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Finite well solution for the E(5) Hamiltonian

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The recently proposed infinite square well E(5) description of nuclei at the critical point of the phase transition from vibrator to γ-soft rotor is extended to include the effects of finite well depth. The evolution of nuclear observables as a function of well depth is studied, and observables sensitive to finite well depth are identified.

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An analytic description of nuclei at the critical point of the second order phase transition from harmonic oscillator structure to rigidly β-deformed γ-soft structure has recently been proposed by Iachello [1]. This description, denoted as E(5) in Ref. [1], arises in the geometrical picture of the nucleus and occurs for a nuclear potential which is an infinite square well in quadrupole deformation space. The E(5) solution was the first example to be identified of a new class of critical point solutions applicable to both nuclear and molecular physics [2,3], for which several experimental examples have already been proposed [4–7].

The E(5) description is of considerable interest for several reasons.

(1) The E(5) solution provides a description of nuclear behavior at the second order critical point of a shape transition. This is a very special point in nuclear parameter space from a theoretical perspective.

(2) The E(5) solution is analytic. Numerical solutions for a wide variety of geometrical potentials (specifically, those which are Taylor expandable) have long been available through the geometric collective model [8], and, in fact, these can closely reproduce the results of E(5) [9]. What is important is the tremendous contribution to the understanding of a phenomenon provided by an analytic solution.

(3) The model allows for an extraordinarily simple (and successful) phenomenological description of a class of nuclei which have historically been the hardest to understand, namely, those at the midpoint of a shape transition.

Experiment indicates that the E(5) description is applicable to real nuclei. Work by Casten and Zamfir [4] and Arias [5] presents ^{134}Ba as a strong candidate for being an E(5) nucleus, and further examples (e.g., ^{102}Pd and ^{104}Ru) are being investigated [10,11].

One of the most pressing open questions in the study of E(5) and related critical point descriptions has been whether or not their features remain valid at finite well depth. The use of an infinite well potential is a convenient calculational approximation. However, actual potentials describing nuclei are expected to be finite, not infinite, in depth, and so it is crucial to assess how sensitive the E(5) results are to well depth, as suggested in Ref. [1]. It is therefore the purpose of this paper to extend the E(5) solution to include the case of a square well of finite depth.

The investigation of finite well depth effects for E(5) is of special interest in another context as well. Key features of the E(5) solutions can be reproduced for an appropriate choice of parameters [4] in the interacting boson model (IBM) [12]. The introduction of finite depth in the E(5) model is closely related to the introduction of a finite boson number along the U(5)-SO(6) transition of the IBM [13].

The E(5) description is obtained by considering the Bohr Hamiltonian [14]

$$H=-\frac{\hbar^2}{2B}\left[\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{1}{\beta^2\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}-\frac{1}{4\beta^2}\sum_\kappa\frac{Q_\kappa^2}{\sin^2\left(\gamma-\frac{2}{3}\pi\kappa\right)}\right]+V(\beta,\gamma) \tag{1}$$

for the special case of a γ-independent infinite square well potential, $V(\beta)=0$ for $\beta<\beta_w$ and infinite elsewhere. A separation of variables can be carried out in the standard way [15] for a γ-independent potential. The eigenfunctions are of the form $f(\beta)\Phi(\gamma,\theta_i)$. The solutions for the "angular" (γ,θ_i) wave functions [16] are common to all γ-soft problems, while the dependence upon the potential $V(\beta)$ is isolated in the "radial" (β) wave function. In terms of the reduced eigenvalue $\varepsilon\equiv(2B/\hbar^2)E$ and reduced potential $v(\beta)\equiv(2B/\hbar^2)V(\beta)$, the equation for $f(\beta)$ is

$$\left[\left(-\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta}+\frac{\tau(\tau+3)}{\beta^2}\right)+v(\beta)\right]f(\beta)=\varepsilon f(\beta), \tag{2}$$

where the separation constant τ assumes the values $\tau=0,1,2,\ldots$. The eigenfunctions of this equation are given [1] in terms of spherical Bessel functions

$$f_{\xi,\tau}(\beta)=\begin{cases}A_{\xi,\tau}\beta^{-1}j_{\tau+1}(\varepsilon_{\xi,\tau}^{1/2}\beta), & \beta\leqslant\beta_w\\ 0, & \beta>\beta_w,\end{cases} \tag{3}$$

where $A_{\xi,\tau}$ is a normalization constant. The eigenvalues are

$$\varepsilon_{\xi,\tau}=\beta_w^{-2}x_{\tau+3/2,\xi}^2, \tag{4}$$

$\xi=1,2,\ldots$, where $x_{\nu,i}$ is the ith zero of the ordinary Bessel function $J_\nu(x)$. Each solution of the β equation results in a multiplet of solutions to the full problem, degenerate with respect to angular momentum according to the usual τ multiplet structure [15,17]. The notation $J^{+}_{\xi,\tau}$ is used to designate these states.

Let us now consider the square well potential of finite depth,

$$V(\beta)=\begin{cases} V_0, & \beta\leqslant\beta_w \\ 0, & \beta>\beta_w, \end{cases} \tag{5}$$

where $V_0<0$. The corresponding reduced potential has depth $v_0\equiv(2B/\hbar^2)V_0$. Bound state solutions can only occur with eigenvalues in the range $v_0<\varepsilon<0$. The finite well potential is piecewise constant as a function of β and so can, as in the infinite well case, be solved in terms of spherical Bessel functions. In the interior of the well ($\beta<\beta_w$), the solution again involves $j_{\tau+1}(\beta)$. In the classically forbidden region exterior to the well ($\beta>\beta_w$), however, the solution set is constructed from spherical Bessel functions of imaginary argument. The linear combination with the correct asymptotic behavior (convergence at $\beta\rightarrow\infty$) involves the modified spherical Bessel function $k_{\tau+1}(\beta)$. The wave function in β is

$$f_{\xi,\tau}(\beta)=\begin{cases} A_{\xi,\tau}\beta^{-1}j_{\tau+1}[(\varepsilon_{\xi,\tau}-v_0)^{1/2}\beta], & \beta\leqslant\beta_w \\ B_{\xi,\tau}\beta^{-1}k_{\tau+1}[(-\varepsilon_{\xi,\tau})^{1/2}\beta], & \beta>\beta_w. \end{cases} \tag{6}$$

The eigenvalues for the finite well are determined by the requirement that $f(\beta)$ be continuous and smooth at the matching point $\beta=\beta_w$. The eigenvalue condition for ε can be obtained in a manner analogous to that for the three-dimensional square well (e.g., Ref. [18]). We define a dimensionless energy variable

$$\eta(\varepsilon)\equiv\left[1-\frac{\varepsilon}{v_0}\right]^{1/2} \tag{7}$$

and a "well size" parameter

$$x_0\equiv(-v_0)^{1/2}\beta_w. \tag{8}$$

The matching condition, expressed in terms of these quantities, is the transcendental equation

$$-\frac{\displaystyle\sum_{i,j=0}^{\tau+2}[c_{(\tau+1)i}e'_{(\tau+1)(j+1)}-c'_{(\tau+1)(i+1)}e_{(\tau+1)j}]x_0^{-i-j}\eta^{-i}[(1-\eta^2)^{1/2}]^{-j}}{\displaystyle\sum_{i,j=0}^{\tau+2}[s_{(\tau+1)i}e'_{(\tau+1)(j+1)}-s'_{(\tau+1)(i+1)}e_{(\tau+1)j}]x_0^{-i-j}\eta^{-i}[(1-\eta^2)^{1/2}]^{-j}}=\tan(x_0\eta), \tag{9}$$

which must be solved numerically for the eigenvalues of ε. The constants c_{ni}, s_{ni}, and e_{ni} are the coefficients in the spherical Bessel function expansions

$$j_n(x)=\left(\sum_{i=1}^{n+1}c_{ni}x^{-i}\right)\cos x+\left(\sum_{i=1}^{n+1}s_{ni}x^{-i}\right)\sin x,$$

$$k_n(x)=\left(\sum_{i=1}^{n+1}e_{ni}x^{-i}\right)e^{-x}, \tag{10}$$

and c'_{ni}, s'_{ni}, and e'_{ni} are defined similarly for the derivative functions $j'_n(x)$ and $k'_n(x)$. Once an eigenvalue $\varepsilon_{\xi,\tau}$ is found, the coefficients $A_{\xi,\tau}$ and $B_{\xi,\tau}$ follow from the matching condition at $\beta=\beta_w$ and the normalization condition $\int_0^\infty\beta^4d\beta|f(\beta)|^2=1$.

The eigenvalue spectrum of the solution depends upon the parameters β_w and v_0 exclusively in the combination x_0, as can be seen from the eigenvalue condition (9). That is, if two wells (β_w,v_0) and (β'_w,v'_0) have the same value for x_0, they will have identical energy spectra, to within an overall normalization factor. Two wells with different x_0 values will have different energy spectra.

Therefore, for a given value of x_0, the numerical solution procedure need only be carried out once, at some "reference" choice of the well width and depth (e.g., $\beta_w=1$), and the solution for any other well of the same x_0 can be deduced analytically. To state the analytical relations explicitly, consider a reference calculation performed at $\beta_w=1$ (and thus $v_0=-x_0^2$), and suppose this calculation produces an eigenvalue ε and normalized wave function $f(\beta)$. Then for a well of the same x_0 but a different width β'_w (and thus $v'_0=-x_0^2/\beta'^2_w$), the corresponding eigenvalue ε' and normalized wave function $f'(\beta)$ are given by the simple rescalings

$$\varepsilon'=\beta'^{-2}_w\varepsilon, \tag{11}$$

$$f'(\beta)=\beta'^{-5/2}_w f(\beta/\beta'_w). \tag{12}$$

Matrix elements of the operator β^m,

$$I\equiv\int_0^\infty\beta^4d\beta f^I(\beta)\beta^m f^{II}(\beta), \tag{13}$$

are of special interest, since they are encountered in the calculation of electromagnetic transition strengths. These rescale as

$$I'=\beta'^{m}_w I. \tag{14}$$

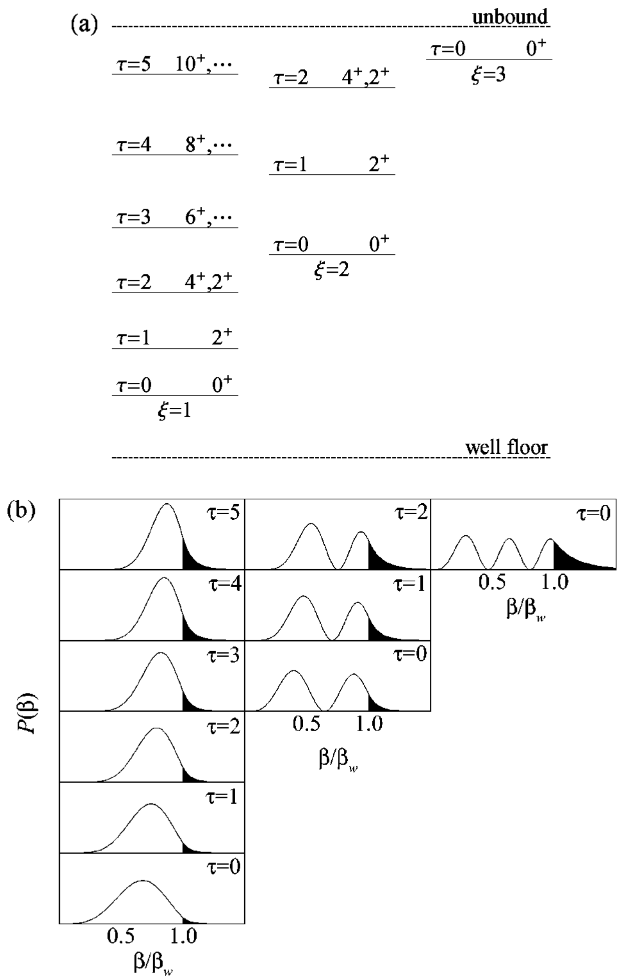

FIG. 1. Bound states of the $x_0=10$ well: (a) excitation energies and (b) probability density functions $P(\beta)\equiv\beta^4|f(\beta)|^2$. The shaded areas under the probability density functions indicate penetration into classically forbidden β values ($\beta>\beta_w$).

The solution for $x_0=10$ is illustrative of the main effects of finite well depth. The level energies and wave functions for this solution are shown in Fig. 1. The main consequences of the finite well depth are not unexpected.

(1) There are only a finite number of bound states. In this case, only members of the lowest few ξ families are bound [Fig. 1(a)]. A summary of the number of bound states for other well sizes is given in Table I.

(2) The wave functions penetrate the classically forbidden region $\beta>\beta_w$. For the highest-lying states, a substantial portion of the probability distribution in β lies outside β_w. This is indicated by the shaded areas of Fig. 1(b).

(3) The eigenvalues are lowered relative to those for the infinite E(5) well of the same β_w [Fig. 2(a)]. This is a natural consequence of the finite well depth: The wave functions are given the freedom to spread into the region $\beta>\beta_w$, and this is analogous in effect to a widening of the well, causing the energies to ''settle'' lower.

Some interesting properties, however, are revealed by an examination of the systematic evolution of the solution with changing well size. A series of calculations ($x_0=5,10,20$) spanning the physical range of interest in the study of nuclei are presented alongside the E(5) solution in Fig. 2. [At a fixed width, x_0 is a measure of the depth of the well, and the infinite E(5) well is obtained in the limit $x_0\rightarrow\infty$.] Although the energy eigenvalues do experience a lowering as the well depth decreases [Fig. 2(a)], it turns out that the level energies are nearly *uniformly* lowered by the same *factor* for all levels in the well, leaving energy *ratios* virtually unchanged. A plot of excitation energies normalized to the first excited state [Fig. 2(b)] reveals these energies to be essentially insensitive to the well depth. Some relevant energy ratios are summarized in Table II.

TABLE I. Number of bound β solutions, by τ quantum number, for selected x_0.

x_0	$\tau=0$	$\tau=1$	$\tau=2$	$\tau=3$	$\cdots$
5	1	1			
10	3	2	2	1	$\cdots$
20	6	5	5	4	$\cdots$
50	15	14	14	14	$\cdots$

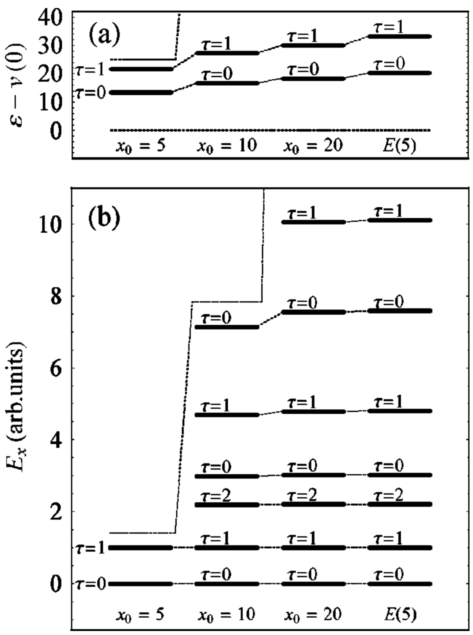

FIG. 2. Evolution of level excitation energies as a function of well parameter x_0 for selected low-lying levels. (a) Absolute eigenvalue relative to floor of well $[\varepsilon-v(0)]$. (b) Excitation energy normalized to the first excited state. The upper dashed line indicates the energy at which the system becomes unbound (top of well).

TABLE II. Excitation energy observables for selected x_0. Excitation energies are normalized to $E_{\xi=1,\tau=1}=1$. The quantity $R_{4/2}$ is defined for each ξ family as $(E_{\tau=2}-E_{\tau=0})/(E_{\tau=1}-E_{\tau=0})$.

	$\xi=1$		$\xi=2$			$\xi=3$	
x_0	$R_{4/2}$	$E_{\tau=0}$	$E_{\tau=1}$	$R_{4/2}$	$E_{\tau=0}$	$E_{\tau=1}$	$R_{4/2}$
10	2.19	2.99	4.69	2.09	7.14		
20	2.20	3.02	4.79	2.12	7.55	10.05	2.08
E(5)	2.20	3.03	4.80	2.12	7.58	10.11	2.09

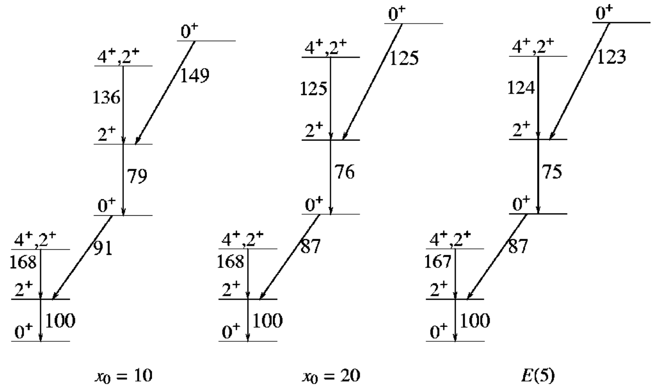

FIG. 3. Evolution of $B(E2)$ strengths as a function of well parameter x_0. Values are normalized to $B(E2;2^+_{1,1}\rightarrow 0^+_{1,0})=100$.

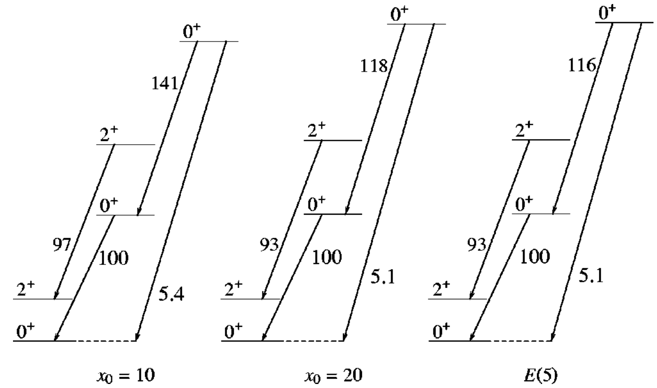

FIG. 4. Evolution of $B(E0)$ strengths as a function of well parameter x_0. Values are normalized to $B(E0;0^+_{2,0}\rightarrow 0^+_{1,0})=100$.

Electromagnetic transition strengths can be calculated from the matrix elements of the collective multipole operators [8]. To leading order in β, the $E2$ and $E0$ transition operators are

$$T(E2;\mu)\propto\beta\left[D^{(2)}_{\mu,0}\cos\gamma+\frac{1}{\sqrt{2}}(D^{(2)}_{\mu,2}+D^{(2)}_{\mu,-2})\sin\gamma\right], \tag{15}$$

$$T(E0;0)\propto\beta^2 \tag{16}$$

and the transition strengths are $B(E\lambda;J_i\rightarrow J_f)=|\langle J_f|T|J_i\rangle|^2/(2J_i+1)$. The evolution of key $E2$ and $E0$ transition strengths is shown in Figs. 3 and 4. The absolute transition strengths are larger at finite well depth than for the infinite well (at the same width β_w), but the increase is, again, largely a uniform overall increase, leaving $B(E2)$ or $B(E0)$ *ratios* nearly unchanged from the E(5) limit.

The uniform reduction of all energies and enhancement of all transition matrix elements do not serve as useful identifying features of finite well depth, since arbitrary energy and transition strength normalizations can be obtained for the infinite E(5) well simply by varying the parameters β_w and B.

Only the very highest energy levels, just short of being unbound, show appreciable deviations from the E(5) normalized energies and transition strengths. The third 0^+ state at $x_0=10$ demonstrates these effects nicely: lowered energy (Fig. 2), enhanced $E2$ transitions (Fig. 3), and enhanced $E0$ transitions (Fig. 4).

The results found for the finite well present a challenge from an experimental viewpoint. There are few clear signatures of finite well depth. Those signatures which are present consist of moderate modifications to energies or transition strengths for high-lying levels, but such levels are typically the least accessible experimentally and most subject to contamination from degrees of freedom outside the collective model framework.

The results are, however, reassuring from a theoretical perspective. They suggest that the E(5) description is "robust" in nature. Key features of the E(5) solutions remain virtually unchanged under radical modification of the depth of the potential, namely, alteration from the ideal infinite well to the realistic finite well likely to be applicable to actual nuclei.

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