Chiral Lagrangians at finite density

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The effective SU(2) chiral Lagrangian with external sources is given in the presence of nonvanishing nucleon densities by calculating the in-medium contributions of the chiral pion-nucleon Lagrangian. As a by-product, a relativistic quantum field theory for Fermi many-particle systems at zero temperature is directly derived from relativistic quantum field theory with functional methods.

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In the limit of massless up and down quarks the QCD Lagrangian is symmetric under the chiral group $SU(2)_L \times SU(2)_R$. One assumes that this symmetry is spontaneously broken to the diagonal subgroup $SU(2)_{L+R}$ giving rise to the appearance of three massless Goldstone bosons which finally acquire small masses due to the nonvanishing mass of the u and d quarks.

This symmetry breaking scenario so much constrains the interactions of the Goldstone bosons that the QCD Green functions can be calculated at low energies as an expansion in powers of momenta and quark masses. This is known as chiral perturbation theory (CHPT) [1,2].

The extension of the theory to the case of low temperature at zero density was considered in Ref. [3]. In this paper we study the case of small densities at zero temperature and derive the corresponding chiral Lagrangian by calculating the in-medium contributions due to the chiral pion-nucleon Lagrangian [4,5] with functional methods. Although we focus our treatment on QCD, the relativistic many-body formalism deduced here for Fermi systems can be applied to processes governed by other dynamical theories, such as the traditional nonrelativistic zero temperature many-body [6,7] quantum theory which stresses the diagrammatic approach. Compared with standard quantum field theory at finite temperature T [8,9] in the grand canonical ensemble, one avoids the use of unknown chemical potentials which themselves have to be calculated in terms of the many-body forces. The former is accomplished by following quantum field theory at $T=0$, considering directly the change of the ground state from the vacuum to one with finite fermionic densities. In the same way one also avoids the nontrivial $T\to 0$ limit due to the so-called anomalous diagrams [10,11]. The price to pay is to rely on the adiabatic hypothesis in order to determine the interacting ground state from that of the free case by turning on the interactions adiabatically.

Let us take first the case of symmetric and unpolarized nuclear matter; the extension of the formalism to the asymmetric and polarized case is straightforward and will be shown below. In the following we take the Heisenberg picture and, closely following scattering theory [12], we consider two ground states $|\Omega_{\rm out}\rangle$ and $|\Omega_{\rm in}\rangle$ which, under the action of any time dependent operator *at asymptotic times* $t\to\pm\infty$, respectively, behave as two symmetric Fermi seas of free protons and neutrons. The Fermi seas are filled up to the corresponding baryonic density,

$$\prod^{N} a^\dagger(\mathbf{p}_n)|0\rangle,$$

where the label n includes also the spin and isospin indices, N is the number of momentum states inside the Fermi sea with Fermi momentum $k_F=(3\pi^2\rho/2)^{1/3}$, ρ is the total nuclear density, and $|0\rangle$ is the vacuum. Our objective is to evaluate the generating functional $\mathcal{Z}[v,a,s,p]$ in the presence of vector v_μ, axial a_μ, scalar s, and pseudoscalar p external fields [2] by working out the transition amplitude $\langle\Omega_{\rm out}|\Omega_{\rm in}\rangle_J$, where the label J just indicates the presence of the aforementioned external sources. In this way by taking functional derivatives of $\mathcal{Z}[v,a,s,p]$ with respect to the external sources one evaluates the in-medium QCD connected Green functions (space-time averages at finite density of the quark currents coupled to the v, a, s, and p sources). To do this we consider the effective chiral Lagrangians $\mathcal{L}=\mathcal{L}_{\pi\pi}+\mathcal{L}_{\bar\psi\psi}+\mathcal{L}_{\bar\psi\psi\bar\psi\psi}+\cdots$ with an increasing number of pairs of nucleon fields $\psi(x)$. We first restrict ourselves to the term with no nucleon fields $\mathcal{L}_{\pi\pi}$ and to that containing two of them $\mathcal{L}_{\bar\psi\psi}=\bar\psi(x)D(x)\psi(x)$, together with the previous external fields. We will discuss later a way to include perturbatively the contributions of Lagrangians with a higher number of nucleons by considering them to arise from bilinear vertices through the exchange of an arbitrary heavy particle. Indeed, although we are talking about CHPT, the only thing that matters for the following derivations is that $\mathcal{L}_{\bar\psi\psi}$ is bilinear in the fermions. Consider now the transition amplitude for the ground states from $t\to-\infty$ to $t\to+\infty$ in the presence of the previous external sources together with Grassmann sources η and $\eta^\dagger$, coupled to the nucleon fields:

$$\langle\Omega_{\rm out}|\Omega_{\rm in}\rangle_{J,\eta,\eta^\dagger}=\int[dU][d\psi][d\psi^\dagger]\langle\Omega_{\rm out}|\psi(+\infty)\rangle \times e^{i\int dx[\mathcal{L}_{\pi\pi}+\bar\psi D\psi+\eta^\dagger\psi+\psi^\dagger\eta]}\langle\psi(-\infty)|\Omega_{\rm in}\rangle \tag{1}$$

with the pion fields described by the 2×2 unitary matrix U.

The ground state functional $\langle\psi(\pm\infty)|\Omega_{\substack{\rm out\\ \rm in}}\rangle$ can be expressed in terms of that of the vacuum by writing the $a(\mathbf{p_n})_{\substack{\rm out\\ \rm in}}$ operators as a function of $\psi(x)$ and its time derivative $\dot\psi(x)$:

$$a(\mathbf{p}_n)_{\substack{\text{out}\\ \text{in}}} = \lim_{t\to\pm\infty} \frac{E(p)}{2m_N} e^{iE(p)t} \int d\mathbf{x} e^{-i\mathbf{p}\mathbf{x}} \overline{u}(\mathbf{p}_n) \times \left[\psi_n(x) + \frac{i}{E(p)} \dot{\psi}_n(x) \right], \tag{2}$$

where as usual in scattering theory for $t\to\pm\infty$ the matrix elements are calculated as if there were no interactions. In the former expression, $E(p)$ is the energy of the nucleon with three-momentum $\mathbf{p}$, m_N is the nucleon mass, and $u(\mathbf{p}_n)$ is a Dirac spinor. Expressing $\dot{\psi}(x)$ in terms of $\psi(x)$ (one way is using the Dirac equation), taking into account that $\psi(x) e^{i\int dy\, \eta^\dagger(y)\psi(y)} = -i[\vec{\delta}/\delta\eta^\dagger(x)] e^{i\int dy\, \eta^\dagger(y)\psi(y)}$, the analogous expression for $\psi^\dagger(x)$, and substituting all that in Eq. (1), one obtains for $\langle \Omega_{\text{out}} | \Omega_{\text{in}} \rangle_{J,\eta,\eta^\dagger}$

$$\begin{aligned}
&\lim_{\substack{t\to+\infty\\ t'\to-\infty}} \int [dU](\det D) e^{i\int dx \mathcal{L}_{\pi\pi}} \\
&\times \left(\prod_n^N \frac{E(p_n)}{2m_N} \int d\mathbf{x}_n e^{ip_n x_n} \overline{u}(\mathbf{p}_n) \left[1 - \frac{i}{E(p_n)} \right.\right. \\
&\times \left.\left. \left(\gamma^0 \sum_{j=1}^{3} \gamma^j \frac{\partial}{\partial x_n^j} + i\gamma^0 m_N \right) \right] \frac{\vec{\delta}}{\delta \eta^\dagger(x_n)} \right) \\
&\times \exp\left(-i \int dx \int dy\, \eta^\dagger(x) D^{-1}(x,y) \gamma^0 \eta(y) \right) \\
&\times \prod_m^N \frac{E(q_m)}{2m_N} \int d\mathbf{y}_m e^{-iq_m y_m} \left[\frac{\overleftarrow{\delta}}{\delta\eta(y_m)} - \frac{i}{E(q_m)} \right. \\
&\times \left. \left(\sum_{k=1}^{3} \frac{\partial}{\partial y_m^k} \frac{\overleftarrow{\delta}}{\delta\eta(y_m)} \gamma^k \gamma^0 + i \frac{\overleftarrow{\delta}}{\delta\eta(y_m)} \gamma^0 m_N \right) \right] \gamma^0 u(\mathbf{q}_m),
\end{aligned} \tag{3}$$

where the integration over the nucleon fields and conjugate momenta is also done. Furthermore we define $x_n = (t, \mathbf{x}_n)$, $y_m = (t', \mathbf{y}_m)$, and $p_n = (E(p_n), \mathbf{p}_n)$, and we do the same analogously for q_m. The action of the spatial derivatives can be readily taken into account by integrating by parts. In this way, they only act on the corresponding exponentials giving rise to three-momenta factors that can be further simplified by applying the Dirac equation on the Dirac spinors. In this way we have

$$\overline{u}(\mathbf{p}_n) \left[1 - \sum_{j=1}^{3} \frac{p_n^j}{E(p_n)} \gamma^0 \gamma_j + \frac{m_N}{E(p_n)} \gamma^0 \right] = \frac{2m_N}{E(p)} u^\dagger(\mathbf{p}_n),$$

$$\left[1 - \sum_{j=1}^{3} \frac{p_n^j}{E(p_n)} \gamma^0 \gamma_j + \frac{m_N}{E(p_n)} \gamma^0 \right] \gamma^0 u(\mathbf{p}_n) = \frac{2m_N}{E(p)} u(\mathbf{p}_n).$$

Applying the previous results to Eq. (3) it simplifies to

$$\begin{aligned}
&\lim_{\substack{t\to+\infty\\ t'\to-\infty}} \int [dU](\det D) e^{i\int dx \mathcal{L}_{\pi\pi}} \\
&\times \left(\prod_n^N \int d\mathbf{x}_n e^{ip_n x_n} u^\dagger(\mathbf{p}_n) \frac{\vec{\delta}}{\delta\eta^\dagger(x_n)} \right) \\
&\times \exp\left(-i \int dx \int dy\, \eta^\dagger(x) D^{-1}(x,y) \gamma^0 \eta(y) \right) \\
&\times \prod_m^N \int d\mathbf{y}_m e^{-iq_m y_m} \frac{\overleftarrow{\delta}}{\delta\eta(y_m)} u(\mathbf{q}_m).
\end{aligned}$$

After acting with the left derivatives $\vec{\delta}/\delta\eta^\dagger(x_n)$ on the exponential depending on the Grassmann sources, the former expression can be recast as

$$\begin{aligned}
&\lim_{\substack{t\to+\infty\\ t'\to-\infty}} \int [dU](\det D) e^{i\int dx \mathcal{L}_{\pi\pi}} \left(\prod_n^N \int d\mathbf{x}_n e^{ip_n x_n} \right. \\
&\times \left. \int dz_{n'} u^\dagger(\mathbf{p}_n) D^{-1}(x_n, z_{n'}) \gamma^0 \eta(z_{n'}) \right) \\
&\times \exp\left(-i \int dx \int dy\, \eta^\dagger(x) D^{-1}(x,y) \gamma^0 \eta(y) \right) \\
&\times \prod_m^N \int d\mathbf{y}_m e^{-iq_m y_m} \frac{\overleftarrow{\delta}}{\delta\eta(y_m)} u(\mathbf{q}_m).
\end{aligned}$$

This result is equal to $e^{i\mathcal{Z}[v,a,s,p]}$ when $\eta\to 0$ and $\eta^\dagger\to 0$.[1] Hence we can equalize to 1 the second exponential from the left and the remaining right derivatives and $\eta(z_n)$ sources have to be paired in order to finish with a nonvanishing result. Thus we have

$$\begin{aligned}
e^{i\mathcal{Z}} = &\lim_{\substack{t\to+\infty\\ t'\to-\infty}} \int [dU](\det D) e^{i\int dx \mathcal{L}_{\pi\pi}} \sum_\sigma \epsilon(\sigma) \prod_n^N \int d\mathbf{x}_n \\
&\times \int d\mathbf{x}_{\sigma_n} e^{ip_n x_n} u^\dagger(\mathbf{p}_n) D^{-1}(x_n, x_{\sigma_n}) \gamma^0 e^{-ip_{\sigma_n} x_{\sigma_n}} u(\mathbf{p}_{\sigma_n})
\end{aligned} \tag{4}$$

with $\epsilon(\sigma)$ the signature of the permutation σ over all the indices—momenta, spin, and isospin. In order to continue let us write the operator $D(x) \equiv D_0(x) - A(x)$ with $D_0(x) = i\gamma^\mu \partial_\mu - m_N$ the Dirac operator for the free motion of the

[1] Without taking the limit $\eta, \eta^\dagger \to 0$, the generating functional contains baryonic sources and taking differential derivatives with respect to them one could directly evaluate in-medium baryonic Green functions, e.g., nucleon propagators. Nevertheless, since for our present purposes the baryons in the medium constitute just a background, we do not consider this case any further.

nucleons. On the other hand, the operator $A(x)$ is completely general although in our case at hand, with CHPT, it is subject to a chiral expansion of powers of soft three-momenta and quark masses. Furthermore, let us note that

$$\lim_{t\to+\infty}\int d\mathbf{x} e^{ipx}u^\dagger(\mathbf{p})D_0^{-1}(x,x')=-ie^{ipx'}\overline{u(\mathbf{p})},$$

$$\lim_{t\to-\infty}\int d\mathbf{x} D_0^{-1}(x',x)\gamma^0 e^{-ipx}u(\mathbf{p})=-ie^{-ipx'}u(\mathbf{p}). \quad (5)$$

This result can be easily obtained by writing $D_0^{-1}(x,x')$ in four-momentum space and then performing the integral over the temporal component of the momentum, taking care of the imposed limits. For instance, let us take the first of the previous equations. Then we have

$$\lim_{t\to+\infty}\int d\mathbf{x} e^{ipx}u^\dagger(\mathbf{p})D_0^{-1}(x,x')$$
$$=\lim_{t\to+\infty}\int d\mathbf{x} e^{ipx}u^\dagger(\mathbf{p})\int\frac{dR}{(2\pi)^4}\frac{e^{-iR(x-x')}(\not R+m_N)}{R^2-m_N^2+i\epsilon},$$

with ϵ a positive infinitesimal. Exchanging the order of the integrations, the spatial one gives rise to $(2\pi)^3\delta(\mathbf{R}-\mathbf{p})$ which fixes $\mathbf{R}$. As a result one has

$$\lim_{t\to+\infty}\int\frac{dR^0}{2\pi}u^\dagger(\mathbf{p})\frac{e^{ip^0t}e^{-iR^0(t-x_0')}e^{-i\mathbf{p}\mathbf{x}'}}{R_0^2-\mathbf{p}^2-m_N^2+i\epsilon}(R^0\gamma^0-\mathbf{p}\gamma+m_N).$$

Since $t\to+\infty$ then $t-x_0'>0$ and we close the integration contour over R^0 with a semicircle of infinite radius on the lower half-plane picking up the pole at $R^0=p^0-i\epsilon=E(p)-i\epsilon$. Applying the Dirac equation to the result, one arrives at Eq. (5).

Then taking into account Eq. (5) and the expansion $D^{-1}\gamma^0=[D_0-A]^{-1}\gamma^0=D_0^{-1}\gamma^0+D_0^{-1}AD_0^{-1}\gamma^0+\cdots,$ we can rewrite Eq. (4) as

$$e^{i\mathcal{Z}}=\int[dU](\det D)e^{i\int dx\mathcal{L}_{\pi\pi}}\sum_\sigma\epsilon(\sigma)\prod_n^N\int d\mathbf{x}_n$$
$$\times\int d\mathbf{x}_{\sigma_n}e^{-i\mathbf{p}_n\mathbf{x}_n}u^\dagger(\mathbf{p}_n)\Big[\delta(\mathbf{x}_n-\mathbf{x}_{\sigma_n})\delta_{n,\sigma_n}$$
$$-i\int dt\int dt' e^{iE(p_n)t}\gamma^0A(x_n,x_{\sigma_n})e^{-iE(p_{\sigma_n})t'}$$
$$-i\int dt\int dt'\int dzdz' e^{iE(p_n)t}\gamma^0A(x_n,z)D_0^{-1}(z,z')$$
$$\times A(z',x_{\sigma_n})e^{-iE(p_{\sigma_n})t'}+\cdots\Big]e^{i\mathbf{p}_{\sigma_n}\mathbf{x}_{\sigma_n}}u(\mathbf{p}_{\sigma_n}), \quad (6)$$

where now both t and t' are integration variables that are the time components of x_n and x_{σ_n}, respectively. The ellipsis just refers to those terms with an increasing number of insertions of the operator A coming from the geometric expansion of $D^{-1}=[D_0-A]^{-1}$ discussed above. Furthermore $A(x,y)$ is defined such that

$$\int dx\overline{\psi}(x)A(x)\psi(x)=\int dxdy\overline{\psi}(x)A(x,y)\psi(y).$$

As a result of Eq. (6) we can simply state that

$$e^{i\mathcal{Z}[v,a,s,p]}=\int[dU](\det D)e^{i\int dx\mathcal{L}_{\pi\pi}}(\widetilde{\det}\mathcal{F}), \quad (7)$$

where the tilde in $\widetilde{\det}\mathcal{F}$ indicates that the determinant has to be taken in the subspace of the Fermi-sea states expanded by the basis functions $e^{i\mathbf{p}_n\mathbf{x}}u(\mathbf{p}_n)$ with $|\mathbf{p}_n|<k_F$. In this notation $\mathcal{F}$ is given by

$$\mathcal{F}\equiv I_3-i\int dt\int dt' e^{iH_0t}\gamma^0A[I_4-D_0^{-1}A]^{-1}e^{-iH_0t'},$$

where $I_3\equiv\delta(\mathbf{x}_n-\mathbf{x}_m)\delta_{n,m}$ and analogously $I_4\equiv\delta(x_n-x_m)\delta_{n,m}$. On the other hand, $e^{-iH_0t'}e^{i\mathbf{p}_n\mathbf{x}'}u(\mathbf{p}_n)=e^{-ip_nx'}u(\mathbf{p}_n)$ and $e^{-i\mathbf{p}_n\mathbf{x}}u^\dagger(\mathbf{p}_n)e^{iH_0t}=e^{ip_nx}u^\dagger(\mathbf{p}_n)$.

In order to obtain from Eq. (7) the contributions of the surrounding medium to the generating functional it is convenient to exponentiate $\widetilde{\det}\mathcal{F}$ as $\exp(\widetilde{\mathrm{Tr}}\ln\mathcal{F})$ (where the tilde has the same meaning as before). Then we have

$$e^{i\mathcal{Z}[v,a,s,p]}=\int[dU](\det D)\exp\Bigg\{i\int dx\mathcal{L}_{\pi\pi}$$
$$+\sum_{r=1}^2\sum_{\alpha=1}^2\int^{k_F}\frac{d\mathbf{p}}{(2\pi)^3 2E(p)}\int d\mathbf{x}$$
$$\times\int d\mathbf{y}e^{-i\mathbf{p}\mathbf{x}}u_{r\alpha}^\dagger(\mathbf{p})\ln[\mathcal{F}]|_{(\mathbf{x}\,\alpha,\mathbf{y}\,\alpha)}e^{i\mathbf{p}\mathbf{y}}u_{r\alpha}(\mathbf{p})\Bigg\}, \quad (8)$$

where we have indicated explicitly the spin r and isospin α indices. It is very appropriate to stress at this point that Eq. (8), although formal, is nonperturbative. From this result we can readily read out the new effective chiral Lagrangian density $\widetilde{\mathcal{L}}_{\pi\pi}$ in the presence of a nuclear density just by equating the expression between curly brackets to $i\int dx\,\widetilde{\mathcal{L}}_{\pi\pi}$.

The perturbative theory is obtained by expanding $\ln\mathcal{F}$ in Eq. (8), so that $e^{i\mathcal{Z}}$ can be written as

$$\int [dU](\det D)$$
$$\times \exp\Bigg\{ i\int dx \mathcal{L}_{\pi\pi} - i\sum_{r=1}^{2}\sum_{\alpha=1}^{2}\int^{k_F}\frac{d\mathbf{p}}{(2\pi)^3 2E(p)}$$
$$\times \int dx\, dy e^{ipx}\overline{u_{r\alpha}}(\mathbf{p})\, A[I_4 - D_0^{-1}A]^{-1}|_{(x\alpha,y\alpha)}$$
$$\times e^{-ipy}u_{r\alpha}(\mathbf{p}) + \frac{1}{2}\sum_{r,s=1}^{2}\sum_{\alpha,\beta=1}^{2}\int^{k_F}\frac{d\mathbf{p}}{(2\pi)^3 2E(p)}$$
$$\times \int^{k_F}\frac{d\mathbf{q}}{(2\pi)^3 2E(q)}\int dx\, dx'\, dy'\, dy e^{ipx}\overline{u_{r\alpha}}(\mathbf{p})$$
$$\times A[I_4 - D_0^{-1}A]^{-1}|_{(x\alpha,x'\beta)}e^{-iqx'}u_{s\beta}(\mathbf{q})e^{iqy'}\overline{u_{s\beta}}(\mathbf{q})$$
$$\times A[I_4 - D_0^{-1}A]^{-1}|_{(y'\beta,y\alpha)}e^{-ipy}u_{r\alpha}(\mathbf{p}) + \cdots \Bigg\}.$$

Finally, taking into account the relation

$$\sum_{r=1,2} u_r(\mathbf{p})\otimes\overline{u_r}(\mathbf{p}) = \not{p} + m_N$$

we arrive at the following expression for the generating functional:

$$e^{i\mathcal{Z}} = \int [dU](\det D)$$
$$\times \exp\Bigg\{ i\int dx \mathcal{L}_{\pi\pi} - i\int^{k_F}\frac{d\mathbf{p}}{(2\pi)^3 2E(p)}\int dx\, dy e^{ip(x-y)}$$
$$\times \mathrm{Tr}(A[I_4 - D_0^{-1}A]^{-1}|_{(x,y)}(\not{p} + m_N))$$
$$+ \frac{1}{2}\int^{k_F}\frac{d\mathbf{p}}{(2\pi)^3 2E(p)}\int^{k_F}\frac{d\mathbf{q}}{(2\pi)^3 2E(q)}$$
$$\times \int dx\, dx'\, dy\, dy'\; e^{ip(x-y)}e^{-iq(x'-y')}$$
$$\times \mathrm{Tr}(A[I_4 - D_0^{-1}A]^{-1}|_{(x,x')}(\not{q} + m_N)$$
$$\times A[I_4 - D_0^{-1}A]^{-1}|_{(y',y)}(\not{p} + m_N)) + \cdots \Bigg\}, \tag{9}$$

where the traces refer both to the isospin and spinor indices. The previous formula implies a double expansion for obtaining the contributions of the nuclear medium to the chiral Lagrangian. One is the standard chiral expansion by expanding the *vacuum* operator $A[I_4 - D_0^{-1}A]^{-1} = A + AD_0^{-1}A + AD_0^{-1}AD_0^{-1}A + \cdots$ together with A itself, $A = A^{(1)} + A^{(2)} + A^{(3)} + \cdots$ in increasing powers of momenta and quark masses valid at low energies. Explicit expressions of A up to $\mathcal{O}(p^3)$ can be obtained from the meson-baryon CHPT Lagrangians given in Ref. [5]. As an example let us consider the lowest order $A^{(1)}(x)$ operator with one derivative from

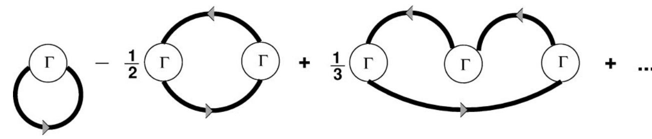

FIG. 1. Diagrammatic expansion of Eq. (9). Every thick line corresponds to the insertion of a Fermi sea and each circle to the insertion of an operator $\Gamma \equiv -iA[I_4 - D_0^{-1}A]^{-1}$.

$$\mathcal{L}_{\psi\psi}^{(1)} = \overline{\psi}(i\gamma^\mu\partial_\mu - \mathring{m}_N I_2 + i\gamma^\mu\Gamma_\mu + i\mathring{g}_A\gamma^\mu\gamma_5\Delta_\mu)\psi, \tag{10}$$

with $\Delta_\mu = \frac{1}{2}u^\dagger\nabla_\mu U u^\dagger$ in terms of the covariant derivative $\nabla_\mu U(x) = \partial_\mu U(x) - i[v_\mu(x) + a_\mu(x)]U(x) + iU(x)[v_\mu(x) - a_\mu(x)]$. Γ_μ is the chiral connection, $\Gamma_\mu = \frac{1}{2}[u^\dagger, \partial_\mu u] - i/2u^\dagger(v_\mu + a_\mu)u - i/2u(v_\mu - a_\mu)u^\dagger$. The Goldstone bosons (the pions) are collected in the 2×2 unitary matrix $u = \exp(i\phi/2f)$, $U = u^2$, and ϕ is given by

$$\phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}.$$

On the other hand, the constants $\mathring{m}_N$, $\mathring{g}_A$, and f refer to the mass, axial coupling of the nucleon, and pion weak decay constant in the SU(2) chiral limit, respectively. Taking into account the definition of the $A(x)$ operator, $A(x) \equiv D_0(x) - D(x)$, with $D_0(x) = i\gamma^\mu\partial_\mu - m_N$, we then obtain from Eq. (10)

$$A^{(1)}(x) = -i\gamma^\mu\Gamma_\mu - i\mathring{g}_A\gamma^\mu\gamma_5\Delta_\mu.$$

The difference of $\mathring{m}_N I_2 - m_N$ is $\mathcal{O}(p^2)$ [4] and then it belongs to $A^{(2)}$. The related $A^{(1)}(x,y)$ operator is just $A^{(1)}(x)\delta(x-y)$. Indeed $A^n(x,y) = A^{(n)}(x)\delta(x-y)$ because of the local character of the CHPT $\mathcal{L}_{\psi\psi}^{(n)}$ Lagrangians.

The other expansion involved in Eq. (9) is an expansion in the number of insertions of *on-shell* fermions belonging to the Fermi sea, schematically indicated in Fig. 1 by a thick solid line. Both expansions of Eq. (9) can be related by giving a chiral power counting to k_F which can be naturally counted as $\mathcal{O}(p)$ [13] since for nuclear saturation density $k_F \simeq 2M_\pi$ with M_π the pion mass. Moreover, the circles labeled by Γ correspond to the nonlocal operator $-iA[I_4 - D_0^{-1}A]^{-1}$. Note that when inserting n Fermi seas from the expansion of the logarithm one picks up a factor $-(-1)^n/n$ where the global minus sign appears due to the fermionic closed loop, n is a combinatoric factor because any cyclic permutation in the trace of n Fermi seas with their associated n Γ operators gives the same result, and finally the sign $(-1)^n$ is a pure in-medium factor that one has to keep in mind and is already present in standard many-body theory [7].

A generally nonlocal vertex Γ comes from the iteration of the A operator with intermmediate free baryon propagators D_0^{-1}, obeying the usual Feynman rules, see Fig. 2. Notice that A was defined from the Lagrangain $\overline{\psi}D\psi$, removing the free term $\overline{\psi}D_0\psi$, and changing the sign to the rest. This is why a minus sign appears in front of A in Fig. 2. On the other hand $iD_0^{-1}(x,y) = i\int[d^4p/(2\pi)^4](\not{p} + m_N)/(p^2 - m_N^2 + i\epsilon)$ is the usual baryon propagator.

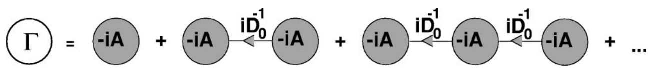

FIG. 2. Expansion of the generalized nonlocal vacuum vertex Γ. Every solid line corresponds to a vacuum baryon propagator and each circle to the insertion of an operator $-iA$ from $\bar{\psi} D \psi$.

Hence a final diagram, when expanding Γ up to the required accuracy, will be a set of $n \geqslant 1$ Fermi-sea insertions, $m \geqslant 0$ free baryon propagators, and $m+n$ vertices $-iA$. First we include the -1 global sign because of the fermionic closed loop, and the combinatoric factor $1/n$ together with the sign $(-1)^n$. Then, following the diagram in the opposite sense to that of the fermionic arrows, we write for each Fermi sea an integral $\int^{k_F} d\mathbf{p}(\not{p}+m)/(2\pi)^3 2E(p)$ with $p^0 = E(\mathbf{p})$, for each vaccum baryon propagator with free momentum p we write $i\int dp/(2\pi)^4(\not{p}+m_N)/(p^2-m_N^2+i\epsilon)$ and for a vertex in momentum space a term $-iA(2\pi)^4$, keeping in mind the energy-momentum conservation at each vertex. Finally we sum over the spin and isospin indices of the fermions. This defines explicitly the Feyman rules in momentum space in order to obtain $i(2\pi)^4$ times the desired connected graph accompanied by the global delta of energy-momentum conservation. Analogous Feynman rules hold of course in configuration space, e.g., see Eq. (9).

An equivalent way to state the previous rules, without including explicitly the integral symbols and factors 2π, is to write the same sign-combinatoric factor $-(-1)^n/n$ and vertices $-iA$ as before. Then for the free nucleons one has just the free propagator $i(\not{p}+m)/(p^2-m^2+i\epsilon)$ and for the Fermi-sea baryons the factor $(2\pi)\delta(p^2-m^2)\theta(p^0)(\not{p}+m)\theta(k_F-|\mathbf{p}|)$. Finally we sum over all the discrete indices attached to the fermions and integrate over all the free four-momenta with the measure $\int d^4p/(2\pi)^4$ after taking into account energy-momentum conservation at each vertex.

Figure 1 fixes the skeleton structure of standard in-medium *vertices* since one still has to consider the pion fields contained in A, over which one has to integrate in Eqs. (8) and (9) to finally obtain the generating functional. That is, from the vertices A as well as from $\mathcal{L}_{\pi\pi}$, one can generate internal as well as external (coupled to the sources) pionic legs denoted by dashed lines in Figs. 3(a) and 3(b). Here one has essentially the same Feynman rules as one has in vaccum in order to proceed in a perturbative way. Simply, for each pionic line with four-momentum q, one writes the vacuum propagator $i\int dq/(2\pi)^4(1/q^2-M_\pi^2+i\epsilon)$, for the first version of the in-medium Feynman rules. For the second, one has $i/(q^2-M_\pi^2+i\epsilon)$. The important remark to keep in mind is that a generalized vertex has properties analogous to those of a standard *local* quantum field theory one, to the effect of determing the numerical factors accompanying the exchange of pion lines inside a given diagram. This can be seen just by applying standard perturbative techniques in path integrals to the action given between brackets in Eq. (8) or more explicitey in Eq. (9). Several examples are discussed in detail in Ref. [13].

Notice also that ultraviolet parts in the integration over running pionic momenta generate local multinucleon vertices and hence a proper treatment of pion loops can only be done

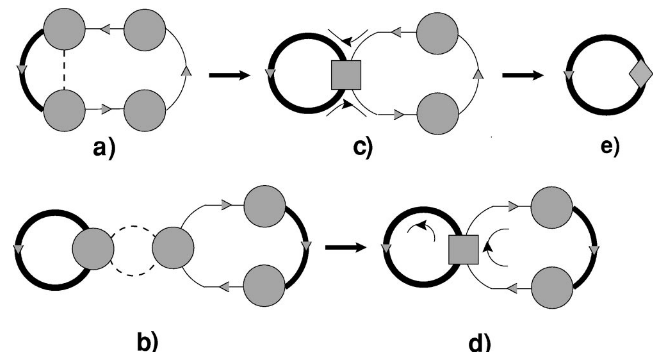

FIG. 3. (a) and (b) represent some typical many-particle diagrams generated from Eqs. (8) and (9). The circle indicates an A operator insertion (which in addition can have attached to it more lines than shown) and the dashed line corresponds to a pion exchange. (c) and (d) arise by considering the ultraviolet divergent part of the pion loops leading to local terms denoted by squares. (e) is a local πN counterterm, indicated by a diamond. For more details see the text.

by including simultaneously local nucleon interactions in order to reabsorb the divergences. This is denoted by the squares in Figs. 3(c) and 3(d). It is important to realize that while in Fig. 3(a) the pion is exchanged inside the same vertex (in the generalized sense of above) in Fig. 3(b) the pions are exchanged between two of them. This implies the presence of only one Dirac trace in Fig. 3(c) (the flow of the Dirac indices along the propagators on both sides of the square is indicated by the arrows of the open solid lines) and two Dirac traces in Fig. 3(d). Thus, when writing the matrix elements corresponding to diagrams of the type in Figs. 3(c) and 3(d) (with local interactions) one has to keep track of the number of closed traces in the spinor indices since each of them will lead to a sign (-1) and its own combinatoric factor $1/n_j$ (with n_j the number of Fermi-sea insertions in the closed Dirac loop) together with the sign $(-1)^{n_j}$. One can also arrive at the same conclusions by considering that the dashed lines originate from the exchange of an arbitrary heavy meson (the only relevant point in our derivations is the bilinear character of the meson-$\bar{N}N$ interaction in the nucleon fields). In this way it is straightforward to realize the presence of a factor 1/2 in front of Fig. 3(d) together with the factors $\Pi_j(-1)^{n_j+1}/n_j$, which here are simply 1. The relation between contact four nucleon interactions at low energies in effective field theories and its saturation by integrating out heavy meson resonances has been established in Ref. [14].

There is still an important difference to be discussed when comparing Figs. 3(c) and 3(d), which in fact is related to the presence of the factor $(\det D)$ in all the formulas from Eq. (3) to Eq. (9). The latter corresponds to contributions to the chiral Lagragian from closed fermion loops in the vacuum and in the spirit of the effective field theories these contributions, coming from states with masses close to or above the chiral scale $\Lambda_\chi \approx M_\rho$, are incorporated in the counterterms of the vacuum effective field theory. In this way we will set $\det D = 1$ in the following. Indeed on the right hand side of the square of Fig. 3(c) we can recognize a momentum loop flowing along the free baryon propagators without any insertion of a Fermi sea. This simply means that since all the momenta that could go into this closed momentum loop are soft because they come from the circles, it just corresponds to vacuum renormalization from heavy particles and is reabsorbed in the low energy counterterms. This is schematically shown in Fig. 3(e) with a diamond corresponding to a higher order πN counterterm. This is consistent as far as one is restricted to the low energy and momentum regime. However, when a Fermi-sea baryon line is present the momentum running though the baryon lines is of the form $p_j = p + Q_j$ with $p = [E(p), \mathbf{p}]$ and $|\mathbf{p}| < k_F$. Thus one is inserting a medium parameter k_F and shifting upward up to around m_N the energy level around which one is perturbating.

We now turn to the generalization of the formalism to the case of asymmetric nuclear matter, with different densities of neutrons, ρ_n, and protons, ρ_p, with Fermi momenta $k_F^{(n)} = (3\pi^2\rho_n)^{1/3}$ and $k_F^{(p)} = (3\pi^2\rho_p)^{1/3}$, respectively. Following the previous derivation of Eq. (8) one can easily convince oneself that the only change is to remove the sum over isospin indices and to distinguish between $\alpha = 1$ (proton) and $\alpha = 2$ (neutron). In this way we will have

$$e^{i\mathcal{Z}[v,a,s,p]} = \int [dU] \exp\Bigg\{ i \int dx \mathcal{L}_{\pi\pi} + \sum_{r=1}^{2} \int^{k_F^{(p)}} \frac{d\mathbf{p}}{(2\pi)^3 2E(p)_1} \int d\mathbf{x} \int d\mathbf{y} e^{-i\mathbf{p}\mathbf{x}} u_{r\,1}^\dagger(\mathbf{p}) \ln[\mathcal{F}]|_{(\mathbf{x}\,1,\mathbf{y}\,1)} e^{i\mathbf{p}\mathbf{y}} u_{r\,1}(\mathbf{p})$$
$$+ \sum_{r=1}^{2} \int^{k_F^{(n)}} \frac{d\mathbf{p}}{(2\pi)^3 2E(p)_2} \int d\mathbf{x} \int d\mathbf{y} e^{-i\mathbf{p}\mathbf{x}} u_{r\,2}^\dagger(\mathbf{p}) \ln[\mathcal{F}]|_{(\mathbf{x}\,2,\mathbf{y}\,2)} e^{i\mathbf{p}\mathbf{y}} u_{r\,2}(\mathbf{p}) \Bigg\}. \tag{11}$$

Notice that we have indicated separately the energies of protons $E(p)_1$ and neutrons $E(p)_2$ with three-momentum $\mathbf{p}$, since the previous equation is valid for the nonisospin limit as well. Nevertheless, in order to simplify the formulas, we will consider in the following the case with equal nucleon masses. We also introduce the 2×2 matrix:

$$n(p) = \begin{pmatrix} \theta(k_F^{(p)} - p) & 0 \\ 0 & \theta(k_F^{(n)} - p) \end{pmatrix} \equiv \begin{pmatrix} n(p)_1 & 0 \\ 0 & n(p)_2 \end{pmatrix} = \hat{n}(p) I_2 + \overline{n}(p)\tau_3 \tag{12}$$

with I_2 the 2×2 unity matrix, τ_3 the usual Pauli matrix diag$(1,-1)$, and $\hat{n}(p) = [n(p)_1 + n(p)_2]/2$ and $\overline{n}(p) = [n(p)_1 - n(p)_2]/2$. Then Eq. (11) can be rewritten as

$$e^{i\mathcal{Z}[v,a,s,p]} = \int [dU] \exp\Bigg\{ i \int dx \mathcal{L}_{\pi\pi} + \int d\mathbf{x} \int d\mathbf{y} \int \frac{d\mathbf{p}}{(2\pi)^3 2E(p)} e^{-i\mathbf{p}(\mathbf{x}-\mathbf{y})} \mathrm{Tr}\{ n(p) \ln[\mathcal{F}]|_{(\mathbf{x},\mathbf{y})} (\not{p} + m) \} \Bigg\}$$
$$= \int [dU] \exp\Bigg\{ i \int dx\, \mathcal{L}_{\pi\pi} - i \int \frac{d\mathbf{p}}{(2\pi)^3 2E(p)} \int dx\, dy\, e^{ip(x-y)} \mathrm{Tr}\{ A[I_4 - D_0^{-1}A]^{-1}|_{(x,y)} (\not{p} + m_N) n(p) \}$$
$$+ \frac{1}{2} \int \frac{d\mathbf{p}}{(2\pi)^3 2E(p)} \int \frac{d\mathbf{q}}{(2\pi)^3 2E(q)} \int dx\, dx'\, dy\, dy'\, e^{ip(x-y)} e^{-iq(x'-y')}$$
$$\times \mathrm{Tr}\{ A[I_4 - D_0^{-1}A]^{-1}|_{(x,x')} (\not{q} + m_N) n(q) A[I_4 - D_0^{-1}A]^{-1}|_{(y',y)} (\not{p} + m_N) n(p) \} + \cdots \Bigg\}. \tag{13}$$

Comparing this equation with Eq. (9), the only difference in the Feynman rules is the inclusion of an isospin matrix $n(p)$ associated to every Fermi-sea insertion.

The case of polarized nuclear matter can be treated in the same way just by doing in Eq. (11) the replacement

$$\sum_{r=1}^{2} \int^{k_F} d\mathbf{p} u_{r\alpha}^\dagger(\mathbf{p}) \cdots u_{r\alpha}(\mathbf{p}) \rightarrow \int^{k_F} d\mathbf{p} u_{r_1\alpha}^\dagger(\mathbf{p}) \cdots u_{r_1\alpha}(\mathbf{p}) + \int^{k_F} d\mathbf{p} u_{r_2\alpha}^\dagger(\mathbf{p}) \cdots u_{r_2\alpha}(\mathbf{p}) \tag{14}$$

since there are two spin states per momentum state.

The present formalism is applied in Ref. [13] to evaluating several quantities relevant for low energy QCD in the nuclear medium. There we will also address in detail the issue of the chiral counting from Eq. (13) in the nuclear medium and the limitations of a plain perturbative treatment of CHPT at finite density. We will mention that instead of a relativistic treatment of the baryons we could also have considered in the same way the nonrelativistic case. This makes more straightforward the evaluation of baryon loops [15] although one has also to take into account recent developments in the field of effective field theories with propagation of relativistic heavy particles [16–18]. In any case the main issue still to be addressed in the medium, as discussed in Refs. [16,14], is to deal with the problem of the large *S*-wave scattering lengths in the nucleon-nucleon scattering which introduces a new extra scale of only ∼10 MeV already in the vaccum case, where consistent power counting schemes have been developed [19]. Some interesting findings in this direction, requiring further consideration, can be found in Ref. [20] for a theory without pions.

To conclude, we have derived the SU(2) chiral Lagrangian with external sources in the presence of nonzero nuclear density by explicitly working out in quantum field theory the in-medium contributions from the πN chiral Lagrangian, Eq. (9) (symmetric nuclear matter), Eq. (13) (asymmetric nuclear matter), and Eq. (14) (asymmetric and unpolarized nuclear matter). Then the perturbative theory was developed and the corresponding Feynman rules were given. Contrarily to standard many-body techniques, the rules and diagrams derived here for the general relativistic case are analogous to the usual ones from vacuum quantum field theory without modification of the baryon propagators, establishing a neat separation between in-medium and vacuum contributions, as indicated in Figs. 1 and 2. Applications of this many-particle relativistic quantum field theory formalism to actual calculations can be found in Ref. [13].

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