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Relativistic approaches to structure functions of nuclei

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We employ a propagator technique to derive a new relativistic $1/|\boldsymbol{q}|$ expansion of the structure function of a nucleus, composed of point nucleons. We exploit nonrelativistic features of low-momentum nucleons in the target and only treat relativistically the nucleon after absorption of a high-momentum virtual photon. The new series permits a three-dimensional reduction of each term and a formal summation of all final state interaction terms. We then show that a relativistic structure function can be obtained from its nonrelativistic analog by a mere change of a scaling variable and the addition of an energy shift. We compare the obtained result with an *ad hoc* generalized Gersch-Rodriguez-Smith theory, previously used in computations of nuclear structure functions.

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I. INTRODUCTION

The major tool for computing nuclear structure functions, as measured in inclusive electron scattering on nuclei, is the impulse (or Born) series (IS) in the residual interaction between the struck nucleon and the remaining spectator nucleus. The lowest-order term of that series is the widely used impulse approximation (IA). Higher-order, final state interaction (FSI) terms are essential for an accurate calculation of the data, but their determination in practice constitutes a formidable problem (see, for instance, Refs. [1–6]).

In the nonrelativistic regime there exists an alternative approach, originally proposed by Gersch, Rodriguez, and Smith (GRS) [7]. There the structure function is expressed in terms of commutators involving the residual interaction and appears, for fixed values of a scaling variable y [7,9], as a series in inverse powers of the three-momentum transfer $|\boldsymbol{q}|$. That theory has extensively been used to compute structure functions (or responses) of quantum gases [8].

If convergent, the GRS and IS approaches, taken to all orders, obviously produce identical results, but this is not the case if these series are truncated at some finite order. An issue is then which of the truncated series is a better approximation to the total structure function. Judged by the lowest-order terms applied to classes of exactly solvable models, the GRS expansion is to be preferred over the IS [5,10–13].

The availability of data obtained with high-energy beams requires a theory valid for the relativistic regime. In the IS, final state interactions are summed by means of a four-dimensional scattering operator, which satisfies a coupled-channel Bethe-Salpeter equation, but their solution remains a complicated, relativistic many-body problem.

As regards the GRS approach, no satisfactory relativistic extension of the nonrelativistic GRS theory has been formulated before. A start has been made by one of the authors, who previously exploited a propagator technique for the desciption of the structure function of composite systems similar to the one used for nonrelativistic systems. Formally exact expressions have been derived for relativistic structure functions [12,13] in terms of four-dimensional integrals over relativistic propagators and scattering operators, as is the case in the relativistic IS treatment of structure functions.

In this paper we develop a relativistic GRS series for structure functions exploiting manifestly nonrelativistic features of the system. There we shall emphasize that only the nucleon which absorbs the virtual photon in inclusive scattering acquires a large momentum and has to be treated relativistically. All others nucleons have nonrelativistic momenta and can be treated accordingly.

We shall show below that the above nonrelativistic features permit an accurate three-dimensional reduction of all terms in the relativistic GRS series for a structure function. This feature gives the GRS series a definite advantage over the IS. For it a three-dimensional reduction is very involved due to negative energy poles in relativistic nucleon propagators.

The outline of this paper is as follows. In Sec. II we rederive the nonrelativistic GRS series, showing the way to a relativistic extension, which is performed in Sec. III. In Sec. IV we exploit nonrelativistic features of the problem and subsequently prove a three-dimensional reduction of the lowest-order and of all higher-order FSI terms of the relativistic GRS series. We demonstrate that the latter can be summed in a closed expression, involving a three-dimensional Lippmann-Schwinger, T matrix. This reduces an evaluation of the relativistic nuclear structure functions to a nonrelativistic problem. In the end we relate our final expressions with approximative representations of the relativistic GRS series, which have been used in a description of nuclear structure functions.

II. NONRELATIVISTIC TREATMENTS OF THE STRUCTURE FUNCTION

A. Impulse series

We start with the nonrelativistic structure function per nucleon, $W(\nu,\boldsymbol{q})$, appropriate to a nucleus of A point nucleons where ν and $\boldsymbol{q}$ are the energy and momentum transferred to the target. In order to simplify the algebra, we restrict our derivation to the case of spinless particles. We focus on the incoherent part of W which dominates for large $|\boldsymbol{q}|$ and exploit its relation to the imaginary part of the forward Compton amplitude:

$$W(\nu,\boldsymbol{q}) = -\frac{1}{\pi}\text{Im}\langle\Phi_A|\mathcal{Q}_1^{\dagger}(\nu,\boldsymbol{q})G_A(E_A,0)\mathcal{Q}_1(\nu,\boldsymbol{q})|\Phi_A\rangle$$
$$\equiv -\frac{1}{\pi}\text{Im}\langle\Phi_A|G_{1,A}(E_A+\nu,\boldsymbol{q})|\Phi_A\rangle. \quad (1)$$

Here Φ_A is the ground state wave function of the target with energy E_A, and $G_A(E_A,0)=(E_A-H_A)^{-1}$, the exact Green's function of the A-nucleon system at rest.

The operator $\mathcal{Q}_1(\mathcal{Q}_1^{\dagger})$ shifts the energy and the momentum of a selected nucleon 1 by ν and $\boldsymbol{q}$ due to the absorption (emission) of a virtual photon. The second line in Eq. (1) defines the corresponding shifted Green's function. The latter is conveniently described, using a decomposition of the target Hamiltonian H_A into a sum of the Hamiltonian H_{A-1} of the $A-1$ nucleon spectator, the kinetic energy K_1 of a nucleon (1), and the residual interaction $V_1=\Sigma_{i\geqslant 2}V_{1i}$, thus

$$G_{1,A}(E_A+\nu,\boldsymbol{q}) = \frac{1}{E_A+\nu-H_{A-1}-K_1(\boldsymbol{q})-V_1+i\eta}, \quad (2)$$

where $K_1(\boldsymbol{q})=(\hat{\boldsymbol{p}}+\boldsymbol{q})^2/2M$ is the kinetic energy operator with the momentum operator $\hat{\boldsymbol{p}}$ shifted by $\boldsymbol{q}$ and M is the nucleon mass. We assume that NN potentials are local, $V_{ij}\equiv V_{ij}(\boldsymbol{r}_i-\boldsymbol{r}_j)$. Consequently $\mathcal{Q}_1^{\dagger}(\nu,\boldsymbol{q})V_1\mathcal{Q}_1(\nu,\boldsymbol{q})=V_1$ such that the interaction is not affected by the shift as is explicit in Eq. (2).

At this point we comment on notation. We distinguish between external parameters E_A, $\boldsymbol{q}$, and ν and variables which depend on the chosen representation of operators. We do not display those variables, unless required for clarity.

The most common treatment of the structure function is the impulse approximation, obtained by taking $V_1\to 0$ in Eq. (2). The shifted Green's function $G_{1,A}(E_A+\nu,\boldsymbol{q})$ in this approximation $G_{1,A}\simeq G_{1,A}^{(0)}$ reads

$$G_{1,A}^{(0)}(E_A+\nu,\boldsymbol{q}) = \frac{1}{E_A+\nu-H_{A-1}-(\hat{\boldsymbol{p}}+\boldsymbol{q})^2/2M+i\eta}. \quad (3)$$

With a relativistic extension in mind, we express the above $G_{1,A}^{(0)}$ as a convolution of Green's functions for the $(A-1)$-nucleon spectator and for the struck nucleon (N),

$$G_{1,A}^{(0)}(E_A+\nu,\boldsymbol{q}) = i\int\frac{dp_0}{2\pi}G_{A-1}(p_0)G_N(E_A+\nu-p_0,\boldsymbol{q}), \quad (4)$$

and where we shall use the spectral representation of G_{A-1}:

$$G_{A-1}(p_0) = \sum_n \frac{|\Phi_{A-1}^{(n)}\rangle\langle\Phi_{A-1}^{(n)}|}{p_0-E_{A-1}^{(n)}+i\eta}. \quad (5)$$

G_N in Eq. (4) stands for the Green's function of the struck nucleon after absorption of the virtual photon. It reads

$$G_N(E_A+\nu-p_0,\boldsymbol{q}) = \frac{1}{E_A+\nu-p_0-(\hat{\boldsymbol{p}}+\boldsymbol{q})^2/2M+i\eta}. \quad (6)$$

Substituting Eq. (4) into Eq. (1) and performing the integration over p_0, one obtains the structure function in the IA:

$$W^{IA}(\nu,\boldsymbol{q}) = \sum_n \int\frac{d\boldsymbol{p}}{(2\pi)^3}|\varphi_A^{(n)}(\boldsymbol{p})|^2 \times\delta\left(E_A+\nu-E_{A-1}^{(n)}-\frac{(\boldsymbol{p}-\boldsymbol{q})^2}{2M}\right) \quad (7a)$$

$$= \int\frac{d\boldsymbol{p}}{(2\pi)^3}\int dE\,\mathcal{P}(\boldsymbol{p},E) \times\delta\left(\nu-E-\Delta-\frac{(\boldsymbol{p}-\boldsymbol{q})^2}{2M}\right), \quad (7b)$$

where $\varphi_A^{(n)}(\boldsymbol{p})=\langle\Phi_{A-1}^{(n)},\boldsymbol{p}|\Phi_A\rangle$ is an overlap amplitude and $\Delta=E_{A-1}-E_A$. We neglect in $E_{A-1}^{(n)}$ the tiny recoil energy of the spectator $p^2/2M_{A-1}$. In Eq. (7b) appears the single-hole spectral function

$$\mathcal{P}(\boldsymbol{p},E) = \sum_n |\varphi_A^{(n)}(\boldsymbol{p})|^2\delta(E-\mathcal{E}_n), \quad (8)$$

with $\mathcal{E}_n\equiv E_{A-1}^{(n)}-E_{A-1}$, the spectator excitation energy.

It will be useful to define the reduced structure function for nonrelativistic systems

$$F(y,\boldsymbol{q}) = (|\boldsymbol{q}|/M)W(\nu,\boldsymbol{q}), \quad (9)$$

where $y\equiv y(\nu,\boldsymbol{q})$ is some scaling variable. After integration in Eq. (7b) over $\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{q}}$ one obtains for the lowest order the IA part of F,

$$F^{IA}(y_0,\boldsymbol{q}) = \frac{1}{4\pi^2}\left[\int_{|y_0|}^{y_0+2q}dp\,p\int_0^{E_{max}}dE\,\mathcal{P}(\boldsymbol{p},E) + \theta(y_0)\int_0^{y_0}dp\,p\int_{E_{min}}^{E_{max}}dE\,\mathcal{P}(\boldsymbol{p},E)\right], \quad (10)$$

with y_0, the IA scaling variable:

$$y_0 = -|\boldsymbol{q}|+\sqrt{2M(\nu-\Delta)}. \quad (11)$$

The integration limits in Eq. (10) are

$$E_{min}^{max}(y_0,p,q) = \frac{y_0\pm p}{M}|\boldsymbol{q}|+\frac{y_0^2-p^2}{2M} \quad (12)$$

and, in particular,

$$\lim_{q\to\infty}E_{max}(y_0,p,q) = \infty. \quad (13)$$

In order to go beyond the IA one expands the total Green's function $G_{1,A}$, Eq. (2), in powers of $V_1G_{1,A}^{(0)}$. Substituting this expansion into Eq. (1) one obtains the impulse series for the structure function

$$W=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|G_{1,A}^{(0)}+G_{1,A}^{(0)}V_1G_{1,A}^{(0)}+G_{1,A}^{(0)}V_1G_{1,A}^{(0)}V_1G_{1,A}^{(0)}+\cdots|\Phi_A\rangle. \tag{14}$$

The first term is the IA and the remainder are FSIs. We now introduce the scattering operator T, which describes the scattering of the knocked-out nucleon from the $(A-1)$-nucleon spectator. It satisfies the Lippmann-Schwinger operator equation $T=V_1+V_1G_{1,A}^{(0)}T$ and clearly permits a formal summation of the FSI terms in Eq. (14). The total structure function thus becomes

$$W=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|G_{1,A}^{(0)}+G_{1,A}^{(0)}TG_{1,A}^{(0)}|\Phi_A\rangle. \tag{15}$$

It is convenient to use the momentum representation for the nucleon and, as in Eqs. (5) and (7), a representation for the spectator states, denoted by n. The Lippmann-Schwinger equation then becomes a set of coupled equations for transition amplitudes $T_{nn'}(E,\boldsymbol{p},\boldsymbol{p}')\equiv\langle\boldsymbol{p},\Phi_{A-1}^{(n)}|T|\Phi_{A-1}^{(n')},\boldsymbol{p}'\rangle$:

$$T_{nn'}(E,\boldsymbol{p},\boldsymbol{p}')=V_{1;nn'}(\boldsymbol{p}-\boldsymbol{p}')+\sum_{n''}\int\frac{d\boldsymbol{p}''}{(2\pi)^3}\frac{V_{1;nn''}(\boldsymbol{p}-\boldsymbol{p}'')T_{n''n'}(E,\boldsymbol{p}'',\boldsymbol{p}')}{E-\mathcal{E}_{n''}-\dfrac{\boldsymbol{p}''^2}{2M}+i\eta}. \tag{16}$$

Here $V_{1;nn'}(\boldsymbol{p}-\boldsymbol{p}')=\langle\boldsymbol{p},\Phi_{A-1}^{(n)}|V_1|\Phi_{A-1}^{(n')},\boldsymbol{p}'\rangle$ and E, the energy in the laboratory frame. In parallel the total reduced response $F(y_0,\boldsymbol{q})$, Eq. (9), reads (we chose the z axis along $\boldsymbol{q}$)

$$F(y_0,\boldsymbol{q})=F^{IA}(y_0,\boldsymbol{q})+\frac{M}{\pi|\boldsymbol{q}|}\mathrm{Im}\sum_{nn'}\int\frac{d\boldsymbol{p}d\boldsymbol{p}'}{(2\pi)^6}\frac{\varphi_A^{(n)}(\boldsymbol{p})T_{nn'}(E_{N,A-1},\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}'+\boldsymbol{q})\varphi_A^{(n')}(\boldsymbol{p}')}{\left(y_0-p_z-\dfrac{M\mathcal{E}_n}{|\boldsymbol{q}|}-\dfrac{\boldsymbol{p}^2-y_0^2}{2|\boldsymbol{q}|}+i\eta\right)\left(y_0-p_z'-\dfrac{M\mathcal{E}_{n'}}{|\boldsymbol{q}|}-\dfrac{\boldsymbol{p}'^2-y_0^2}{2|\boldsymbol{q}|}+i\eta\right)}, \tag{17}$$

with

$$E_{N,A-1}=\nu-\Delta=\frac{(y_0+|\boldsymbol{q}|)^2}{2M} \tag{18}$$

the off-shell energy of the nucleon-spectator amplitudes.

B. GRS series

The expansion (14) of a structure function in powers of the residual interaction V_1 is not the only possible perturbative approach. In this section we shall expand the shifted Green's function $G_{1,A}(E_A+\nu,\boldsymbol{q})$ in a different operator $\tilde{V}=V_1+K_1(0)+H_{A-1}-E_A\equiv-G_{1,A}^{-1}(E_A,0)$, for which by definition $\tilde{V}|\Phi_A\rangle=0$. Then using the identity

$$G_{1,A}(E_A+\nu,\boldsymbol{q})=\frac{1}{G_{1,A}^{-1}(E_A+\nu,\boldsymbol{q})-G_{1,A}^{-1}(E_A,0)-\tilde{V}}, \tag{19}$$

the shifted Green's function $G_{1,A}(E_A+\nu,\boldsymbol{q})$ permits the expansion

$$G_{1,A}=\tilde{G}_{1,A}(1+\tilde{V}\tilde{G}_{1,A}+\tilde{V}\tilde{G}_{1,A}\tilde{V}\tilde{G}_{1,A}+\cdots), \tag{20}$$

where

$$\tilde{G}_{1,A}\equiv\tilde{G}_{1,A}(\nu,\boldsymbol{q})=\frac{1}{G_{1,A}^{-1}(E_A+\nu,\boldsymbol{q})-G_{1,A}^{-1}(E_A,0)} \tag{21a}$$

$$=\frac{1}{[G_{1,A}^{(0)}(E_A+\nu,\boldsymbol{q})]^{-1}-[G_{1,A}^{(0)}(E_A,0)]^{-1}}. \tag{21b}$$

Again we assume V_1 to be local and it therefore cancels out in Eq. (21b). Expressing $\tilde{G}_{1,A}$ as a convolution (cf. Eq. (4), Eq. (21b) becomes

$$\tilde{G}_{1,A}(\nu,\boldsymbol{q})=i\int\frac{dp_0}{2\pi}G_{A-1}(p_0)\frac{1}{G_N^{-1}(E_A+\nu-p_0,\boldsymbol{q})-G_N^{-1}(E_A-p_0,0)}=i\int\frac{dp_0}{2\pi}G_{A-1}(p_0)\tilde{G}_N(\nu,\boldsymbol{q}). \tag{22}$$

Since G_N^{-1}, Eq. (6), is linear in the energy argument, the spectator energy p_0 and E_A cancel in the denominator in Eq. (22). Thus in contrast to $G_{1,A}^{(0)}$, Eqs. (4)–(6), $\tilde{G}_{1,A}$ does not depend on the excitation energy $E_{A-1}^{(n)}$ of the spectator. Using Eq. (5), one performs the p_0 integral in Eq. (22) with the result

$$\tilde{G}_{1,A}(\nu,\boldsymbol{q})\equiv\tilde{G}_N(\nu,\boldsymbol{q})=\frac{M}{|\boldsymbol{q}|}\frac{1}{y_W-\hat{p}_z+i\eta}, \tag{23}$$

where y_W is the GRS-West scaling variable [7,9]:

$$y_W=\frac{M}{|\boldsymbol{q}|}\left(\nu-\frac{q^2}{2M}\right). \tag{24}$$

Substitution of the series (20) for $G_{1,A}$ into Eq. (1), and use of Eq. (23) there, manifestly produces a power series in $\tilde{V}/|\boldsymbol{q}|$ (the GRS series) for the nuclear response

$$W(\nu,\boldsymbol{q})=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|\tilde{G}_N+\tilde{G}_N\tilde{V}\tilde{G}_N+\tilde{G}_N\tilde{V}\tilde{G}_N\tilde{V}\tilde{G}_N\cdots|\Phi_A\rangle \tag{25a}$$

$$=\sum_{j=0}^{\infty}\left(\frac{M}{|\boldsymbol{q}|}\right)^{j+1}F_j(y_W), \tag{25b}$$

with coefficients F_j, which are functions of the scaling variable y_W. The lowest-order GRS term $(j=0)$ is the asymptotic limit $\boldsymbol{q}\to\infty$, of the reduced structure function Eq. (9),

$$F_0^{GRS}(y_W)=\int n(p)\,\delta(y_W-p_z)\frac{d^3p}{(2\pi)^3}=\frac{1}{4\pi^2}\int_{|y_W|}^{\infty}n(p)p\,dp. \tag{26}$$

Above $n(p)$ is the nucleon momentum distribution, which is related to the spectral function Eq. (8) by

$$n(p)=\int_0^{\infty}\mathcal{P}(p,E)dE. \tag{27}$$

We remark that the leading terms in the impulse and GRS series, Eqs. (10) and (26), are quite different. However, using $\lim_{|\boldsymbol{q}|\to\infty}(y_W-y_0)=0$ and Eqs. (13) and (27), one finds that in the limit $|\boldsymbol{q}|\to\infty$, $F^{IA}\to F_0^{GRS}$.

Consider next higher-order terms $\langle\Phi_A|\tilde{G}_N(\tilde{V}\tilde{G}_N)^n|\Phi_A\rangle$ in the series (25). Since $[\tilde{V},\tilde{G}_N]=[V_1,\tilde{G}_N]$ and also $\tilde{V}|\Phi_A\rangle=0$, each of those terms can be expressed by commutators, involving the residual interaction V_1 and the kinetic energy operator K_1 of the struck nucleon, and not $\tilde{V}=H_A-E_A$. For instance,

$$\tilde{V}\tilde{G}_N|\Phi_A\rangle=[V_1,G_N^{(0)}]|\Phi_A\rangle,$$

$$\tilde{V}\tilde{G}_N\tilde{V}\tilde{G}_N|\Phi_A\rangle=\{[V_1,\tilde{G}_N]^2+[(V_1+K_1),[V_1,\tilde{G}_N]]\}|\Phi_A\rangle,\quad\cdots. \tag{28}$$

From Eq. (25) one then finds for the corresponding reduced structure function, Eq. (9):

$$F^{GRS}=-\frac{|\boldsymbol{q}|}{\pi M}\mathrm{Im}\langle\Phi_A|\tilde{G}_N+[\tilde{G}_N,V_1]\tilde{G}_N+[\tilde{G}_N,V_1]\tilde{G}_N[V_1,\tilde{G}_N]+\cdots|\Phi_A\rangle. \tag{29}$$

Equation (29) is the GRS series for the response function which, using a coordinate-time representation, was first derived in Ref. [7]. For instance, the leading FSI term $F_1(y_W)$ reads

$$F_1^{GRS}(y_W)=\frac{1}{\pi}\mathrm{Im}\sum_{nn'}\int\frac{d\boldsymbol{p}d\boldsymbol{p}'}{(2\pi)^6}\frac{\varphi_A^{(n)}(\boldsymbol{p})V_{1,nn'}(\boldsymbol{p}-\boldsymbol{p}')(p_z'-p_z)\varphi_A^{(n')}(\boldsymbol{p}')}{(y_W-p_z+i\eta)(y_W-p_z'+i\eta)^2} \tag{30a}$$

$$=-i\int_{-\infty}^{\infty}\frac{ds}{2\pi}e^{isy_W}\int\int d\boldsymbol{r}_1d\boldsymbol{r}_2\rho_2(\boldsymbol{r}_1-s\hat{\boldsymbol{q}},\boldsymbol{r}_2;\boldsymbol{r}_1,\boldsymbol{r}_2)\int_0^s d\sigma[V_{12}(\boldsymbol{r}-\sigma\hat{\boldsymbol{q}})-V_{12}(\boldsymbol{r}-s\hat{\boldsymbol{q}})], \tag{30b}$$

with ρ_2 is the two-particle density matrix.

In spite of the increasing complexity of the commutators in the series (29), it has been demonstrated in Ref. [12] that, like Eq. (15) for the IS, *all* FSI terms in the GRS series, Eq. (29), can be summed in a closed expression,

$$F=-\frac{|\boldsymbol{q}|}{\pi M}\mathrm{Im}\langle\Phi_A|\tilde{G}_N+G_{1,A}^{(0)}\tilde{G}_N^{-1}[\tilde{G}_N,T]G_{1,A}^{(0)}|\Phi_A\rangle, \tag{31}$$

with $G_{1,A}^{(0)}$, given by Eq. (3). A derivation of Eq. (31) is given in the Appendix.

The characteristic feature of the expression (31) is the commutator $[\tilde{G}_N,T]$, involving T, Eq. (16), which describes the scattering of the struck nucleon and the spectator. That commutator has a simple form in the momentum representation

$$\langle\boldsymbol{p}|\tilde{G}_N^{-1}[\tilde{G}_N,T]|\boldsymbol{p}'\rangle=\langle\boldsymbol{p}|T|\boldsymbol{p}'\rangle\frac{p_z-p_z'}{y_W-p_z'+i\eta}. \tag{32}$$

Using the spectral representation of the Green's function $G_{1,A}^{(0)}$, Eqs. (4) and (5), one rewrites Eq. (31) as

$$F(y_W,\boldsymbol{q})=F_0^{GRS}(y_W)+\frac{M}{\pi|\boldsymbol{q}|}\mathrm{Im}\sum_{nn'}\int\frac{d\boldsymbol{p}d\boldsymbol{p}'}{(2\pi)^6}\frac{\varphi_A^{(n)}(\boldsymbol{p})(p_z'-p_z)T_{nn'}(\tilde{E}_{N,A-1};\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}'+\boldsymbol{q})\varphi_A^{(n')}(\boldsymbol{p}')}{\left(y_W-p_z-\frac{M\Delta_n(p)}{|\boldsymbol{q}|}+i\eta\right)\left(y_W-p_z'-\frac{M\Delta_{n'}(p')}{|\boldsymbol{q}|}+i\eta\right)(y_W-p_z'+i\eta)},\tag{33}$$

where

$$\Delta_n(p)=\Delta+\mathcal{E}_n+\frac{p^2}{2M}\tag{34}$$

and

$$\tilde{E}_{N,A-1}=\nu-\Delta=\frac{(y_W+|\boldsymbol{q}|)^2}{2M}-\Delta-\frac{p^2}{2M}\tag{35}$$

is the off-shell energy of T in the laboratory frame.

Expansion of the integrand (33) in powers of $1/|\boldsymbol{q}|$ for constant y_W generates the entire GRS series, Eq. (29). For instance, the leading FSI term of the GRS series $F_1^{GRS}(y_W)$, Eq. (30), is retrieved from Eq. (33) by the replacement $T\to V_1$ and disregarding $M\Delta_n(p)/|\boldsymbol{q}|$. Likewise one assembles terms of higher order in $1/|\boldsymbol{q}|$, all appearing as sums over n. Those may in fact be evaluated and ultimately produce, as in the original presentation of the GRS theory, coefficients F_j in terms of off-diagonal density matrices [7] [cf. Eq. (30b) for F_1].

The expressions (17) and (33) permit a comparison of the total FSI contributions in the IS and GRS series. Both contain nucleon-spectator transition amplitudes, which are strongly peaked for small momentum transfers $p_z'-p_z$. However, the same momentum transfer also appears as a factor in the numerator of Eq. (33) and thus reduces FSIs in the GRS series.

An additional suppression of FSIs in that series comes from the different off-shell energies, Eqs. (18) and (35). From those one finds, for $y_0=y_W$, $\tilde{E}_{N,A-1}<E_{N,A-1}$; i.e., the energy of the GRS amplitude is farther from the energy shell than is IS amplitude. Since the complete expressions for the structure functions are identical, the forwarded arguments indicate that the leading GRS term F_0^{GRS} is a better approximation to the total structure function than is the corresponding F^{IA}. Experimental evidence is deferred to the end of Sec. IV.

III. RELATIVISTIC NUCLEAR STRUCTURE FUNCTION

In Sec. II we have used an unconventional propagator technique to rederive the GRS series, primarily because the same will now be shown to lead to the desired relativistic generalization of the GRS series, Eq. (29).

We start with the relativistic nuclear structure function $W_{\mu\nu}$. As in the previous case we consider for simplicity scalar nucleons and photons. This implies that we restrict ourself to the longitudinal component of the structure function $W=[(\boldsymbol{q}^2-\nu^2)/\boldsymbol{q}^2]W_{00}$ (see, for instance, [9,14]). We presume that the techniques which we shall present below will also be applicable for nucleons and photons with spin.

The relativistic nuclear structure function is then again given by the imaginary part of the forward Compton amplitude. The latter can always be written as a sum of two terms, which represent the IA and FSI contributions (Fig. 1):

$$W(q)=-\frac{1}{\pi}\mathrm{Im}\{\Gamma_A G_N(P_A)[G_{1,A}^{(0)}(P_A+q)+G_{1,A}^{(0)}(P_A+q)\times T(P_A+q)G_{1,A}^{(0)}(P_A+q)]G_N(P_A)\Gamma_A\}.\tag{36}$$

G_N and $G_{1,A}^{(0)}$ are propagators for, respectively, a nucleon and the noninteracting nucleon-spectator system, with four-momentum P_A+q. As before we display in Eq. (36) only the external parameters $P_A=(M_A,0)$ and $q=(\nu,\boldsymbol{q})$. Only when necessary do we make explicit the four-momenta of target nucleons. Those appear, for example, in G_N,

$$G_N(P_A)\equiv G_N(P_A-p)=\frac{1}{(P_A-p)^2-M^2+i\eta},\tag{37}$$

and likewise in $G_{1,A}^{(0)}$,

$$G_{1,A}^{(0)}(P_A+q)\equiv G_{1,A}^{(0)}(P_A+q,p)=iG_{A-1}(p)G_N(P_A+q-p),\tag{38}$$

where G_{A-1} is the propagator of the fully interacting spectator. The operator $T=T(P_A+q)$ in Eq. (36) again describes elastic and inelastic scattering of the N-spectator subsystems and satisfies the Bethe-Salpeter equation [cf. Eq. (16)]

$$T(P_A+q)=V_1(P_A+q)[1+G_{1,A}^{(0)}(P_A+q)T(P_A+q)].\tag{39}$$

The effective interaction V_1 is defined as the sum of all irreducible contributions, which drive the scattering operator in Eq. (39).

We still have to define the target-spectator-N vertex function Γ_A in Eq. (36) (see also Fig. 1). It appears in the residue of the bound state pole of the scattering operator $T(P)$:

$$\Gamma_A(p)\Gamma_A(p')=\lim_{P^2\to M_A^2}(P^2-M_A^2)\langle p|T(P)|p'\rangle.\tag{40}$$

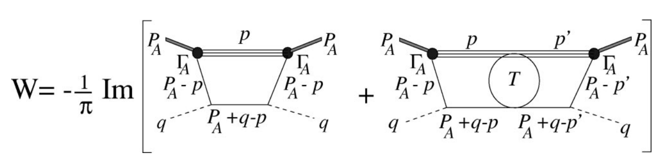

FIG. 1. Nuclear structure function expressed as the imaginary part of the forward Compton amplitude. The first diagram represents the IA and the second one FSIs.

One then derives from the Bethe-Salpeter equation (39) with the four-momentum P_A as the argument,

$$\Gamma_A(p)=i\int\langle p|V_1(P_A)|p'\rangle G_{A-1}(p')\times G_N(P_A-p')\Gamma_A(p')\frac{d^4p'}{(2\pi)^4}, \tag{41}$$

which is the Dyson equation, satisfied by Γ_A. The latter can be rewritten in a form similar to the Schrödinger equation,

$$\{[G_{1,A}^{(0)}(P_A)]^{-1}-V_1(P_A)\}G_{1,A}^{(0)}(P_A)\Gamma_A=0, \tag{42}$$

with $G_{1,A}^{(0)}(P_A)\Gamma_A$ a relativistic target wave function.

Next we link in a standard way the Green's function of the fully interacting A-nucleon target with $G_{1,A}^{(0)}$ and the scattering operator [cf. Eqs. (2) and (15)]:

$$G_{1,A}(P_A+q)=G_{1,A}^{(0)}(P_A+q)[1+T(P_A+q)G_{1,A}^{(0)}(P_A+q)]=\frac{1}{[G_{1,A}^{(0)}(P_A+q)]^{-1}-V_1(P_A+q)}. \tag{43}$$

The momentum q of the virtual photon in the argument of the total Green's function is ultimately the one absorbed by nucleon 1. Equation (36) can then be rewritten as

$$W(q)=-\frac{1}{\pi}\mathrm{Im}[\Gamma_A G_N(P_A)G_{1,A}(P_A+q)G_N(P_A)\Gamma_A]. \tag{44}$$

Equations (43) and (44) are the relativistic analogs of Eqs. (1) and (2). Whereas the latter have been derived by explicit use of a Hamiltonian, this is not so for the former.

The above equations serve as the starting point for various perturbative approaches for the structure function. First one expands $G_{1,A}(P_A+q)$ in powers of V_1 which produces the four-dimensional relativistic IS [cf. Eq. (14)]

$$W(q)=-\frac{1}{\pi}\mathrm{Im}\{\Gamma_A G_N(P_A)[G_{1,A}^{(0)}(P_A+q)+G_{1,A}^{(0)}(P_A+q)\times V_1(P_A+q)G_{1,A}^{(0)}(P_A+q)+\cdots]G_N(P_A)\Gamma_A\}. \tag{45}$$

As for the nonrelativistic case [see paragraph before Eq. (28)] we next look for a different expansion of W in powers of an operator $\tilde{V}$ which annihilates the target ground state. A choice which satisfies this requirement is provided by the bracketed operator in Eq. (42):

$$\tilde{V}(P_A)=V_1(P_A)-[G_{1,A}^{(0)}(P_A)]^{-1}. \tag{46}$$

Using Eq. (46) we then rewrite $G_{1,A}(P_A+q)$, Eq. (43), as

$$G_{1,A}(P_A+q)=\frac{1}{[G_{1,A}^{(0)}(P_A+q)]^{-1}-[G_{1,A}^{(0)}(P_A)]^{-1}-V_1(P_A+q)+V_1(P_A)-\tilde{V}(P_A)}. \tag{47}$$

For further evaluation we assume that the interaction between the N and the spectator is the sum of local pair potentials, each depending only on the four-momentum transfer:

$$\langle p_1,p_2,\ldots,p_k,\ldots|V_1|p_1',p_2',\ldots,p_k',\ldots\rangle=\sum_{k\geqslant 2}V_{1k}(p_1-p_1')\,\delta^{(4)}(p_1-p_k-p_1'+p_k'). \tag{48}$$

As a consequence $V_1(P_A+q)-V_1(P_A)=0$, in Eq. (47). Expanding there $G_{1,A}(P_A+q)$ in powers of $\tilde{V}$ and substituting the result into Eq. (44), one obtains [cf. Eq. (25a)]

$$W(q)=-\frac{1}{\pi}\mathrm{Im}\{\Gamma_A G_N(P_A)[\tilde{G}_{1,A}(P_A,q)+\tilde{G}_{1,A}(P_A,q)\tilde{V}(P_A)\tilde{G}_{1,A}(P_A,q)+\cdots]G_N(P_A)\Gamma_A\}, \tag{49}$$

with

$$\tilde{G}_{1,A}(P_A,q)=\frac{1}{[G_{1,A}^{(0)}(P_A+q)]^{-1}-[G_{1,A}^{(0)}(P_A)]^{-1}}=iG_{A-1}(p)\frac{1}{G_N^{-1}(P_A+q-p)-G_N^{-1}(P_A-p)}\equiv G_{A-1}(p)\tilde{G}_N(P_A-p,q). \tag{50}$$

For clarity we made explicit the four-momentum of the struck nucleon.

We now evaluate the modified Green's function of the struck nucleon, $\tilde{G}_N$ in Eq. (50). Using Eq. (37) one obtains

$$\tilde{G}_N(P_A-p,q)=\frac{1}{(P_A-p+q)^2-(P_A-p)^2+i\eta}=\frac{1}{2(M_A-p_0)\nu-2p_z|\boldsymbol{q}|-Q^2+i\eta}, \tag{51}$$

with $Q^2=\boldsymbol{q}^2-\nu^2$ and where the negative z axis has been chosen in the direction of the momentum the virtual photon. One notes that in contrast to the nonrelativistic case, Eqs. (22) and (23), the quadratic dependence on energy in the relativistic propagator, Eq. (37), causes the spectator energy p_0 to persist in Eq. (51).

Next one exploits Eq. (42) in order to replace $\tilde{V}$ in each term of this series by commutators involving the residual interaction, V_1 [12,13]. For instance, the leading FSI term [the second term of the expansion (49)] becomes

$$W_1^{GRS}(q)=-\frac{1}{\pi}\mathrm{Im}\int\frac{d^4p\,d^4p'}{(2\pi)^8}\Gamma_A(p)G_N(P_A-p)G_{A-1}(p)\times[\tilde{G}_N(P_A-p,q),V_1(p-p')]\times\tilde{G}_N(P_A-p',q)G_{A-1}(p')G_N(P_A-p')\Gamma_A(p'), \tag{52}$$

where we made explicit the momentum of the struck nucleon, but left implicit variables chosen to represent the spectator nucleons. The entire series formally acquires the same form as its nonrelativistic GRS counterpart, Eq. (29), but each term contains four-dimensional integrals over intermediate four-momenta.

The exact evaluation of these terms, as well of those in the relativistic IS, constitutes a formidable many-body problem. We now discuss minimal assumptions which lead to considerable simplifications.

IV. THREE-DIMENSIONAL REDUCTION

A. Nonrelativistic limit for target wave functions

We start this section with the observation that nucleons in ground states of nuclei and in not too highly excited states have on the average three-momenta $\langle\boldsymbol{p}^2\rangle^{1/2}\lesssim p_F\approx 0.3$ GeV, with p_F the Fermi momentum. The above are thus essentially nonrelativistic systems. Examples are the struck nucleon before the absorption of the virtual photon, the nucleons in the target nucleus at rest and in the spectator, which recoils with momentum $\boldsymbol{p}$. Only particles or subsystems with momenta containing q are truly relativistic. As Fig. 1 shows, this applies only to the recoiling nucleon with momentum $p+q\approx q$.

We thus apply nonrelativistic limits to all quantities which contain low-momentum nucleons. Those are the propagators $G_N(P_A-p)$ and $G_{A-1}(p)$, Eqs. (37) and (38) [cf. Eqs. (5) and (6)],

$$G_N(P_A-p)\simeq\left(\frac{1}{2M}\right)\frac{1}{M_A-p_0-M-\boldsymbol{p}^2/2M+i\eta} \tag{53}$$

and

$$G_{A-1}(p)\simeq\sum_n\left(\frac{1}{2M_{A-1}}\right)\frac{|\Phi_{A-1}^{(n)},\boldsymbol{p}\rangle\langle\boldsymbol{p},\Phi_{A-1}^{(n)}|}{p_0-M_{A-1}-\mathcal{E}_n+i\eta}. \tag{54}$$

In the same limit one can use for the residual interaction $V_1(p-p')\approx V_1(\boldsymbol{p}-\boldsymbol{p}')$ and for the vertex function $\Gamma_A(p)\approx\Gamma_A(\boldsymbol{p})$. After substitution of the above limits into Eq. (41) we consider the integration over p_0. One notes that the Green's functions $G_N(P_A-p)$ and $G_{A-1}(p)$ have poles in the complex p_0 plane which lie on different sides of the real axis. One may thus close the integration contour around the spectator pole and perform the p_0 integration. The result is

$$\left(E_A-H_{A-1}-K_1-\frac{1}{4M_{A-1}M}V_1\right)\Phi_A=0, \tag{55}$$

with

$$\Phi_A=\frac{1}{(8M_{A-1}M^2)^{1/2}(E_A-H_{A-1}-K_1)}\Gamma_A. \tag{56}$$

Equation (55) is now a standard three-dimensional Schrödinger equation for the target bound state wave function, with effective residual interaction $(1/4M_{A-1}M)V_1$.

B. Reduction of relativistic impulse series

We consider the relativistic IS and first apply the above nonrelativistic limits to all quantities, depending on nucleons with low momenta. This we illustrate below on the IA for the structure function, $W^{IA}=-(1/\pi)\mathrm{Im}[\Gamma_A G_N G_{1,A}^{(0)} G_N\Gamma_A]$, which is the first term of the IS, Eq. (45). Explicitly,

$$W^{IA}(\nu,\boldsymbol{q})=-\frac{1}{\pi}\mathrm{Im}\sum_n\int\frac{d^3p}{(2\pi)^3}\int\frac{dp_0}{2\pi}\frac{i(8M_{A-1}M^2)^{-1}|\langle\Gamma_A(\boldsymbol{p})|\Phi_{A-1}^{(n)}\rangle|^2}{\left(M_A-p_0-M-\frac{\boldsymbol{p}^2}{2M}+i\eta\right)^2(p_0-M_{A-1}-\mathcal{E}_n+i\eta)}\frac{1}{(M_A+\nu-p_0)^2-e_{\boldsymbol{q}-\boldsymbol{p}}^2+i\eta}, \tag{57}$$

with $e_{\boldsymbol{p}}=\sqrt{M^2+\boldsymbol{p}^2}$.

One observes that the above-mentioned spectator pole, $p_0=M_{A-1}+\mathcal{E}_n-i\eta$ and the negative energy nucleon pole in the relativistic propagator $G_N(P_A+q)$ at $p_0=M_A+\nu+e_{\boldsymbol{q}-\boldsymbol{p}}-i\eta$ lies both in the lower half of the complex p_0 plane. One ought to include the two above-mentioned poles, but we first disregard the one with negative energy and compute only the residue of the spectator pole, leading to

$$W^{IA}(\nu,\boldsymbol{q})=\sum_n \int \frac{d^3p}{2e_{\boldsymbol{q}-\boldsymbol{p}}(2\pi)^3}|\varphi_A^{(n)}(\boldsymbol{p})|^2\delta(\nu-\Delta-\mathcal{E}_n+M-e_{\boldsymbol{q}-\boldsymbol{p}}). \tag{58}$$

Next we introduce the reduced relativistic structure function

$$\mathcal{F}(\overline{y_0},\boldsymbol{q})\equiv 2|\boldsymbol{q}|W(\nu,\boldsymbol{q}), \tag{59}$$

where the factor $2|\boldsymbol{q}|$ has been adjusted to produce the correct nonrelativistic limit, Eq. (9), of $\mathcal{F}$. Integration over $\cos(\boldsymbol{p},\boldsymbol{q})$ leads to

$$\mathcal{F}^{IA}(\overline{y_0},\boldsymbol{q})=\frac{1}{4\pi^2}\left[\int_{|\overline{y_0}|}^{2q+\overline{y_0}}pdp\int_0^{\overline{E}_{max}}\mathcal{P}(p,E)dE+\theta(\overline{y_0})\int_0^{\overline{y_0}}pdp\int_{\overline{E}_{min}}^{\overline{E}_{max}}\mathcal{P}(p,E)dE\right]. \tag{60}$$

It has the same form as the nonrelativistic IA, Eqs. (10), where the scaling variable and the integration limits have been replaced by relativistic ones

$$\overline{y_0}=-|\boldsymbol{q}|+\sqrt{2M(\nu-\Delta)+(\nu-\Delta)^2}, \tag{61}$$

$$\overline{E}_{min}^{max}(q,\overline{y_0},p)=e_{\overline{y_0}+|\boldsymbol{q}|}-e_{p\pm|\boldsymbol{q}|}. \tag{62}$$

In contrast with Eq. (13)

$$\lim_{|\boldsymbol{q}|\to\infty}\overline{E}_{max}(q,\overline{y_0},p)=\overline{y_0}+p; \tag{63}$$

i.e., the asymptotic limit of the maximum excitation energy is finite. Apart from the order of the p,E integration, Eq. (60) is identical to the result of Ref. [1].

There actually is no difficulty in computing the residue of the above-neglected negative energy pole in the IA. However, the same for higher-order FSI terms seriously complicates a three-dimensional reduction. Rather than elaborating on this point, we proceed towards our major goal; namely, the three-dimensional reduction of the relativistic GRS series.

C. Reduction of the relativistic GRS series

We thus consider the relativistic GRS series, Eqs. (49) and (50), and start with its lowest-order term $W_0^{GRS}=-(1/\pi)\mathrm{Im}[\Gamma_A G_N \tilde{G}_{1,A} G_N \Gamma_A]$. Applying the above nonrelativistic limit one obtains

$$W_0^{GRS}(\nu,\boldsymbol{q})=-\frac{1}{\pi}\mathrm{Im}\sum_n\int\frac{d^3p}{(2\pi)^3}\frac{dp_0}{2\pi}\frac{i(8M_{A-1}M^2)^{-1}|\langle\Gamma_A(\boldsymbol{p})|\Phi_{A-1}^{(n)}\rangle|^2}{\left(M_A-p_0-M-\frac{\boldsymbol{p}^2}{2M}+i\eta\right)^2(p_0-M_{A-1}-\mathcal{E}_n+i\eta)}\frac{1}{2(M_A-p_0)\nu-2\boldsymbol{pq}-Q^2+i\eta}. \tag{64}$$

In contrast to the relativistic IS, the modified propagator $\tilde{G}_N$ has only one pole in the lower half of the complex p_0 plane and simple calculus produces

$$W_0^{GRS}(\nu,\boldsymbol{q})=\sum_n\int\frac{d^3p}{(2\pi)^3}|\varphi_A^{(n)}(\boldsymbol{p})|^2\delta[2(M-\Delta-\mathcal{E}_n)\nu-2\boldsymbol{pq}-Q^2]. \tag{65}$$

Integration over $\cos(\boldsymbol{p},\boldsymbol{q})$ then yields

$$\mathcal{F}_0^{GRS}(y_G,\boldsymbol{q})=\frac{1}{4\pi^2}\left[\int_{|y_G|}^{\infty}pdp\int_0^{\tilde{E}_{max}}\mathcal{P}(p,E)dE+\theta(y_G)\int_0^{y_G}pdp\int_{\tilde{E}_{min}}^{\tilde{E}_{max}}\mathcal{P}(p,E)dE\right], \tag{66}$$

where

$$y_G=\frac{M}{|\boldsymbol{q}|}\left[\nu\left(1-\frac{\Delta}{M}\right)-\frac{Q^2}{2M}\right] \tag{67}$$

is a relativistic generalization of y_W, Eq. (24), derived in Refs. [12,13]. The integration limits in Eq. (66) are

$$\tilde{E}_{min}^{max}(q,y_G,p)=\frac{(y_G\pm p)|\boldsymbol{q}|}{\nu}. \tag{68}$$

In particular,

$$\lim_{|\boldsymbol{q}|\to\infty}\tilde{E}_{max}(q,y_G,p)=y_G+p. \tag{69}$$

Since $y_G\to\overline{y_0}$ in the asymptotic limit, the above $\tilde{E}_{max}$ and its analog $\overline{E}_{max}$ in the IA, Eq. (63), coincide.

Equation (66), the first term of the relativistic GRS series, is seen to contain the spectral function, Eq. (8) It does not resemble its nonrelativistic counterpart, Eq. (26), which contains exclusively the momentum distribution $n(p)$. The latter is due to the independence of the nonrelativistic nucleon propagator $\tilde{G}_N$, Eq. (23) on p_0, in contrast to its relativistic counterpart, Eqs. (50) and (51). The upper limit $\tilde{E}_{max}$, Eq. (68), is always finite. Consequently Eq. (26) is not the nonrelativistic limit of $\mathcal{F}_0^{GRS}$, Eq. (66). In fact, the latter resembles more the corresponding expression for the IA, Eq. (60).

Since the momentum distribution $n(p)$, Eq. (27), is a simpler function than the spectral function $\mathcal{P}(p,E)$ which depends on two variables, it is of practical interest to compare expressions for the maximum excitation energy of the spectator. One thus concludes from Eqs. (62) and (68) that in the nonrelativistic regime $\nu \ll M$, $E_{max}^{GRS} = \tilde{E}_{max} \gg E_{max}^{IA}$. The replacement $\tilde{E}_{max} \to \infty$ [and consequently $\mathcal{P}(p,E)$ by $n(p)$ in Eq. (66)] is therefore less of an offense in the GRS case than $\bar{E}_{max} = E_{max}^{IA} \to \infty$ in the IA expression, Eq. (60).

We now turn to FSI terms in the relativistic GRS series, Eq. (49), for instance the dominant FSI term:

$$W_1^{GRS}(\nu,\boldsymbol{q}) = -\frac{1}{\pi}\mathrm{Im}\sum_n \int \frac{d^4p\,d^4p'}{(2\pi)^8} \frac{(8M^2M_{A-1})^{-1}\langle\Gamma_A(\boldsymbol{p})|\Phi_{A-1}^{(n)}\rangle}{\left(M_A - p_0 - M - \frac{\boldsymbol{p}^2}{2M} + i\eta\right)(p_0 - M_{A-1} - \mathcal{E}_n + i\eta)}$$
$$\times \frac{[2\nu(p_0' - p_0) - 2|\boldsymbol{q}|(p_z' - p_z)]V_{1;nn'}(\boldsymbol{p}-\boldsymbol{p}')}{[2(M_A - p_0)\nu - 2p_z|\boldsymbol{q}| - Q^2 + i\eta][2(M_A - p_0')\nu - 2p_z'|\boldsymbol{q}| - Q^2 + i\eta]^2}$$
$$\times \frac{\langle\Phi_{A-1}^{(n')}|\Gamma_A(\boldsymbol{p}')\rangle}{(p_0' - M_{A-1} - \mathcal{E}_{n'} + i\eta)\left(M_A - p_0' - M - \frac{\boldsymbol{p}'^2}{2M} + i\eta\right)}. \qquad (70)$$

As in Eq. (64) for W_0^{GRS}, one reduces W_1^{GRS}, Eq. (70), by performing the p_0, p_0' integrations over the isolated spectator poles and the result for the corresponding reduced structure function, Eq. (59), becomes

$$\mathcal{F}_1^{GRS}(y_G,\boldsymbol{q}) = -\frac{1}{\pi}\mathrm{Im}\sum_{nn'} \int \frac{d^3\boldsymbol{p}\,d^3\boldsymbol{p}'}{(2\pi)^6} \frac{\varphi_A^{(n)}(p)\left[\frac{\nu}{|\boldsymbol{q}|}(\mathcal{E}_n - \mathcal{E}_{n'}) - (p_z - p_z')\right]V_{1;nn'}(\boldsymbol{p}-\boldsymbol{p}')\varphi_A^{(n')}(p')}{\left(y_G - p_z - \frac{\nu}{|\boldsymbol{q}|}\mathcal{E}_n + i\eta\right)\left(y_G - p_z' - \frac{\nu}{|\boldsymbol{q}|}\mathcal{E}_{n'} + i\eta\right)^2}. \qquad (71)$$

Upon neglect of the small relativistic corrections $(\nu/|\boldsymbol{q}|)(\mathcal{E}_n - \mathcal{E}_{n'})$ in the numerator, one compares Eq. (71) with its nonrelativistic analog, Eq. (30a). The latter turns into the former upon the following replacements of the scaling variable and Green's function of the recoiling nucleon, Eq. (23):

$$y_W \to y_G,$$
$$\tilde{G}_N \to \tilde{G}_N^r(\nu,\boldsymbol{q},\boldsymbol{p}) = \left(\frac{M}{|\boldsymbol{q}|}\right)\frac{1}{y_G - p_z - \frac{\nu}{|\boldsymbol{q}|}\mathcal{E}_n + i\eta}. \qquad (72)$$

At this point we return to the nonrelativistic kinetic energy $p^2/2M_{A-1}$ which has been neglected above and is valid for all but the lightest spectators. It is actually straightforward to include that energy, which amounts to replacing y_G, Eq. (67), by the A-dependent scaling variable

$$y_G^A = \frac{2y_G}{1 + \sqrt{1 + (2\nu y_G/M_{A-1}|\boldsymbol{q}|)}}. \qquad (73)$$

One notes that the energy shift $(\nu/|\boldsymbol{q}|)\mathcal{E}_n$ in the propagator, $\tilde{G}_N$, Eq. (72), puts a finite upper limit to the maximum excitation energy of the spectator in $\mathcal{F}_1^{GRS}$, Eq. (71). The same has been discussed for the lowest order term $\mathcal{F}_0^{GRS}$, Eq. (66), and occurs in all higher FSI terms. Those energy shifts should therefore be retained in the denominator of Eq. (71). We neglected, however, their differences in the numerator of the same equation.

The above three-dimensional reduction can be extended straightforwardly to all higher-order terms of the relativistic GRS series, leading to the result

$$\mathcal{F}^{GRS}(y_G^A,|\boldsymbol{q}|) = \sum_{j=0}^{\infty}\left(\frac{M}{\boldsymbol{q}}\right)^j \mathcal{F}_j^{GRS}(y_G^A,|\boldsymbol{q}|). \qquad (74)$$

It obviously differs from its nonrelativistic analog, Eq. (25b), by the q dependence of its expansion coefficients, which is due to the $(\nu/|\boldsymbol{q}|)\mathcal{E}_n$ term in Eq. (72).

A special case is the deuteron target D for which $\mathcal{E}_n = 0$. This enables to reinstate in Eq. (74) the q-independent expansion coefficients $\mathcal{F}_j^{GRS}(y_G^D,|\boldsymbol{q}|) \equiv \mathcal{F}_j^{GRS}(y_G^D)$ and the construction of the reduced relativistic structure function, Eq. (74), from the nonrelativistic one [cf. Eqs. (9) and (25)]

$$\mathcal{F}^{REL}(y_G^D,|\boldsymbol{q}|) = F^{NR}(y_W \to y_G^D,|\boldsymbol{q}|) \qquad (75)$$

or, alternatively,

$$W^{REL}(\nu,\boldsymbol{q}) = \frac{\nu}{\tilde{\nu}}W^{NR}(\tilde{\nu},\boldsymbol{q}), \qquad (76a)$$

$$\tilde{\nu} = \left(1 - \frac{\Delta}{M} + \frac{\nu}{2M} - \frac{(y_G^D)^2}{2M^2}\right)\nu. \tag{76b}$$

Equation (75) implies that a calculation of the FSI part of $\mathcal{F}$ requires the solution of a three-dimensional, instead of a more complicated four-dimensional, scattering equation.

It would be desirable to reach a similar simplification for targets with $A \geqslant 3$. A hint as to how to proceed comes from a comparison of the dominant FSI term F_1, Eq. (30a), of the nonrelativistic GRS series with the summed FSI, Eq. (33).

It is shown in the Appendix that for the nonrelativistic case, the summed FSI expression [the second term in Eq. (33)] is obtained from F_1, Eq. (30a), by $V_1 \to T$ and addition of $M\Delta_n(\boldsymbol{p})/|\boldsymbol{q}|$ to the propagators. On account of the similarity of F_1 and $\mathcal{F}_1$ [cf. Eqs. (30a) and (71)], we conjecture that the relativistic summed FSIs are similarly generated from $\mathcal{F}_1^{GRS}$, Eq. (71). The final result is

$$\mathcal{F}^{GRS}(y_G,\boldsymbol{q}) = \mathcal{F}_0^{GRS}(y_G,\boldsymbol{q}) - \frac{M}{\pi|\boldsymbol{q}|}\mathrm{Im}\sum_{n,n'}\int\frac{d^3\boldsymbol{p}d^3\boldsymbol{p}'}{(2\pi)^6} \times \frac{\varphi_A^{(n)}(\boldsymbol{p})(p_z'-p_z)T_{nn'}(\tilde{E}_{N,A-1};\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}'+\boldsymbol{q})\varphi_A^{(n')}(\boldsymbol{p}')}{\left(y_G - p_z - \frac{\nu\mathcal{E}_n}{|\boldsymbol{q}|} - \frac{M\Delta_n(\boldsymbol{p})}{|\boldsymbol{q}|} + i\eta\right)\left(y_G - p_z' - \frac{\nu\mathcal{E}_{n'}}{|\boldsymbol{q}|} - \frac{M\Delta_{n'}(\boldsymbol{p}')}{|\boldsymbol{q}|} + i\eta\right)\left(y_G - p_z' - \frac{\nu\mathcal{E}_{n'}}{|\boldsymbol{q}|} + i\eta\right)}. \tag{77}$$

We emphasize again the occurrence of three-dimensional N-spectator transition amplitudes $T_{nn'}$ for off-shell energy [cf. Eq. (35)]

$$\tilde{E}_{N,A-1} = \frac{(y_G + |\boldsymbol{q}|)^2}{2M} - \Delta - \frac{p^2}{2M}. \tag{78}$$

The occurrence of a T operator usually indicates the summation of GRS terms in V_1, which is mandatory if the latter is singular. As was the case for the nonrelativistic case, Eq. (33), the relativistic expression for the structure function in terms of the scattering operator T, Eq. (77) is more general than the entire GRS series.

Equations (75) and (77) are our main results. They are the outcome of very accurate three-dimensional reduction of the relativistic structure functions of targets composed of point particles. The reduction is a direct consequence of the separation in slow and fast target nucleons. Compared with a nonrelativistic GRS theory, FSI interactions are summed by means of a three-dimensional scattering operator. Relativistic effects are only manifest in a relativistic scaling variable and in an additional energy shift in N propagators.

In spite of the role $(\nu/|\boldsymbol{q}|)\mathcal{E}_n$ plays in the the limits of the excitation energies of the spectator, we wish to explore closure over those excitations in Eq. (77), replacing state-dependent quantities by suitable averages, $\mathcal{E}_n \to \langle\mathcal{E}\rangle$. This leads to an operator T, which describes the scattering of nucleon 1 from $A-1$, fixed spectator nucleons (see, for instance, Ref. [15]). In particular one may expand T in a Watson series of scattering operators t for nucleon pairs and retain only the lowest-order term. The result is

$$\mathcal{F}^{GRS}(y_G,\boldsymbol{q}) \approx \mathcal{F}_0^{GRS}(y_G,\boldsymbol{q}) - \frac{M}{\pi|\boldsymbol{q}|}\mathrm{Im}\int\frac{d^3\boldsymbol{p}d^3\boldsymbol{p}'}{(2\pi)^6}\rho_2(\boldsymbol{p},\boldsymbol{p}';\boldsymbol{q}) \times \frac{(p_z'-p_z)\langle\boldsymbol{p},-\boldsymbol{p}+\boldsymbol{q}|t(\tilde{E}_{N,A-1}|\boldsymbol{p}',-\boldsymbol{p}'+\boldsymbol{q})\rangle}{\left(y_G - p_z - \frac{\nu\langle\mathcal{E}\rangle}{|\boldsymbol{q}|} - \frac{M\langle\Delta(\boldsymbol{p})\rangle}{|\boldsymbol{q}|} + i\eta\right)\left(y_G - p_z' - \frac{\nu\langle\mathcal{E}\rangle}{|\boldsymbol{q}|} - \frac{M\langle\Delta(\boldsymbol{p}')\rangle}{|\boldsymbol{q}|} + i\eta\right)\left(y_G - p_z' - \frac{\nu\langle\mathcal{E}\rangle}{|\boldsymbol{q}|} + i\eta\right)}. \tag{79}$$

Above $\rho_2(\boldsymbol{p},\boldsymbol{p}',\boldsymbol{q})$ is the two-particle density matrix in the momentum representation: i.e., the Fourier transform of the same in coordinate representation [cf. Eq. (30b)].

Clearly, for sufficiently high-energy transfers ν, nucleons may be excited, and this is manifest in the nucleon structure functions F^N. Those dynamical features should be built into a theory and an example is a generalized convolution of structure functions for nucleons and for a nucleus, composed of point nucleons [16]:

$$F_2^A(x,Q^2) = \int_x^A dz f^{PN}(z,Q^2) F_2^N\left(\frac{x}{z},Q^2\right). \tag{80}$$

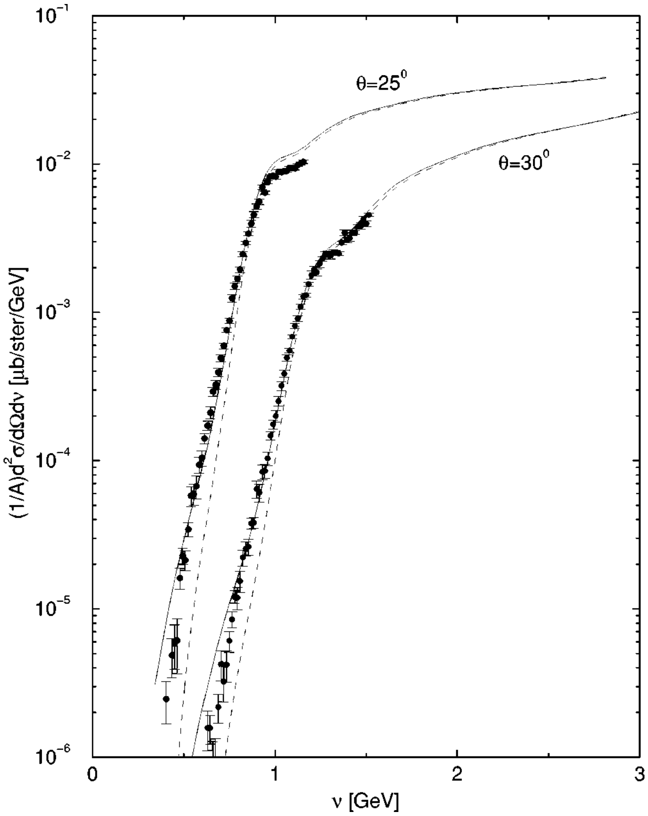

FIG. 2. Double-differential cross section for $e+{}^4\mathrm{He}\rightarrow e+X$ inclusive scattering as a function of ν. The dashed line is the IA and the solid line corresponds to the lowest term of the relativistic GRS series.

Here $f^{PN}(x,Q^2)$ is $\mathcal{F}^{GRS}(q,y_G^A)$, Eqs. (73) and (77), expressed in the Bjorken variable x and Q^2. The above equations (79) and (80) come close to the expression used in actual calculations of nuclear structure functions [17].

We conclude this section by stating that for the same reasons as for the nonrelativistic case forwarded at the end of Sec. II, the relativistic GRS series is expected to show better convergence than the IS series. We shall provide proof, using inclusive scattering of 3.595 GeV electrons on ^{4}He, for which the nuclear input can be computed with high precision. Figure 2 thus shows cross sections for two scattering angles $\theta=25°,30°$ as a function of the energy loss in a standard way related to the ^{4}He structure function, Eq. (80).

Inspection shows that, except for the smallest ν, the drawn lines representing the lowest-order GRS prediction using Eq. (66) nearly account for the data [18]. In contrast the dashed lines for the IA based on Eq. (60) show sizable discrepancies. Details can be found in Ref. [19]. Similar evidence comes from D data [12,20].

V. SUMMARY

In this paper we studied a relativistic GRS series for structure functions of a nucleus composed of point nucleons. The latter we simplified, exploiting nonrelativistic features of all quantities there, which are related to slow target nucleons, and only treated relativistically the nucleon which absorbs the high momentum of the transferred photon in inclusive scattering. Our focus is on an accurate three-dimensional reduction of the expression, which is possible through a specific feature of a modified nucleon propagator: namely, its linear momentum dependence. This is in contrast with the standard relativistic IS.

In the case of a deuteron target, the above three-dimensional reduction leads to a perfect correspondence between the relativistic and the corresponding nonrelativistic expressions for its structure function. The derivation of that mapping does not employ light-cone kinematics which has similar features, but without the need to relax spherical symmetry [21].

For targets with $A\geqslant 3$ no such mapping can be proved rigorously. We emphasized though the close correspondence between the nonrelativistic and relativistic dominant FSI terms and then conjectured that the same correspondence holds between the nonrelativistic and relativistic summed FSI contributions. The latter can then be calculated, using three- instead of a more complicated four-dimensional scattering equation. The above rests on the assumption that the driving term of the four-dimensional Bethe-Salpeter scattering equation is local and given by a sum of pair interactions, as has also been assumed in the nonrelativistic case.

Our final remarks regarded the application of the obtained results. In spite of the proven reduction and correspondence, it is still nontrivial to solve multichannel scattering of a nucleon from a fully interacting $A-1$ nucleon spectator. We mentioned the approximation where many-body transition operators are replaced by a sum of scattering operators for a pair of nucleons. In addition we recalled the incorporation of nucleons with internal dynamics. Both features are about the basis of previously performed computations [17].

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APPENDIX: FINAL STATE INTERACTION IN THE GRS EXPANSION

In the following we expand on the derivation of Eq. (31) for the summed FSI contribution, which was previously given in Ref. [12]. Consider nonrelativistic structure function given by the GRS series, Eqs. (25a). Using Eq. (20) [with $\tilde{G}_{1,A}\equiv\tilde{G}_N$, Eq. (23)], we can rewrite the FSI part of the structure function as

$$W_{FSI}(\nu,\boldsymbol{q})=-\frac{1}{\pi}\mathrm{Im}\left\langle \Phi_A\left|\sum_{n=1}^{\infty}\tilde{G}_N(\tilde{V}\tilde{G}_N)^n\right|\Phi_A\right\rangle$$
$$=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|\tilde{G}_N\tilde{V}G_{1,A}|\Phi_A\rangle, \qquad \text{(A1)}$$

where $G_{1,A} \equiv G_{1,A}(E_A+\nu,\boldsymbol{q})$ is the total Green's function after the absorption of the virtual photon, Eq. (2). Using the Lippmann-Schwinger equation (16) one can express $G_{1,A}$ in terms of the scattering operator T,

$$G_{1,A}=(1+G_{1,A}^{(0)}T)G_{1,A}^{(0)}, \tag{A2}$$

where $G_{1,A}^{(0)} \equiv G_{1,A}^{(0)}(E_A+\nu,\boldsymbol{q})$, Eq. (3). Substituting Eq. (A2) into Eq. (A1) and using $\tilde{V}=H_A-E_A=V_1-g_0^{-1}$ with $g_0 \equiv G_{1,A}^{(0)}(E_A,0)$, we obtain

$$W_{FSI}=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|\tilde{G}_N(V_1-g_0^{-1})(1+G_{1,A}^{(0)}T)G_{1,A}^{(0)}|\Phi_A\rangle. \tag{A3}$$

One easily finds from Eqs. (21) and (23) that $\tilde{G}_N-G_{1,A}^{(0)} =\tilde{G}_N g_0^{-1}G_{1,A}^{(0)}$. Then using $V_1(1+G_{1,A}^{(0)}T)=T$ we can rewrite Eq. (A3) as

$$W_{FSI}=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|G_{1,A}^{(0)}TG_{1,A}^{(0)}-g_0^{-1}\tilde{G}_N G_{1,A}^{(0)}|\Phi_A\rangle. \tag{A4}$$

At the last step we use the following relation between the scattering operator T and the target wave function, valid for any local interactions:

$$\langle\Phi_A|g_0^{-1}=\langle\Phi_A|G_{1,A}^{(0)}\tilde{G}_N^{-1}T. \tag{A5}$$

The relation can easily obtained by multiplying the Lippmann-Schwinger equation $T=V_1(1+G_{1,A}^{(0)}T)$ by $\langle\Phi_A|$ and using the Schrödinger equation for the target wave function: $\langle\Phi_A|V_1=\langle\Phi_A|g_0^{-1}$. Then Eq. (A4) can be rewritten as

$$W_{FSI}=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|G_{1,A}^{(0)}TG_{1,A}^{(0)}-G_{1,A}^{(0)}\tilde{G}_N^{-1}T\tilde{G}_N G_{1,A}^{(0)}|\Phi_A\rangle$$

$$=-\frac{1}{\pi}\mathrm{Im}\langle\Phi_A|G_{1,A}^{(0)}\tilde{G}_N^{-1}[\tilde{G}_N,T]G_{1,A}^{(0)}|\Phi_A\rangle, \tag{A6}$$

thus obtaining the desired expression for the summed FSI contribution in the GRS series.

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