C} P \right)^k_{\wedge}, \tag{24}$$

where $P = \overline{H} - \overline{H}'$ and $C$ is any operator satisfying $[C,P] = [\overline{H}', P]$. In Eq. (24) we introduce the notation in which the mark "$_\wedge$" indicates the position where the operator with hat (the first $C$) is to be evaluated. The derivation of this formula and the positional operator formalism are sketched in Appendix. It is important to stress that Eq. (24) holds true only for a nondegenerate spectrum of $\overline{H}$, while otherwise divergence problems can be encountered.

In the majority of cases it is difficult (if not impossible) to derive explicit expressions for the hermitized images of physical operators using the general transformation in Eq. (23). At first this difficulty seems to put serious restrictions on the use of the mapping technique described above. Fortunately, the calculation of *matrix elements* of physical operators can be performed without really knowing the hermitized images in the operator form, by using the obvious identity $\langle \psi_1|O|\psi_2\rangle = (\psi_1^{\mathrm{L}}|\overline{O}|\psi_2^{\mathrm{R}})$ or its modification

$$\langle \psi_1|O|\psi_2\rangle = \sqrt{(\psi_1^{\mathrm{L}}|\overline{O}|\psi_2^{\mathrm{R}})(\psi_2^{\mathrm{L}}|\overline{O^+}|\psi_1^{\mathrm{R}})^*}, \tag{25}$$

that both directly result from Eqs. (14) and (15) (the second identity is usually favored in practical calculations as we will see in Sec. VII D). The evaluation of the hermitization transformation is turned here into another nontrivial task—finding the left and right images of general physical states. However, this can already be accomplished for certain sets of states, namely those generated by some creation operators from the real vacuum, i.e., for states having the form $|\psi\rangle = X^+|0\rangle$ (where $X^+$ represents, e.g., a sequence of single-fermion and/or bifermion creation operators). Then one can write $|\psi^{\mathrm{R}}) = \overline{X^+}|0)$ and $(\psi^{\mathrm{L}}| = (0|\overline{X}$ with $|0)\equiv|0^{\mathrm{R}}) = |0^{\mathrm{L}})$. In this way, one can evaluate—using only the ideal images of state vectors and operators—the complete set of matrix elements of the given real operator in an appropriate real basis (the single-particle basis, for instance). The goal of the mapping can thus be achieved [26,30,31].

### B. Bosonization

The necessity for a similarity transformation following the mapping described in the Sec. III appears even before considering the hermitization problem. This is evident from Eq. (18) where the ideal image of the real pair creation operator contains the ideal pair creation operator, $A^i \mapsto \mathcal{A}^i + R^i$ with $R^i = gB^i - \frac{1}{2}c_{jl}^{ik}B^jB^lB_k - \chi_{\mu\sigma}^i\chi_j^{\nu\sigma}B^j\alpha^\mu\alpha_\nu$. While all terms in $R^i$ translate the creation of a bifermion in the real space into the creation of a boson in the ideal space (this can be accompanied by an interaction with another boson or fermion), $\mathcal{A}^i$ just introduces an equivalent ideal fermion pair. The real pairs are thus not truly bosonized by the mapping. In particular, the real-bifermion state $A^i|0\rangle$ is transformed into a superposition of ideal-bifermion and boson states, $(\mathcal{A}^i + gB^i)|0)$, and real fermion-fermion interactions are exactly transmitted to the ideal Hamiltonian, where the additional boson and boson-fermion terms (see Sec. V) only obscure the original problem.

This difficulty can be again overcome with the aid of the formalism sketched in Appendix. Indeed, when considering the operator $A^jA_j$ and its image $\mathcal{A}^jB_j + R^jB_j$, we see that the unwanted part containing $\mathcal{A}^j$ does not affect the spectrum of the image. This follows from the fact that while the $R^jB_j$ operator is diagonal in the basis characterized by numbers of ideal bosons and fermions, the $\mathcal{A}^jB_j$ term has an upper off-diagonal block structure in the same basis. We, therefore, anticipate the existence of a similarity transformation $S_{\mathrm{B}}$ with the following properties:

$$S_{\mathrm{B}}(\mathcal{A}^i + R^i)S_{\mathrm{B}}^{-1} = R^i, \tag{26}$$

$$S_{\mathrm{B}}B_iS_{\mathrm{B}}^{-1} = B_i. \tag{27}$$

The form of $S_{\mathrm{B}}^{-1}$ is given by Eq. (A2) in the Appendix with $O' = R^jB_j$ and $P = \mathcal{A}^jB_j$. However, it can be shown [18] that $[O',P] = [-\mathcal{C}_{\mathrm{F}},P]$, where $\mathcal{C}_{\mathrm{F}} = \mathcal{A}^l\mathcal{A}_l$ is the Casimir operator of the ideal fermion core algebra, $[\mathcal{C}_{\mathrm{F}},[\mathcal{A}_i,\mathcal{A}^j]] = 0$, conserving the total number of ideal fermions, $\mathcal{N} = \alpha^\mu\alpha_\mu$. In agreement with Eqs. (24) and (A6) we thus arrive at

$$S_{\mathrm{B}}^{-1} = \sum_{k=0}^{k_{\max}}\left(\frac{1}{\mathcal{C}_{\mathrm{F}} - \hat{\mathcal{C}}_{\mathrm{F}}}\mathcal{A}^jB_j\right)^k_{\wedge} = \exp\left[\frac{\mathcal{N} - \hat{\mathcal{N}}}{2(\mathcal{C}_{\mathrm{F}} - \hat{\mathcal{C}}_{\mathrm{F}})}\mathcal{A}^jB_j\right]_{\wedge}, \tag{28}$$

where the upper bound of the sum, $k_{\max}$, reflects the finiteness of the fermionic space. It is clear that $k_{\max} \leqslant \Omega$ and that the real cutoff depends on the numbers $\mathcal{N}$ and $N_{\mathrm{B}}$ of the ideal fermions and bosons present in the state to be transformed (in this way also the higher-order terms in expansion of the exponential naturally vanish). Let us stress again that due to the limitations mentioned above and in Appendix, there is no general guarantee that Eq. (28) converges. This is further illustrated in Sec. VII, where the convergence requirement will set some limits upon the states to be transformed.

In order to obtain the transformed images of general physical operators, we also need to determine the form of the inverse similarity transformation $S_{\mathrm{B}}$. As the expansion of $S_{\mathrm{B}}^{-1}$ in Eq. (28) consists of terms which increase the number of ideal fermions by $\Delta\mathcal{N} = 2k = 0, +2, +4, +6\cdots$, the same must hold true also for the $S_{\mathrm{B}}$. If we define

$$S_k = \left(\frac{1}{\mathcal{C}_{\mathrm{F}} - \hat{\mathcal{C}}_{\mathrm{F}}}\mathcal{A}^jB_j\right)^k_{\wedge}, \tag{29}$$

i.e., if we rewrite Eq. (28) as

$$S_{\mathrm{B}}^{-1} = 1 + S_1 + S_2 + S_3 + \cdots, \tag{30}$$

we find that

$$S_{\mathrm{B}} = 1 + \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 + \cdots \tag{31}$$

with

$$\tilde{S}_k = \sum_{n=1}^{k}(-)^n\sum_{k_1+k_2+\cdots+k_n=k}S_{k_1}S_{k_2}\cdots S_{k_n}. \tag{32}$$

In particular, $\tilde{S}_1 = -S_1$, $\tilde{S}_2 = -S_2 + S_1^2$, ... , cf. Ref. [18]. These expressions enable one to evaluate the similarity transformation $S_{\mathrm{B}}XS_{\mathrm{B}}^{-1}$ of an arbitrary operator $X$, which changes the number of ideal fermions by a specific value $\Delta\mathcal{N}$, as a series where individual terms correspond to $\Delta\mathcal{N}$, $\Delta\mathcal{N}+2$, $\Delta\mathcal{N}+4$, etc. We use these expansions in Sec. VI when discussing the general form of transformed single-fermion images.

## V. CONSERVATION OF THE NUMBER OF IDEAL PARTICLES

One of the most interesting questions immediately arising from the previous considerations concerns the link to superalgebras of the type $U(M/2\Omega)$ known from phenomenological boson-fermion models of nuclear structure [3–5,9]. The use of these dynamical superalgebras on the phenomenological level is motivated by the fact that they provide a direct generalization of the unitary bosonic and fermionic algebras that proved to be relevant and successful in the description of collective states in both even and odd (odd-$A$ or odd-odd) nuclei [1,2,9]. In fact, generators of the proton-neutron superalgebra $U^{\pi}(6/2\Omega_{\pi})\otimes U^{\nu}(6/2\Omega_{\nu})$ produce a class of related Hamiltonians that seems general enough to simultaneously describe low-energy spectra in quartets of nuclei whose nucleon (proton and/or neutron) numbers differ by one [5,9].

The key feature of the $U(M/2\Omega)$ superalgebras is that their generators conserve the total number of bosons plus fermions, $N_{BF}=N_B+\mathcal{N}$ (where $N_B=B^iB_i$). We thus enquire whether this also holds for the Hamiltonian mapped from a microscopic real-fermion Hamiltonian. Let us stress that this property cannot be deduced from the conservation of the number of real fermions $N$ by the original nuclear Hamiltonian since $N$ corresponds to $2N_B+\mathcal{N}$ on the boson-fermion level, as dictated by the fermion-boson mapping [Eq. (54) below]. It is nevertheless clear from Eqs. (18)–(22) that any fermionic many-body Hamiltonian composed of operators belonging to the collective algebra, e.g.,

$$H=u+v^j_iA^iA_j+w^i_j[A_i,A^j]=(u+gw^i_i)-(\chi^j_{\mu\sigma}\chi^{\nu\sigma}_i w^i_j)a^{\mu}a_{\nu}+\left(\frac{1}{4}\chi^i_{\mu\nu}\chi^{\pi\sigma}_j v^j_i\right)a^{\mu}a^{\nu}a_{\sigma}a_{\pi} \quad (33)$$

[where $v^j_i=(v^i_j)^*$ and $w^j_i=(w^i_j)^*$ are arbitrary coefficients associated with two- and one-body interactions, respectively, and $u=u^*$ is an additive constant], is mapped onto an ideal Hamiltonian that indeed conserves the total number of ideal particles. This conclusion remains unchanged even after the similarity transformation in Eqs. (26) and (27). The resulting Hamiltonian keeps the same ideal-fermion mean field as the original real Hamiltonian (33), but the fermion-fermion interaction is replaced by boson-involving terms that describe a boson mean field and boson-boson plus boson-fermion interactions

$$H\mapsto(u+gw^i_i)-(\chi^j_{\mu\sigma}\chi^{\nu\sigma}_i w^i_j)\alpha^{\mu}\alpha_{\nu}+(gv^j_i-c^{jk}_{il}w^l_k)B^iB_j-\left(\frac{1}{2}c^{mk}_{ij}v^l_m\right)B^iB^jB_kB_l-(\chi^k_{\mu\sigma}\chi^{\nu\sigma}_i v^j_k)B^i\alpha^{\mu}B_j\alpha_{\nu}. \quad (34)$$

The image in Eq. (34) is still non-Hermitian in both interaction terms, but, as discussed in Sec. IV A, the similarity transformation $S_H$ can be chosen such that it does not affect the particle number conservation. (In the matrix representation connected with fixed particle numbers the Hamiltonian has a block-diagonal structure that can be preserved by a suitably selected transformation $S_{H0}$.) This result holds true for *any* collective algebra we decide to start with.

The image (34) conserves numbers of bosons and fermions *separately*, which is the structure known from the interacting boson-fermion model [2]. It seems, therefore, that $U^B(M)\otimes U^F(2\Omega)$ could be equally well chosen as the dynamical algebra on the phenomenological level instead of $U(M/2\Omega)$. Note that the choice of the phenomenological dynamical algebra (superalgebra) is more or less a matter of convenience; it certainly does not result from the mapping procedure that only constructs a boson-fermion realization of the original collective superalgebra. If, nevertheless, the $U(M/2\Omega)$ dynamical superalgebra is employed, it must be decomposed into the above product of boson and fermion unitary algebras in the very first step of any relevant dynamical-symmetry chain. This indeed happens in the phenomenological model [5] used to analyze experimental data [6–8]. From this point of view, the hitherto discussed supersymmetric description of neighboring even-even, odd-odd, and odd-$A$ nuclei relies just on the use of the IBFM with a single set of parameters, which is a natural expectation based on the mapping of the same microscopic Hamiltonian (33) acting on spaces with various real-fermion numbers (see also Ref. [19] in this regard). At the same time we note that the above considerations are not in contradiction with the recently proposed possibility [32–34] that a $U(n/m)\subset U(M/2\Omega)$ supergroup may in fact constitute a real invariance symmetry of the nuclear Hamiltonian, without reference to an underlying dynamical symmetry, giving rise to boson-fermion "supermultiplets" in neighboring nuclei.

We conclude this section by the remark that *general* one-plus two-body Hamiltonians (for instance, those containing general single-particle terms $\varepsilon^{\nu}_{\mu}a^{\mu}a_{\nu}$) do *not* have to conserve the number of ideal particles after the mapping. This probably misled the authors of Ref. [16] who ascribed the conservation property to only the Schwinger type of mapping while it was alleged to fail for mappings that associate bosons with fermion pairs. However, from the above discussion we see that the ideal-particle number *is* indeed conserved in the generalized Dyson mapping as far as the mapped collective algebra represents the dynamical algebra of the fermionic Hamiltonian [for instance, if the single-particle terms are given only by the commutators in Eq. (6)]. An enquiry about the most general set of fermion Hamiltonians (beyond the preselected dynamical algebra) that conserve $N_{BF}$ after the mapping is hampered by the following difficulties: (i) the ideal image of the Hamiltonian does not have to commute with $N_{BF}$ in the whole ideal space, but only in the physical subspace, and (ii) there are no obvious candidates for fermion space counterparts to the observables associated with $N_B$ and $\mathcal{N}$.

## VI. SINGLE-FERMION IMAGES

While the action of the similarity transformation (28) on the bifermion images is by construction guaranteed to yield the compact results (26) and (27), the expressions for transformed single-fermion images can only be determined in an expanded form, using Eqs. (29)–(32). Denoting the "bare" single-fermion images appearing in Eqs. (20) or (21) by $X$, one can write

$$S_{\mathrm{B}}XS_{\mathrm{B}}^{-1}=X+X_1+X_2+X_3+\cdots, \tag{35}$$

where individual terms are determined by

$$X_k=[X,S_k]+\sum_{n=2}^{k}(-)^{n-1}\times\sum_{k_1+k_2+\cdots+k_n=k}S_{k_1}S_{k_2}\cdots S_{k_{n-1}}[X,S_{k_n}], \tag{36}$$

or recursively from

$$X_1=[X,S_1],\quad X_2=[X,S_2]-S_1X_1,\ \ldots,$$
$$X_k=[X,S_k]-S_1X_{k-1}-S_2X_{k-2}-\cdots-S_{k-1}X_1. \tag{37}$$

In the expressions above $X$ can, in principle, be any physical operator. With $X=\alpha_\mu$ the term $X_k$ changes the number of ideal fermions by $\Delta\mathcal{N}=2k-1$, while with $X=\alpha^\mu+\chi_j^{\mu\nu}B^j\alpha_\nu\equiv X'+X''$ we have $\Delta\mathcal{N}=2k+1$ for $X_k'$ and $\Delta\mathcal{N}=2k-1$ for $X_k''$. The transformed images of both annihilation and creation operators thus contain terms with $\Delta\mathcal{N}=-1,+1,+3,+5\cdots$ [18].

The series (35) for transformed single-fermion images comprises (i) the operators contained in the bare single-fermion images, i.e., $\alpha_\mu$ or $\alpha^\mu$ and $\chi_i^{\mu\nu}B^i\alpha_\nu$, and (ii) those in the similarity transformations $S_{\mathrm{B}}^{-1}$ and $S_{\mathrm{B}}$, i.e., $\mathcal{A}^iB_i$ and $(\mathcal{C}_{\mathrm{F}}-\hat{\mathcal{C}}_{\mathrm{F}})$. In any term of the series, there can be only one operator from (i) and an arbitrary combination (no restrictions to multiplicity) of operators from (ii). To determine the physical interpretation of these expressions by inspection, we first commute $(\mathcal{C}_{\mathrm{F}}-\hat{\mathcal{C}}_{\mathrm{F}})$ from all places of its occurrence to the respective positional marks and then to the right-hand side in all terms of the series. It then turns out that the resulting formulas can be decomposed into building blocks representing some elementary processes: (a) Processes corresponding to the ideal-fermion creation,

$$[\mathcal{C}_{\mathrm{F}},[\mathcal{C}_{\mathrm{F}}\cdots[\mathcal{C}_{\mathrm{F}},\alpha^\mu]\cdots]\cdots]_n,$$
$$[\mathcal{C}_{\mathrm{F}},[\mathcal{C}_{\mathrm{F}}\cdots[\mathcal{C}_{\mathrm{F}},\chi_i^{\mu\nu}B^i\alpha_\nu]\cdots]\cdots]_n\quad n=0,1,2\ldots. \tag{38}$$

Here, the $n=0$ terms emerge as just $\alpha^\mu$ and $\chi_i^{\mu\nu}B^i\alpha_\nu$, the $n=1$ terms as $\chi_i^{\mu\nu}\mathcal{A}^i\alpha_\nu$ and $\chi_i^{\mu\nu}\chi_{\sigma\nu}^j\alpha^\sigma\mathcal{A}_jB^i$, etc. These expressions can be interpreted as processes that encompass single-fermion creation, coupling of the created fermion into a pair, and the bifermion–boson transformations (all fermions of course being of the ideal type). (b) Processes corresponding the ideal-fermion annihilation,

$$[\mathcal{C}_{\mathrm{F}},[\mathcal{C}_{\mathrm{F}}\cdots[\mathcal{C}_{\mathrm{F}},\alpha_\mu]\cdots]\cdots]_n\quad n=0,1,2\ldots, \tag{39}$$

which appear just as the Hermitian conjugate of the first commutator in Eq. (38) and receive analogous diagrammatic interpretations. (c) Background processes accompanying (a) and (b),

$$[\mathcal{C}_{\mathrm{F}},[\mathcal{C}_{\mathrm{F}}\cdots[\mathcal{C}_{\mathrm{F}},\mathcal{A}^iB_i]\cdots]\cdots]_n\quad n=0,1,2\ldots, \tag{40}$$

that denote various forms of the boson decomposition into ideal fermions and bifermions; for $n=0$ we have just $\mathcal{A}^iB_i$, while the $n=1$ term is $\chi_{\nu\pi}^i\chi_j^{\nu\sigma}\mathcal{A}^j\alpha^\pi\alpha_\sigma B_i$, etc. While expressions from Eq. (38) or (39) Appear only once in each term of the series, those from Eq. (40) generally have a multiple occurrence.

We have already seen that the most general transformed images of the single-fermion creation and annihilation operators contain terms that change the number of ideal fermions by $\Delta\mathcal{N}=-1,+1,+3,+5\cdots$. In view of the elementary processes (a)–(c), the actual value of $\Delta\mathcal{N}$ in a given term is determined by the number of repetitions of the background processes (40). Relative weights appearing with increasing $\Delta\mathcal{N}$ are expected to decrease according to the increasing power of the denominator in Eq. (29) [cf. Eqs. (65) and (69) below]. In addition, terms corresponding to large $\Delta\mathcal{N}$ are not likely to play a significant role in matrix elements for low-energy nuclear states as the decomposition of bosons into separate noncollective fermions is associated with higher energy excitations. As argued in Ref. [18], it may be plausible to cut off the terms with $\Delta\mathcal{N}\geqslant+3$. We will see in Sec. VII that for some algebras these terms can vanish identically. Even with the restriction $\Delta\mathcal{N}\leqslant+1$, however, the most general formula built of terms from Eqs. (38)–(40) comprises an infinite series (terms with all $n$'s). The situation is much simplified if $\mathcal{C}_{\mathrm{F}}$ is just a function of $\mathcal{N}$ (or number operators associated with some fermionic subspaces). Then, evaluating only the $\Delta\mathcal{N}=\pm1$ terms, one gets

$$S_{\mathrm{B}}\alpha_\mu S_{\mathrm{B}}^{-1}=\alpha_\mu+\chi_{\mu\nu}^i\alpha^\nu B_i+\mathcal{A}^i\alpha_\mu B_iF(\mathcal{N})+\cdots, \tag{41}$$

$$S_{\mathrm{B}}(\alpha^\mu+\chi_i^{\mu\nu}B^i\alpha_\nu)S_{\mathrm{B}}^{-1}=\chi_i^{\mu\nu}B^i\alpha_\nu+\alpha^\mu+\chi_i^{\mu\nu}\chi_{\nu\sigma}^j\alpha^\sigma B^iB_j+\chi_i^{\mu\nu}\mathcal{A}^i\alpha_\nu F'(\mathcal{N})+\chi_i^{\mu\nu}\mathcal{A}^j\alpha_\nu B^iB_jF(\mathcal{N})+\cdots, \tag{42}$$

where $F(\mathcal{N})$ and $F'(\mathcal{N})$ are some functions of the ideal-fermion number(s) that are directly related to the form $\mathcal{C}_{\mathrm{F}}=f(\mathcal{N})$. This general result is illustrated by specific examples in Sec. VII C.

It is clear that even after the transformation (35) the images of real-fermion creation and annihilation operators are not Hermitian conjugated; cf. Eqs. (41) and (42). As the general hermitization transformation, Eq. (23), of the single-fermion images in an operator form seems intractable, at least in the general case, one has to turn to the evaluation of the single-fermion matrix elements in a specific basis by the method described in Sec. IV A, see Eq. (25). This step, of course, critically depends on the concrete form of single-fermion images after the bosonization transformation. Examples are given in Sec. VII D.

We conclude this section by noting that the general results discussed here may suggest expressions suitable for the determination of single-nucleon transfer amplitudes within the phenomenological superalgebraic models. In fact, on the phenomenological level only very general considerations, such as tensorial properties and effective particle number properties, relate to this question, as the transfer operators are beyond the $U(M/2\Omega)$ superalgebra. A link to microscopic models is thus essential. For instance, we can recall the formula fitted to experimental data in Refs. [6,7] and compare it with the most general form discussed above. We see immediately that although the image of the single-fermion creation operator given by Eq. (3) in Ref. [6] contains terms describing plausible processes with $\Delta\mathcal{N}=+1$, some other possibly relevant terms are missing.

## VII. EXAMPLES: SU(2) AND SO(4) MAPPINGS

### A. Definition of the algebras

In this section, we illustrate the general considerations of the previous sections by simple examples concerning fermions in a single shell with total angular momentum $j$ (half integer). Accordingly we consider a set of $2\Omega=2j+1$ single-particle states created by $a^{\mu}\equiv a^{\dagger}_{j\mu}$ with $\mu=-j\cdots+j$. We will deal with the simplest algebras based on these operators, namely, the SU(2) seniority algebra [35] and the extended SO(4) algebra [16,36].

In the SU(2) case we introduce only one type of fermion pair, namely,

$$A^1\equiv S^{\dagger}=\frac{1}{2}\sum_{\mu}(-)^{j-\mu}a^{\mu}a^{-\mu}. \tag{43}$$

SO(4) contains the pair (43) and another one given by

$$A^2=\frac{1}{2}\left(\sum_{|\mu|\leqslant\Omega/2}(-)^{j-\mu}a^{\mu}a^{-\mu}-\sum_{|\mu|>\Omega/2}(-)^{j-\mu}a^{\mu}a^{-\mu}\right). \tag{44}$$

In the notation of Eq. (4) we can write

$$\chi^1_{\mu\nu}=\begin{cases}(-)^{j-\mu} & \text{for} \quad \nu=-\mu\\ 0 & \text{for} \quad \nu\neq-\mu,\end{cases} \tag{45}$$

$$\chi^2_{\mu\nu}=\begin{cases}(-)^{j-\mu} & \text{for} \quad \nu=-\mu,|\mu|\leqslant\Omega/2\\ -(-)^{j-\mu} & \text{for} \quad \nu=-\mu,|\mu|>\Omega/2\\ 0 & \text{for} \quad \nu\neq-\mu\end{cases} \tag{46}$$

and from Eq. (5) we get $g=\Omega$.

$A^1$ and $A_1$ together with the commutator $[A_1,A^1]=\Omega-N$ (where $N=a^{\mu}a_{\mu}$ is the real-fermion number operator) close the SU(2) algebra with the only structure constant $c^{11}_{11}=2$. In the extended case we introduce an operator

$$Q=\sigma^{\mu}_{\nu}a^{\nu}a_{\mu}, \tag{47}$$

where $\sigma^{\mu}_{\nu}=\pm\delta^{\mu}_{\nu}$ with the upper (lower) sign valid for $|\mu|\leqslant\Omega/2$ ($|\mu|>\Omega/2$). The bifermion operators together with $N$ and $Q$ then close the SO(4) algebra, $[A_1,A^1]=[A_2,A^2]=\Omega-N,[A_1,A^2]=-Q$, with the following structure constants:

$$c^{11}_{11}=c^{22}_{22}=c^{11}_{22}=c^{22}_{11}=c^{12}_{12}=c^{21}_{21}=c^{12}_{21}=c^{21}_{12}=2,$$
$$c^{12}_{11}=c^{21}_{22}=c^{12}_{22}=c^{21}_{11}=c^{11}_{12}=c^{22}_{21}=c^{11}_{21}=c^{22}_{12}=0. \tag{48}$$

It is evident that by dividing the fermionic space into two subspaces, the first one spanned by single-fermion states with $|\mu|\leqslant\Omega/2$ and the other by states with $|\mu|>\Omega/2$, the SO(4) algebra can be decomposed into a tensor product of two independent SU(2) algebras. Accordingly define the following transformation of bifermion operators:

$$A^{<}=\frac{1}{2}(A^1+A^2)=\frac{1}{2}\sum_{|\mu|\leqslant\Omega/2}(-)^{j-\mu}a^{\mu}a^{-\mu}, \tag{49}$$

$$A^{>}=\frac{1}{2}(A^1-A^2)=\frac{1}{2}\sum_{|\mu|>\Omega/2}(-)^{j-\mu}a^{\mu}a^{-\mu}. \tag{50}$$

Both $A^{<}$ and $A^{>}$ are just the $S^{\dagger}$-type bifermions in the respective subspaces, cf. Eq. (43). We have $[A_{<},A^{<}]=(\Omega/2-N_{<})$, $[A_{>},A^{>}]=(\Omega/2-N_{>})$, and $[A_{<},A^{>}]=0$, where $N_{<}$ and $N_{>}$ are fermion number operators associated with both the subspaces: $N_{<}=(N+Q)/2$, $N_{>}=(N-Q)/2$. The only nonzero structure constants are $c^{<<}_{<<}=c^{>>}_{>>}=2$. Let us note that the new bifermion states are not generally normalized to a common factor: $\langle 0|A_{<}A^{<}|0\rangle=\Omega_{<}$, $\langle 0|A_{>}A^{>}|0\rangle=\Omega_{>}$ with $\Omega_{<}=\Omega_{>}=\Omega/2$ for $\Omega$ even, but $\Omega_{<}=(\Omega+1)/2$, $\Omega_{>}=(\Omega-1)/2$ for $\Omega$ odd. A common normalization for odd $\Omega$ would introduce some additional factors that we skip here for the sake of simplicity.

### B. Mapping of the even sector

By the straightforward application of Eqs. (18) and (26) we get

$$A^1\mapsto B^1(\Omega-N_{\rm BF}) \tag{51}$$

for the SU(2) algebra and

$$A^1\mapsto B^1(\Omega-N_{\rm BF})-B^2(\mathcal{Q}+B^iB_{i'}), \tag{52}$$

$$A^2\mapsto B^2(\Omega-N_{\rm BF})-B^1(\mathcal{Q}+B^iB_{i'}), \tag{53}$$

for the SO(4) algebra. Here we introduce boson creation and annihilation operators $B^i$ and $B_i$ with $i=1$ for SU(2) and $i=1,2$ for SO(4). We also define $N_{\rm B}=B^iB_i$, $\mathcal{N}=\alpha^{\mu}\alpha_{\mu}$, $N_{\rm BF}=N_{\rm B}+\mathcal{N}$, and $\mathcal{Q}=\sigma^{\mu}_{\nu}\alpha^{\nu}\alpha_{\mu}$. In the SO(4) case the summation convention is used such that $B^iB_{i'}$ stands for $B^1B_2+B^2B_1$. From Eq. (22) it follows that

$$N\mapsto\mathcal{N}+2N_{\rm B} \tag{54}$$

for both the SU(2) and SO(4), and

$$Q\mapsto\mathcal{Q}-2B^iB_{i'} \tag{55}$$

for the SO(4).

Instead of $A^1$ and $A^2$ we can also map the bifermions from Eqs. (49) and (50). The result is then

$$A^{<}\mapsto B^{<}(\Omega/2-N_{\rm BF<}), \tag{56}$$

$$A^{>} \mapsto B^{>}(\Omega/2 - N_{\mathrm{BF}>}), \tag{57}$$

$$N_{<} \mapsto \mathcal{N}_{<} + 2N_{\mathrm{B}<}, \tag{58}$$

$$N_{>} \mapsto \mathcal{N}_{>} + 2N_{\mathrm{B}>}, \tag{59}$$

where $N_{\mathrm{B}<} = B^{<}B_{<}$, $N_{\mathrm{B}>} = B^{>}B_{>}$, $\mathcal{N}_{<} = (\mathcal{N} + \mathcal{Q})/2$, $\mathcal{N}_{>} = (\mathcal{N} - \mathcal{Q})/2$, $N_{\mathrm{BF}<} = N_{\mathrm{B}<} + \mathcal{N}_{<}$, and $N_{\mathrm{BF}>} = N_{\mathrm{B}>} + \mathcal{N}_{>}$. Both SO(4) results, Eqs. (52)–(55) and (56)–(59), can be combined using the bosonic counterpart of the transformation in Eqs. (49) and (50),

$$B^{<} = B^{1} + B^{2}, \quad B^{>} = B^{1} - B^{2},$$

$$B_{<} = \frac{1}{2}(B_1 + B_2), \quad B_{>} = \frac{1}{2}(B_1 - B_2), \tag{60}$$

which results from the linearity of mapping. It should be noted that the new boson creation and annihilation operators in Eq. (60), unlike $B^{i}$ and $B_{i}$ with $i = 1,2$, are not related by the Hermitian conjugation—a result of nonunitarity of the mapping. If, in contrary, $B^{\bullet}$ and $B_{\bullet}$, where $\bullet$ denotes $<$ and $>$, were chosen to be Hermitian conjugated, the same would not hold for $B^{i}$ and $B_{i}$.

The mapping of the most general one- plus two-body Hamiltonian (33), evaluated for the two algebras under discussion, yields

$$H \mapsto (u + \Omega w_1^1) - w_1^1\mathcal{N} + (v_1^1 - 2w_1^1)N_{\mathrm{B}} + v_1^1 N_{\mathrm{B}}(\Omega - N_{\mathrm{BF}}) \tag{61}$$

for SU(2) and

$$\begin{aligned} H \mapsto {} & (u + \Omega w_i^i) - w_i^i\mathcal{N} - w_{i'}^{i}\mathcal{Q} + (v_i^i - 2w_i^i)N_{\mathrm{B}} \\ & + (v_{j'}^{j} + 2w_{j'}^{j})B^iB_{i'} + v_i^j B^iB_j(\Omega - N_{\mathrm{BF}}) - (v_2^1 B^1B_1 \\ & + v_1^2 B^2B_2 + v_1^1 B^2B_1 + v_2^2 B^1B_2)(\mathcal{Q} + B^iB_{i'}) \end{aligned} \tag{62}$$

for the SO(4). We note that whereas the mapped SU(2) Hamiltonian is manifestly Hermitian, the SO(4) Hamiltonian is not, because of its last term.

### C. Similarity transformations and mapping of the odd sector

Let us finally focus on the form of single-fermion images for both algebras. The similarity transformation (28) depends on the form of the Casimir operator $\mathcal{C}_{\mathrm{F}}$. For the SU(2) algebra we can introduce the seniority quantum number $v$ [35] such that

$$\mathcal{C}_{\mathrm{F}} \equiv \mathcal{A}^1\mathcal{A}_1 = \frac{1}{4}[(\Omega - v)(\Omega + 2 - v) - (\Omega - \mathcal{N})(\Omega + 2 - \mathcal{N})] \tag{63}$$

in the seniority eigenbasis. Because $\mathcal{A}^1 B_1$ does not change $v$ (the number of fermions *not* coupled in pairs), the first term in Eq. (63) does not contribute in Eq. (28) and one gets [19]

$$\mathcal{C}_{\mathrm{F}} - \hat{\mathcal{C}}_{\mathrm{F}} \to \frac{1}{2}(\mathcal{N} - \hat{\mathcal{N}})\left(\Omega + 1 - \frac{\mathcal{N} + \hat{\mathcal{N}}}{2}\right). \tag{64}$$

The operators $S_k$ in Eq. (29) thus read as

$$S_k = \frac{1}{k!}(\mathcal{A}^1 B_1)^k \frac{(\Omega - \mathcal{N} - k)!}{(\Omega - \mathcal{N})!}, \tag{65}$$

which is equivalent to the known expression [19]

$$S_{\mathrm{B}}^{-1} = \frac{\left(\Omega - \dfrac{\mathcal{N} + \hat{\mathcal{N}}}{2}\right)!}{(\Omega - \hat{\mathcal{N}})!} \exp(\mathcal{A}^1 B_1)_{\wedge} \tag{66}$$

[cf. Eq. (A5) in Appendix]. It is instructive to note that the above expressions for the similarity transformation converge under limited conditions only. Consider the case of $\mathcal{N} \leqslant \Omega$. Then we see that the expression in Eq. (65) diverges for $\Omega - \mathcal{N} + 1 \leqslant k \leqslant N_{\mathrm{B}}$ (the upper limit follows from the fact that $S_k$ gives just zero if it attempts to annihilate too many bosons). So the divergence problems are avoided if

$$\mathcal{N} + N_{\mathrm{B}} = \frac{N + \mathcal{N}}{2} \leqslant \Omega. \tag{67}$$

Beyond the validity of Eq. (67) the forms (65) and (66) of the similarity transformation is invalid and another derivation would be required [see the remark below Eq. (A1) in Appendix].

In the SO(4) case the construction of a similarity transformation turns out to be more difficult as the denominator in Eq. (28) cannot be expressed as a function of $\mathcal{N}$. We can, however, use the SU(2)⊗SU(2) type of mapping, Eqs. (56)–(60), for which the analogy with the single SU(2) case can be fully exploited. The Casimir operator $\mathcal{C}_{\mathrm{F}} = 2(\mathcal{A}^{<}\mathcal{A}_{<} + \mathcal{A}^{>}\mathcal{A}_{>})$ then reads as a sum of two terms of the form (63). Again, seniorities corresponding to both subspaces are not affected by $\mathcal{A}^iB_i = 2(\mathcal{A}^{<}B_{<} + \mathcal{A}^{>}B_{>})$ and the following substitution can be used within Eq. (28):

$$\begin{aligned} \mathcal{C}_{\mathrm{F}} - \hat{\mathcal{C}}_{\mathrm{F}} \to {} & (\mathcal{N}_{<} - \hat{\mathcal{N}}_{<})\left(\Omega_{<} + 1 - \frac{\mathcal{N}_{<} + \hat{\mathcal{N}}_{<}}{2}\right) + (\mathcal{N}_{>} - \hat{\mathcal{N}}_{>}) \\ & \times \left(\Omega_{>} + 1 - \frac{\mathcal{N}_{>} + \hat{\mathcal{N}}_{>}}{2}\right). \end{aligned} \tag{68}$$

From this expression we find that

$$\begin{aligned} S_k = {} & \sum_{k_< + k_> = k} \frac{1}{k_<!\,k_>!}(\mathcal{A}^{<}B_{<})^{k_<}(\mathcal{A}^{>}B_{>})^{k_>} \\ & \times \frac{(\Omega_< - \mathcal{N}_< - k_<)!\,(\Omega_> - \mathcal{N}_> - k_>)!}{(\Omega_< - \mathcal{N}_<)!\,(\Omega_> - \mathcal{N}_>)!}, \end{aligned} \tag{69}$$

with the summation going from $k_<, k_> = 0$ to $k$. In analogy with Eq. (66) we also have

$$S_{\mathrm{B}}^{-1}=\frac{\left(\Omega_{<}-\frac{\mathcal{N}_{<}+\hat{\mathcal{N}}_{<}}{2}\right)!}{(\Omega_{<}-\hat{\mathcal{N}}_{<})!}\frac{\left(\Omega_{>}-\frac{\mathcal{N}_{>}+\hat{\mathcal{N}}_{>}}{2}\right)!}{(\Omega_{>}-\hat{\mathcal{N}}_{>})!}\exp(\mathcal{A}^{<}B_{<}+\mathcal{A}^{>}B_{>})_{\wedge}. \quad (70)$$

For $\mathcal{N}_{<}\leqslant\Omega_{<}$ and $\mathcal{N}_{>}\leqslant\Omega_{>}$ the convergence of Eq. (69) requires, in analogy with Eq. (67), $N_{\mathrm{B}<}\leqslant\Omega_{<}-\mathcal{N}_{<}$ and $N_{\mathrm{B}>}\leqslant\Omega_{>}-\mathcal{N}_{>}$.

Now we can evaluate the similarity transformation of the images of single-fermion annihilation and creation operators. We already know that in general the resulting series contains terms changing the number of ideal fermions by $\Delta\mathcal{N}=-1,+1,+3,\ldots$. In the SU(2) case, however, all terms with $\Delta\mathcal{N}\geqslant+3$ vanish. Indeed, it can be shown that with $S_k$ from Eq. (65) we have

$$[\alpha_{\mu},S_k]-S_{k-1}[\alpha_{\mu},S_1]=0 \quad (71)$$

for $k=2,3,\ldots$, which together with Eq. (37) implies that $X_k=0$ for $k\geqslant 2$ with $X=\alpha_{\mu}$ being the bare image of the annihilation operator. In the bare image of the creation operator, $X=\alpha^{\mu}+B^1\tilde{\alpha}_{\mu}\equiv X'+X''$, where $\tilde{\alpha}_{\mu}=(-)^{j-\mu}\alpha_{-\mu}$, the first term changes the ideal-fermion number by $+1$ and the second by $-1$. The condition for the cancellation of $\Delta\mathcal{N}\geqslant+3$ terms in the transformed image, therefore, reads as $X'_k=-X''_{k+1}$, i.e.,

$$[\alpha^{\mu},S_k]+[B^1\tilde{\alpha}_{\mu},S_{k+1}]-S_k[B^1\tilde{\alpha}_{\mu},S_1]=0 \quad (72)$$

for $k=1,2,3,\ldots$. Again, it can be proven from Eq. (65) that Eq. (72) is valid. For the transformed SU(2) single-fermion images we finally obtain [19]

$$a_{\mu}\mapsto\alpha_{\mu}+\tilde{\alpha}^{\mu}B_1\frac{1}{\Omega-\mathcal{N}}+\mathcal{A}^1B_1\alpha_{\mu}\frac{1}{(\Omega+1-\mathcal{N})(\Omega-\mathcal{N})}, \quad (73)$$

$$a^{\mu}\mapsto B^1\tilde{\alpha}_{\mu}+\alpha^{\mu}\frac{\Omega-N_{\mathrm{BF}}}{\Omega-\mathcal{N}}-\mathcal{A}^1\tilde{\alpha}_{\mu}\frac{\Omega-N_{\mathrm{BF}}}{(\Omega+1-\mathcal{N})(\Omega-\mathcal{N})}. \quad (74)$$

[In analogy to $\tilde{\alpha}_{\mu}$ we define $\tilde{\alpha}^{\mu}=(-)^{j-\mu}\alpha^{-\mu}$.]

To derive the single-fermion images in the SO(4) case, one first shows that Eqs. (71) and (72) are again fulfilled with $S_k$ from Eq. (69), if $B^1$ in Eq. (72) is replaced by $B^{\bullet}\equiv B^{<}$ or $B^{>}$ according to whether $|\mu|\leqslant\Omega/2$ or $|\mu|>\Omega/2$, respectively. This means that the series for the transformed single-fermion images $S_{\mathrm{B}}\alpha_{\mu}S_{\mathrm{B}}^{-1}$ and $S_{\mathrm{B}}(\alpha^{\mu}+B^{\bullet}\tilde{\alpha}_{\mu})S_{\mathrm{B}}^{-1}$ both terminate at the terms with $\Delta\mathcal{N}=+1$. We thus obtain

$$a_{\mu}\mapsto\alpha_{\mu}+\tilde{\alpha}^{\mu}B_{\bullet}\frac{1}{\Omega_{\bullet}-\mathcal{N}_{\bullet}}+\mathcal{A}^{\bullet}B_{\bullet}\alpha_{\mu}\frac{1}{(\Omega_{\bullet}+1-\mathcal{N}_{\bullet})(\Omega_{\bullet}-\mathcal{N}_{\bullet})}, \quad (75)$$

$$a^{\mu}\mapsto B^{\bullet}\tilde{\alpha}_{\mu}+\alpha^{\mu}\frac{\Omega_{\bullet}-N_{\mathrm{BF}\bullet}}{\Omega_{\bullet}-\mathcal{N}_{\bullet}}-\mathcal{A}^{\bullet}\tilde{\alpha}_{\mu}\frac{\Omega_{\bullet}-N_{\mathrm{BF}\bullet}}{(\Omega_{\bullet}+1-\mathcal{N}_{\bullet})(\Omega_{\bullet}-\mathcal{N}_{\bullet})}, \quad (76)$$

where the bullet $\bullet$ stands for $>$ or $<$, depending on which subspace $\mu$ belongs to. Equations (75) and (76) are direct analogues of the single-fermion images in the SU(2) case, cf. Eqs. (73) and (74).

### D. Fermion and bifermion transfer matrix elements

To demonstrate the utility of the results derived above, we calculate matrix elements of single-fermion and fermion-pair transfer operators using the ideal boson-fermion images. We start with the SU(2) case, where we consider the following three normalized fermionic states:

$$|\psi_0\rangle=C_0(A^1)^{N/2}|0\rangle,$$
$$|\psi_1\rangle=C_1a^{\mu}(A^1)^{N/2}|0\rangle,$$
$$|\psi_2\rangle=C_2(A^1)^{N/2+1}|0\rangle \quad (77)$$

($N$ or $N+2$ are even numbers of paired fermions). The matrix elements of the single-fermion and fermion-pair transfer operators between these states depend just on the normalization constants $C_0$, $C_1$, and $C_2$ and one readily finds

$$\langle\psi_1|a^{\mu}|\psi_0\rangle=\langle\psi_0|a_{\mu}|\psi_1\rangle=\frac{C_0}{C_1}=\sqrt{\frac{2\Omega-N}{2\Omega}}, \quad (78)$$

$$\langle\psi_2|A^1|\psi_0\rangle=\langle\psi_0|A_1|\psi_2\rangle=\frac{C_0}{C_2}=\frac{1}{2}\sqrt{(2\Omega-N)(N+2)}. \quad (79)$$

The results given in Eqs. (78) and (79) are reproduced in the ideal space, using Eq. (25) with the single-fermion and fermion-pair images $\overline{a_{\mu}}$ from Eq. (73), $\overline{a^{\mu}}$ from Eq. (74), $\overline{A_1}=B_1$, and $\overline{A^1}$ from Eq. (51). The left and right ideal states corresponding to Eq. (77) read as follows:

$$|\psi_0^{\mathrm{R}})=C_0(\overline{A^1})^{N/2}|0)=C_0^{\mathrm{R}}(B^1)^{N/2}|0),$$
$$(\psi_0^{\mathrm{L}}|=(0|(\overline{A_1})^{N/2}C_0^*=(0|(B_1)^{N/2}C_0^{\mathrm{L}},$$
$$|\psi_1^{\mathrm{R}})=C_1\overline{a^{\mu}}(\overline{A^1})^{N/2}|0)=C_1^{\mathrm{R}}\alpha^{\mu}(B^1)^{N/2}|0),$$
$$(\psi_1^{\mathrm{L}}|=(0|(\overline{A_1})^{N/2}\overline{a_{\mu}}C_1^*=(0|(B_1)^{N/2}\alpha_{\mu}C_1^{\mathrm{L}}\cdots \quad (80)$$

(the images of $|\psi_2\rangle$ are analogous to the ones of $|\psi_0\rangle$). It is clear that the coefficients $C_i^{\mathrm{R}}$ and $C_i^{\mathrm{L}}$ carry information on the specific construction of the images in Eq. (80) from the real states, in particular, information on the fermionic normalization constants. This seems to undermine the practical implementation of the mapping procedure because once the mapping of a particular algebra has been established, one certainly wants to be able to perform all the calculations solely on the ideal boson-fermion level. Here we come to the reason why Eq. (25) is more convenient from the phenomenological viewpoint than the seemingly simpler identity $\langle\psi_1|O|\psi_2\rangle=(\psi_1^{\rm L}|\overline{O}|\psi_2^{\rm R})$: with Eq. (25) the results depend just on the products $C_i^{\rm L}C_i^{\rm R}$ (with $i=1,2$) that can be easily determined using only the *bosonic* (ideal) normalization condition $(\psi_i^{\rm L}|\psi_i^{\rm R})=1$. We see, therefore, that the matrix elements (78) and (79) can be calculated on the purely *phenomenological* level—i.e., using only ideal boson-fermion states with no explicit reference to their real-fermionic ancestors—provided that we know microscopically based ideal images of (bi)fermion creation and annihilation operators. This again emphasizes the importance of the construction carried out in Secs. VI and VII C.

In the SO(4) case one can proceed in a close analogy with SU(2). It turns out that it is much easier to work in the collective basis created by pairs $A^<$ and $A^>$ rather than $A^1$ and $A^2$. We thus define fermionic states

$$\begin{aligned}|\psi_0\rangle&=C_0(A^>)^{N/2-k}(A^<)^k|0\rangle,\\ |\psi_1\rangle&=C_1a^\mu(A^>)^{N/2-k}(A^<)^k|0\rangle,\\ |\psi_2\rangle&=C_2(A^>)^{N/2-k}(A^<)^{k+1}|0\rangle,\\ |\psi_2'\rangle&=C_2'(A^>)^{N/2-k+1}(A^<)^k|0\rangle.\end{aligned}\tag{81}$$

Note that we now have two possibilities, $|\psi_2\rangle$ and $|\psi_2'\rangle$, of building a paired $(N+2)$-fermion states from $|\psi_0\rangle$. With the aid of the left and right ideal states corresponding to Eq. (81) [similar to those in Eq. (80)] and the operator images in Eqs. (56), (57), (75), and (76), it is now simple to verify that Eq. (25) yields

$$\langle\psi_1|a^\mu|\psi_0\rangle=\langle\psi_0|a_\mu|\psi_1\rangle=\begin{cases}\sqrt{\dfrac{\Omega_<-k}{\Omega_<}} & \text{for } |\mu|\leqslant\Omega/2\\ \sqrt{\dfrac{2\Omega_>-N+2k}{2\Omega_>}} & \text{for } |\mu|>\Omega/2,\end{cases}\tag{82}$$

and

$$\langle\psi_2|A^<|\psi_0\rangle=\langle\psi_0|A_<|\psi_2\rangle=\sqrt{\frac{(\Omega-2k)(k+1)}{2}},\tag{83}$$

$$\langle\psi_2'|A^>|\psi_0\rangle=\langle\psi_0|A_>|\psi_2'\rangle=\frac{1}{2}\sqrt{(\Omega-N+2k)(N-2k+2)},\tag{84}$$

$$\langle\psi_2'|A^<|\psi_0\rangle=\langle\psi_0|A_<|\psi_2'\rangle=\langle\psi_2|A^>|\psi_0\rangle=\langle\psi_0|A_>|\psi_2\rangle=0.\tag{85}$$

These results can be checked by evaluating the fermionic normalization constants. Let us point out that the calculation would be much more involved if we chose to use the collective basis created by $A^1$ and $A^2$. Since ideal images of these operators contain both $B^1$ and $B^2$ [see Eqs. (52) and (53)] the mapped collective states (right images) would combine various numbers of type-1 and type-2 bosons (only the total boson number being constant).

## VIII. CONCLUSIONS

We investigated various aspects of the generalized Dyson mapping that transforms fermionic shell-model superalgebras into the ideal boson-fermion space [17–19]. The main motivation for this review was the recent experimental verification [6–8] of the phenomenological boson-fermion supersymmetric model [5] and the resulting renewed interest in its microscopic foundations. Along with presenting some particular new results we found it useful also to summarize in a compact form the main principles of the underlying mathematical formalism and the hurdles that remain.

While in the standard Dyson mapping only the collective algebra of fermion-pair operators is transformed into the ideal space, yielding a set of purely bosonic images, the generalized Dyson mapping transforms also the single-fermion creation and annihilation operators, i.e., the whole superalgebra defined in Sec. II. As a result, ideal-fermion operators enter the images of physical observables in addition to the boson operators. The mapping procedure outlined here makes use of the generalized Usui operator (17), which has the advantage of providing in a relatively straightforward manner a first set of simple formulas—Eqs. (18)–(22)—for the images of the operators involved in the superalgebra. However, it turns out that some additional transformations are needed to accomplish the physically motivated bosonization and unitarity of the mapping. The general form of these transformations was discussed in Sec. IV, while in Appendix we provided technical insight into the formalism used for their derivation.

We studied, in particular, the "bosonization" similarity transformations, see Eqs. (28)–(32). Without these transformations, the main aim of the mapping—replacement of the fermionic correlations by simpler bosonic correlations—would not be achieved, since all fermion-fermion interactions would be exactly reproduced in the ideal-particle space. The action of the bosonization similarity transformation on a general operator was determined in the expanded form of Eqs. (35)–(37). These expressions represent a new result compared to previous work on this subject. However, to use them in general for deriving closed expressions might still be elusive unless the Casimir operator $\mathcal{C}_{\rm F}$ of the ideal-fermion core algebra turns out to depend solely on the number $\mathcal{N}$ of ideal fermions in the whole space or its specific subspaces. If this condition is fulfilled, the calculations can be carried out further and one derives, e.g., the explicit form of transformed single-fermion images in Eqs. (41) and (42). These results are already of importance to hint at suitable expressions for nucleon transfer operators in phenomenological supersymmetric models.

A particularly interesting question, related directly to the microscopic justification of phenomenological supersymmetric models, concerns the conservation of the total number of ideal particles (fermions plus bosons). It was shown in Sec. V that this number is indeed a natural integral of motion if the even sector of the mapped superalgebra is chosen properly, i.e., so that it fully represents the real fermionic Hamiltonian. From the point of view of the generalized Dyson mapping, the origin of the phenomenological U$(M/2\Omega)$ dynamical superalgebra seems to have a sound microscopic basis. Since numbers of ideal bosons and fermions turn out to be conserved separately, the present method also advocates the decomposition of the phenomenological dynamical superalgebra into the product of bosonic and fermionic algebras in the first step of the relevant dynamical-symmetry chains [5]. However, the realization of truly supersymmetric predictions that are not specifically connected with dynamical symmetries, as discussed in Refs. [32–34], is not excluded.

To illustrate the general technique outlined in this paper, we investigated in Sec. VII concrete examples of mapping the SU(2) and SO(4) collective superalgebras. The results for the seniority SU(2) model were derived earlier [19], but we reconsidered them from a more general point of view and to facilitate the analysis of the SO(4) case, originally discussed by Kaup and Ring [16]. As the SO(4) algebra can also be written as the product $\mathrm{SU}(2)_{<}\otimes\mathrm{SU}(2)_{>}$, it provides an interesting insight into the link between boson images of the two different realizations, as, e.g., in Eq. (60). Both the SU(2) and SO(4) models exemplify the relative simplicity of the Dyson mapping that follows from "bare" operator images, while they also point to technical difficulties associated with similarity transformations. For the specific superalgebras studied here the bosonization similarity transformation leads to the closed expressions for single-fermion images given in Eqs. (73)–(76). In more general cases, however, the transformed images may involve more complicated series, where convergence becomes an issue. This problem must be ovecome for an optimal comparison with the phenomenological framework. Note also that unlike the models considered here, the dynamical definition of collective fermion pairs (bosons) requires attention beyond the algebraic definitions (see, e.g., Ref. [10]).

One of the main remaining obstacles in the quantitative microscopic analysis of phenomenological supersymmetric models is associated with the nonunitarity of the generalized Dyson mapping. While this property obscures some aspects of a direct comparison with phenomenology, we also documented that on the matrix-element level the formalism can already be implemented in a way which closely resembles the phenomenological application. This was illustrated through the use of Eq. (25) leading to the examples in Sec. VII D.

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## APPENDIX: POSITIONAL OPERATOR CALCULUS

Consider an operator $O=O'+P$ where $O$ and $O'$ are isospectral and denote $O|\psi_i\rangle=o_i|\psi_i\rangle$ and $O'|\psi_i'\rangle=o_i|\psi_i'\rangle$. In general, $O$ does not have to be Hermitian, so that the $|\psi_i\rangle$ are not necessarily orthogonal, while $O'$ *is* Hermitian, implying that its eigenvectors $|\psi_i'\rangle$ form an orthonormal basis. The transformation connecting the two sets of eigenvectors, $S|\psi_i\rangle=|\psi_i'\rangle$, transforms away the $P$ term of $O$, i.e., $S(O'+P)S^{-1}=O'$. This is the property of similarity transformations required in Sec. IV. The form of $S^{-1}$ can be determined from the ordinary perturbative series expressing $|\psi_i\rangle$ in terms of $|\psi_i'\rangle$ (with $P$ treated as a perturbation). The isospectrality condition is often guaranteed by the fact that $P$ has the upper (lower) off-diagonal block structure in the basis $|\psi_i'\rangle$. In that case $\langle\psi_i'|P|\psi_i'\rangle=0$ and the expansion reads as follows:

$$|\psi_i\rangle=\sum_{k=0}^{\infty}\left(\frac{1}{o_i-O'}P\right)^k|\psi_i'\rangle. \tag{A1}$$

Note that the terms with $k>0$ are to be evaluated only in case of $[O',P]\neq 0$, otherwise they are equal to zero. It must be stressed that Eq. (A1) is derived using the perturbation theory for nondegenerated cases. Its applicability is thus not quite universal and the convergence conditions should be determined in each particular case.

In fact, Eq. (A1) defines the action of $S^{-1}$ on any vector via its expansion in the eigenbasis $|\psi_i'\rangle$. To avoid the explicit reference to the basis, the idea of positional operators was introduced in Refs. [29,37]. Eq. (A1) can be rewritten as

$$\begin{aligned} S^{-1}&=\sum_{k=0}^{\infty}\sum_i\left(\frac{1}{o_i-O'}P\right)^k|\psi_i'\rangle\langle\psi_i'|\\ &=\sum_{k=0}^{\infty}\left(\frac{1}{\hat{O}'-O'}P\right)^k_{\wedge}, \end{aligned} \tag{A2}$$

where the hat above the operator $O'$ means that in the expansion of each term on the rhs of Eq. (A2) this operator must be evaluated at the position indicated by "$\wedge$."

One can develop a general calculus suitable for handling expressions such as the one in Eq. (A2). In fact, any hatted operator is treated as an ordinary $c$-number during the evaluation, i.e., it may freely travel to any place as far as its true position is marked. Any part of $O'$ that commutes with $P$ cancels with the corresponding part of $\hat{O}'$, so that $O'$ in Eq. (A2) can be replaced by any operator $C$ that satisfies $[C,P]=[O',P]$. We thus arrive at Eq. (24). With no further assumption upon commutation relations between the operators involved, the evaluation of terms such as $f(\hat{C}-C)AB_{\wedge}$ must unavoidably deal with the decomposition of the function $f(x)$ into a series, which usually leads to rather complicated expressions. For example, for $f(x)=\Sigma_{n=0}^{\infty}f_nx^n$ one can derive

$$\begin{aligned} f(C-\hat{C})AB_{\wedge}=&f(C-\hat{C})A_{\wedge}B+\sum_{n=0}^{\infty}f_n\sum_{k=0}^{n}(-)^{n-k}\binom{n}{k}\\ &\times([C^k,A][B,C^{n-k}]+A[C^k,B]C^{n-k}). \end{aligned} \tag{A3}$$

However, great simplification can be achieved if $A,B,C$ conform to commutation relations such as $f(c-C)A=Ag(c-C)$, where $g(x)$ and $f(x)$ are some interrelated functions and $c$ an arbitrary constant.

Let us consider an important special case with $C=aN^2+bN+c$, where the operator $N$ fulfills the condition $NP=P(N+m)$ (with $m$ a positive integer) and $a,b,c$ are constants. $P$ is a ladder operator for $N$ and we assume eigenvalues of $N$ within the range from 0 to $n_{\max}>0$. Since $N$ here represents the fermion-number operator and $n_{\max}$ the shell capacity, the above condition is satisfied if $P$ creates $m$ fermions and $C$ is a quadratic function of $N$. The sum in Eq. (A2) terminates at $k_{\max}=\lfloor n_{\max}/m \rfloor$. Moreover, for some values of the constants the series can be formally summed, yielding

$$S^{-1}=\frac{\left(2\hat{N}+\frac{b}{a}\right)!^m}{\left(N+\hat{N}+\frac{b}{a}\right)!^m}\exp\left(-\frac{a}{m}P\right)_\wedge \tag{A4}$$

for $b/a>0$ and

$$S^{-1}=\frac{\left(-\frac{b}{a}-m-N-\hat{N}\right)!^m}{\left(-\frac{b}{a}-2\hat{N}\right)!^m}\exp\left(\frac{a}{m}P\right)_\wedge \tag{A5}$$

for $b/a<-3k_{\max}m$. Here, $!^m$ stands for the "factorial over $m$," i.e., $x!^m=x(x-m)(x-2m)\cdots(x \bmod m)$ for $x>0$ and $x!^m:=1$ for $x\leqslant 0$. One may verify Eqs. (A4) and (A5) from the relation $\hat{C}-C=(\hat{N}-N)[a(N+\hat{N})+b]$, commuting the first term to the right and the second term to the left. (The constraints on $b/a$ ensure that the factorial-like terms contain only positive numbers; otherwise the above formulas can be used in a restricted subspace only.) Various specific realizations of Eqs. (A4) and (A5) can be found in Refs. [18,19,29,37]. If $C$ cannot be expressed as a quadratic function of $N$, but $[C,N]=0$ still holds, one obtains

$$S^{-1}=\exp\left[-\frac{\hat{N}-N}{m(\hat{C}-C)}P\right]_\wedge . \tag{A6}$$


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