Geometric relation between centrality and the impact parameter in relativistic heavy-ion collisions

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We show, under general assumptions which are well satisfied in relativistic heavy-ion collisions, that the geometric relation of centrality c to the impact parameter b, namely, $c \simeq \pi b^2/\sigma_{\rm inel}$, holds to a very high accuracy for all but most peripheral collisions. More precisely, if $c(N)$ is the centrality of events with a multiplicity higer than N, then b is the value of the impact parameter for which the average multiplicity $\overline{n}(b)$ is equal to N. The corrections to this geometric formula are of the order $(\Delta n(b)/\overline{n}(b))^2$, where $\Delta n(b)$ is the width of the multiplicity distribution at a given value of b; hence they are very small. In other words, the centrality effectively measures the impact parameter.

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Data from relativistic heavy-ion collisions (SPS, RHIC) are typically categorized by introducing *centrality*, c, defined as the percentile of events with the largest number of produced particles (as registered in detectors), or the largest number of participants (as determined from zero-degree calorimeters). We denote this number generically as n. Results of measurements, such as multiplicities [1,2], $p_\perp$ spectra [3–5], the elliptic flow coefficient v_2 [6,7], the HBT radii [8], etc., are then presented for various centralities. From an experimental viewpoint the centrality is a good, unambiguous criterion, allowing one to divide the data. On the other hand, theoreticians need to assign an impact parameter b to a given centrality. The impact parameter is in a sense more basic, since it determines the initial geometry of the collision and appears across the formalism. Theoretical calculations in heavy-ion physics input b in order to obtain predictions. Having done the calculation, the question arises as to which centrality data the model results should be compared to. For this purpose one typically applies the Glauber model in order to compute the number of wounded nucleons or binary collisions at a given b, which are subsequently related to multiplicities or number of participants [9,10]. Since these are measured in the experiment, one is able to identify b with c.

In this paper we argue that such an effort is not necessary, since, under general assumptions which hold very well in relativistic heavy-ion collisions, we have, to a very high precision, the relation

$$c(N) \simeq \frac{\pi b(N)^2}{\sigma_{\rm inel}} \quad \text{for} \quad b < \overline{R}, \tag{1}$$

where $\sigma_{\rm inel}$ is the total inelastic nucleus-nucleus cross section, and $\overline{R}$ is of the order of the sum of the radii of the colliding nuclei. The centrality $c(N)$ is the centrality of events with a multiplicity higher than N, while $b(N)$ is the value of the impact parameter for which the average multiplicity $\overline{n}(b)$ is equal to N. As will be shown, Eq. (1) holds to a high accuracy for all but most peripheral collisions. Note that it is geometric in nature, and does not explicitly involve the variable n needed to categorize the data (multiplicities, number of participants, number of binary collisions, etc.). At first glance, this fact may seem a bit surprising.

One can explain the geometric nature of Eq. (1), and the fact that it does not explicitly depend on n, with the following pedagogical example. Consider a competition where archers are shooting at a target of radius R, each of them once. The archers are very poor, such that they shoot randomly. They are paid according to their aim: the more central the aim, the higher the reward. We are not allowed to watch the competition; hence do not know which spot on the target has been hit, but we review the reward records later. Suppose a large number of archers scored (here we take only ten in order to write down the results explicitly), and are ranked according to their prizes, which are \$100, \$100, \$50, \$50, \$50, \$10, \$10, \$10, \$10, and \$10. The two archers who received the highest prize (\$100 in this case) had to hit the bull's eye. Since these comprise 20% of all archers, and they were shooting randomly, we can immediately determine (neglecting the statistical error) the radius b of the bull's eye, since 20% is the ratio of the area of the bull's eye to the total area of the target: $20\% = \pi b^2/(\pi R^2)$. Therefore, $b = R\sqrt{20\%}$. Now imagine another competition is held, with all rules the same but the prizes assigned differently to the rings of the target. Suppose the ten archers got \$500, \$500, \$100, \$100, \$100, \$50, \$50, \$50, \$50, and \$50. Again, we can determine that the 20% of the highest rewards correspond to hitting the central spot, and can determine its radius b exactly as before. Note that in the determination of b we are not using the actual values of the rewards at all—the function used can be any monotonic function of the centrality. The rewards are only used to *categorize* the data. Once this is done, we can identify the c "most central" archers and determine b according to Eq. (1), irrespective of the function used for categorizing. Our example can be translated into heavy-ion collisions in the following way: archery competition–heavy-ion experiment, archer that scored–event, rewards in competition I–number of participants, rewards in competition II–multiplicity of produced particles, percentile of highest-scoring archers–centrality, radii of rings on the target–impact parameters.

The above example shows the essence of our argument, valid for the classical physics of relativistic heavy-ion collisions. There are, however, two additional features which need to be considered. First, a collision at a particular impact parameter b produces values of n which are statistically distributed around some mean value $\overline{n}(b)$ with a distribution width $\Delta n(b)$. As we will show, Eq. (1), formally valid at $\Delta n(b) \ll \overline{n}(b)$, is accurate even for a realistically large value of $\Delta n(b)$, such as obtained from statistical models of particle production. Second, there are boundary effects near $b \sim R$—at lower values of b the inelastic cross section is the cross section for colliding *black disks*, whereas at the boundary the target gradually becomes transparent.

We now proceed with a formal derivation. Let $P(n)$ denote the probability of obtaining a value n for the categorizing function (multiplicity of produced particles, number of participants, number of binary collisions, etc.). For simplicity of language we call this *multiplicity*, bearing in mind it could be any of these quantities. The centrality c is defined as the cumulant of $P(n)$, namely,

$$c(N) = \sum_{n=N}^{\infty} P(n). \tag{2}$$

Thus $c(N)$ is the probability of obtaining an event with a multiplicity larger than or equal to N. A particular value of the multiplicity n may be collected from collisions with various impact parameters b'; thus we can write

$$c(N) = \sum_{n=N}^{\infty} \int_0^{\infty} \frac{2\pi b' db'}{\sigma_{\text{inel}}} \rho(b') P(n|b'), \tag{3}$$

where $2\pi b' db'$ is the area of the ring between impact parameters b' and $b'+db'$, the quantity $\rho(b')$ is the probability of an event (inelastic collision) at impact parameter b', and $P(n|b')$ is the conditional probability of producing multiplicity n provided the impact parameter is b'. The function $\rho(b')$ is unity for b' below R, and drops smoothly to zero at b' around R, reflecting the washed-out shape of the nuclear density functions at the edges. The interpretation of Eq. (3) is clear: the probabilities for hitting the ring between b' and $b'+db'$, the probability for an event to occur at b', and the probability to produce multiplicity n (provided the event occurred at b') are multiplied, as requested by the classical nature of the problem. Since we have $\sum_{n=1}^{\infty} P(n|b') = 1$, and, by definition, $\int_0^{\infty} 2\pi b' db' \rho(b') = \sigma_{\text{inel}}$, we verify the proper normalization in Eq. (3), namely, $c(1)=1$. Furthermore, for heavy nuclei we may use the continuity limit $\sum_{n=N}^{\infty} \to \int_N^{\infty} dn = \int_0^{\infty} dn\, \theta(n-N)$.

The function $P(n|b')$ is not known; however, by the statistical nature of the particle production, and from the experience of various models, we expect that for large values of n it is narrowly peaked around an average value $\overline{n}(b')$. Thus we begin our study by taking the limit of an infinitely narrow distribution, $P(n|b') = \delta[n - \overline{n}(b')]$. In this case,

$$c(N) = \int_0^{\infty} dn\, \theta(n-N) \int_0^{\infty} \frac{2\pi b' db'}{\sigma_{\text{inel}}} \rho(b') \delta[n - \overline{n}(b')]$$
$$= \int_0^{\infty} \frac{2\pi b' db'}{\sigma_{\text{inel}}} \rho(b') \theta[\overline{n}(b') - N]. \tag{4}$$

Since $\overline{n}(b')$ is a monotonically *decreasing* function of b', we have $\theta[\overline{n}(b') - N] = \theta[b(N) - b']$, where $b(N)$ is the solution of the equation $\overline{n}(b) = N$. Therefore,

$$c(N) = \int_0^{\infty} \frac{2\pi b' db'}{\sigma_{\text{inel}}} \rho(b') \theta[b(N) - b']$$
$$= \int_0^{b(N)} \frac{2\pi b' db'}{\sigma_{\text{inel}}} \rho(b') = \frac{\sigma_{\text{inel}}[b(N)]}{\sigma_{\text{inel}}}, \tag{5}$$

where $\sigma_{\text{inel}}(b(N))$ is the inelastic cross section accumulated from $b' \leq b(N)$. Equation (5) is a generalization of formula (1). In Ref. [11] it was quoted in the context of the Glauber model. We note that although c and b depend implicitly on N, their relation does not explicitly involve N.

We now turn to a quantitative analysis of dispersion effects. Assume that

$$P(n|b') = \frac{1}{\Delta n(b') \sqrt{2\pi}} \exp\left(- \frac{[n - \overline{n}(b')]^2}{2 \Delta n(b')^2} \right), \tag{6}$$

which is a good approximation when $\Delta n(b') < \overline{n}(b')$. Then

$$c(N) = \int_0^{\infty} \frac{2\pi b' db'}{\sigma_{\text{inel}}} \rho(b') \left\{ \frac{1}{2} \left[\text{erf}\left(\frac{\overline{n}(b') - N}{\sqrt{2} \Delta n(b')} \right) + 1 \right] \right\}. \tag{7}$$

For small $\Delta n(b')$ the function in curly brackets resembles the function $\theta[\overline{n}(b') - N]$, washed out over the range $\Delta n(b')$. Thus we introduce the function

$$d(x) = \frac{1}{2} \left[\text{erf}\left(\frac{x}{\sqrt{2} \Delta n} \right) + 1 \right] - \theta(x). \tag{8}$$

The integral of $d(x)$ with a regular function $f(x)$ can be expanded in even powers of Δn as follows (this is analogous in spirit to the Sommerfeld expansion of the Fermi-Dirac distribution function at low temperatures):

$$\int dx f(x) d(x) = - \sum_{j=1,3,5,\ldots} a_j (\Delta n)^{j+1} \left. \frac{d^j f(x)}{dx^j} \right|_{x=0}, \tag{9}$$

with the coefficients

$$a_j = \frac{1}{j!} \int_{-\infty}^{\infty} dx x^j d(x) = \frac{2^{(j+3)/2} \Gamma\left(\frac{j}{2} + 1 \right)}{\sqrt{\pi} (j+1)!},$$

$$a_1 = 1, \quad a_3 = \frac{1}{4}, \quad a_5 = \frac{1}{24}, \quad a_7 = \frac{1}{192}, \ldots . \tag{10}$$

We rewrite the integral in Eq. (7) as $\int 2b' db' = \int d\overline{n} db'^2 / d\overline{n}$, and use expansion (9) to obtain

$$c(N)=\frac{\sigma_{\rm inel}[b(N)]}{\sigma_{\rm inel}} -[\Delta n(b(N))]^2\frac{d}{d\bar{n}}\left(\frac{\pi\rho[b(\bar{n})]}{\sigma_{\rm inel}}\frac{db^2(\bar{n})}{d\bar{n}}\right)\Bigg|_{\bar{n}=N}-\cdots . \tag{11}$$

For inner b, where $\rho(b(\bar{n}))\simeq 1$, the correction term is proportional to $d^2[b^2(\bar{n})]/d\bar{n}^2$. In the models considered below this quantity is proportional to $1/\bar{n}^2$, and as a result $c(N)=\sigma_{\rm inel}[b(N)]/\sigma_{\rm inel}+O(\Delta n^2/\bar{n}^2)$, quantitatively showing that the geometric identification [Eq. (1) or (5)] is good for narrow distributions.

In order to illustrate the above results and to obtain more detailed numerical estimates for the corrections, we consider two models: a model inspired by the *wounded-nucleon* model [12], and the optical limit of the *Glauber model* [13] for the binary collisions. A combination of these models has been used to explain the observed hadron multiplicities produced in RHIC [14]. We look at the Au+Au reaction, with the nucleus density profile $\rho_A(r)$ described be the standard Woods-Saxon function with the radius $r_0=(1.12A^{1/3}–0.86A^{-1/3})$ fm, with $A=197$, and the width parameter $a=0.54$ fm. The nucleus-nucleon thickness function is given by $T_A(s)=\int_{-\infty}^{\infty}dz\rho_A(\sqrt{s^2+z^2})$, and the average number of wounded nucleons is

$$\bar{n}(b)=2A\int_0^\infty s ds\int_0^{2\pi}d\varphi T_A(\sqrt{s^2+b^2+2sb\cos\varphi}) \times\{1-[1-\sigma T_A(s)]^A\}, \tag{12}$$

where, following Ref. [14], we take $\sigma=40$ mb as the nucleon-nucleon inelastic cross section. The total nucleus-nucleus cross section obtained in this model is $\sigma_{\rm inel}=7.05$ b. The expressions for the dispersion of wounded nucleons produced at a given b is very complicated. Instead of computing multidimensional integrals, we explore, for our illustrative purpose, two cases: $\Delta n\sim\bar{n}$, and $\Delta n\sim\sqrt{\bar{n}}$. Led by the sample numerical results for the distributions given in Fig. 1 of Ref. [15], we take (i) $\Delta n=\bar{n}/10$, or (ii) $\Delta n=\sqrt{\bar{n}}$. In Fig. 1 we show the results of computing $c(N)$ according to Eqs. (7) and (12) with $\rho(b')=\theta(\sqrt{\sigma_{\rm inel}/\pi}-b')$, and for the choices (i) and (ii) (dot-dashed and dashed lines, respectively). These are compared to $\pi b(N)^2/\sigma_{\rm inel}$ (solid line), where $b(N)$ is defined as the solution of the equation $\bar{n}(b)=N$. The curves overlap within the width of the line, except for tiny regions at very low N ($N<2$), corresponding to very peripheral collisions, and at large N, corresponding to b around 0. The discrepancy at large N follows from the fact that $c(N)$ evaluated exactly continues to be nonzero till the maximum value of wounded nuclei, $N=2A$, whereas $b(N)$ by construction goes to zero at $N=\bar{n}(b=0)\approx 377$. This effect is visible in Fig. 1 only for choice (i) for the widths.

We can treat the dependence on N as parametric, and plot $c[b(N)]$ vs $b(N)$. The result is shown in Fig. 2(a). Again, the model curves for $c(b)$ for choices (i) and (ii) overlap with the curve $\pi b^2/\sigma_{\rm inel}$ except for very peripheral ($b>14$ fm) and very central ($b<2$ fm) collisions. This behavior directly reflects the behavior of Fig. 1. The size of the correction of Eq. (11) is, at intermediate b, of the order of 10^{-3}.

As another illustrative example we consider the Glauber model of nucleus-nucleus collisions and analyze binary collisions, $n=n_{coll}$. We use the optical limit of the model, which results in simple expressions. In this model

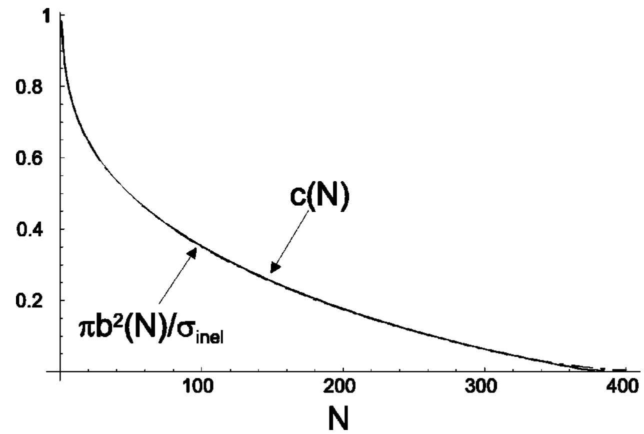

FIG. 1. Centrality in models (i) and (ii) (dot-dashed and dashed lines), and the function $\pi b(N)^2/\sigma_{\rm inel}$ (solid line), plotted as functions of the number of participants, N.

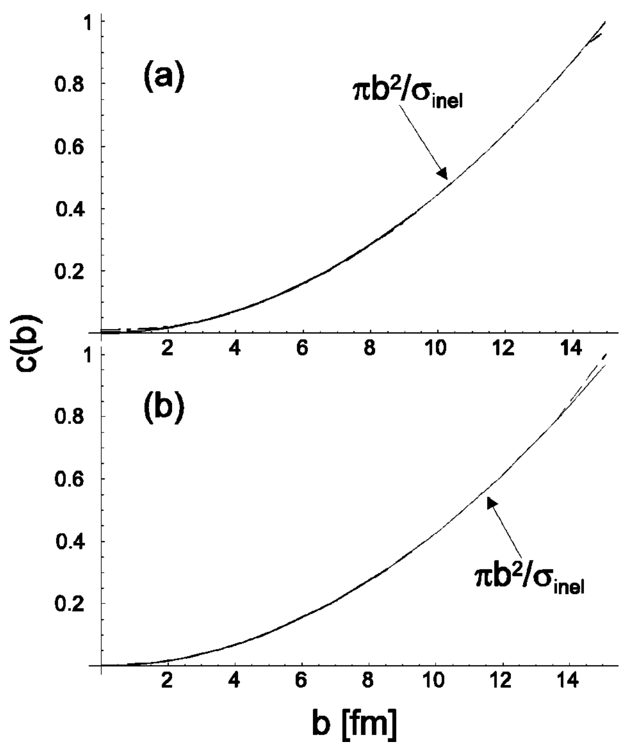

FIG. 2. (a) Centrality as a function of the impact parameter for models (i) (dot-dashed line) and (ii) (dashed line). (b) The same for the Glauber model for binary collisions (dashed line). The solid line shows $\pi b^2/\sigma_{\rm inel}$.

$$c(N)=\sum_{n=N}^{A^2}\int_0^\infty \frac{2\pi b' db'}{\sigma_{\rm inel}} P_G(n,b') =\sum_{n=N}^{A^2}\int_0^\infty \frac{2\pi b' db'}{\sigma_{\rm inel}} \binom{A^2}{n}[T(b')\sigma]^n \times[1-T(b')\sigma]^{A^2-n}, \tag{13}$$

where for $P_G(n,b')$ we have used the formula for the probability of the occurrence of n inelastic baryon-baryon collisions at an impact parameter b' [13] [note that P_G plays the role of the product $\rho(b')P(n|b')$ from the previous discussion]. Here $T(b)$ is the nucleus-nucleus thickness function:

$$T(b)=\int_0^\infty ds\int_{-\infty}^\infty dz_A\int_{-\infty}^\infty dz_B\int_0^{2\pi}d\varphi \times\rho(\sqrt{s^2+z_A^2})\rho(\sqrt{s^2+b^2+2sb\cos\varphi+z_B^2}). \tag{14}$$

The sum in Eq. (13) can be carried out exactly, yielding, with the notation $x=T(b')\sigma$, the expression

$$c(N)=\int_0^\infty \frac{2\pi b' db'}{\sigma_{\rm inel}}\binom{A^2}{n}(1-x)^{A^2}x^N \times {}_2F_1\left(1,N-A^2;N+1;\frac{x}{x-1}\right). \tag{15}$$

We perform the integration in Eq. (15) numerically. On the other hand, the average number of collisions at a fixed value of the impact parameter b is $\overline{n}(b^2)=A^2T(b)\sigma$. Repeating the steps of the previous example results in the identification $c(N)=c[\overline{n}(b^2)]=c[A^2T(b)\sigma]$. This function is compared to $\pi b^2/\sigma_{\rm inel}$ in Fig. 2(b). The agreement is excellent, and the two curves are indistinguishable except for very peripheral collisions ($b>13.5$ fm).

With a Gaussian parametrization of the thickness function, the model can be treated analytically a bit further. We use $T(b')=1/(2\pi\beta^2)\exp[-b'^2/(2\beta^2)]$, with $\beta=4.6$ fm, which leads to a quite good approximation of the exact thickness function. Then

$$c(N)=\frac{2\pi\beta^2}{\sigma_{\rm inel}}\sum_{n=N}^{A^2}\binom{A^2}{n}\int_0^{\sigma/(2\pi\beta^2)}dy\, y^{n-1}(1-y)^{A^2-n} =\frac{2\pi\beta^2}{\sigma_{\rm inel}}\sum_{n=N}^{AB}\frac{1}{n}I_{\sigma/(2\pi\beta^2)}(n,1+A^2-n), \tag{16}$$

where $I_z(a,b)=B_z(a,b)/B_1(a,b)$, and $B_z(a,b)$ is the incomplete beta function. For large A^2 and small $\sigma/(2\pi\beta^2)$ the function $I_{\sigma/(2\pi\beta^2)}(n,1+A^2-n)$ is well approximated by the step function $\theta[A^2\sigma/(2\pi\beta^2)-n]$. Replacing the sum by the integral in Eq. (16), we find the leading expression

$$c(N)=-\frac{2\pi\beta^2}{\sigma_{\rm inel}}\ln\left(\frac{2\pi\beta^2}{AB\sigma}N\right). \tag{17}$$

On the other hand, $b^2(N)=-2\beta^2\ln[2\pi\beta^2N/(AB\sigma)]$, which immediately results in Eq. (1). Since $\Delta n^2=A^2T(b)\sigma[1-T(b)\sigma]\simeq\overline{n}$, the correction of Eq. (11) becomes $-2\pi\beta^2/\sigma_{\rm inel}(\Delta n/N)^2\simeq-2\pi\beta^2/\sigma_{\rm inel}(1/N)\simeq-0.2/N$, and hence is very small at large N.

As already mentioned, there were attempts [14] to explain the multiplicity of produced particles through a combination of the *wounded* nucleon model [12], associated with soft processes, and production proportional to the number of binary nucleon-nucleon collisions, associated with hard physics. The folding of the distributions of wounded nucleons, n_w, or number of collisions, $n_{\rm coll}$, with the distribution of particles produced in an elementary event (by the wounded nucleon or in a single binary collision), may result in a broadening effect in the observed distribution of the multiplicity of the produced particles, n. However, we expect this broadening to be negligible in the ratio $\Delta n/\overline{n}$, which is the quantity controlling the accuracy of Eq. (1). In particular, for the wounded nucleon model [15] one has $(\Delta n/\overline{n})^2=2(\Delta_H)^2/(\overline{n}_w\overline{n}_H^2)+(\Delta n_w/\overline{n}_w)^2$, where the subscript $_H$ refers to the nucleon-nucleon collision. Assuming $(\Delta_H)^2\sim\overline{n}_H$, we find that the contribution from the first term is smaller than that from the second term already for moderately large $\overline{n}_w$, and $\Delta n/\overline{n}\simeq\Delta n_w/\overline{n}_w$. This indicates that Eq. (1) remains very accurate when the multiplicities of produced particles are used as the centrality criterion.

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