SU(3) normal mode theory

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A normal mode analysis of rotating equilibrium densities is made in the su(3) mean field approximation. The su(3) self-consistent mean field solutions are stable equilibria for rotations of a triaxial body about a short or long principal axis and unstable equilibria for rotations about the middle axis. The wobbling frequencies are determined for the short and long axis cases. Bands of stable equilibrium solutions are found for tilted rotation in a principal plane. The normal modes for tilted rotors are oscillations of the axis lengths together with wobbling off the principal plane. The frequencies for these normal modes are determined.

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I. INTRODUCTION

This article contains three main sections. The Introduction presents an overview of su(3) mean field theory and establishes notation. Section II formulates normal mode theory for su(3) and analyzes the stability of principal axis and tilted su(3) rotors. The Conclusion addresses the physical implications of the mean field method and future applications.

Mean field theory was formulated for the su(3) algebra and applied to the description of rotational bands in two recent articles [1,2]. The first one found rotating equilibrium solutions as the critical points of physically relevant su(3) energy functionals. The second one developed time-dependent su(3) mean field theory and derived the su(3) mean field Hamiltonian from the energy functional. The equilibrium solutions of the dynamical equations of [2] coincide with the critical points calculated in [1]. In this paper the time-dependent theory is used to study solutions in the neighborhood of equilibria.

The su(3) mean field theory provides an approximation to su(3) irreducible representations. The latter are relevant to the oscillator shell model [3] and the interacting boson model [4]. In the mean field approximation, the mathematical analysis of an su(3) irreducible representation, whatever its dimension, involves only operations with 3×3 matrices from the group and the algebra. The physical interpretation of the su(3) mean field results is simpler than that of the su(3) shell model results in part due to the derivation of analytical formulas for the deformation and the energy as functions of the angular momentum.

Mean field and normal mode theories may be formulated as an alternative to shell model irreducible representations for any Lie algebra model of nuclear structure. The algebra u(n) of one-body Hermitian operators in an n-dimensional valence space is the motivating example: a fully antisymmetric irreducible representation of u(n) is a shell model subspace, the mean field theory of u(n) is Hartree-Fock, and normal mode theory for u(n) is the random phase approximation (RPA) [5–7]. In fact, Hartree-Fock and RPA theories are special cases of the general framework for several reasons; e.g., the idempotent densities are in one-to-one correspondence with the Slater determinants and the idempotent densities are a minimal dimension U(n) coadjoint orbit. The su(3) theory does not possess such special properties and, therefore, it is a paradigm for a mean field theory of a general Lie algebra. In particular, the mathematical methods employed for su(3) are applicable to any Lie algebra.

A "state" in su(3) mean field theory is a density matrix defined as a Hermitian traceless 3×3 matrix $\rho = q - \frac{1}{2} i l$, where q is a real symmetric traceless matrix and l is an antisymmetric matrix. The real part q is interpreted as the quadrupole moment expectation and the imaginary part l is the angular momentum expectation. The components of the angular momentum pseudovector $\vec{l}$ are related to the entries of the antisymmetric matrix l via $l_{ij} = \varepsilon_{ijk} l_k$. The dual space su$(3)^*$ of the Lie algebra consists of all such traceless Hermitian density matrices.

The su(3) mean field approximation restricts the model densities to a surface in the dual space su$(3)^*$. The surface may be characterized by two equivalent conditions, either as an orbit of the SU(3) group action or as a level surface of the su(3) Casimirs. Each one of these conditions provides a significant component of the theory.

A group element g in SU(3) transforms a density ρ via the coadjoint action, $\mathrm{Ad}^*_g \rho \equiv g \rho g^{-1}$. The coadjoint orbit $\mathcal{O}_\rho$ of the density ρ consists of ρ and all densities $\mathrm{Ad}^*_g \rho$ as g ranges over the group SU(3). Each orbit contains a unique real diagonal matrix

$$\varrho = \frac{1}{3}\begin{pmatrix} -\lambda+\mu & 0 & 0 \\ 0 & -\lambda-2\mu & 0 \\ 0 & 0 & 2\lambda+\mu \end{pmatrix}, \tag{1}$$

where λ, μ are non-negative real numbers. The surface of the model densities is a coadjoint orbit

$$\mathcal{O}_\varrho = \{\rho = g \varrho g^{-1} \in \mathrm{su}(3)^* \,|\, g \in \mathrm{SU}(3)\}. \tag{2}$$

A coadjoint orbit $\mathcal{O}_\varrho$ is a symplectic manifold or phase space [8–10]. An energy functional $\mathcal{E}[\rho]$ of the density ρ in conjuction with the symplectic structure on a coadjoint orbit determines a unique mean field Hamiltonian $h[\rho]$. For each density ρ in $\mathcal{O}_\varrho$, the mean field Hamiltonian is an element of the su(3) Lie algebra, and, viewed geometrically, $h[\rho]$ is a tangent vector to the coadjoint orbit at ρ. The su(3) mean field Hamiltonian was derived explicitly from the energy functional in [2]. The Hamiltonian dynamics on $\mathcal{O}_\varrho$ is a finite-dimensional Lax system [11]

$$i\frac{d\rho}{dt}=[h[\rho],\rho]. \tag{3}$$

The orbit surface is also a level surface of the su(3) Casimir functions. There are two independent Casimirs $C_2(\rho)=\mathrm{tr}(\rho^2)$ and $C_3(\rho)=\mathrm{tr}(\rho^3)$, which are SU(3) invariants, $C_k(\rho)=C_k(g\rho g^{-1})$, $k=2,3$. The values of the Casimirs on $\mathcal{O}_\varrho$ are

$$\mathcal{C}_2(\varrho)=\frac{2}{3}(\lambda^2+\lambda\mu+\mu^2),$$

$$\mathcal{C}_3(\varrho)=\frac{1}{9}(2\lambda^3+3\lambda^2\mu-3\lambda\mu^2-2\mu^3). \tag{4}$$

A point $\tilde{\rho}=\tilde{q}-\frac{1}{2}iI$ of the dual space for which the quadrupole moment $\tilde{q}=\mathrm{diag}(q_1,q_2,q_3)$ is diagonal represents a density in the principal axis frame. The space $\mathcal{M}_\varrho$ is defined to be the subset of all principal axis densities contained in $\mathcal{O}_\varrho$. The density $\tilde{\rho}$ in the principal axis frame is a point of $\mathcal{M}_\varrho$ if and only if the two Casimir identities are satisfied:

$$\sum_k q_k^2+\frac{1}{2}\vec{I}\cdot\vec{I}=\mathcal{C}_2(\varrho), \tag{5}$$

$$\sum_k q_k^3-\frac{3}{4}\sum_k q_k I_k^2=\mathcal{C}_3(\varrho). \tag{6}$$

The rotation group SO(3) is a subgroup of SU(3). Each SO(3) orbit of a density $\rho=q-\frac{1}{2}il$ contains a principal axis frame density $\tilde{\rho}=\tilde{q}-\frac{1}{2}iI=\mathrm{Ad}_R^*\rho$ in $\mathcal{M}_\varrho$ for some rotation R. The rotation R transforms the laboratory frame density ρ in $\mathcal{O}_\varrho$ into the principal axis frame density $\tilde{\rho}$ in $\mathcal{M}_\varrho\subset\mathcal{O}_\varrho$.

The dynamical system (3) on $\mathcal{O}_\varrho$ determines an equivalent dynamical system on $\mathcal{M}_\varrho$,

$$i\frac{d\tilde{\rho}}{dt}=[h_\Omega[\tilde{\rho}],\tilde{\rho}], \tag{7}$$

where $h_\Omega[\tilde{\rho}]=Rh[\rho]R^T+i\Omega$ is the mean field Routhian and $\Omega=\dot{R}R^T$ is the angular velocity of the principal axis frame. When the mean field Hamiltonian is a polynomial in q and l, the projection to the body-fixed frame is simply $Rh[\rho]R^T=h[\tilde{\rho}]$.

For example, the density in $\mathcal{M}_\varrho$ corresponding to a rotation with angular momentum I about the short axis is

$$\tilde{\rho}=\begin{pmatrix} q_- & 0 & iI/2\\ 0 & -(\lambda+2\mu)/3 & 0\\ -iI/2 & 0 & q_+\end{pmatrix}, \tag{8}$$

where $q_\pm=(\lambda+2\mu)/6\pm\sqrt{\lambda^2-I^2}/2$ for $0\leqslant I\leqslant\lambda$ [1]. This density is a point on the orbit surface $\mathcal{O}_\varrho$ since $\rho=\mathrm{Ad}_g^*\varrho$ for

$$g=\begin{pmatrix} g_{11} & 0 & g_{13}\\ 0 & 1 & 0\\ g_{31} & 0 & g_{33}\end{pmatrix}, \tag{9}$$

where the nonzero entries of the matrix g in SU(3) are

$$\begin{aligned} g_{11}&=\{(\lambda+\sqrt{\lambda^2-I^2})/2\lambda\}^{1/2},\\ g_{13}&=i\{(\lambda-\sqrt{\lambda^2-I^2})/2\lambda\}^{1/2},\\ g_{31}&=iI/\{2\lambda(\lambda+\sqrt{\lambda^2-I^2})\}^{1/2},\\ g_{33}&=I/\{2\lambda(\lambda-\sqrt{\lambda^2-I^2})\}^{1/2}. \end{aligned} \tag{10}$$

Alternatively, the density $\tilde{\rho}$ is proven to be a point of $\mathcal{M}_\varrho$ by verifying the Casimir equations (5) and (6). When the energy functional is

$$\mathcal{E}[\tilde{\rho}]=A_1I_1^2+A_2I_2^2+A_3I_3^2, \tag{11}$$

where A_k are real constants, the mean field Routhian $h_\Omega[\tilde{\rho}]$ vanishes for rotation about one principal axis [2]. The density $\tilde{\rho}$ of Eq. (8) is a rotating equilibrium solution because the time rate of change $d\tilde{\rho}/dt$ vanishes.

II. LINEARIZED EQUATIONS OF MOTION

In this section the linearized equations of motion on $\mathcal{M}_\varrho$ are derived. For triaxial systems, applications are made to rotations about the three principal axes and to tilted rotations in a principal plane. The stability of each equilibrium rotating density is determined. For the stable normal modes, formulas for the oscillation frequencies are reported.

Suppose the principal frame density $\tilde{\rho}^{(0)}$ in $\mathcal{M}_\varrho$ is rotating in equilibrium with constant angular velocity $\Omega^{(0)}$. This density is a self-consistent solution to the equation

$$[h_{\Omega^{(0)}}[\tilde{\rho}^{(0)}],\tilde{\rho}^{(0)}]=0. \tag{12}$$

In a neighborhood of the equilibrium point in $\mathcal{M}_\varrho$, the density and angular velocity are perturbed to $\tilde{\rho}(t)=\tilde{\rho}^{(0)}+\epsilon\delta\tilde{\rho}(t)$ and $\Omega(t)=\Omega^{(0)}+\epsilon\delta\Omega(t)$, respectively, where ϵ is a small real number. The mean field Hamiltonian is approximated in a Taylor series expansion by $h[\tilde{\rho}]=h[\tilde{\rho}^{(0)}]+\epsilon\delta h[\tilde{\rho}^{(0)};\delta\tilde{\rho}]$. The linearization of the equation of motion (7) yields the dynamical equation

$$i\frac{d}{dt}\delta\tilde{\rho}=[\delta h[\tilde{\rho}^{(0)};\delta\tilde{\rho}]+i\,\delta\Omega,\tilde{\rho}^{(0)}]+[h_{\Omega^{(0)}}[\tilde{\rho}^{(0)}],\delta\tilde{\rho}]. \tag{13}$$

The perturbation $\delta\Omega$ of the angular velocity is chosen so that $\tilde{\rho}(t)$ is in the principal axis frame; i.e., the imaginary parts of the off-diagonal components of the right-hand side of Eq. (13) are zero.

The mean field Hamiltonian $h[\tilde{\rho}]$ in the principal axis frame is derived from the energy functional (11) in [2]. Let

q_k and I_k denote the Cartesian components of the quadrupole moment and angular momentum of the equilibrium principal axis frame density $\tilde{\rho}^{(0)}$. The corresponding Cartesian components of the perturbed density $\delta\tilde{\rho}$ are denoted by δq_k and δI_k. The matrix elements of the linearized mean field Hamiltonian for the functional (11) are

$$\delta h[\tilde{\rho}^{(0)};\delta\tilde{\rho}]_{ij}=\frac{A_i-A_j}{q_i-q_j}(I_i\delta I_j+I_j\delta I_i)-2i\epsilon_{ijk}A_k\delta I_k - \frac{A_i-A_j}{(q_i-q_j)^2}I_iI_j(\delta q_i-\delta q_j) \tag{14}$$

for $i\neq j$, while the diagonal components δh_{ii} are zero.

A. Rotation about a principal axis

The linearized equations simplify for rotation about a principal axis, since two of the angular momentum components and the mean field Routhian vanish at the equilibrium principal axis density. Thus, the second commutator on the right hand side of Eq. (13) and the third term on the right hand side of Eq. (14) are zero. Using Eq. (14) and the equilibrium densities reported in [2], the linearized dynamical equations (13) result in no vibration of the axis lengths and a proportionality between the perturbed angular velocity and the perturbed angular momentum components:

$$\frac{d}{dt}\delta q_k=0,$$

$$\delta\Omega_k=2A_k\delta I_k. \tag{15}$$

The proportionality relation implies that the perturbed angular velocity components obey the Euler dynamical equations [2]. The linearized dynamical equations for the perturbed angular momentum components depend on whether the rotation is about the short, long, or middle axis.

1. Short axis rotation

The equilibrium density for the short axis rotation is given in Eq. (8). In this case, $I_2=I$, $I_1=I_3=0$, $\Omega_2=2A_2I_2$, and the linearized equations for the angular momenta are

$$\frac{d}{dt}\delta I_1=(A_3-A_2)u\,\delta I_3,$$

$$\frac{d}{dt}\delta I_2=0, \tag{16}$$

$$\frac{d}{dt}\delta I_3=-(A_1-A_2)u\,\delta I_1,$$

where

$$u=\frac{8\mu(\lambda+\mu)I}{4\mu(\lambda+\mu)+I^2}. \tag{17}$$

For $I=0$ or $\mu=0$ the density is static because $u=0$. When A_1 and A_3 are bigger than A_2, there is a harmonic wobbling about the equilibrium rotation axis with frequency

$$\omega=\frac{2I\sqrt{(A_1-A_2)(A_3-A_2)}}{1+\dfrac{I^2}{4\mu(\lambda+\mu)}},\quad 0<I\leqslant\lambda. \tag{18}$$

The numerator of this formula is the wobbling frequency predicted for asymmetric rotors with large I by Bohr and Mottelson [12]. The denominator is a correction factor imposed by the su(3) model.

2. Long axis rotation

When the body rotates about its long principal axis, the result is similar to the short axis case. With the convention of Eq. (1), the rotation is about the third axis. If A_1 and A_2 are smaller than A_3, the wobbling frequency is

$$\omega=\frac{2I\sqrt{(A_3-A_1)(A_3-A_2)}}{1+\dfrac{I^2}{4\lambda(\lambda+\mu)}},\quad 0<I\leqslant\mu. \tag{19}$$

For $I=0$ or $\lambda=0$ the density is static.

3. Middle axis rotation

Consider a rotation about the middle or one-axis. The dynamical equations for the perturbed angular momentum components are

$$\frac{d}{dt}\delta I_1=0,$$

$$\frac{d}{dt}\delta I_2=(A_1-A_3)v\,\delta I_3, \tag{20}$$

$$\frac{d}{dt}\delta I_3=(A_2-A_1)v\,\delta I_2,$$

where

$$v=\frac{8\lambda\mu I}{4\lambda\mu-I^2},\quad 0\leqslant I<\sqrt{4\lambda\mu}. \tag{21}$$

For middle axis rotation, A_1 is bracketed between A_2 and A_3. Thus, the equilibrium density is an unstable hyperbolic point.

B. Tilted rotation

Suppose the rotation axis is tilted into the 1-3 principal plane and $I_2=0$ [13,14]. The rotating equilibrium densities in $\mathcal{M}_\varrho$ are parametrized by the real roots of an eighth-degree polynomial, Eq. (4.9) of [1]. In terms of a root q_1 there are formulas for the other quadrupole moment components q_2,q_3 and the nonzero angular momentum components I_1,I_3, Eqs. (4.10) and (4.11) of [1]. The tilted equilibrium densities are independent of the inertial parameters A_k,

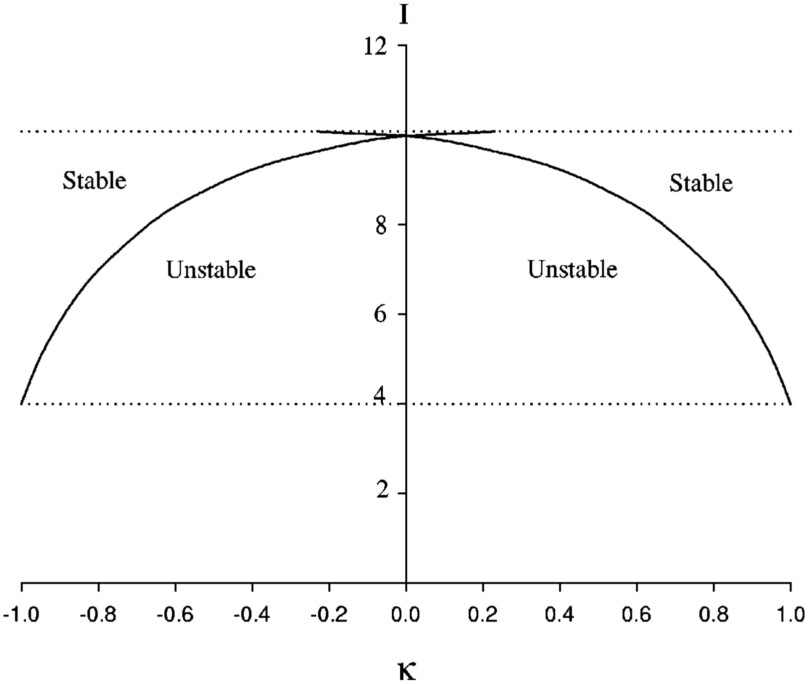

FIG. 1. For tilted asymmetric su(3) rotors the regions of stability are shown in the I-κ plane, where I is the angular momentum and κ is an inertial parameter.

$$\tilde{\rho}=\begin{pmatrix} q_1 & -iI_3/2 & 0 \\ iI_3/2 & q_2 & -iI_1/2 \\ 0 & iI_3/2 & q_3 \end{pmatrix}. \tag{22}$$

In equilibrium the angular velocity vector is aligned with the angular momentum vector, $\Omega^{(0)}=AI$, where the reciprocal of the moment of inertia is

$$A=\frac{(A_3-A_1)(I_3^2-I_1^2)}{2(q_3-q_1)^2}+2\frac{[A_1(q_3-q_2)-A_3(q_1-q_2)]}{(q_3-q_1)}. \tag{23}$$

The Routhian $h_{\Omega^{(0)}}[\tilde{\rho}^{(0)}]$ is a Hermitian matrix with zero diagonal entries; its matrix elements in the upper triangular block are

$$h_{\Omega^{(0)}}[\tilde{\rho}^{(0)}]_{ij}=\left(\frac{A_3-A_1}{q_3-q_1}\right)\times\begin{cases} iI_3u, & (ij)=(12), \\ I_1I_3, & (ij)=(13), \\ iI_1v, & (ij)=(23), \end{cases} \tag{24}$$

where

$$u=\frac{[4(q_1-q_3)(q_2-q_3)-I^2]}{6q_2},$$

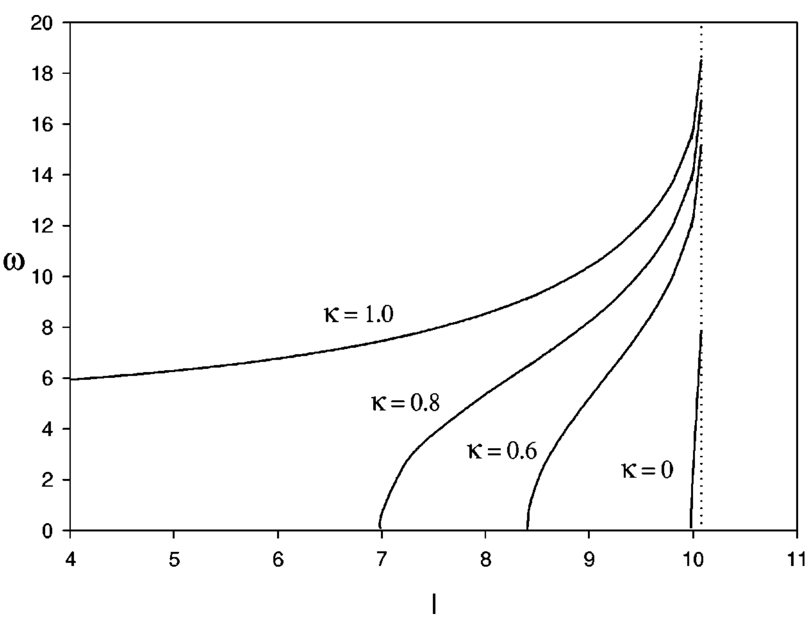

FIG. 2. For tilted asymmetric su(3) rotors, the wobbling-vibrational frequency ω, in units of $|A_1-A_3|$, is plotted versus the angular momentum I for various values of the inertial parameter κ.

$$v=\frac{[4(q_2-q_1)(q_3-q_1)-I^2]}{6q_2}. \tag{25}$$

The linearized system (13) shows that the angular momentum components in the principal plane are stationary,

$$\frac{d}{dt}\delta I_1=\frac{d}{dt}\delta I_3=0. \tag{26}$$

But there is a wobbling off the principal plane coupled to a vibration of the axis lengths:

$$\frac{d}{dt}(\delta q_1-\delta q_2)=3I_1I_3q_3r\,\delta I_2/p,$$

$$\frac{d}{dt}(\delta q_2-\delta q_3)=3I_1I_3q_1r\,\delta I_2/p,$$

$$\frac{d}{dt}\delta I_2=(A_3-A_1)I_1I_3/(6q_2p)\times[s(\delta q_1-\delta q_2)+t(\delta q_2-\delta q_3)], \tag{27}$$

where

$$p=(q_1-q_2)(q_1-q_3)(q_2-q_3),$$

$$r=(q_3-q_1)(A_1-A_2)+(q_2-q_1)(A_3-A_1),$$

$$s=(q_2-q_3)(4q_2^2-32q_1q_2-8q_1^2+I^2),$$

$$t=(q_2-q_1)(4q_2^2-32q_2q_3-8q_3^2+I^2). \tag{28}$$

One of the eigenvalues of the linearized system (27) is zero. The square of the nonzero wobbling-vibrational frequency is

$$\omega^2=(A_1-A_3)^2I_1^2I_3^2[\kappa(q_3-q_1)+3q_2]/(q_2p^2)\times[(q_3-q_2)\times(q_2^2-8q_1q_2-2q_1^2)+(q_1-q_2)(q_2^2-8q_2q_3-2q_3^2)-3q_2I^2/4], \tag{29}$$

where the asymmetry parameter [15]

$$\kappa=(2A_2-A_1-A_3)/(A_1-A_3). \tag{30}$$

When ω^2 is positive, the wobbling-vibrational normal mode is stable.

As an illustration, consider the $(\lambda,\mu)=(8,4)$ coadjoint orbit. There is a band of tilted triaxial equilibrium densities beginning with a noncollective prolate spheroid at $I=\mu=4$ and ending at $I\approx 10.08$; see Table 4 of [1]. In Fig. 1 the zones of stability in the I-κ plane are shown. For the stable normal modes, a plot of the frequency in units of $|A_1-A_3|$ versus the angular momentum I is drawn in Fig. 2 for several values of the inertial parameter κ. When κ is less than unity, the band of tilted equilibrium rotors consists of both stable and unstable configurations. For example, if κ equals 0.8, then the rotational band densities from $I=4$ to about $I=7$ are unstable. At $I\approx 7$, a band of stable equilibrium states emerges and continues up to $I\approx 10.08$.

III. CONCLUSION

The normal mode theory of su(3) achieves a clear physical picture of vibrational and wobbling motion of a rotating nucleus in the su(3) approximation. Indeed the su(3) densities corresponding to the wobbling and vibrational collective modes are determined in this article as explicit functions of the time. The analytic formulas for the normal mode frequencies give the energies of the collective excitations. The derivation of the RPA from time-dependent Hartree-Fock theory has a similar physical appeal [16].

In a shell model description of rotational bands, the existence of su(3) dynamical symmetry imposes the stringent requirement that the wave functions of band members be vectors from a single irreducible representation of su(3). For some light deformed nuclei, the amplitude of such wave functions can be concentrated in a single leading irreducible representation. But in medium mass and heavy deformed nuclei, the wave function is expected to be a superposition of vectors from many irreducible su(3) representations. The mixing is caused primarily by spin-orbit and pairing forces which break su(3) symmetry. Nevertheless, the measured dependence of the quadrupole deformation of band members on the angular momentum may be consistent with su(3) dynamical symmetry. Although the experimental evidence does not imply that the wave functions belong to a single irreducible representation, it does mean that the su(3) densities of band members share a common value for the su(3) quadratic Casimir function. Thus, even when su(3) shell model dynamical symmetry is broken strongly, the fundamental ansatz of su(3) mean field theory can be verified experimentally. Similarly the normal mode theory for su(3) may apply for the su(3) density even though the wave functions of collective vibrational and wobbling states are not drawn from a single irreducible representation space.

In future investigations, the mean field method and its associated normal mode theory will be applied to other Lie algebra nuclear structure models. Because the dimension of a Lie algebra's dual space of densities equals the dimension of the algebra, the mean field and normal mode approximations are expected to be especially useful in applications to high- or infinite-dimensional representations.

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